Impact-Parameter Amplitudes for Large-Angle Scatterings and Particle Correlations

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Impact-parameter amplitudes for potential scatterings are calculated in the various approximations, and the results are compared with the exact phase-shift analysis up to the second diffraction minimum. General trends of the approximate cross sections are given which should be taken into account in the study of correlation effect in nucleon-nucleus scattering experiments. An improved form of the amplitude for momentum transfer up to the second diffraction maximum is given. Parametrization of the nuclear correlations in terms of the two-nucleon amplitude is discussed.

I. INTRODUCTION

HIGH-ENERGY scattering experiments are often analyzed by the impact-parameter method, with the amplitude given in the form[1]

$$f(k,q)=\int_0^\infty bdb\,J_0(qb)B(b,k)\,,\qquad(1.1)$$

where

$$B(b,k)=ik[1-e^{i\chi(b,k)}],\qquad(1.2)$$

$$\chi(b,k)=-k^{-1}\int_{-\infty}^{\infty}V_{\rm eff}dZ\,,\qquad \mathbf{r}=\mathbf{b}+\mathbf{Z}.\qquad(1.3)$$

In (1.1), we have used $c=m=\hbar=1$, and $V_{\rm eff}$ is an effective interaction defined through the representation (1.1) itself. The fact that the simple form (1.1), derived[1] in the approximation of small momentum transfer $\mathbf{q}$ and using a wave equation of some sort, may have a more general validity[2,3] and satisfy the high-energy unitarity makes it very attractive for further theoretical study. Composite system scatterings have also been treated by Glauber,[4] using (1.1) and several additional assumptions, as discussed in Sec. IV.

On the other hand, it is not a simple matter to relate $V_{\rm eff}$ of (1.3) to the potential V which appears, for example, in the Schrödinger scattering equation. Such a connection for a local potential V has been shown in a roundabout way by Blankenbecler and Goldberger[3] using the dispersion relations for the amplitude, but for practical purpose of improving the representation it would be desirable to find a more *direct* connection. The difficulty in such an attempt seems to arise from the nonlocal nature of the scattering operator $\mathfrak{I}=V+VG_0^{(+)}\mathcal{J}$. We note that the unitarity condition gives Im$\mathfrak{I}$ in the dispersion relations in an essentially separable form, while Re$\mathfrak{I}$ is not.

Here we are concerned with the accuracy of various approximations used in the past and possible improvements at large angles. This problem is of some interest, since recent nucleon-nucleus scattering analyses[5–7] seem to require improved representations, and also a better understanding of the accuracy is desired in the variational formulation proposed recently.[8] Several improved forms of amplitudes for the potential scattering have been given by Schiff[9] and by Saxon and Schiff.[10] Among others, their formulas at large angles contain factors which depend on $\beta=q/2k$, where $\mathbf{q}$ is the momentum transfer, such that B of (1.2) is now a function of β. More recently, Feshbach[11] has given still another form of amplitudes which is similar in form to (1.1). Most of these formulas are extremely effective in reproducing the large forward-diffraction peak, but, as $\mathbf{q}$ gets large, with $q=2k(1-\cos\theta)$, the agreement with the partial-wave calculation is not as good.

We report here the result of a simple computer experiment we have carried out to ascertain the general trend of deviations each approximate amplitude makes from the "exact" amplitude. This will in turn help to correctly determine the parameters involved in the theory. The problem may be less critical if one takes (1.1) as a "correct" representation of f and uses the inverse Fourier-Bessel transform of f, rather than the form B with particular V. The multiple-diffraction theory of Glauber[4] is such a case. However, the result presented here may still have some general validity even in those cases without potentials. We emphasize, however, that although the representation (1.1) is simple and conceptually appealing, the form of B with q dependence ($\beta\neq0$) to be discussed in Sec. II is equally valid and perhaps more tractable within the potential scattering theory.

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II. IMPACT-PARAMETER REPRESENTATIONS AND POTENTIAL SCATTERINGS

We collect here some elementary formulas used in our calculation for potential scatterings, most of which are well known.[1] For simplicity, we consider the Schrödinger equation with a local Gaussian-type potential. We have

$$[T+V-E]\Psi=0, \tag{2.1}$$

with

$$T=-\tfrac{1}{2}\nabla_r^2, \qquad E=\hbar^2k^2/2m=\tfrac{1}{2}k^2, \tag{2.2}$$

where $m=\hbar=c=1$. As usual, we set

$$\Psi=e^{i\mathbf{k}_i\cdot\mathbf{r}}\Phi(\mathbf{r}), \tag{2.3}$$

and obtain

$$(\nabla_r^2+2i\mathbf{k}_i\cdot\nabla_r-2V)\Phi(\mathbf{r})=0. \tag{2.4}$$

The amplitude is given by

$$f(\mathbf{k}_f\cdot\mathbf{k}_i)=-\frac{2m}{4\pi\hbar^2}\int e^{-i\mathbf{k}_f\cdot\mathbf{r}}V(r)\Psi(\mathbf{r})d^3r$$

$$=-\frac{1}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{r}}V\Phi(\mathbf{r})d^3r, \tag{2.5}$$

where we have the usual asymptotic behavior of Ψ given by

$$\Psi(\mathbf{r})\underset{r\to\infty}{\longrightarrow}e^{i\mathbf{k}_i\cdot\mathbf{r}}+f(\Theta)e^{ikr}/r, \tag{2.6}$$

i.e.,

$$\Phi(\mathbf{r})\to 1 \quad \text{as} \quad |\hat{k}_i\cdot\mathbf{r}|\to-\infty. \tag{2.7}$$

The Born amplitude is obtained from (2.5) by setting

$$\Phi(\mathbf{r})=1 \tag{2.8}$$

and thus

$$f_B(k,q)=-(2\pi)^{-1}\int e^{i\mathbf{q}\cdot\mathbf{r}}Vd^3r. \tag{2.9}$$

If we choose the Z axis to be parallel to $\mathbf{k}_i$ (system I), and take the XZ plane to be the scattering plane, then we have

$$(\mathbf{k}_i-\mathbf{k}_f)\cdot\mathbf{r}=\mathbf{q}\cdot\mathbf{r}=kZ_{\rm I}(1-\cos\Theta)-kb_{\rm I}\sin\Theta\cos\phi_{\rm I}, \tag{2.10}$$

where

$$|\mathbf{k}_i|=|\mathbf{k}_f|=k, \qquad \mathbf{k}_i-\mathbf{k}_f\equiv\mathbf{q}, \qquad 0\leq q\leq 2k$$

$$\mathbf{r}=\mathbf{b}_{\rm I}+\mathbf{Z}_{\rm I}, \qquad \cos\Theta=\hat{k}_i\cdot\hat{k}_f.$$

The $\phi_{\rm I}$ dependence of the integrand of (2.8) is contained only in (2.10), so that we can perform the $d\phi_{\rm I}$ integration of $d^3r=b_{\rm I}db_{\rm I}dZ_{\rm I}d\phi_{\rm I}$ using

$$J_0(a)=(2\pi)^{-1}\int_0^{2\pi}e^{ia\cos\phi}d\phi, \tag{2.11}$$

and obtain

$$f_{B\rm I}=-\int_0^\infty b_{\rm I}db_{\rm I}\int_{-\infty}^\infty dZ_{\rm I}\,V(r)J_0(2kb_{\rm I}\sin\tfrac{1}{2}\Theta\cos\tfrac{1}{2}\Theta)\times\exp(2ikZ_{\rm I}\sin^2\tfrac{1}{2}\Theta),$$

$$\equiv\int_0^\infty b_{\rm I}db_{\rm I}\,B_{B\rm I}(b_{\rm I},q,k)J_0[b_{\rm I}q(1-\beta^2)^{1/2}], \tag{2.12}$$

where

$$B_{B\rm I}=-\int_{-\infty}^\infty dZ_{\rm I}\,V[(b_{\rm I}^2+Z_{\rm I}^2)^{1/2}]e^{i2\beta^2Z_{\rm I}k},$$

$$\beta=q/2k. \tag{2.13}$$

We used in (2.12) and (2.13)

$$1-\cos\Theta=q^2/2k^2=2\beta^2, \qquad q=2k\sin\tfrac{1}{2}\Theta$$

$$\cos\tfrac{1}{2}\Theta=(1-\beta^2)^{1/2}, \qquad \sin\Theta=(q/k)(1-\beta^2)^{1/2}. \tag{2.14}$$

On the other hand, we can equally well choose the Z axis to be parallel to the vector $\mathbf{k}_i+\mathbf{k}_f$ (system II) and again the XZ plane to be the scattering plane. Then we have

$$\mathbf{q}\cdot\mathbf{r}=-qb_{\rm II}\cos\phi_{\rm II},$$

$$d^3r=b_{\rm II}db_{\rm II}dZ_{\rm II}d\phi_{\rm II}, \tag{2.15}$$

and thus

$$f_{B\rm II}=-(2\pi)^{-1}\int_0^\infty b_{\rm II}db_{\rm II}\int_{-\infty}^\infty dZ_{\rm II}\,V(r)\int_0^{2\pi}d\phi_{\rm II}\times\exp(-iqb_{\rm II}\cos\phi_{\rm II})$$

$$=\int_0^\infty b_{\rm II}db_{\rm II}\,J_0(qb_{\rm II})B_{B\rm II}(k,b_{\rm II}), \tag{2.16}$$

where

$$B_{B\rm II}=-\int_{-\infty}^\infty dZ_{\rm II}\,V[(b_{\rm II}^2+Z_{\rm II}^2)^{1/2}], \tag{2.17}$$

and noting that $J_0(a)$ is an even function of a. Although (2.12) and (2.16) look quite different in detail, they originate from (2.8) which is *independent* of the choice of coordinates, and thus we have

$$f_{B\rm I}=f_{B\rm II}=f_B. \tag{2.18}$$

We were not able to give a direct transformation from (2.16) to (2.12) in any simple way, but the accurate numerical analysis on the computer checks the validity of (2.18), which was in turn used to check the accuracy of the subsequent calculation. It is significant to note that these two β-dependent factors *cancel*, although they have individually a very large effect on f for $q\lesssim\frac{1}{4}$.

We can improve on (2.8) by solving (2.4) after dropping the ∇_r^2 term since, for $\mathbf{q}$ not too large, Φ should be a slowly varying function of $\mathbf{r}$. Thus

$$(2i\mathbf{k}_i\cdot\nabla_r-2V)\Phi_0=0, \tag{2.19}$$

with the boundary condition (2.7). Here, (2.7) is simple only if we take the system I, in which the Z axis is parallel to $\mathbf{k}_i$. Thus, we have

$$ik(\partial/\partial Z_{\mathrm{I}})\Phi_{0\mathrm{I}}=V\Phi_{0\mathrm{I}}$$

and obtain

$$\Phi_{0\mathrm{I}}=\exp\left[-\frac{i}{k}\int_{-\infty}^{Z_{\mathrm{I}}}dZ_{\mathrm{I}}'\,V[(b_{\mathrm{I}}^2+Z_{\mathrm{I}}'^2)^{1/2}]\right]. \quad (2.20)$$

Substitution of (2.20) into (2.5) gives

$$f_{0\mathrm{I}}=\int_0^\infty b_{\mathrm{I}}db_{\mathrm{I}}\,J_0[qb_{\mathrm{I}}(1-\beta^2)^{1/2}]B_{0\mathrm{I}}(q,k,b_{\mathrm{I}}), \quad (2.21)$$

where

$$B_{0\mathrm{I}}=-\int_{-\infty}^\infty dZ_{\mathrm{I}}\,V(r)\Phi_{0\mathrm{I}}(b_{\mathrm{I}},Z_{\mathrm{I}},k)e^{i2\beta^2Z_{\mathrm{I}}k}. \quad (2.22)$$

(2.21) is the form closely related to the ones studied by Schiff,[9] Saxon and Schiff,[10] and more recently by Ross.[7]

Obviously, the system II will be cumbersome in satisfying the boundary condition (2.7), and thus corresponding solution of (2.19) will not be simple. Thus, we do not consider $\Phi_{0\mathrm{II}}$ for this case explicitly, but make a further approximation on (2.21), following the original derivation by Glauber.[1]

If we make a small-angle-scattering approximation at high energies, then, in (2.21) and (2.22),

$$(1-\beta^2)^{1/2}\rightarrow1,\qquad e^{i2\beta^2Zk}\rightarrow1, \quad (2.23)$$

which is reasonable from the point of view of its effect cancelling in the Born amplitude. Thus, we get

$$f_G=\int_0^\infty b_{\mathrm{I}}db_{\mathrm{I}}\,J_0(qb_{\mathrm{I}})B_G(b_{\mathrm{I}},k), \quad (2.24)$$

where

$$B_G=-\int_{-\infty}^\infty dZ_{\mathrm{I}}\,V(r)\Phi_{0\mathrm{I}}=-ik(e^{i\chi_G(b)}-1), \quad (2.25)$$

$$\chi_G(b)=-k^{-1}\int_{-\infty}^\infty VdZ'. \quad (2.26)$$

This is the Glauber amplitude,[1] and it is obtained here as a small-angle approximation to (2.21). Since both (2.21) and (2.24) are approximate, it is not clear *a priori* which one is better since errors made in successive approximations may cancel. It is the main purpose of this paper to compare these two formulas for a specific case, and see if one can make a general statement of the range of validity for each of these two formulas. Note that, unlike in the Born amplitude f_B, the correction factors dependent on β in (2.21) are significant, and *do not cancel* out because $\Phi_{0\mathrm{I}}$ of (2.20) already contains the crucial information (2.7) with the choice of the coordinate system I.

More recently, Feshbach[11] has considered a slight modification of the form (2.24) by replacing V in (2.26) by

$$-V/k\rightarrow-V_H/k=(k^2-2V)^{1/2}-k$$
$$\approx-(V/k)[1+V/2k^2+\cdots], \quad (2.27)$$

and thus

$$f_H=\int_0^\infty bdb\,J_0(qb)B_H(k,b), \quad (2.28)$$

where

$$B_H\equiv-\int_{-\infty}^\infty dZ_{\mathrm{I}}\,V_H\exp\left(-\frac{i}{k}\int_{-\infty}^{Z_{\mathrm{I}}}V_HdZ_{\mathrm{I}}'\right)$$
$$=-ik(e^{i\chi_H}-1), \quad (2.29)$$

$$\chi_H(b)=-k^{-1}\int_{-\infty}^\infty V_H(r)dZ_{\mathrm{I}}'. \quad (2.30)$$

We also define $f_{\mathrm{I}0H}$ by

$$f_{\mathrm{I}0H}=\int_0^\infty bdb\,J_0[qb(1-\beta^2)^{1/2}]B_{0H}(k;b,q),$$

where

$$B_{0H}=-\int_{-\infty}^\infty dZ_{\mathrm{I}}\,V_H\exp\left(-\frac{i}{k}\int_{-\infty}^{Z_{\mathrm{I}}}V_HdZ_{\mathrm{I}}'\right)e^{2ik\beta^2Z_{\mathrm{I}}}.$$

The approximations involved in the derivation of (2.28) are identical to the case with (2.24), in addition to the fact that, in (2.29), the multiplicative V_H is put in instead of the usual V in order to have the form (2.30). To first order in $V/(2k^2)$, (2.24) and (2.28) are the same, and there is again no *a priori* reason why the substitution (2.27) should improve the amplitude, since the β-dependent terms have been neglected in both formulas. Since, as in f_B, the β-dependent effects may nearly cancel and the approximation (2.23) is very difficult to estimate, we decided to look at a simple model case to examine the formulas (2.21), (2.24), and (2.28). The result will be discussed below.

The form (2.24) or (2.28) is obviously much simpler than (2.21), and the fact that (2.24) with V replaced by an effective V_{eff} may have a more general validity makes it of some interest to search for a better V_{eff}. V_H of (2.28) is one such form. For this purpose, we go back to (2.4) and write it as

$$ik(\partial/\partial Z_{\mathrm{I}})\Phi=V\Phi+(-\tfrac{1}{2}\nabla_r^2)\Phi,$$
$$=V\Phi+[\Phi^{-1}(-\tfrac{1}{2}\nabla_r^2)\Phi]\Phi,$$
$$\equiv(V+V_T)\Phi, \quad (2.31)$$

where we have assumed that Φ^{-1} exists. Since V_T is now a local operator like V, we have

$$\Phi=\exp\left[-\frac{i}{k}\int_{-\infty}^{Z_{\mathrm{I}}}(V+V_T)dZ_{\mathrm{I}}'\right], \quad (2.32)$$

and thus we have the *exact* amplitude given by

$$f=\int_0^\infty bdb\,J_0[qb(1-\beta^2)^{1/2}]B(q,b,k), \quad (2.33)$$

where

$$B(q,b,k)=-\tfrac{1}{2}\int_{-\infty}^{\infty}dZ_{\rm I}\,V(r)e^{i2\beta^2Z_{\rm I}k}\Phi(k,b,Z_{\rm I}).\quad(2.34)$$

Obviously, (2.32) is extremely nonlinear in Φ and (2.33) is perhaps useless from a practical point of view. Besides, the form (2.34) does not give (1.2) in any simple way. From (1.2), we have

$$B(b,k)\approx-\tfrac{1}{2}\int_{-\infty}^{\infty}dZ\,V_{\rm eff}\exp\left(-\frac{i}{k}\int_{-\infty}^{Z}dZ'\,V_{\rm eff}\right).\quad(2.35)$$

In order to compare (2.34) with (2.35), we simply make the approximation (2.23), and further require that

$$V\Phi\approx V_{\rm eff}\exp\left(-\frac{i}{k}\int_{-\infty}^{Z}V_{\rm eff}dZ'\right).\quad(2.36)$$

Equation (2.36) can be rewritten as

$$\ln V-\frac{i}{k}\int_{-\infty}^{Z}(V+V_T)dZ'\approx\ln V_{\rm eff}-\frac{i}{k}\int_{-\infty}^{Z}V_{\rm eff}dZ'.$$

Using the fact that $V_{\rm eff}^\dagger=V_{\rm eff}$ and $V^\dagger=V$, we have

$$\ln V+k^{-1}\int_{-\infty}^{Z}{\rm Im}V_TdZ'\approx\ln V_{\rm eff},$$

$$\int_{-\infty}^{Z}(V+{\rm Re}V_T)dZ'\approx\int_{-\infty}^{Z}V_{\rm eff}dZ',$$

or

$$V_{\rm eff}\approx V\exp\left(k^{-1}\int_{-\infty}^{Z}{\rm Im}V_TdZ'\right),\quad(2.37a)$$

$$\approx V+{\rm Re}V_T,\quad(2.37b)$$

where V_T is defined by (2.31). In the approximation (2.23), we than expect from (2.37) that

$${\rm Re}V_T\approx V\left[\exp\left(k^{-1}\int_{-\infty}^{Z}{\rm Im}V_TdZ'\right)-1\right].\quad(2.38)$$

Obviously (2.38) should be dependent on the approximation (2.23). If we make $\Phi\approx\Phi_{0\rm I}$ in V_T, for example, then

$$V_T=(\Phi^{-1}[-\tfrac{1}{2}\nabla_r^2]\Phi)\approx(\Phi_{0\rm I}^{-1}[-\tfrac{1}{2}\nabla_r^2]\Phi_{0\rm I}),$$
$$=\tfrac{1}{2}\left\{\left(\frac{\partial U}{\partial b}\right)^2+\left(\frac{\partial U}{\partial Z}\right)^2-i\left[\left(\frac{\partial^2U}{\partial b^2}\right)+\left(\frac{\partial^2U}{\partial Z^2}\right)+\frac{1}{b}\frac{\partial U}{\partial b}\right]\right\},\quad(2.39)$$

where

$$U\equiv-k^{-1}\int_{-\infty}^{Z}VdZ'.$$

Apparently $\Phi_{0\rm I}$ is not a very good approximation to Φ for large q, and thus the form (2.39) would not be very good for a large-angle region. However, if we assume that at least the Z-dependent part of Φ is reasonably well reproduced by (2.39), then, with

$${\rm Re}V_T\approx\tfrac{1}{2}(\partial U/\partial Z)^2,$$
$${\rm Im}V_T\approx-\tfrac{1}{2}(\partial^2U/\partial Z^2),\quad(2.40)$$

we immediately satisfy (2.38) to first order in ${\rm Im}V_T/k$, and that

$${\rm Re}V_T\approx V^2/2k^2,\quad(2.41)$$

as given by (2.27). Therefore, the above discussion seems to bring out more explicitly the various approximations used in (2.24) and (2.28). Further indication that (2.39) is not quite right comes from the form given by Schiff and by Saxon and Schiff, where we essentially have $\beta\neq0$ and

$$V_T\sim(\Phi_fT\Phi_i).\quad(2.42)$$

However, (2.42) involves both k and q, so that it is not clear at present whether a form simpler than (2.42) may be possible. Still another indication of the failure of (2.39) comes from the variational method proposed recently.[8] Equation (2.39) results if the d function defined there is replaced by a simpler form, as

$$\Phi_{0\rm I}^{-1}T\,|\,s\rangle(1/d)\langle s\,|\,T\,|\,\Phi_{0\rm I}\rangle\approx\Phi_{0\rm I}^{-1}T\Phi_{0\rm I},\quad(2.43)$$

where

$$d=\langle s\,|\,T\,|\,s\rangle+\frac{i}{k}\left\langle s\left|T\Phi_{0\rm I}\int_{-\infty}^{Z_{\rm I}}dZ'\,\Phi_{0\rm I}^{-1}T\right|s\right\rangle$$
$$\approx\langle s\,|\,T\,|\,s\rangle.$$

$|\,s\rangle$ is a trial function. However, for many cases, explicit calculation[12] shows that the second term in d is not negligible compared to the first term.

As a conclusion, we emphasize that due to the complicated cancellation of the effect of β in (2.21), the various approximate amplitudes are only good to first order in V, and it is not clear how well higher-order effects are reproducible. Therefore, physical interpretations of the experiments which rely on the higher-order effect in the representations (2.21), (2.24), and (2.28) should require additional care. This is essentially the same conclusion reached by various authors.[7,11] We further define several forms of approximate amplitudes which will be explicitly studied in the next section. They are

$$f_A=f_0+\frac{i}{2k}\int_0^{\infty}bdb\,J_0[qb(1-\beta^2)^{1/2}]\times\int_{-\infty}^{\infty}dZ_{\rm I}\,V(r)e^{2i\beta^2kZ_{\rm I}}\Phi_0\int_{-\infty}^{Z_{\rm I}}dZ_{\rm I}'V_{TA}\quad(2.44)$$

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[12] Y. Hahn, Phys. Rev. (to be published).

and

$$f_{AH}=-\tfrac{1}{2}\int_0^\infty bdb\, J_0[qb(1-\beta^2)^{1/2}]$$

$$\times\int_{-\infty}^{\infty} dZ_{\rm I}\, V(r)\, e^{2i\beta^2 kZ_{\rm I}}\left(1-\frac{i}{k}\int_{-\infty}^{Z_{\rm I}} dZ_{\rm I}'\, V_{TA}\right)$$

$$\times\exp\left[-\frac{i}{k}\int_{-\infty}^{Z_{\rm I}} dZ_{\rm I}'V\left(1-\frac{i}{k}\int_{-\infty}^{Z_{\rm I}'} dZ_{\rm I}''V_{TA}\right)\right], \quad (2.45)$$

where

$$V_T\approx V_{TA}=\Phi_{0\rm I}^{-1}T\Phi_{0\rm I}, \quad (2.46)$$

and

$$f_{AG}=f_A \quad \text{with} \quad \beta=0, \quad (2.47)$$

$$f_{AHG}=f_{AH} \quad \text{with} \quad \beta=0. \quad (2.48)$$

The forms (2.44) and (2.45) give nearly the same result and are the improved amplitudes for q up to the second diffraction maximum.

III. RESULT OF CALCULATION

The numerical calculation we have carried out is very trivial but seems to show several interesting features of the various approximate amplitudes. For simplicity, we choose the form

$$V(r)=\tfrac{1}{2}Ge^{-Ar^2}(1+\rho r^2), \quad (3.1)$$

where $m=\hbar=1$, and $r^2=b^2+Z^2$. The ρ-dependent term is added to readily reproduce the diffraction peaks and valleys near the angles where experimental cross sections also have them. The parameters used in the calculation are $A=0.20$, $\rho=0.30$, $k=2.0$, and the strength constant G is tried with $G=-0.10$, -0.20, and -0.40. Needless to say, the form (3.1) may be an oversimplification; when high-energy particles are scattered off complex nuclei, many inelastic channels are open which give rise to an effective interaction $V_{\rm eff}$ with appreciable imaginary part. This is also true for the nucleon-nucleon interaction above the production threshold and with $\mathbf{L}\cdot\mathbf{S}$ and tensor forces. Some estimates of a large $\mathrm{Im}V_{\rm eff}$ on the scattering amplitudes at larger angles were made, for example, by Feshbach[1] for a simple model. We have not carried out such calculations, and thus our discussion is limited in this respect.

The point of view taken in this paper is that, for a given form of the interaction $V_{\rm eff}$, which may or may not be complex, an approximate form of the amplitude is acceptable if it is capable of reproducing the correct values at all q. Even if the presence of a large $\mathrm{Im}V_{\rm eff}$ may affect the large q behavior of the amplitude, the ambiguity of determing the correct parameters of $V_{\rm eff}$ using an approximate formula still remains. Besides, it is not clear whether a large $\mathrm{Im}V_{\rm eff}$ in the nucleon-nucleon interaction, for example, would not contradict the additivity assumption of phases in the Glauber's multiple-scattering theory.[4] We are also neglecting here possible effects due to nonlocality and energy dependence of the effective interactions. The exchange effect may also be important. In this respect, the discussion given below is specifically for V of the form (3.1) with particular set of parameters, but the conclusion and the general behavior of the amplitude we are interested in may have a larger region of validity.

A. Born Amplitude and the Partial-Wave Analysis

The two different forms $f_{B\rm I}$ of (2.12) and $f_{B\rm II}$ of (2.16) were calculated for all momentum transfer, $0\leq q\leq 2k$. To the accuracy of the computer program, we find that these two forms are identical. Since the individual factors in (2.12) which depend on β have strong effects on f, a delicate cancellation is taking place. We also used (2.18) as a numerical check in our later, less elaborate calculations of other quantities. The function f_B is real and extremely good at small q. The exact amplitude f_B is obtained for each k and q by the partial-wave method, which required $l\leq 15$ for $k\leq 2.0$. Because of a large cancellation among these partial waves at large q, we restrict our discussion to regions of $q\leq 2.0$, where we have an accuracy of 1 part in 10^3, while near $q\approx 0$ we have at least 1 part in 10^5.

We next evaluated $f_{0\rm I}$ and f_G of (2.21) and (2.24), and the result is again compared with the exact f_E, as given in Table I. The parameters used are $A=0.20$, $\rho=0.30$, and $G=-0.10$, -0.20, and -0.40. The agreement of $|f_G|^2$ and $|f_{0\rm I}|^2$ with $|f_E|^2$ is excellent for $G=-0.10$, and less so for $G=-0.20$ and $G=-0.40$, as expected. It is more significant, however, that each approximate amplitude deviates from the exact one in some specific way.

First of all, $|f_{0\rm I}|^2$ is consistently *lower* than $|f_E|^2$ even in the region where $\mathrm{Re}f_{0\rm I}$ itself changes sign. It is appreciably better than $|f_G|^2$ in the region between the first minimum and the second maximum, but decreases too fast beyond the second maximum. On the other hand, $|f_G|^2$ seems to have *less sharp* peaks and valleys than $|f_E|^2$ and thus crosses over $|f_E|^2$ at several points, but it follows the exact value rather closely.

We have also evaluated $|f_H|^2$ of (2.29) keeping the V^2 term in (2.27). It gives slightly improved values near the first diffraction minimum, and over-all values are lower than $|f_G|^2$, which is too high in that region. We did also try $|f_{\rm I0H}|^2$, which is the same as $|f_{0\rm I}|^2$ with V replaced by (2.27); the result is not as good as $|f_H|^2$. This indicates that some of the variations we are studying fall within the error made in the $\beta=0$ assumption, and it is difficult to make a correct error estimate.

Thus, the amplitudes with $\beta\neq 0$ are consistently *lower* than $|f_E|^2$ and approach the correct value near the rising part of the peaks, while $\beta=0$ makes the $|f|^2$ less sharp in its diffraction oscillations. These general trends would be useful in actual applications of the forms $f_{0\rm I}$, f_G, and f_H. That is, fits to experiment

TABLE I. Differential cross sections $|f|^2$ for a modified Gaussian potential. The values for each momentum transfer q correspond to f_B, f_G, f_{0I}, f_H, and f_E, respectively. The parameters used are $k=2.0$, $A=0.20$, and $\rho=0.30$.

q	$G=-0.10$	$G=-0.20$	$G=-0.40$
0.0	2.592	10.37	41.48
	2.588	10.30	40.31
	2.588	10.30	40.31
			39.20
	2.605	10.44	41.50
0.8	0.2083	0.8330	3.332
	0.2080	0.8669	3.261
	0.2012	0.7751	2.849
			3.110
	0.2044	0.8118	3.155
1.2	0.00203	0.00812	0.03247
	0.00214	0.00993	0.06110
	0.00159	0.00555	0.02566
			0.05417
	0.00186	0.00763	0.04330
1.4	0.00033	0.00132	0.00528
	0.00039	0.00225	0.02004
	0.00052	0.00316	0.02424
			0.01942
	0.00048	0.00307	0.02624
1.6	0.00098	0.00392	0.01566
	0.00100	0.00422	0.02050
	0.00109	0.00482	0.02310
			0.01991
	0.00111	0.00508	0.02662
1.8	0.00059	0.00238	0.00950
	0.00060	0.00246	0.01090
	0.00060	0.00238	0.00930
			0.01028
	0.00064	0.00271	0.01250
2.0	0.00020	0.00080	0.00321
	0.00020	0.00084	0.00389
	0.00018	0.00066	0.00209
			0.00353
	0.00020	0.00082	0.00360

TABLE II. Dependence of $|f|^2$ on the potential parameters. ↑, 0, and ↓ denote the values of $|f|^2$ increased, unchanged, and decreased, respectively.

	A	↑	↑	↑	0	0	0	↓	↓	↓
q	ρ	↑	0	↓	↑	0	↓	↑	0	↓
0.0		↓	↓	↓	↑	0	↓	↑	↑	↑
0.4		↑	↑	↑	0	0	↓	↓	↓	↓
0.8		↑	↑	↑	0	0	0	↓	↓	↓
1.2		↑	↑	↑	0	0	0	0	↓	↓
1.4		↓	↓	↓	↑	0	↓	↑	↑	↑
1.6		↓	↓	↓	↑	0	↓	↑	↑	0
1.8		↑	0	↓	↑	0	↓	↑	↓	↓
2.0		↑	↑	↑	↑	0	↓	↓	↓	↓

TABLE III. The real and imaginary parts of the amplitudes as well as $|f|^2$ are given. For each q, we have f_G (fitted), f_{0I} (fitted), f_{I0H}, and f_E. The parameters used are $G=-0.40$, $k=2.0$, $A=0.20$, and $\rho=0.30$.

q	Re f	Im f	$\|f\|^2$
1.0	0.6119	0.4239	0.5540
	0.6461	0.3703	0.5545
	0.5627	0.3254	0.4226
	0.6340	0.3899	0.5540
1.2	0.0381	0.2125	0.0466
	0.0717	0.1754	0.0359
	0.0325	0.1474	0.0228
	0.0653	0.1976	0.0433
1.4	−0.1600	0.0715	0.0307
	−0.1469	0.0498	0.0241
	−0.1504	0.0368	0.0240
	−0.1484	0.0650	0.0262
1.6	−0.1631	−0.0058	0.0266
	−0.1631	−0.0117	0.0267
	−0.1486	−0.0140	0.0223
	−0.1629	−0.0092	0.0266
1.8	−0.1045	−0.0363	0.0122
	−0.1069	−0.0285	0.0122
	−0.0898	−0.0253	0.0087
	−0.1052	−0.0378	0.0125
2.0	−0.0500	−0.0392	0.0040
	−0.0509	−0.0231	0.0031
	−0.0388	−0.0188	0.0019
	−0.0464	−0.0380	0.0036

TABLE IV. The approximate amplitudes using $V_T \approx V_{TA}$. For each value of q, the column contains $f_{AG}, f_A, f_{AHG}, f_{AH}$, and f_E. The parameters used are $k=2.0$, $A=0.20$, $\rho=0.30$, and $G=-0.10$ and -0.40.

	Re f		Im f		$\|f\|^2$	
q	$G=-0.40$	$G=-0.10$	$G=-0.40$	$G=-0.10$	$G=-0.40$	$G=-0.10$
0.0	6.308	1.612	1.320	0.0837	41.54	2.607
	6.308	1.612	1.320	0.0837	41.54	2.607
	6.287	1.612	1.392	0.0848	41.47	2.607
	6.287	1.612	1.392	0.0848	41.47	2.607
	6.305	1.612	1.318	0.0834	41.50	2.605
0.4	4.686	1.200	1.115	0.0705	23.20	1.444
	4.634	1.196	1.112	0.0709	22.71	1.436
	4.666	1.200	1.180	0.0714	23.17	1.444
	4.607	1.196	1.174	0.0719	22.61	1.436
	4.633	1.195	1.105	0.0703	22.69	1.432
0.8	1.783	0.4601	0.6560	0.0411	3.609	0.2134
	1.657	0.4521	0.6366	0.0417	3.149	0.2061
	1.765	0.4600	0.7037	0.0418	3.612	0.2134
	1.623	0.4517	0.6715	0.0423	3.084	0.2058
	1.662	0.4502	0.6275	0.0409	3.155	0.2044
1.2	0.1642	0.0473	0.2395	0.0147	0.0843	0.00245
	0.0604	0.0403	0.1974	0.0142	0.0426	0.00182
	0.1525	0.0472	0.2632	0.0150	0.0925	0.00246
	0.0366	0.0400	0.2030	0.0144	0.0426	0.00181
	0.0653	0.0408	0.1976	0.0140	0.0433	0.00186
1.6	−0.1436	−0.0315	0.0214	0.0012	0.0211	0.00099
	−0.1519	−0.0327	−0.0102	0.0003	0.0232	0.00107
	−0.1480	−0.0315	0.0240	0.0013	0.0225	0.00099
	−0.1559	−0.0328	−0.0136	0.0003	0.0245	0.00107
	−0.1629	−0.0333	−0.0093	0.0004	0.0266	0.00111
2.0	−0.0763	−0.0155	−0.0349	−0.0019	0.0070	0.00024
	−0.0356	−0.0131	−0.0265	−0.0017	0.0020	0.00018
	−0.0746	−0.0155	−0.0434	−0.0021	0.0075	0.00024
	−0.0339	−0.0131	−0.0245	−0.0017	0.0018	0.00017
	−0.0464	−0.0141	−0.0380	−0.0023	0.0036	0.00020

should be regarded as excellent if the above trend is reasonably well reproduced.

Obviously, if one defines V_{eff} using a particular representation, either (2.21), (2.24), or (2.29), then $|f_E|^2$ can be reproduced. Table III shows a rough fit which is obtained using the behavior of $|f|^2$ as functions of the parameters G, A, and ρ, as given in Table II. For $k=2.0$, we have roughly

$$\begin{aligned} &f_E; \quad G=-0.400, \quad A=0.200, \quad \rho=0.300, \\ &f_G; \quad G=-0.385, \quad A=0.190, \quad \rho=0.315, \\ &f_{0I}; \quad G=-0.415, \quad A=0.210, \quad \rho=0.330. \end{aligned} \tag{3.2}$$

Since we picked the larger $|G|$ value, the effect discussed above is exaggerated, but again the trend seems to persist as G and k are varied.

B. Effect of V_T

The form (2.44) and (2.45) with V_T in the approximation (2.46) is evaluated and also a similar form with $\beta=0$. The result is very encouraging, except in the region beyond the second maximum, and this indicates that (2.39) is probably not a good approximation. The form (2.35) with $V_{\text{eff}}=V+\text{Re}V_T$, where V_T is given by (2.39), is also tried, and the result seems to indicate that the β-dependent factor is important as soon as the $\partial/\partial b$ factors in T are included. Finally, some attempts were made to replace the factor $\exp(i2\beta^2kZ)$ by an equivalent β-independent form. The β dependence in J_0 is so strong, however, that it was difficult to cancel it in any other way. The result is given in Table IV.

Thus, unless one is willing to go to more involved large-angle formulas, such as those given by Schiff, the accuracy involved in the various amplitudes is about the same near the second maximum, and it would be difficult to extract any other new effects of the same order of their difference in an unambiguous way.

IV. PARTICLE CORRELATIONS

We consider in this section the nucleon scattering from a composite target nucleus using the representations of the two-particle amplitude discussed in Sec. II. Since the nucleons inside the target nucleus are bound and interacting with each other, the scattering amplitude should contain information of this correlation as well as the off-energy-shell properties of the two-nucleon amplitude. They show up at high energies for large momentum transfer q as corrections to the uncorrelated amplitudes. Thus, in order to isolate these effects in a convincing manner, one has to have first of all a reasonably efficient representation of the amplitude at moderately large angles, and also a reliable theory to take the correlation into account. We consider in some detail the formulation proposed recently[8] and compare it with the multiple diffraction theory of Glauber[4] and its modification by Feshbach.[11] We restrict our discussion to elastic scattering.

The Glauber theory[4] assumes the representation (1.1) and writes the nucleon-nucleon elastic scattering amplitude in the form

$$F(k,q)=\int_0^\infty bdb\,J_0(qb)B(k,b), \tag{4.1}$$

where

$$B(k,b)=ik\int d\tau\,|\psi_0(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_A)|^2\delta^{(3)}(A^{-1}\sum_i \mathbf{r}_i)\times\Gamma(\mathbf{b},\mathbf{b}_1,\cdots,\mathbf{b}_A), \tag{4.2}$$

$$\Gamma\equiv1-e^{i\chi},\qquad d\tau=d^3r_1\cdots d^3r_A.$$

ψ_0 is the target ground-state wave function and χ depends on the positions of all A target nucleons. The essential simplifying assumption of Glauber is that

$$\chi=\sum_{i=1}^A\chi_i(\mathbf{b}-\mathbf{b}_i,k), \tag{4.3}$$

and χ_i in turn can be expressed as

$$\Gamma_i(\mathbf{b}-\mathbf{b}_i,k)\equiv1-e^{i\chi_i}=(2\pi ik)^{-1}\int e^{-i\mathbf{q}\cdot(\mathbf{b}-\mathbf{b}_i)}\times f_i(\mathbf{q},\mathbf{k})d\mathbf{q}_\perp. \tag{4.4}$$

In (4.4), f_i is the two-nucleon amplitude which is assumed given. Substitution of (4.3) into (4.1) gives immediately the multiple-scattering expansion

$$F=F_1+F_2+\cdots, \tag{4.5}$$

where

$$F_1=\frac{ik}{2\pi}\sum_i\int d^2b\,e^{i\mathbf{q}\cdot\mathbf{b}}\int\delta(A^{-1}\sum_i\mathbf{r}_i)\Gamma_i|\psi_0|^2d\tau, \tag{4.6}$$

$$F_2=-\frac{ik}{2\pi}\sum_{i>j}\int d^2b\,e^{i\mathbf{q}\cdot\mathbf{b}}\int\delta(A^{-1}\sum_i\mathbf{r}_i)\Gamma_i\Gamma_j|\psi_0|^2d\tau,\quad\text{etc.} \tag{4.7}$$

As it stands, F_2 contains some correlations coming from the c.m. δ function and also the averaging over the target ground state. However, the approximation (4.3) may be too crude.

An improved form of (4.3) has been given recently by Feshbach,[11] who writes

$$\chi=\sum_i\chi_i+\sum_{i>j}\omega_{ij}+\cdots. \tag{4.8}$$

Then, F can be written again in a series of the form (4.5), with F_1 unchanged, but F_2 containing an additional unfactorizable term, as

$$F_2\rightarrow-\frac{ik}{2\pi}\sum_{i>j}\int d^2b\,e^{i\mathbf{q}\cdot\mathbf{b}}\int d\tau\,\delta(A^{-1}\sum_i\mathbf{r}_i)\times[\Gamma_i\Gamma_j-H_{ij}]|\psi_0|^2, \tag{4.9}$$

where

$$H_{ij}=1-e^{i\omega_{ij}}.$$

If the interaction between the incoming projectile and the target nucleons are additive, with

$$V=H-H_0=\sum_{i=1}^A V_i(\mathbf{r}-\mathbf{r}_i), \tag{4.10}$$

then the approximation which led to the Glauber formula (2.25) also gives precisely the form (4.3) with each χ_i given essentially by (2.26). Obviously, in the present case the neglect of the β factor and $\nabla\mathbf{r}^2$ term in the equation may have a more serious effect, but it is extremely difficult in this way to untangle the approximations involved in the impact parameter representation and in the correlations.

If we assume that the energy transfer during each collision is negligible compared to k^2, then the inelastic channels projected by the operator Q is negligible, and the "static" part gives[8]

$$B=B_0+B_P+B_Q,$$

$$=B_0+B_1\approx B_0, \qquad (4.11)$$

where

$$B_0\approx ik(1-e^{i\chi_0^P}), \qquad (4.12)$$

$$\chi_0^P=-k^{-1}\int^{\infty} dZ\langle PVP\rangle$$

$$=-k^{-1}\int_{-\infty}^{\infty} dZ\, A\int d\tau\,|\psi_0|^2\delta(A^{-1}\sum_i \mathbf{r}_i)\times V_i(\mathbf{r}-\mathbf{r}_i), \qquad (4.13)$$

and

$$B_1\approx -\frac{i}{k}\int_{-\infty}^{\infty} dZ\Big\langle PV(P\Phi_0+Q)\int_{-\infty}^{Z} dZ'(P\Phi_0^{-1}+Q)\times\Lambda\Big[\Lambda+\frac{i}{k}\Lambda(P\Phi_0+Q)\int_{-\infty}^{Z} dZ'(P\Phi_0^{-1}+Q)\Lambda\Big]^{-1}\Lambda P\Phi_0\Big\rangle. \qquad (4.14)$$

In (4.14) we have

$$P=\psi_0\rangle\langle\psi_0, \qquad Q=1-P, \qquad QP=0,$$

and

$$\Lambda=(H-E_0)-PVP, \qquad E=E_0+\tfrac{1}{2}k^2.$$

B_1 can be written in a more compact form as

$$B_1\approx\int_{-\infty}^{\infty} dZ\left\langle PVS\frac{1}{1-S}\Lambda P\Phi_0\right\rangle,$$

where

$$S\equiv-\frac{i}{k}(P\Phi_0+Q)\int_{-\infty}^{Z} dZ'(P\Phi_0^{-1}+Q)\Lambda.$$

The approximation involved in (4.12) and (4.14) is simply $\beta=0$, and $P\Phi_0$ is given by

$$P\Phi_0=\exp\left(-\frac{i}{k}\int_{-\infty}^{Z}\langle PVP\rangle dZ'\right). \qquad (4.15)$$

That is, in the static approximation (4.11), χ_0^P is automatically additive if V is given by (4.10). If we define

$$\gamma\equiv1-e^{i\chi_0^P/A}=\langle P\gamma P\rangle, \qquad (4.16)$$

where γ and χ_0^P are functions of $\mathbf{b}$ and $\mathbf{k}$ only, then

$$B_0=ik[A\gamma-\tfrac{1}{2}A(A-1)\gamma^2+\cdots]. \qquad (4.17)$$

Equation (4.17) may be compared with (4.7), since they involve

$$B\sim\sum_i\langle\Gamma_i\rangle-\sum_{i>j}\langle\Gamma_i\Gamma_j\rangle+\cdots, \qquad (4.7')$$

$$B_0\sim A\langle\gamma\rangle-\tfrac{1}{2}A(A-1)\langle\gamma\rangle^2+\cdots. \qquad (4.17')$$

Obviously (4.7′) does not allow the ith target nucleon to be scattered more than once during a complete scattering, while (4.17′) may include such collisions because the second term involving $\langle\gamma\rangle^2$, for example, does not distinguish the target nucleons. $\Gamma_i\cdot\Gamma_j$ is a separable two-particle operator and carries some correlations as

$$\langle\Gamma_i\Gamma_j\rangle=\langle\Gamma_i(P+Q)\Gamma_j\rangle$$

$$=\langle\Gamma_i\rangle\langle\Gamma_j\rangle+\langle\Gamma_iQ\Gamma_j\rangle. \qquad (4.18)$$

According to Feshbach, one would like to generalize (4.18) by adding further correlation terms H_{ij} of (4.9).

The static amplitude B_0 does not include the Q-space effect by definition, but eventually one is interested in parameterizing the complete F in terms of the two-nucleon amplitude f. This can be simply carried out by replacing χ_0^P by an effective χ_{eff} involving both single- and double-nucleon interactions. How such a modification based on the forms B_0 and also how B_1 are related to f_i is not entirely clear.

We now study the leading terms (4.7′) and (4.19′). If we assume that χ_i is reasonably small at high energies, then we have

$$\langle\Gamma_i\rangle=\langle1-e^{i\chi_i}\rangle$$

$$=-i\langle\chi_i\rangle-\langle\chi_i^2\rangle+\cdots, \qquad (4.19)$$

and

$$\langle\gamma\rangle=1-e^{i\langle\chi_i\rangle}$$

$$=-i\langle\chi_i\rangle-\langle\chi_i\rangle^2+\cdots. \qquad (4.20)$$

Obviously, the difference between $\langle\Gamma_i\rangle$ and $\langle\gamma\rangle$ is, to second order,

$$(\chi)_{\text{fl}}=\langle\chi_i^2\rangle-\langle\chi_i\rangle^2, \qquad (4.21)$$

which gives the fluctuation of the single-nucleon scattering phase within the nucleus, as in the compound-elastic scattering. For small energy transfer during the collision, however, we expect that $(\chi)_{\text{fl}}$ may be small, and then γ is related to f_i by (4.4). In this case the series for B_0 is extremely simple.

To first order in $\langle\chi_i\rangle$, the various theories proposed so far all give the identical result, but there is much variation in the second-order effect. When they are small, it is not easy to distinguish between the $\beta=0$ approximations and the difference among the second-order expressions, and that makes it difficult to analyze the experiments correctly.

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