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Exact Metric for a Nonrotating Mass with a Quadrupole Moment*

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Erez and Rosen have given a method for calculating an exact static metric for a nonrotating, cylindrically symmetric mass distribution possessing multipole moments of arbitrary order; however, the explicit expressions they give for the metric components arising from a combination of monopole and quadrupole moments contain errors. In this paper, we sketch the Erez-Rosen method and then state the corrected form of the metric for this experimentally significant case of a mass distribution with monopole and quadrupole moments.

I. INTRODUCTION

RECENTLY proposed experimental tests of general relativity require a knowledge of the predicted orbit of a particle in the gravitational field of a body possessing a quadrupole moment.[1,2] The orbital calculations have been carried out either by treating the problem as a Newtonian one except for the inclusion of a Schwarzschild precession,[2] or by calculating an approximate nonspherical metric from a set of linearized gravitational equations.[3] It seems to have been largely overlooked that Erez and Rosen[4] have previously given a method for calculating an *exact* metric for a body possessing a multipole moment of a given order, and, in particular, have given expressions for an exact metric for a body possessing only monopole and quadrupole moments. Unfortunately, the expressions given by Erez and Rosen for the metric for this latter case contain errors, and lead to incorrect asymptotic behaviors for certain metric components. In this paper we first summarize the Erez-Rosen technique for calculating the metric for a body with a multipole moment, since their method seems to be little known. We then give the correct form of the metric obtained by the Erez-Rosen technique for the case of a body possessing both monopole and quadrupole moments, for apparently the proper statement of this metric does not now appear in the literature.

II. METHOD OF EREZ AND ROSEN

Weyl and Levi-Civita[5] have shown that static, axially symmetric solutions of the Einstein field equations for empty space can be obtained by writing the line element in cylindrical coordinates as

$$ds^2=e^{2\psi}c^2dt^2-e^{2\gamma-2\psi}(d\rho^2+dz^2)-\rho^2e^{-2\psi}d\phi^2\,, \qquad (1)$$

where ψ and γ satisfy the equations

$$\psi_{\rho\rho}+(1/\rho)\psi_\rho+\psi_{zz}=0\,, \qquad (2)$$

$$\gamma_\rho=\rho(\psi_\rho{}^2-\psi_z{}^2),\quad \gamma_z=2\rho\psi_\rho\psi_z\,. \qquad (3)$$

By changing to the prolate ellipsoidal coordinates

$$\lambda=(r_++r_-)/2m,\quad \mu=(r_+-r_-)/2m\,,$$

where

$$r_\pm{}^2=\rho^2+(z\pm m)^2\,,$$

and m is for the moment simply a positive constant, Erez and Rosen transform the equations for ψ and γ to the form

$$[(\lambda^2-1)\psi_\lambda]_\lambda+[(1-\mu^2)\psi_\mu]_\mu=0\,, \qquad (4)$$

$$\begin{aligned}\gamma_\lambda&=[(1-\mu^2)/(\lambda^2-\mu^2)]\\&\quad\times[\lambda(\lambda^2-1)\psi_\lambda{}^2-\lambda(1-\mu^2)\psi_\mu{}^2-2\mu(\lambda^2-1)\psi_\lambda\psi_\mu]\,,\\ \gamma_\mu&=[(\lambda^2-1)/(\lambda^2-\mu^2)]\\&\quad\times[\mu(\lambda^2-1)\psi_\lambda{}^2-\mu(1-\mu^2)\psi_\mu{}^2+2\lambda(1-\mu^2)\psi_\lambda\psi_\mu]\,.\end{aligned} \qquad (5)$$

Equation (4) is separable, and if one puts

$$\psi=\Lambda(\lambda)M(\mu)\,,$$

then one finds that Λ and M must satisfy Legendre equations in their respective variables. The product solutions yielding a space-time that is flat at infinity are then of the form

$$\psi_l=\text{const}\;P_l(\mu)Q_l(\lambda),\quad l=0,1,2,\cdots \qquad (6)$$

where P_l is the lth Legendre polynomial and Q_l is the lth Legendre function of the second kind. The general solution to Eq. (4) describing a space-time which is flat at infinity can be constructed as a linear combination of the solutions (6), and Eqs. (5) then yield the corresponding γ by direct but tedious integration. Finally, Erez and Rosen make the second change of coordinates

$$r=m(\lambda+1),\quad \theta=\cos^{-1}\mu\,. \qquad (7)$$

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[1] See, e.g., R. H. Dicke, Astron. J. **70**, 395 (1965); I. Goldberg, Phys. Rev. **149**, 1010 (1966); I. Goldberg and R. Madey, Nuovo Cimento **41**, 1 (1966); I. Goldberg and E. Marx, *ibid.* **44**, 91 (1966); J.-P. Richard, Astronaut. Acta **13**, 353 (1967).
[2] R. H. Dicke, Nature **202**, 432 (1964).
[3] H. G. L. Krause, in *The Use of Artificial Satellites for Geodesy*, edited by G. Veis (North-Holland Publishing Co., Amsterdam, 1963), p. 69; N. Rosen and H. Shamir, Rev. Mod. Phys. **29**, 429 (1957), also calculate an approximate metric.
[4] G. Erez and N. Rosen, Bull. Res. Council Israel **8F**, 47 (1959).
[5] H. Weyl, Ann. Physik **54**, 117 (1918); **59**, 185 (1919); T. Levi-Civita, Rend. Accad. Nazl. Lincei **28**, 3 (1919).

In terms of the new variables one finds that

$$ds^2=e^{2\psi}c^2dt^2-e^{2\gamma-2\psi}\left[\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)dr^2+(r^2-2mr+m^2\sin^2\theta)d\theta^2\right]-e^{-2\psi}(r^2-2mr)\sin^2\theta d\phi^2. \quad (8)$$

If ψ is now taken to be of the form (6) with $l=0$, say

$$\psi_0=\tfrac{1}{2}\ln[(\lambda-1)/(\lambda+1)]=-P_0(\mu)Q_0(\lambda),$$

then one easily calculates that

$$\gamma_0=\tfrac{1}{2}\ln[(\lambda^2-1)/(\lambda^2-\mu^2)],$$

and the corresponding metric components determined from (8) are just those of the Schwarzschild field with $2m$ being the Schwarzschild radius. More generally, Erez and Rosen point out that if ψ is taken in the form

$$\psi=\psi_0+q_l\psi_l, \quad (9)$$

then from the asymptotic properties of the Q_l one finds that, for large r,

$$g_{00}=e^{2\psi}=1+2\left(-\frac{m}{r}-\mathfrak{M}_l\frac{P_l(\cos\theta)}{r^{l+1}}+O(r^{-l-2})\right),$$

where $\mathfrak{M}_l$ is related to q_l and the Newtonian gravitational constant G by

$$q_l=\frac{(2l+1)!!}{l!}\frac{G\mathfrak{M}_l}{c^2m^{l+1}}.$$

Thus, in the Newtonian limit, the gravitational field is that of a mass possessing a multipole moment of order l.

III. METRIC FOR A MASS POSSESSING A QUADRUPOLE MOMENT

If one chooses $l=2$ in Eq. (9),[6] then ψ can be written (with $q_2\equiv q$)

$$\psi=-P_0(\mu)Q_0(\lambda)-qP_2(\mu)Q_2(\lambda)$$
$$=\tfrac{1}{2}\ln\frac{\lambda-1}{\lambda+1}+q\tfrac{1}{2}(3\mu^2-1)\times\left[\tfrac{1}{4}(3\lambda^2-1)\ln\left(\frac{\lambda-1}{\lambda+1}\right)+\tfrac{3}{2}\lambda\right]. \quad (10)$$

By substituting this form for ψ into Eqs. (5), integrating, and choosing the constant of integration so that γ tends to zero for large λ, one finds

$$\gamma=\tfrac{1}{2}\ln\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)+q\left\{\ln\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)+(\mu^2-1)\left[\tfrac{3}{2}\lambda\ln\left(\frac{\lambda-1}{\lambda+1}\right)+3\right]\right\}$$
$$+q^2\left\{\tfrac{1}{2}\ln\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)+(1-\mu^2)\left[\frac{9}{64}(\lambda^4-2\lambda^2+1)\ln^2\left(\frac{\lambda-1}{\lambda+1}\right)+\tfrac{3}{16}(3\lambda^3-5\lambda)\ln\left(\frac{\lambda-1}{\lambda+1}\right)+\tfrac{3}{16}(3\lambda^2-4)\right]\right.$$
$$\left.+\mu^2(\mu^2-1)\left[\frac{9}{64}(9\lambda^4-10\lambda^2+1)\ln^2\left(\frac{\lambda-1}{\lambda+1}\right)+\frac{9}{16}(9\lambda^3-7\lambda)\ln\left(\frac{\lambda-1}{\lambda+1}\right)+\frac{9}{16}(9\lambda^2-4)\right]\right\}. \quad (11)$$

Expression (11) differs from the result of Erez and Rosen in several of the numerical coefficients. The γ given in their paper is, because of its incorrect numerical coefficients, not actually a solution of Eqs. (5). It has rather drastically erroneous asymptotic behavior, and leads, for instance, to orbital equations in which the effect of the quadrupole moment does not approach the Newtonian quadrupole moment effect even in the limit of weak fields and small quadrupole moments. The corrected form of γ given in Eq. (11) eliminates these difficulties.[7]

Because of the complexity of the expressions for ψ and γ, it is useful to expand the q-dependent parts of these quantities in inverse powers of r. One can easily verify that

$$\psi=\tfrac{1}{2}\ln\left(1-\frac{2m}{r}\right)-qP_2(\cos\theta)\left(\frac{2}{15}\frac{m^3}{r^3}+\frac{2}{5}\frac{m^4}{r^4}+\frac{32}{35}\frac{m^5}{r^5}+\frac{40}{21}\frac{m^6}{r^6}+\cdots\right), \quad (12)$$

$$\gamma=\tfrac{1}{2}\ln\left(\frac{r^2-2mr}{r^2-2mr+m^2\sin^2\theta}\right)+q[\tfrac{1}{10}(5\cos^4\theta-6\cos^2\theta+1)m^4/r^4$$
$$+\tfrac{2}{5}(5\cos^4\theta-6\cos^2\theta+1)m^5/r^5+(1/21)(7\cos^6\theta+105\cos^4\theta-135\cos^2\theta+23)m^6/r^6+\cdots]$$
$$+q^2[(1/150)(25\cos^6\theta-39\cos^4\theta+15\cos^2\theta-1)m^6/r^6+\cdots]. \quad (13)$$

[6] The $l=1$ (dipole) contribution can be eliminated by an appropriate choice of the origin of the coordinate system.

[7] *Note added in proof*. This corrected form of γ has also been obtained recently by J. Winicour, A. Janis, and E. Newman, Phys. Rev. **176**, 1507 (1968). We thank Professor Janis for bringing this fact to our attention.

By using Eqs. (12) and (13), one can easily see that the metric components g_{00} and g_{11} may be written in the form

$$g_{00}=\left(1-\frac{2m}{r}\right)\left\{1-qP_2(\cos\theta)\left(\frac{4}{15}\frac{m^3}{r^3}+\frac{4}{5}\frac{m^4}{r^4}+\frac{64}{35}\frac{m^5}{r^5}+\frac{80}{21}\frac{m^6}{r^6}+\cdots\right)+q^2[P_2(\cos\theta)]^2\left(\frac{8}{225}\frac{m^6}{r^6}+\cdots\right)+\cdots\right\} \quad (14)$$

and

$$g_{11}=-\left(1-\frac{2m}{r}\right)^{-1}\left\{1+\frac{4}{15}q\frac{m^3}{r^3}P_2(\cos\theta)\right.$$
$$+q\left[\tfrac{1}{5}(5\cos^4\theta-1)\frac{m^4}{r^4}+\frac{4}{35}(35\cos^4\theta-18\cos^2\theta-1)\frac{m^5}{r^5}+\frac{2}{21}(7\cos^6\theta+105\cos^4\theta-75\cos^2\theta+3)\frac{m^6}{r^6}+\cdots\right]$$
$$\left.+q^2\left[\frac{1}{225}(75\cos^6\theta-99\cos^4\theta+33\cos^2\theta-1)\frac{m^6}{r^6}+\cdots\right]+\cdots\right\}. \quad (15)$$

Equation (14) shows that the quadrupole moment $Q(=\mathfrak{M}_2)$ of the mass distribution is related to q by the equation $q=\frac{15}{2}GQ/c^2m^3$. Expressions for g_{22} and g_{33} may easily be determined from those for g_{00} and g_{11}, so they will not be stated explicitly.

Because of their incorrect expression for γ, the expansion for g_{11} in inverse powers of r given by Erez and Rosen differs markedly from Eq. (11). Their expansion contains terms proportional to q and q^2 which fall off only as r^{-2} for large r, and lead to much too large a dependence of the metric on q at large distances. The approximate metric given by Krause,[3] on the other hand, is found by comparison with Eqs. (14) and (15) to be correct within its range of validity (through terms in r^{-3}).[8]

[8] Even though Krause's treatment of the effect of the quadrupole moment on the metric is only approximate, while that given here is exact, it should be noted that Krause's calculation takes into account the possibility that the mass whose field is being considered is rotating. The results given here are valid only for nonrotating masses.

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Presymmetry. II*

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The association between laboratory procedures and self-adjoint operators (observables), usually implicit and allusive, is made explicit by the introduction of two collections of nonmathematical objects; the observation procedures and the state-preparing procedures. By a process of idealization and extrapolation from empirical facts, these two collections are turned into mathematical sets ("hardware spaces") with the structure of a star algebra $\mathcal{O}$ and a convex linear set $\mathcal{S}$, respectively. There is a many-one mapping from $\mathcal{O}$ into the space $\mathfrak{A}$ of observables (the operators on Hilbert space), and physical laws or solutions of equations of motion are embodied in the mapping $\Phi:\mathcal{O}\to\mathfrak{A}$. Certain material motions of observation instruments and state-preparing instruments induce automorphisms of the hardware spaces but in general, not automorphisms of $\mathfrak{A}$. An extension of space-time invariance theory to accelerations $[\mathbf{x}\to\mathbf{x}+\sum_2^\infty(n!)^{-1}\mathbf{r}_n t^n]$, to external symmetry-breaking fields, and to subsystems under the influence of other subsystems becomes possible. The main results of this paper are a derivation of Newton's second law and of the gravitational equivalence principle for nonrelativistic quantum mechanics from invariance and causality principles.

1. INTRODUCTION AND GENERAL ASSUMPTIONS

The purpose of this study is to extend the powerful considerations of physical invariance to systems that are not free, but instead are influenced by external fields, and to subsystems influenced by fields created by other subsystems.

An example of the kind of question that such a theory can answer is: Why do the observables of a single particle in an external field (or the observables of a single distinguishable particle in a many-body system with or without external field) satisfy the canonical commutation relations? A preliminary answer to this question was given in a previous paper.[1] The present paper reformulates and extends the assumptions and attempts to

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[1] H. Ekstein, Phys. Rev. 153, 1397 (1967).