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Space-Time Code*

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It is known that the entire geometry of many relativistic space-times can be summed up in two concepts, a space-time measure μ and a space-time causal or chronological order relation C, defining a causal measure space. On grounds of finiteness, unity, and symmetry, we argue that macroscopic space-time may be the classical-geometrical limit of a causal quantum space. A tentative conceptual framework is provided. Mathematical individuals that naturally form causal spaces are symbol sets or words, taken in the order of their generation. The natural extension of this purely logical concept to quantum symbols is formulated. The problem is posed of giving finite quantum rules for the generation of quantum symbol sets such that the order of generation becomes, in the classical limit, the causal order of space-time—as it were, to break the space-time code. The causal quantum spaces of three simple codes are generated for comparison with reality. The singulary code (repetitions of one digit) gives a linearly ordered external world of one time dimension and a circular internal space. The binary code gives the future null cone of special relativity and a circular internal space. The causal quantum space of word pairs in the binary code gives the solid light cone $t>(x^2+y^2+z^2)^{1/2}$ of special relativity and an internal space $U(2,C)$ suitable for the description of charge and isospin. In the classical limit, there is full translational and proper Lorentz invariance except at the boundary of the light cone, where the classical-geometrical limit fails. Plausible consequences of this model for cosmology and elementary particles are discussed. There is a quantum of time $\tau \lesssim \hbar/m_\mu c^3$ and a space-time complementarity relation $\Delta t\Delta x\Delta y\Delta z \gtrsim \tau^4$.

I. INTRODUCTION

UNTIL we find a satisfactory theory of space-time structure, we shall be beset by the dilemma of the discrete versus the continuous, the dilemma already posed by Riemann,[1] in much the following terms:

(a) A discrete manifold has finite properties, whereas a continuous manifold does not. Natural quantities are to be finite. The world must be discrete.

(b) A discrete manifold possesses natural internal metrical structure, whereas a continuous manifold must have its metrical structure imposed from without. Natural law is to be unified. The world must be discrete.

(c) A continuous manifold has continuous symmetries, whereas a discrete manifold does not. Nature possesses continuous symmetries. The world must be continuous.

The third argument is especially serious for rotational and Lorentz symmetry, which are much more difficult to counterfeit than translational symmetry. Subgroups can be found as dense as desired in the translation group that are not everywhere dense, but I do not think they exist for the rotation or Lorentz groups.

Since Riemann a new approach to this dilemma has become available. The same question about matter, asked for two millenia—Is it continuous or is it discrete? —has at last been answered in this century: No. Matter is made neither of discrete objects nor waves but of quanta. In most familiar terms, a quantum is an object whose coordinates form a noncommutative algebra. Most fundamentally put, a quantum is an object whose class calculus[2,3] is neither a discrete nor a continuous Boolean algebra, but an algebra which is not even Boolean, being nondistributive. This nondistributive class calculus is the lattice of subspaces of a separable Hilbert space, and is naturally imbedded in (and defines) the algebra of operators on that Hilbert space. A quantum manifold is a third possibility for space-time too. This possibility would pass us cleanly between the horns of Riemann's dilemma:

(a) A quantum manifold, like a discrete one, has better convergence than a continuous manifold—remember Planck and the black body.

(b) A quantum manifold, like a discrete one, is born with internal structure and is even more unified, being coherent.[2]

(c) A quantum manifold, like a continuous one, possesses continuous symmetry groups.

The intrinsic structure Riemann meant for a discrete manifold must have been like a chessboard or honeycomb, a tesselation or graph in which the germs of a topology and a metric are present in the concepts of incidence and number. The world he faced was one three-dimensional continuum, space, changing in

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[1] B. Riemann, in *Collected Works* [1892], edited by H. Weber (Dover Publications, Inc., New York, 1953), p. 285.

[2] For the algebra of quantum classes, see J. von Neumann, in *Mathematical Foundations of Quantum Mechanics* (Dover Publications, Inc., New York, 1936), Chap. III, Sec. 5; J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1968). The first person I know of to suggest that space-time is a quantum space is H. Snyder, Phys. Rev. **71**, 38 (1947). Commutation relations of Snyder's form $[x^\mu,x^\nu]=\text{const}\times J^{\mu\nu}$ are valid for our x^μ, but there must be a difference in sign somewhere: We get a discrete time and a continuous space; he gets the reverse. A deep scepticism concerning the continuum nature of space-time is again expressed by R. P. Feynman, *The Character of Physical Laws* (M.I.T. Press, Cambridge, Mass., 1968).

[3] The language of quantum logic still varies from speaker to speaker, so we summarize the local dialect in the Appendix.

another one-dimensional continuum, time. Here, too, we have a better point of departure than Riemann. Since Einstein we have been confronted by just one four-dimensional continuum. More important, the structure of this continuum is not that of a changing metric space but that of a space with an order relation between its points, causal or chronological precedence C. Alas, most of classical geometry ran off on the wrong road after dimension 1, from the point of view of relativity. The number line has been many things to many people: a metric geometry, a number field, an ordered space, and so forth. Any of these exist in higher dimension. The development of higher-dimensional metric geometry flourished in the recognition that the world is a kind of higher-dimensional line and in the mistaken belief that the essential surviving property of the line is its metric structure. The important thing about the line for us is its order. The world *is* like a line, but in respect to the order structure, not the metric structure. For example, the topology of space-time must be based on order intervals $aCxCb$, not metric balls $d(x,a)<r$. Points at 0 pseudometric separation can be as far apart topologically as the stars. Evidently the relation pCp' (p is causally prior to p') is a simpler thing than an indefinite numerical distance function $d(p,p')$. We are unlikely to find an indefinite metric by counting squares on a space-time chessboard, and we are much better off hunting for structures that are born with order. The causal order C determines the conformal structure of space-time, or nine of the ten components of the metric. The measure on space-time fixes the tenth component.

The simplest mathematical objects I can think of that are born with order and measure are composite objects, sets of one kind or another ordered by inclusion and measured by counting. This leads to the idea that each point of space-time is some kind of assembly of some kind of thing, and a point that is later is a point that is greater, regarded as such an assembly. The things have to be quanta if they are to be finite in kind, yet possess continuous symmetries, and we could just as well call them chronons, since their creation is to be the passage of time. However, they should not be ascribed mass or other mechanical properties, which are to emerge in a higher order of things, and it is safest to call them "digits." This reminds us of their abstract quality, forestalls meaningless questions, and implies that their disjoint kinds are finite, like the binary or decimal digits. The basic object of the ordered quantum space—a quantum set of quantum digits—is then called a quantum "word." We set the problem of "breaking the space-time code": finding finite quantum rules for word generation such that the order of generation gives the causal order of space-time and thus the entire geometrical structure of space-time, in the classical limit. Since Euclid, classical geometry has been a development in classical logic, and here we attempt a quantum geometry which is a development in quantum logic; as it were, geometrizing quanta rather than quantizing geometry.

This approach, which attributes to space-time points an intricate internal structure, seems upside down from the point of view of general relativity, and general relativity seems upside down from here, seems too complicated a theory of too simple a thing to be fundamental rather than phenomenological. What is too complicated about general relativity is the delicate vertical structure of laws that would have had to be legislated on the first day of creation: set theory holding up topology, holding up differential manifolds, holding up pseudometric geometry, with a precarious topper of quantization. What is too simple about general relativity is the space-time point. It looks as if a point might be an enormously complicated thing. Each point, as Feynman once put it, has to remember with precision the values of indefinitely many fields describing indefinitely many elementary particles; has to have data inputs and outputs connected to neighboring points; has to have a little arithmetic element to satisfy the field equations; and all in all might just as well be a complete computer. Maxwell made his machine out of gears and idlers, Feynman is inclined towards digital rather than analog components, and we attempt the synthesis of quantum automata out of quantum elements. But the laws of these complex structures should be simple.

The code seems as if it might be simple. If it were much simpler, there would be no space-time, just a one-dimensional time continuum. We all knew that the Lorentz group was as simple as could be—had Cartan's A_1 rating, so to spead—and presumably we were puzzled by the less compelling nature of the space-time signature $+1-3$, which is supposed to be theoretically prior. Perhaps here is the real reason for the dimension and signature of space-time. Working out the causal order of three simple codes, the first (words in the singulary code, repetitions of a single digit) gives the linear space-time, the second (words in the binary code) already gives the future null cone $t=+(x^2+y^2+z^2)^{1/2}$, and the third (word pairs in the binary code, admittedly chosen with the first two examples as a guide) gives the solid light cone $t\geq+(x^2+y^2+z^2)^{1/2}$, with the Poincaré group as symmetry group as long as we stay away from the bounding cone where the classical limit breaks down and normal conceptions of space-time fail.

II. ORDER OF SPACE-TIME

The classical space-times of special and much of general relativity may each be described completely by a measure space M and a partial ordering pCp' (p "causes" p', i.e., causally precedes p') of the points

p, p' of M.[4] The measure space gives the set theory (class lattice) and measure theory of space-time and finally determines $\sqrt{(-g)}$. The causal relation pCp' means that p' is in the closed future light cone emanating from p and gives the topology, differential manifold structure, and conformal geometry of space-time, finally determining $g_{\mu\nu}$. Following Penrose, we call a space with a causal order a causal space.

Here we consider new space-times likewise endowed with measure and order, but quantum[3] rather than classical. With a quantum object q we shall associate the $*$ algebra $A(q)$ (algebra with an anti-automorphism $a\rightarrow a^*$, the adjoint) of operators on Hilbert space, thought of as containing the algebra of coordinates of an individual object, a quantum. We suppose that the space-time coordinates of a point p in space-time form the same kind of structure as the phase-space coordinates (dynamical variables) of an elementary quantum-mechanical system:

A space-time point is a quantum object.

We must indicate briefly how this hypothesis fixes the class theory and measure of space-time. The q(uantum) classes of a quantum object q are the Hermitian idempotents σ of its $*$ algebra:

$$\sigma^2=\sigma^*=\sigma\in A(q).$$

The *inclusion* $\sigma\subset\sigma'$ of quantum classes is expressed by $\sigma'\sigma=\sigma$. Two quantum classes σ and σ' are *compatible*, $\sigma\leftrightarrow\sigma'$, when the idempotents σ and σ' commute: $\sigma\sigma'=\sigma'\sigma$. The complement of a quantum class σ is $1-\sigma\equiv\sim\sigma$. *Disjoint* $\sigma\perp\sigma'$ means $\sigma\subset\sim\sigma'$. The *conjunction* $\sigma\cap\sigma'$ is the greatest q class included in both σ and σ' (in the sense of $\subset$). The *adjunction* $\sigma\cup\sigma'$ is the smallest q class including both σ and σ'. The measure of a σ is $\mathrm{tr}\sigma$ and is normalized so that the two lowest values it assumes are 0 and 1. A quantum class σ is a *singlet* when $\mathrm{tr}\sigma=1$. Then $\mathrm{tr}\sigma$ counts the maximal number of disjoint singlets s, s', $\cdots$, in σ,

$$\mathrm{tr}s=1,\quad s\perp s',\quad s,\, s'\subset\sigma.$$

A frame means a maximal set of disjoint singlets. The minimal $*$ algebra containing symbols g obeying relations ρ is called the $*$ algebra generated by g and ρ, $*\,\mathrm{alg}(g;\rho)$. The things we call $*$ algebras are required to contain the number 1.

A singlet—or, equivalently, a vector in the Hilbert space $H(p)$ on which the $*$ algebra $A(p)$ acts—represents a maximally precise determination of location in space-time.

[4] Spaces getting their topology from an order relation are in G. Birkhoff, *Lattice Theory* (American Mathematical Society, Providence, R. I., 1948). The important unifying role of this concept for relativity has been stressed by E. C. Zeeman and R. Penrose. Cf. E. C. Zeeman, J. Math. Phys. **5**, 490 (1964); E. H. Kronheimer and R. Penrose, Proc. Cambridge Phil. Soc. **63**, 481 (1967). Certain ideas of D. Bohm [*Special Theory of Relativity* (W. A. Benjamin, Inc., New York, 1965)] are suggestive of our approach.

Quantum Relations

How is the causal relation between points to be described if points are quantum objects?

Examples of quantum relations between two quantum objects from elementary quantum mechanics are:

eR_1e': Electrons e and e' have the same L_z.

BR_2B': Baryons B and B' are bound into a triplet-s internal state.

pR_3p': Particle p has a greater kinetic energy than particle p'.

Such relations are represented by subspaces of the direct product of two Hilbert spaces. The projection operators for these subspaces are Hermitian idempotents of the direct product of the two $*$ algebras.

Classically, and we suppose quantally as well, a relation R between two things p and p' is a class of pairs (p,p'), and pRp' means the same as $\{(p,p')\in R\}$. When p is a quantum, the pair (p,p') is described by the direct product $A(p_1)\times A(p_2)$ of the $*$ algebra $A(p)$ with itself, the direct square, which is the $*$ algebra generated by two commuting replicas of $A(p)$. If σ is in $A(p)$, we shall show which of the two points $p(1)$ and $p(2)$ of $\times$ is intended by a symbolic argument, like $\sigma(1)$. Thus, if $\{s,s',\cdots\}$ is a frame for $\times$, $\{s(1)s'(2)\}$ is a frame for $A(1)\times A(2)$.

For a quantum space-time, any quantum relation between two points $p(1)$ and $p(2)$ is a Hermitian idempotent of the direct square, expressible as a linear combination of products like $a(1)b(2)$, where a and b are quantities in $A(p)$. In particular, we suppose that the causal relation C is such.

The quantum theory of relations is more complicated than the classical theory because of the possibility of coherent superposition and the impossibility of perfect identity.

Quantum Identity

Classically, the causal order is a reflective relation. We might wish to speak, therefore, of reflexive quantum relations. Classically, that R is reflexive means

$$\{p=p'\}\subset pRp'.$$

This could be transcribed to quantum theory if we knew what $p=p'$ means. Classically, the relation $p=p'$ is described by the diagonal set in the Cartesian product:

$$\{1=2\}=\bigcup_x\{x(1)x(2)\}.$$

Since we have no underlying set of x's to sum over, this definition can only be suggestive for the quantum case. For example, if s ranges over all singlets in $A(p)$, then

$$\bigcup_s s(1)\cap s(2)=\tfrac{1}{2}(1+X),$$

where 1 is the unit operator in $A(p)$ and X is the exchange operator. This is the least projection including all states of the form $s(1)\cap s(2)$ (both quanta in the same state), yet does not imply the equality for the two systems of any nontrivial quantity.

We shall have to settle for equality with respect to some complete set of commuting quantities. By *an* equality relation we shall mean one of the form

$$E=\bigcup_s s(1)s(2),$$

where $\{s\}$ is a frame for Q.

By a reflexive quantum relation R, we mean one that follows from some quantum equality E:

$$E\subset R.$$

This behaves well in the classical limit.

Classically, the causal relation is asymmetric. Symmetry of a relation S has obvious quantum meaning: $pSp'=p'Sp\equiv pS^Tp'$. Asymmetry of A will mean

$$pAp'\cap p'Ap\subset pEp',$$

where E is an equality relation. Antisymmetry will mean $pAp'\perp p'Ap$.

Precedence Relations

Classically, the causal relation is transitive. Transitivity is a concept of three-variable logic and the direct cube $M(1)\times M(2)\times M(3)$:

$$p(1)Tp(2)\cap p(2)Tp(3)\subset p(1)Tp(3).$$

We can define a quantum partial ordering, at last, as a reflexive, antisymmetric, transitive q relation. This concept does not behave well under conjunction. We wish the causal relation to agree with the conjunction of all observers' time orderings by time coordinates. The conjunction of two reflexive relations which are not compatible is not reflexive, but can even be null. Therefore, we shall not assume that the causal order relation is a partial ordering.

The classical concept whose quantum translation is well behaved in this respect is that of *precedence* (relation), a transitive antisymmetric relation. The causal structure of classical space-time, moreover, is as well described by the *causal precedence relation*

$$1C2=\{\Delta x^0>0\}\cap\{\Delta x^\mu\Delta x_\mu>0\},$$

where $\Delta x^\mu=x^\mu(2)-x^\mu(1)$, as by a similar expression with $\leq$ instead of $<$, which would be a partial ordering.

Coherent Relations

Call a quantum relation R *coherent* when it is incompatible with any separate identity relation for either member:

$$pRp'\leftrightarrow pEp'',\quad p''Ep'.$$

The true quantum relations are coherent. That two quanta are bound into the ground state of the hydrogen atom is a coherent relation between them. The causal relation between space-time points proposed later is a coherent relation.

Internal Coordinates

While classically the causal relation C is a partial ordering, we have seen that this concept does not generalize well to quantum theory, and we have softened the requirements on the quantum relation C. As further rationalization, there is abundant indication that the physical relation C is not asymmetric at all. Call a unitary transformation U of M an *internal* symmetry if U leaves C unaffected in the sense that

$$U(1)(1C2)U(1)^*=U(2)(1C2)U(2)^*=1C2.$$

This is not to be confused with the milder concept of a symmetry transformation, which leaves C invariant in the sense that

$$U(1)U(2)(1C2)U(2)^*U(1)^*=1C2.$$

There are many transformations in elementary quantum mechanics that are internal in this sense. For example, isospin rotation and charge conjugation seem to have no effect upon causal dependency, do not affect the metrical relations between objects in space-time. This is why they are called internal. Dropping the asymmetry of C makes room for such transformations in the geometrical foundations of physics. Such quantum orderings correspond approximately to higher-dimensional geometries in classical theories of space-time. We posit, in brief, that:

C, the causal relation between space-time points, is a quantum precedence relation.

We call a quantum object with a precedence relation, a *causal quantum space*.

Quantum Coordinates

By a coordinate in a classical space-time we mean a real function on the measure space. For a quantum object q, we mean by a coordinate an arbitrary self-adjoint q-number in the $*$ algebra $A(q)$. If f is such a coordinate and σ is a class, classical or quantum, f takes on a definite value on σ if and only if

$$f\sigma=\lambda\sigma,$$

and then λ is the value. In general a coordinate f has an expectation value on each σ, given by

$$\langle f\rangle=\mathrm{tr}\sigma f/\mathrm{tr}\sigma.$$

An *external coordinate system* means a set of coordinates $\{x\}$ that determine the causal relation in the sense that $p(1)Cp(2)$ can be expressed in terms of the $\{x(1),x(2)\}$. [More precisely, everything that commutes with the $x(1)$, $x(2)$ commutes with C.]

An *internal coordinate* y is one compatible with the causal relation or, equivalently, one that generates an internal symmetry.

These definitions should be taken with care. An internal coordinate can be part of an external coordinate system. An external coordinate as such is not defined.

Every coordinate x determines a binary relation $x(1)<x(2)$ ("less in x") represented by the same element of $A(1)\times A(2)$ as the function $\theta(x(2)-x(1))$, where θ is the step function

$$\theta(x)=1, \quad x>0$$
$$=0, \quad x\leq 0.$$

Evidently this is a precedence relation and is compatible both with $x(1)$ and $x(2)$.

Let us call a coordinate t a time coordinate if it is greater for points which are later—more exactly, if $1C2\subset\{t(1)<t(2)\}$.

The time coordinates make up a convex subset of the $*$ algebra $A(p)$. Call the extreme points of this convex set *pure* times (following the old terminology for density matrices, which likewise form a convex set in their $*$ algebra).

III. ORDER OF ALGORITHMS

The most general finite model for the causal relation that I can think of is that of an *algorithm*. An algorithm is a finite class or alphabet of *digits* d, finite sequences of which are called *words* w; a finite class of *operations* o, finite sequences of which are called *programs*; and a basic ternary relation $g(w_2ow_1)$ expressing the idea that operation o transforms word w_1 into word w_2. Further restrictions are generally placed upon this relation. If the alphabet has n digits, the algorithm is called n-ary. The causal relation C defined by an algorithm is one between words, w_1Cw_2 meaning that there exists a program transforming w_1 into w_2. An algorithm is a special case of the more general concept of automaton.

We now consider the extension of quantum logic required to define a quantum algorithm.

Quantification As Second Quantization

There is more to logic than the $(\subset,\cap,\cup,\sim)$ calculus of classes. The theory of automata requires us to consider further elements of logical structure expressing the relation of the individual to the collective. Given the class calculi of some systems ("individuals"), we are led inevitably to new systems (ensembles of such individuals). Pairs are the simplest example, but here we are more concerned with ensembles of unspecified numbers of individuals. Now, instead of being confined to "yes or no" questions A, B, $\cdots$, we can also ask "how many" questions: "How many elements of the ensemble are in the class A?" When we move from "yes or no" to "how many," we "quantify" the calculus of classes. The common quantifiers in classical logic are the existential quantifiers $\bigcup_x$ and the universal quantifier $\bigcap_x$. [The corresponding operators on propositions are written $(\exists x)$ and $(\forall x)$.] If P is any quality or class of individuals and x designates an individual, then $P(x)$ designates the proposition about the ensemble that the individual x belongs to the class P, and also the class of ensembles for which this proposition is true. Then $\bigcup_x P(x)$ is the class of ensembles containing some P individual, and $\bigcap_x P(x)$ is the class of ensembles in which all individuals are P. These are both qualities of the ensemble, which itself is regarded as a new individual. In terms of $\bigcap_x$ or $\bigcup_x$, it is possible to define the numerical quantifier N_x so that $N_xP(x)$ means the number of elements in the ensemble with the quality P. Any one of these three quantifiers (existential, universal, or numerical) serves to define the others:

$$\bigcup_x P(x)=\{N_xP(x)\neq 0\},$$
$$\bigcap_x P(x)=\{N_x\sim P(x)=0\}.$$

What logicians accomplish by quantification, physicists accomplish by "second-quantization," in which the number of individuals becomes a quantum variable and no special number is imposed.[5]

Before developing the quantum theory of the relation between the individual and the collective, we must particularize the classical concept slightly. Consider a classical object c with universal class I. Depending on context, we are apt to mean at least three different things by the expression "an ensemble of n c's":

(a) The n *sequence* of c's, an ordered n-uple of objects isomorphic to c, is the object c^n whose universal class is I^n, the nth power of I, with cardinality

$$|I^n|=|I|^n.$$

The generic sequence of c's is the object seq c, which is an n sequence for some n, the disjoint union

$$\text{seq } c=\bigcup_n c^n.$$

The cardinality of seq I is infinite if $|I|>0$.

(b) The n *series* of c's, an unordered n-tuple of objects isomorphic to c, is the object $c^{[n]}$ with universal class $I^{[n]}$ obtained from I^n by identifying with respect to permutations of the n objects, or is the symmetrized nth power, with cardinality

$$|I^{[n]}|=(i+n-1)!/(i+1)!n!$$

($i\equiv|I|$). The generic series of c's is the object ser c, which is an n series for some n:

$$\text{ser } c=\bigcup_n c^{[n]}.$$

The cardinality of ser c is infinite if $|I|>1$.

(c) The n *set* of c's, a set of n c's, is the object $c^{\{n\}}$ with universal class $I^{\{n\}}$ obtained from I^n by identifying with respect to permutations of the n objects

[5] This was noted independently by E. W. Beth, in *The Concept and the Role of the Model*, edited by H. Freundenthal (D. Reidel Publishing Co., Dordrecht, Holland, 1961), and by D. Finkelstein, J. M. Jauch, and D. Speiser, CERN Report Nos. 59-7, 59-9, and 59-17, 1959 (unpublished).

and deleting sequences with two or more identical elements, or is the antisymmetrized nth power, with cardinality

$$|I^{\{n\}}|=i!/n!(i-n+1)!.$$

The generic set of c's is the object set c which is an n set for some n:

$$\text{set } c=\bigcup_n c^{\{n\}}.$$

The cardinality of set c is 2^i, and set c is usually written 2^c.

For reasons which will be clear to quantum physicists, I have chosen to carve the series and the set out of the sequence. Logically it might have been more natural to take the set concept as fundamental and define a series and a sequence as mappings $N\rightarrow I$ and $I\rightarrow N$, respectively, where N is the set of natural numbers.

Now each of these classical methods of aggregation, *sequence*, *series*, and *set*, can be performed for a quantum object q as well as for a classical object c, in such a way as to yield a new quantum object, with its own algebra of quantities and Hilbert space. Indeed, the substitution $c\rightarrow q$ makes I become a Hilbert space and makes the above descriptions of seq, ser, and set become valid definitions of the ensembles of "Maxwell-Boltzmann" objects, "Bose-Einstein" objects, and "Fermi-Dirac" objects, respectively. All products of sets are replaced by direct products of Hilbert spaces in this process; unions are replaced by direct sums. The discovery of the Fermi-Dirac statistics of the electron is the discovery that the physical object of many-electron theory is not a sequence or a series of electrons (for instance) but a set of electrons, in the quantum sense. The three operators, seq, ser, and set can be read as standing for the three familiar kinds of second-quantization processes or, as we will henceforth call them, quantification processes.

There is a uniform way to generate the algebra of quantities for these three quantified theories from the Hilbert space $I=H(q)$ of one object q. Each vector ψ of I is thought of as a q creator, creating an additional q in the class ψ, and is an element in the algebra being constructed. The algebra is, in fact, that generated from the ψ's by the processes of linear combinations, products, and adjoints with the following relations:

$$\psi^*\varphi=\psi*\varphi; \qquad \text{(seq)}$$

$$\psi^*\varphi=\varphi\psi^*+\psi*\varphi, \qquad \psi\varphi=\varphi\psi; \qquad \text{(ser)}$$

and

$$\psi^*\varphi=-\varphi\psi^*+\psi*\varphi, \quad \psi\varphi=-\varphi\psi. \quad \text{(set)}$$

Here ψ and φ are generic elements of the Hilbert space I, $\psi*\varphi$ is their inner product in this space, and [,] and { , } designate commutators and anticommutators, respectively. The numerical quantifier $N_qP(q)$ for any one-object quality p is uniformly defined by

$$N_qP(q)\equiv\sum\varphi_m p_{mn}\varphi_n^*,$$

where φ_m ranges over an orthonormal basis for I, and $p_{mn}=\varphi_m{}^*p\varphi_n$.

It remains to give the Hilbert spaces of the objects seq q, ser q, and set q an explicit representation. This is uniformly done by selecting a vector v_0 to be the vector describing the null ensemble $\varnothing$ in that Hilbert space,

$$\varphi^*v_0=0,$$

$$\varnothing=v_0v_0{}^*,$$

and letting the algebra already defined act on v_0:

$$A(\text{seq } q)v_0=H(\text{seq } q),$$

$$A(\text{ser } q)v_0=H(\text{ser } q),$$

and

$$A(\text{set } q)v_0=H(\text{set } q).$$

Now the quantification process can be iterated as often as the physics makes necessary. It is perfectly meaningful to regard, for example, a set of q's as one object and consider a series of such objects, which is ser set q. In general, we can deal with arbitrary iterations

$$Q_1Q_2\cdots Q_nq,$$

where the Q_i are any of the three kinds of quantification we have defined. (Others exist but seem unnecessary.) Evidently, the logical adjunction of all these iterates—call it $\sum q$—is a palette broad enough for the expression of most of the concepts ever used in physical logic. Since $\sum q$ is the adjunction of all the nth-quantified theories, it may be called the $\aleph_0$-quantified theory. But it too is but the beginning of its own hierarchy.

Quantum Algorithms

We can now transpose the concept of a classical algorithm into quantum logical terms. By a quantum algorithm we mean: a finite quantum object d called a digit, a finite assembly of which we call a quantum word w; a finite quantum object o called an operation, a finite assembly of which we call a quantum program; and a basic ternary quantum relation $g(w_2ow_1)$ expressing the idea that quantum operation o transforms quantum word w_1 into quantum word w_2. Of course, a finite quantum object is simply one described in a finite-dimensional Hilbert space, and quantum words (programs) are described in the manner familiar from the many-quantum problem.[3] $A(d)$ and $A(o)$ will represent the $*$ algebras of the corresponding Hilbert spaces $H(d)$ and $H(o)$. We shall not define the causal quantum relation of a quantum algorithm in the most general case, because of the extreme simplicity of the algorithms we employ in the present work.

Some Classical Algorithms

Let us designate by Z_n the integers modulo n. The classical algorithm defined by digits Z_n, operations Z_n, and ternary relation $c(w_2\delta w_1)=$(adjoining digit δ to

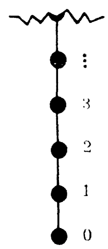

FIG. 1. Quantized time axis. This graph defines the causal order of a causal quantum space whose classical limit is the real time axis. The quantum space is isomorphic to the linear harmonic oscillator and is the space of a character in the quantum singulary code. A dot (●) is an element of a frame in the Hilbert space. An upward line segment implies a causal order relation. In the classical limit, this space becomes the positive time axis with the usual causal order and measure and an internal space in the form of a circle arising from the possibility of coherent superpositions of adjoint elements.

word w_1 generates word w_2) we shall call the *n-ary code*. For example, the words of the singulary code ($n=1$) in their causal order are $\varnothing$, 0, 00, 000, $\cdots$, where $\varnothing$ is the null set of 0's, and are represented graphically with their causal order in Fig. 1. A classical algorithm in which the order of digits in a word is ignored we shall call *symmetric*. The words of the symmetric binary code are $\varnothing$; 0; 1; 00, 01, 11; 000, $\cdots$, and are represented graphically with their causal order in Fig. 2. The words of the binary symmetric code correspond also to the positions accessible to one man in the game of checkers (draughts) on a semi-infinite chessboard, with the obvious causal relation.

Some Quantum Algorithms

A quantum algorithm with alphabet of measure n, $|H(d)|=n$, is n-ary. The *quantum n-ary code* is the quantum n-ary algorithm in which for each kind of digit (ray in H^n) there is one operation, the adjoining of a digit of that kind. The quantum word of this code is

$$w=\operatorname{seq} d\,,$$

a sequence of digits. The quantum n-ary symmetric

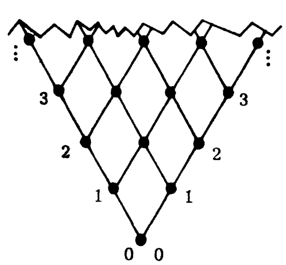

FIG. 2. Quantized null cone. This graph indicates the causal order of a causal quantum space whose classical limit is the future null cone of special relativity. The quantum space is isomorphic to the two-dimensional harmonic oscillator and is the space of a word in the binary symmetric quantum code. Dots (●) and line segments mean the same as in Fig. 1. In the classical limit, the space becomes the three-dimensional future null cone of special relativity with its usual causal order and measure and an internal space in the form of a circle. The apparent doubling of dimensions arises from the possibility of coherent superpositions of near elements. While no element in the graph is Lorentz-invariant, the Lorentz group can act on the space through unitary transformations that leave the causal order invariant—in fact, by the Majorana representation of the proper Lorentz group. In this model, the exact isotropy of space-time is a quantum effect due to the existence of coherent superpositions of near elements in the diagram, while incoherent superpositions then provide points off the null cone and account for the homogeneity of space-time.

code is similar, except that we apply Bose-Einstein quantification:

$$w=\operatorname{ser} d\,.$$

The vectors δ of $H(d)$, which already have the repertory of operations that define a Hilbert space, are given in addition the operations of multiplication and adjoint of a $*$ algebra, viz., $\delta_1\,\delta_2$ and δ^*. Then $A(w)$ is the $*$ algebra generated by $H(d)$ and the commutation relations

$$\delta_1{*}\delta_2-\delta_2\delta_1{}^*=\tau(\delta_1{*}\delta_2)$$

and

$$\delta_1\delta_2-\delta_2\delta_1=0\,,$$

using the inner product ($*$) of $H(d)$. A vector δ given such algebraic structure is also called a creator of a digit in the singlet δ of d. The factor τ is put in to help define the classical limit $\tau\rightarrow 0$. It appears later as a quantum of time.

The null word $\varnothing$ (more correctly, the property of having zero digits) is the singlet of $A(w)$ defined by $\delta^*\varnothing=0$ for all δ. The explicit power-series solution of this equation gives

$$\varnothing=[\sin(2\pi t/\tau)]/(2\pi t/\tau)\,,$$

where $t=\sum\delta\delta^*$, summed over an orthonormal basis $\{\delta\}$ of $H(d)$.

We now define the order of generation of a quantum code as a quantum relation. As for any quantum object, the quantum classes of words, the Hermitian idempotents of w, are ordered by inclusion:

$$\sigma\subset\sigma' \quad\text{means}\quad \sigma\sigma'=\sigma\,.$$

This has absolutely nothing to do with the relation C we seek, which is not of word classes but of words. The individual word happens itself to be an assembly of other individuals, and this is therefore a problem in a higher-order quantum logic than the simple class calculus of $\subset$, $\cup$, $\cap$, $\sim$. The classical limit of the relation C is quite clear: The individual is still a kind of set, and the partial ordering of such sets defines a partial ordering C of such individuals. For example, there is no doubt about when one set of classical coordinates and momentum values

$$(x_1,p_1;\,x_2,p_2;\,\cdots;\,x_m,p_m)$$

is part of another such set

$$(x_1',p_1';\,x_2',p_2';\,\cdots;\,x_n',p_n')\,.$$

It is when the individual cannot be represented by a point in a classical set that any problem arises. When is one real (FD) electron ensemble part of another?

Each creator δ creates a certain kind of digit connected with that creator, and each $\delta\delta^*=\tau n_\delta$ is a quantum coordinate for w, with n_δ counting how many of the δ kind of digit appear in the word. A well-defined precedence relation between w's is $n_\delta(2)>n_\delta(1)$, expressing the relation of having less of the δ kind of digit. We define the quantum ordering by digit count, $w(1)Cw(2)$, by

$$C=\bigcap_\delta\{n_\delta(2)>n_\delta(1)\}. \quad (1)$$

Of course, the same expression defines the classical order relation of the classical symmetric n-ary code. The conjunction is taken over all δ. The null word $\varnothing$, the "digit vacuum," is part of every word in the sense that

$$\varnothing(w_1)\subset w_1Cw_2,$$

and no other word is part of the null word:

$$w_1Cw_2\cap\varnothing(w_2)\subset\varnothing(w_1).$$

Singulary Code

We first compute the causal order of quantum words in one digit, recasting harmonic-oscillator theory.

The * algebra $A(d)$ of the digit is now the complex plane K, the idempotents of K being 0 ("d does not exist") and 1 ("d exists"). The * algebra of words $A(w)$ is generated by one creator δ_0 and the relation

$$\delta_0{}^*\delta_0-\delta_0\delta_0{}^*=\tau.$$

The digit count (times τ) is the operator

$$t=\delta_0\delta_0{}^*,$$

and the causal order C of this world is, by (1),

$$C=\theta(t(2)-t(1)).$$

Then the null word $\varnothing$, the element of $A(w)$ which obeys

$$\varnothing^2=\varnothing^*=\varnothing$$

and

$$\delta_0{}^*\varnothing=0,$$

is uniquely defined by these conditions as the power series

$$\varnothing=\frac{\sin 2\pi t/\tau}{2\pi t/\tau}.$$

This element of the * algebra indeed annihilates the singlet $t=\tau,\ 2\tau,\ \cdots$, and preserves the singlet $t=0$. [It helps to recall that $(\sin 2\pi n)/2\pi n$ is the projector on the ground state of the harmonic oscillator, where n is the number operator.] A frame for w is $\{\delta^{*n}\varnothing\}$, and the causal ordering of these singlets is shown in Fig. 1.

This is evidently a trivial kind of partially ordered space for our purposes, with too simple a structure. It is not quite as simple as Fig. 1 would suggest. Figure 1 shows a line, and this space has to be called two-dimensional, I think. I have not yet been able to formulate the concept of dimension for causal quantum spaces, but in the classical limit the algebra becomes that of complex functions on K, and the underlying set and measure theory is that of the complex plane. The ordering C is by radius, and a neighborhood of a point is an annulus centered on the origin. Therefore, points at the same distance from the origin cannot be separated causally. The radius $r=\sqrt{t}$ is an external coordinate system. It is a Newtonian sort of world, since there is an absolute time, and since causal effects can propagate with infinite speed in one of the coordinates, the "internal coordinate," which may be taken to be a polar angle θ. The transformations $\delta_0\rightarrow e^{i\alpha}\delta_0$ are internal symmetries, and they are generated by the time t.

Binary Code

Let the digit now have two states. A creator δ is a two-component vector $\delta^\alpha c_\alpha$, $\alpha=0,\ 1$, where c_0, c_1 are complex numbers and δ^0, δ^1 are unit vectors. For any such δ there is an associated coordinate $n_\delta=\delta\delta^*$ and precedence relation $n_\delta(2)>n_\delta(1)$. The ordering C of the w's is to be, by (1),

$$\begin{aligned}C&=\bigcap_\delta(n_\delta(2)>n_\delta(1))\\&=\bigcap_\delta\theta(\delta\delta^*(2)-\delta\delta^*(1))\\&=\bigcap_\delta\theta(\Delta\delta\delta^*).\end{aligned}$$

Here and in the following, $\Delta f\equiv f(2)-f(1)$. Let c be a generic complex two vector (c_α):

$$C=\bigcap_c\theta(c_\alpha\Delta\delta^\alpha\delta^{\beta*}c_\beta{}^*).$$

Thus is the order of two words $w(1)$ and $w(2)$ determined by the differences of the respective quantities

$$\delta^\alpha\delta^{\beta*}=T^{\alpha\beta*}\equiv(T).$$

T^{00*} counts 0's, T^{11*} counts 1's, and T^{12*} tells about coherent superpositions of 0 and 1. The relation $w(1)Cw(2)$ means that for all $c\neq0$, $c\Delta Tc^*>0$. In the classical limit, where the four quantities $T^{\alpha\beta*}$ commute, the conclusion is swift. The condition on ΔT is invariant under $\Delta T\rightarrow\Lambda\Delta T\Lambda^*$, where Λ is any matrix of $GL(2,C)$; for Λ simply shuffles the c's. The condition is therefore a function of the invariants of ΔT under $GL(2,C)$, which reduce to the signs of the eigenvalues of ΔT. The eigenvalues must be non-negative. That is, C means

$$\mathrm{tr}\Delta T>0,\quad \det\Delta T>0.$$

Since the quantities $\Delta T^{\alpha\beta*}$ are not Hermitian, we introduce, for convenience only, coordinates $txyz$ through

$$T=\begin{pmatrix}t+z & x+iy\\ x-iy & t-z\end{pmatrix}.$$

The relation C becomes

$$\Delta t>0,\quad \Delta t^2-\Delta x^2-\Delta y^2-\Delta z^2>0.$$

The first nontrivial code we try, the binary symmetric or chessboard code, yields the order of the future null cone of special relativity. (We have $t^2-x^2-y^2-z^2=0$ as an identity in the classical limit.) The (t,x,y,z) are an external coordinate system. The fourth coordinate [loosely speaking, the one missing from the (t,x,y,z)] is totally irrelevant to the causal ordering C, and is readily shown to be an angle invariant under Λ.

The transformation $T \longrightarrow \Lambda T \Lambda^*$, which leaves the ordering C invariant in the classical limit, leads us to suspect that the quantum C might be Lorentz-invariant. Is there a unitary transformation $U(\Lambda)$ of $H(w)$ that leaves C invariant and sends $T \longrightarrow \Lambda T \Lambda^*$?

For unitary Λ (spatial rotations), $U(\Lambda)$ exists, but not for Hermitian Λ (boosts). (Proof: The operator $\Lambda = \mathrm{diag}(\lambda, 1/\lambda)$ changes the commutator $[t,x]$ unless $\tau = 0$.)

Majorana Model

We readily modify the binary symmetric code to form a Lorentz-covariant causal quantum space with the same algebra and classical limit. Define operators

$$\psi^0 = (\delta^0 - i\delta^{1*})/\sqrt{2}$$

and

$$\psi^1 = (\delta^1 + i\delta^{0*})/\sqrt{2}$$

and their adjoints. The δ^α were like creators of spin-$\frac{1}{2}$ bosons. The ψ^α's obey the Majorana commutation relations

$$[\psi^0,\psi^1] = -i, \quad [\psi^\beta,\psi^{\alpha*}] = 0.$$

These relations are invariant under the $SL(2,C)$ transformation $\psi \longrightarrow \Lambda\psi$. Since the ψ's generate the $*$ algebra, there exists a unitary transformation $U(\Lambda)$ accomplishing this transformation:

$$U\psi U^* = \Lambda\psi.$$

We define the new causal precedence

$$1C2 = \bigcap_\psi [\psi\psi^*(1) < \psi\psi^*(2)]. \qquad (2)$$

Covariant external coordinates are now

$$x^{\alpha\beta*} = \psi^\alpha\psi^{\beta*} = x^\mu \sigma_\mu{}^{\alpha\beta*}.$$

Because of the Majorana relations, we have

$$x^\mu x_\mu = -\tau^2,$$

which approaches $x^\mu x_\mu = 0$ when $\tau \longrightarrow 0$.

The point is that this Lorentz invariance of $A(w)$ and C guarantees that of the measure in the classical limit, which must therefore approach the relativistic measure on the light cone,

$$d\mu = dx\, dy\, dz/t.$$

The space might have been conformally flat and still have metrical curvature. It is flat.

IV. ORDER OF WORD PAIRS

The calculations of the causal measure space following from the symmetric binary code and the Majorana model immediately suggest candidates for the causal measure space of special relativity. For example, by the causal quantum space W of word pairs in the symmetric binary code we mean the object $W = w(1) \times w(2)$, where $w(1)$ and $w(2)$ are words in the symmetric binary code, with the order relation C based on the sums of the digit counts for $w(1)$ and $w(2)$. W is defined by a ternary algorithm, the third digit serving as punctuation. Since each w separately gives a future null vector in the classical limit, $w(1) \times w(2)$ gives the sum of two future null vectors for an external coordinate system, and such a sum ranges over the solid cone

$$t \geq \sqrt{(x^2+y^2+z^2)}.$$

The causal order being Minkowskian, we simply have to calculate the measure to see if the space is metrically as well as conformally flat. By Lorentz covariance, the trace in the classical limit

$$\mathrm{tr} f = \int (d\delta) f(\delta_j{}^\beta)$$

must have the form

$$\mathrm{tr} f = \int (dx) \rho(x^2) f(x^\mu).$$

Here $(d\delta)$ is the product of four elements of area, one from each of the complex $\delta_i{}^\alpha$ planes, and therefore is of degree 8 in the δ, while (dx) is the Minkowski measure, of degree 4 in the x^μ, which are of degree 2 in the δ. Therefore, ρ is of degree 0. Since there are no constant lengths left in the theory when $\tau \longrightarrow 0$, we must have $\rho(x^2) = \text{const}$. The measure is Minkowskian; the space is flat.

The internal space can be represented as the collection of complex numbers $\delta_i{}^\alpha$ that all map into one x^μ. Without loss of generality, the point $x^\mu = (1,0,0,0)$ can be taken, and a simple calculation shows the internal space has the structure of $U(2,C)$. The word in the beginning, of course, is $\varnothing\varnothing$.

Space-Time Complementarity Relation

The space-time coordinates of this model obey a complementarity relation. The canonical volume element $(d\delta)$, the product of four complex differentials $d\delta_i{}^\alpha$ and their complex conjugates, directly gives the number dn of disjoint unit quantum sets in a region of classical space:

$$dn = (d\delta)/(2\pi\tau)^4.$$

The relation between δ and the coordinates x and y is most simply expressed by regarding δ as a 2×2 complex matrix in the classical limit. Then

$$x = \delta\delta^{\mathbf{H}},$$

where $\delta^{\mathbf{H}}$ is the transposed matrix of complex-conjugate elements: $\mathbf{H} = \mathbf{CT}$. Let the unique polar factorization of δ be

$$\delta = \xi y,$$

with ξ positive definite and y unitary. Then y makes a suitable internal coordinate and the full coordinate transformation is

$$x=\delta\delta^{\mathsf{H}}, \quad y=\delta(\delta\delta^{\mathsf{H}})^{-1/2},$$

and conversely

$$\delta=x^{1/2}y.$$

Thus integrals can be transformed according to

$$\int(d\delta)\sim\int(dx)\int(dy),$$

where $\sim$ conceals a pure numeric and (dy) is an element of volume in $U(2,C)$. For functions $f(x)$ of the external coordinates alone,

$$\int(d\delta)f\sim\int(dx)f.$$

Thus the number of disjoint singlets in a cell $\Delta t\Delta x\Delta y\Delta z$ is

$$\Delta n\sim\Delta t\Delta x\Delta y\Delta z/\tau^4.$$

The minimum space-time volume per singlet is thus $\Delta t\Delta x\Delta y\Delta z\sim\tau^4$.

V. ORDER OF SPACE-TIME

The path from fundamental principles to reasonable models has been short, and the conceptual economy great; the principles deserve further study. We must propose that the structures we ordinarily identify as single points of space-time are, or are reached by, approximately incoherent assemblies of binary elements to which ordinary quantum principles apply and that a point we regard as later is a greater assembly. Some immediate consequences, we have seen, are the four-dimensionality of space-time, the signature $+1-3$ of its pseudometric structure, the existence of an origin and a bounding null surface for the universe at which the classical approximation breaks down, the Poincaré invariance of special relativity away from the boundary region, and the existence of a certain internal space. At the present epoch T, the number of binary elements in the sets must be on the order of T/τ, where τ is the quantum of time we have introduced. Although the whole realm of dynamics remains to be opened to this kind of exploration, we suspend the development of our principles at this point to estimate the size of τ, quite unrigorously.

There are already three possible traces of the chronon size about us.

Mass Spectrum

Whatever matter is, when it moves in the pinball machine of Fig. 2, it moves in a periodic system. Even near the classical limit, this period shows up as a periodicity of t of size τ and the four periodicities of the internal space $U(2,C)$. In the space of propagation vectors of matter, there will therefore be bands of transmission and bands of rapid attenuation by Bragg scattering. Since the system is Lorentz-invariant, the band structure must be Lorentz-invariant, and therefore will be made up of mass shells. These presumably are shown in the mass spectrum of the elementary particles. The size of the first gaps will be $\sim\tau^{-1}$. Taking the muon as typical, we have

$$\tau\sim 1/m_\mu.$$

In any case, $\tau\lesssim 1/m_e$.

Size of Nucleus

There seems no experimental obstacle to ascertaining that an event—say, a photoproduction—took place in a nuclear volume $\sim r_0^3$ in the transit time r_0 of a photon. It follows that

$$r_0^4>\tau^4, \quad r_0>\tau.$$

If N events could be localized in that region, it would lower the bound on τ to

$$\tau\lesssim r_0N^{1/4}.$$

High-Energy Cross Sections

Look at two billiard balls in their center-of-mass frame, approaching each other with Einstein factors $\gamma\gg1$. If their transverse cross section is $\sigma\sim a^2$, their longitudinal radius shrinks to a/γ, and the maximum time of contact is likewise $t\sim a/\gamma$. The maximum four-volume of intersection, thought of as defining the event of head-one collision, is then σ^2/γ^2. For sufficiently large γ, this must eventually become less than τ^4, and the conjunction or intersection of the two world tubes vanishes, $A\cap B=0$, though they are not disjoint, $A\perp B$. The event occurs with reduced amplitude, then, when

$$\gamma>\sigma/\tau^2.$$

ACKNOWLEDGMENTS

Several conversations seem to have influenced this work strongly: David Bohm said the world is a pattern of elementary causations. Richard P. Feynman said space-time points were like digital computers. Peter G. Bergmann, in a lecture, said the signature of space-time must come from 2×2 Hermitian matrices. Roger Penrose said all along that everything must be made of spinors. I came to believe them. It is not their responsibility. The work was finished during a sabbatical with support from Yeshiva University and the International Centre for Theoretical Physics, for which I am respectively and pleasurably indebted to Abe Gelbart and Abdus Salam.

APPENDIX

We summarize in this Appendix the concepts of quantum logic occurring throughout the paper, which should be consulted for details.

We label objects with symbols $a, b, \cdots, z, a_1, \cdots, Z_n$, which are merely object names and have no algebraic meaning. What are variously called properties, qualities, or attributes of an object we call its *classes*. A class then, like a set in phase space, is a virtual ensemble, and may be large even though there is but one object. We reserve the word *set* for a certain kind of real ensemble discussed later. What really defines an object x is its calculus of classes $L(x)$, which is at least a complemented ($\sim$), compatible ($P\subset Q \Rightarrow P \leftrightarrow Q$) lattice. $A(x)$, the $*$algebra of the object x, is defined by $L(x)$. For quantum objects we take $A(x)$ to be the algebra of all bounded operators on some separable, usually finite-dimensional Hilbert space $H(x)$, a Hilbert space of the object x. $H(x)$ is also the unit operator in that space, since we use the same symbol frequently for a linear subspace and the projector thereon; and $H(x)$, as projector, also represents the universal class in $L(x)$.

It is customary to distinguish between propositions and classes, but when the object of discourse is clearly stated, this would be an unnecessary duplication of symbolism, and instead we frequently identify the class C and the proposition $x\in C$. For example, if the object p is a point and x^μ are its coordinates, $x^0=0$ designates, depending on context, a proposition about p and a p class (the spacelike surface with this equation). Sometimes for greater clarity we indicate when an equation should be read as a class by braces: $\{x^0=0\}$. In particular, the causal relation pCp' will designate strictly the class of point pairs (pp') for which p is causally relevant to p, and loosely the proposition $(pp')\in C$.

However, if P and Q are quantum classes, then $P\subset Q$ is best regarded as a classical proposition and not a quantum class at all.

Table I outlines the structure we presently call quantum logic.

TABLE I. Some concepts of quantum logic.

Concept	Notation	Representation
	I. The Individual	
Object, system	$a, b, \cdots, A, B, \cdots$	
Class	$P, Q, \cdots$	Subspace, projector
Inclusion	$\subset$	Inclusion
Universal class	$I, H(a), I_a$	Unit operator, Hilbert space
Null class	0	Zero operator
Or, adjunction	$\cup$	Span
And, conjunction	$\cap$	Intersection
Not, complement	$\sim$	Orthocomplement
Disjoint	$\perp$	Orthogonal
Measure of P	$\|P\|$	Dimension, trace
Quantity, coordinate, variable	$p, q, \cdots$	Normal operator
Function of a quantity	$f(P), P^2, \cdots$	Cf. functional calculus
Point	σ	Singlet, pure state
	II. The Collective	
Sum, disjunction of objects	$a+b, \sum a_n$	Direct sum
Product, pair	$ab, \Pi a_m$	Direct product
Theory of binary relations		
Relation	R, aRb	Subspace of $I_a \times I_b$
Transpose	$R^T, aR^Tb \equiv bRa$	Exchange subspace
Antisymmetry	$R \perp R^T$	
Transitivity	$aRb \cap bRc \subset aRc$	
Precedence	Transitive antisymmetric relation	
Assemblies		
Set of a's	set a, 2^a	(FD) Antisymmetric tensors on I_a
Series of a's	ser a	(BE) Symmetric tensors on I_a
Sequence of a's	seq a	(MB) Tensors on I_a
Quantifiers		
Numerical	$NP, N_aP(a)$	$\sum (P\psi)\psi^*$
Universal	$\bigcap P, \bigcap_a P(a), \forall_a P(a)$	$N(\sim P)=0$
Existential	$\bigcup P, \bigcup_a P(a), \exists a P(a)$	$NP\neq 0$