ACKNOWLEDGMENT

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APPENDIX

Using the convention $\tau_3^{(i)}p(i)=p(i)$, $\tau_3^{(i)}n(i)=-n(i)$, the third component of the nucleon isospin operator applied to the isospin functions of ^{3}He given by Eqs. (7) and (11) yields

$$\tau_3^{(1)}\zeta'=\zeta',$$
$$\tau_3^{(2)}\zeta'=-(1/\sqrt{3})\zeta''+(\sqrt{\tfrac{3}{2}})\zeta^s,$$
$$\tau_3^{(3)}\zeta'=+(1/\sqrt{3})\zeta''-(\sqrt{\tfrac{3}{2}})\zeta^s,$$
$$\tau_3^{(1)}\zeta''=-\tfrac{1}{3}\zeta''-\tfrac{2}{3}\sqrt{2}\zeta^s,$$
$$\tau_3^{(2)}\zeta''=-(1/\sqrt{3})\zeta'+\tfrac{2}{3}\zeta''+\tfrac{1}{3}\sqrt{2}\zeta^s,$$
$$\tau_3^{(3)}\zeta''=+(1/\sqrt{3})\zeta'+\tfrac{2}{3}\zeta''+\tfrac{1}{3}\sqrt{2}\zeta^s.$$

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Independent-Particle-Model Energy-Level Formula*

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The Morse function is assumed to approximate a nonlocal or velocity-dependent nucleon-nuclear potential. Analytic neutron and proton wave functions and eigenvalue formulas are obtained for all states of all nuclei, using a modified version of an analytic perturbation method due to Pekeris. The eigenvalue formula is in approximate agreement with the following experimental data: (1) the last-particle binding energies of neutrons and protons, (2) the neutron and proton magic numbers, (3) the positions of the S-wave-size resonances in total-neutron-cross-section data, and (4) the recent experimental work on deeply bound inner particle states by Amaldi *et al.* The eigenvalue formula gives results that are consistent with recent Hartree-Fock calculations.

I. INTRODUCTION

PREVIOUSLY, Green, Darewych, and Berezdivin[1] (GDB) noted the possibility of using the Morse potential to arrive at analytic eigenvalues and analytic wave functions for single-particle states in velocity-dependent nuclear potentials. Their results compared favorably with the numerical solutions of Wyatt, Wills, and Green.[2] The procedure of GDB involved fitting the effective energy-dependent potential together with the centrifugal potential by an approximating Morse potential.[3] In the present work, the numerical fitting procedure used in the earlier study is replaced by a modified version of a technique used by Pekeris.[4] Thus, one arrives at explicit formulas for the effective state-dependent Morse-function parameters and for the eigenvalues. This should greatly enhance the utility of the approximate method.

II. VELOCITY-DEPENDENT POTENTIALS

In this work, velocity-dependent potentials of the form

$$V(\mathbf{r},\mathbf{p})=-V_0\xi_0(\mathbf{r})-(\delta\hbar^2/8m)\times[\nabla^2\xi(\mathbf{r})+2\nabla\cdot\xi(\mathbf{r})\nabla+\xi(\mathbf{r})\nabla^2] \quad (1)$$

are considered. Here, $\xi_0(r)$ is the static form factor, which may differ from $\xi(r)$, the velocity-dependent form factor. If use is made of the wave function $R(r)Y_l^m(\theta,\phi)$ and the definitions

$$x=r/a, \quad \epsilon_w^2=-W/E_0, \quad R(x)=G(x)/x,$$
$$E_0=\hbar^2/2ma^2, \quad \epsilon_0^2=V_0/E_0, \quad X(x)=(1+\delta\xi)^{1/2}G(x); \quad (2)$$

it is possible to transform the usual Schrödinger equation for such potentials into the form

$$X''-v(x,\epsilon_w)X-[\epsilon_w^2/(1+\delta)]X=0, \quad (3)$$

where $v(x,\epsilon_w)$ is the energy-dependent potential

$$v(x,\epsilon_w)=-\frac{\epsilon_0^2\xi_0}{1+\delta\xi}+\frac{1}{4}\frac{\delta\xi''}{1+\delta\xi}-\frac{1}{4}\frac{\delta^2\xi'^2}{(1+\delta\xi)^2}+\frac{\delta\xi'}{2x(1+\delta\xi)}+\frac{l(l+1)}{x^2}+\frac{\delta(1-\xi)\epsilon_w^2}{(1+\delta\xi)(1+\delta)}. \quad (4)$$

In this paper, an analytic method is developed for the approximate solution of Eq. (3) in certain cases of interest in nuclear physics. Form factors ξ_0 and ξ are assumed, for which the first four terms of Eq. (4) become a Morse function. Then all remaining terms are treated by the quadratic perturbation method.

III. MATHEMATICAL PROPERTIES OF THE MORSE POTENTIAL

The form of the Morse function used in this calculation is[3]

$$v_m(x)=\alpha_0^2\{\exp[-2(x-x_0)/d]-2\exp[-(x-x_0)/d]\}. \quad (5)$$

* Work supported in part by the Air Force Office of Scientific Research Grant No. AFOSR-68-1397.
† National Science Foundation Trainee.
[1] A. E. S. Green, G. Darewych, and R. Berezdivin, Phys. Rev. **157**, 929 (1967).
[2] P. J. Wyatt, J. G. Wills, and A. E. S. Green, Phys. Rev. **119**, 1031 (1960).
[3] P. M. Morse, Phys. Rev. **34**, 57 (1929).
[4] C. L. Pekeris, Phys. Rev. **45**, 98 (1934).

Now substituting this into Eq. (3) for $v(x, \epsilon_w)$ and rewriting in terms of the variable $y = \exp[-(x-x_0)/d]$, one obtains

$$\frac{y^2}{d^2}\frac{d^2\chi}{dy^2} + \frac{y}{d^2}\frac{d\chi}{dy} \quad \left[\frac{\epsilon_w^2}{1+\delta} + \alpha_0^2(y^2 - 2y)\right]\chi = 0; \tag{6}$$

and now defining

$$b = 2\epsilon_w d/(1+\delta)^{1/2}, \qquad k = 2\alpha_0 d, \qquad z = ky,$$
$$F(z) = e^{z/2} z^{-b/2}\chi(z) \tag{7}$$

and making the appropriate transformations, one arrives at the eigenvalue and eigenfunction

$$W = -E_0(1+\delta)[\alpha_0 - (n-\tfrac{1}{2})/d]^2, \tag{8}$$

$$F = c_0 F(-n \mid b+1 \mid z)$$
$$= c_0 \sum_{j=0}^{n-1} \frac{(-1)^j (n-1)!\,\Gamma(b+1)}{(n-j-1)!\,j!\,\Gamma(b+j+1)} z^j. \tag{9}$$

The normalization constant c_0 cannot be given explicitly as the unspecified quantity $(1+\delta\xi)$ is contained in an integral. If $\delta = 0$, the constant c_0 is given by

$$c_0 = \left[\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} (-1)^{j+l} \times \frac{d[(n-1)!]^2\Gamma(b+j+l)\,\Gamma^2(b+1)}{(n-j-1)!(n-l-1)!\,j!\,l!\,\Gamma(b+j+1)\,\Gamma(b+l+1)}\right]^{-1/2}. \tag{10}$$

The analytic wave function and eigenvalue apply only for s states. Pekeris[4] has given a method for handling the centrifugal perturbations by using the quadratic representation

$$v_p = c_1 + c_2 y + c_3 y^2. \tag{11}$$

Pekeris evaluated the constants using a particular

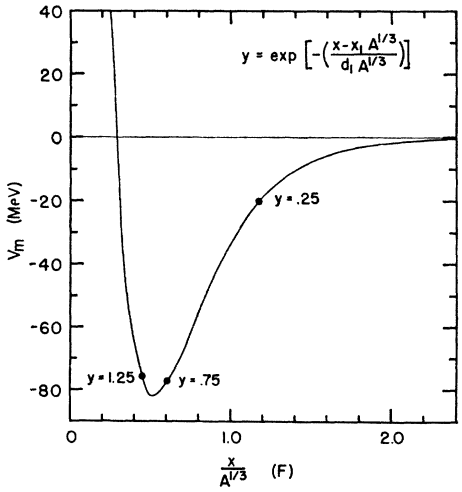

FIG. 1. A Morse function approximates the nonlocal nucleon-nuclear potential. The fitting points for perturbations are indicated on the plot.

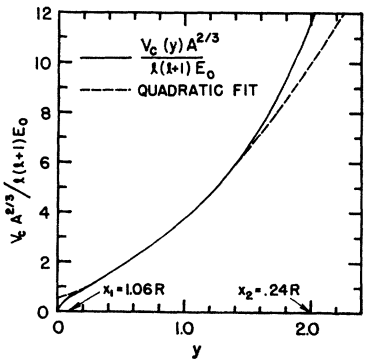

FIG. 2. The centrifugal potential and its quadratic fit in y space. Note that $y=0$ corresponds to $x=\infty$, and $y\simeq 4.85$ corresponds to $x=0$. The points at which the fit deteriorates (x_1 and x_2) are given as functions of the nuclear radius (R).

prescription, which is well suited to diatomic molecules but not nuclei. In the present work, the quadratic representation is used for all perturbations although varying prescriptions for the c's are employed.

In the presence of a number of quadratic perturbations, the eigenvalue is given by

$$W = 20.73[(A+1)/A](1+\delta) \times \left[\Omega_1 - \left(\frac{\alpha_0^2 - \frac{1}{2}\Omega_2}{(\alpha_0^2+\Omega_3)^{1/2}} - \frac{n-\frac{1}{2}}{d}\right)^2\right] \text{MeV}, \tag{12}$$

when $a = 1$ F and $\Omega_i = \Sigma_j c_{ij}$. The wave function is of the same form as Eq. (9), except that now

$$z = 2(\alpha_0^2+\Omega_3)^{1/2} dy. \tag{13}$$

Equation (12) is our basic eigenvalue formula. Let us now consider the evaluation of the perturbation terms.

IV. CENTRIFUGAL POTENTIAL

Expressing the centrifugal potential in terms of the variable y, one has $V_c = l(l+1)g(y)$, where

$$g(y) = (x_0 - d\ln y)^{-2}. \tag{14}$$

This function must be approximated by a quadratic perturbation. In the work of Pekeris, the constants were determined by expanding $V_c(y)$ in a Taylor series about the point $y=1$, and then evaluating c_1, c_2, and c_3 from the expansion. For this work in which the centrifugal effect is proportionally much larger, the function $V_c(y)$ is fitted at three strategically located positions of y: Specifically, $y=0.25$, which corresponds approximately to the surface of the velocity-dependent nucleon-nuclear potential, and $y=0.75$ and 1.25, which correspond to the points on either side of the potential minima (see Fig. 1). Figure 2 shows the function $V_c(y)$ and the quadratic approximating function. Clearly, these functions agree quite well except near the origin and in the region outside the nucleus.

In order to determine quantitatively the accuracy of eigenvalues obtained from the quadratic method, they have been compared with numerical calculations of the

TABLE I. Accuracy of the quadratic perturbation method for the centrifugal potential. The differences in S states are probably due to rounding errors in the calculation of input parameters for the codes.

S-P levels	$A=50$ ABACUS	$A=50$ Q.P.	$A=210$ ABACUS	$A=210$ Q.P.
$1S$	−48.14	−48.15	−57.26	−57.30
$1P$	−38.31	−38.81	−53.17	−53.30
$1D$	−22.95	−23.96	−45.51	−46.07
$1F$	−6.51	−7.08	−35.64	−36.72
$1G$			−24.55	−26.22
$1H$			−13.22	−15.19
$1I$			−2.55	−3.92
$2S$	−11.03	−11.01	−28.57	−28.55
$2P$	−5.88	−5.29	−25.38	−25.44
$2D$			−19.78	−19.82
$2F$			−12.81	−12.55
$2G$			−5.51	−4.35
$3S$			−9.75	−9.71
$3P$			−7.71	−7.49

eigenvalues of some higher-angular-momentum states using the ABACUS Code.[5] Results of these calculations for useful ranges of the parameters x_0 and d are given in Table I. As can be seen, the results are quite good.

To serve as a general nucleon-nuclear potential, the parameters x_0 and d of the Morse potential are assumed to depend upon A as follows:

$$x_0=x_1A^{1/3}, \qquad d=d_1A^{1/3}. \tag{15}$$

The well-depth parameter α_0^2 is maintained independent of A. To ensure that $2s$, $3s$, and $4s$ eigenvalues are zero at mass numbers 12, 55, 155, respectively, which are approximately the positions of the s-wave-size resonances, we impose the relation

$$d_1=0.656/\alpha_0. \tag{16}$$

The resulting centrifugal constants are

$$\begin{aligned} c_1&=l(l+1)[1.87g(0.25)-1.25g(0.75)+0.375g(1.25)], \\ c_2&=l(l+1)[-4g(0.25)+6g(0.75)-2g(1.25)], \\ c_3&=l(l+1)[2g(0.25)-4g(0.75)+2g(1.25)]. \end{aligned} \tag{17}$$

Substitution of these into Eq. (12) yields an eigenvalue formula for nl states as a function of A, with parameters α_0, δ, d_1, x_1. Following the suggestion of Frahn and Lemmer,[6] the value $\delta=1$ is used, and Eq. (16) establishes the value of d_1; hence only α_0 and x_1 are considered free. By fixing α_0 and varying x_1, one may find values for x_1 at which the last-particle separation energies for neutrons are approximately correct over the entire range of mass numbers. One may, in fact, do this over a considerable range of the well-depth parameter α_0.

V. SPIN-ORBIT, SYMMETRY, COULOMB, AND VELOCITY-DEPENDENT PERTURBATIONS

A spin-orbit interaction of the Thomas type is not feasible, since the static and velocity-dependent form factors are not specified. Cohen[7] has cited experimental data which indicate that the spin-orbit potential should be proportional to $A^{-2/3}$. A simple volume spin-orbit interaction was finally chosen

$$V_{so}=A_{so}LA^{-2/3}\alpha_0^2(y^2-2y), \tag{18}$$

where A_{so} is an adjustable parameter, and

$$L=\tfrac{1}{2}l \quad \text{for} \quad j=l+\tfrac{1}{2}, \qquad L=-\tfrac{1}{2}(l+1) \quad \text{for} \quad j=l-\tfrac{1}{2}. \tag{19}$$

Calculations were made using this form for a broad range of well-depth parameters α_0, always maintaining x_1 such that the last particle separation energies for neutrons were reasonable. The constant A_{so} was adjusted in these calculations in an attempt to force the single-particle levels to group in such a way as to give the magic numbers. The search for magic numbers put some definite limits upon the parameter α_0. An α_0 which corresponds approximately to $\alpha_0^2E_0\simeq 45$ MeV gave the best results. Figure 3 shows a plot of the neutron last-particle states as a function of A along the line of β stability. The smooth curve is the numerical curve due to Green which fits the experimental data. The magic number breaks are also indicated.

Proton single-particle states may be considered if one adds, as a perturbation, an electrostatic potential from some assumed charge distribution of the nucleus. For simplicity a Coulomb potential due to a uniformly charged sphere of radius $R=1.2A^{1/3}$ F was assumed. The same quadratic perturbation method may be used, but the electrostatic potential must be divided by $1+\delta\xi$ before it is added to the effective velocity-depen-

TABLE II. Model parameters used in calculations.

Parameter symbol	Parameter description	Numerical value
α_0	Well-depth parameter	2.0
X_1	Potential-minima radial parameter	0.52
d_1	Well-diffuseness parameter	0.328
δ	Velocity-dependence parameter	1.0
A_{so}	Spin-orbit-strength parameter	0.50
A_s	Symmetry-strength parameter	90.0

[5] E. H. Auerbach, Brookhaven National Laboratory Report No. BNL 6562, 1962 (unpublished).

[6] W. E. Frahn and R. H. Lemmer, Nuovo Cimento 5, 1564 (1957).

[7] B. L. Cohen, Am. J. Phys. 33, 1011 (1965).

TABLE III. Quadratic perturbation constants.

Centrifugal	Spin orbit	Coulomb	Symmetry	Energy-dep.
$C_{11}=0.535$		$C_{13}=0.0755$		$C_{15}=-0.0482$
$C_{21}=1.668$	$C_{22}=-4.0$	$C_{23}=0.0985$	$C_{24}=-8.683$	$C_{25}=0.0607$
$C_{31}=1.525$	$C_{32}=2.0$	$C_{33}=-0.0151$	$C_{34}=4.342$	$C_{35}=-0.0124$

dent potential. The quadratic perturbation constants were thus chosen to ensure the following reasonable properties of ξ:

$$\xi\to 0 \quad \text{as} \quad r\to\infty,$$
$$\xi\to 1 \quad \text{as} \quad y=1, \quad \text{and} \quad \xi\to 0 \quad \text{as} \quad r\to 0. \tag{20}$$

These characteristics of ξ, or similar ones for $\xi(0)\neq 0$, are necessary for the velocity-dependent potential, Eq. (4), to have the repulsive core that is characteristic of the Morse function.

As expected,[8] a symmetry term must be added to account for the last-particle separation energies of protons. A symmetry interaction of the form

$$V_{\text{sym}}{}^0=t_z A_s[(N-Z)/A](y^2-2y) \tag{21}$$

was chosen, where A_s is an adjustable parameter, and $t_z=+\frac{1}{2}$ for protons and $t_z=-\frac{1}{2}$ for neutrons. Here A_s is adjusted so as to give correct last-particle separation energies for both neutrons and protons and the parameters shown in Fig. 3 are readjusted. The last term of Eq. (4) $\delta/(1+\delta)(1+\delta\xi)^{-1}(1-\xi)\epsilon_w{}^2$ may also be treated by the quadratic perturbation method using relations (20). Inclusion of this term will require an iteration of Eq. (12), in order to solve for W. Convergence is rapid as the term is tailored to be small over the important regions of the nucleus.

VI. WAVE FUNCTIONS AND THE EIGENVALUE FORMULA

In view of the perturbations discussed in Secs. IV and V, one may now write the single-particle eigenvalue formula as a function of the state variables A, Z, n, l, j, and t_z using Eq. (12). The expressions for Ω_1, Ω_2, and Ω_3 may be put in the form

$$\Omega_1=C_{11}\frac{l(l+1)}{A^{2/3}}+C_{13}(t_z+\tfrac{1}{2})\frac{Z-1}{A^{1/3}}+C_{15}\frac{\delta}{1+\delta}W, \tag{22}$$

$$\Omega_2=C_{21}\frac{l(l+1)}{A^{2/3}}+C_{22}\frac{L}{A^{2/3}}+C_{23}(t_z+\tfrac{1}{2})\frac{Z-1}{A^{1/3}}+C_{24}t_z\frac{A-2Z}{A}+C_{25}\frac{\delta}{1+\delta}W, \tag{23}$$

$$\Omega_3=C_{31}\frac{l(l+1)}{A^{2/3}}+C_{32}\frac{L}{A^{2/3}}+C_{33}(t_z+\tfrac{1}{2})\frac{Z-1}{A^{1/3}}+C_{34}t_z\frac{A-2Z}{A}+C_{35}\frac{\delta}{1+\delta}W. \tag{24}$$

A set of model parameters α_0, x_1, d_1, δ, A_{so}, and A_s and

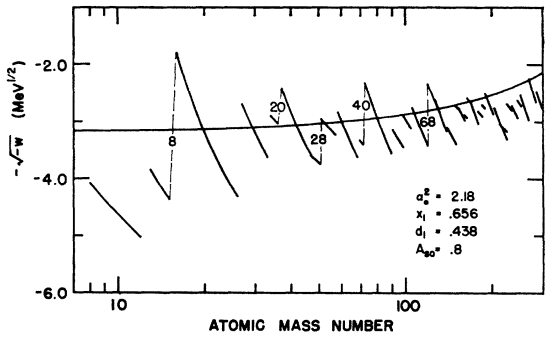

FIG. 3. The square root of neutron single-particle energy levels for the last-particle states along the line of β stability. Magic number breaks are indicated.

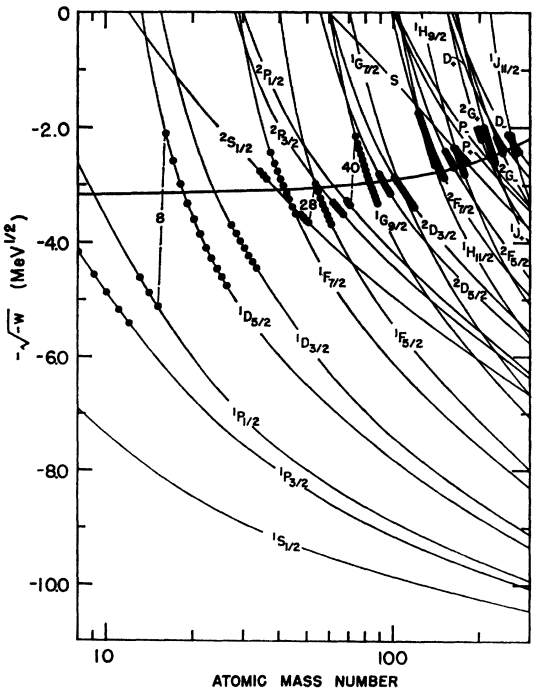

FIG. 4. The square root of neutron single-particle energy levels along the line of β stability. Magic numbers and last-particle binding energies are indicated. Because of a lack of space the j quantum numbers for some states are denoted by (+) or (−). Also the N quantum number is deleted for the $N=3$ levels.

[8] A. E. S. Green, T. Sawada, and D. S. Saxon, *The Nuclear Independent Particle Model* (Academic Press Inc., New York, 1968).

TABLE IV. Comparison of the author's model (MG) with experimental proton states (Ref. 9).

State	O^{16} Expt	O^{16} MG	Al^{27} Expt	Al^{27} MG	S^{32} Expt	S^{32} MG	Ca^{40} Expt	Ca^{40} MG
$1S_{1/2}$	43±5	57.5	57±7	69	80±8	73	77±14	77
$1P_{3/2}$	19±1	30		46		51		58
			32±3		43±7		32±4	
$1P_{1/2}$	12±1	21		38		44		50

the corresponding parameters C_{ij} are given in Tables II and III.

In Figs. 4 and 5, the neutron and proton single-particle states are plotted as functions of atomic-mass number. In order to place more emphasis on the high-lying states, the square root of the absolute value of the eigenvalue has been chosen as the ordinate. The last-particle states, indicated by closed black circles, cluster quite well around the line of average last-particle separation energies. The magic numbers are indicated where space permits. In Fig. 4, the magic number 20 for neutrons is not strong, owing to the height of the $2S_{1/2}$ state. The magic number 40 looks stronger than 50, because of the closeness of the levels $1G_{9/2}$ and $2D_{5/2}$. The magic number 82 fails to occur because the $3S_{1/2}$ state is too high; likewise the magic number 126 fails only because the $4S_{1/2}$ state (not shown) is much too high. In Fig. 5 the proton magic numbers 20, 28, 82 all have similar problems with S states.

There exist some direct experimental data on single-particle levels for moderately large nuclei, and although such data have not been considered in the variation of the model parameters, it is interesting to compare the results. In Table IV calculations with the authors' model (MG) are compared with experimental proton levels for O^{16}, Al^{27}, S^{32}, and Ca^{40}, which were obtained from studies of the $(P, 2P)$ and $(e, e'P)$ reactions.[9] Note the agreement is better for the heavier nuclei. In Table V the neutron levels of O^{16} and Ca^{40} are compared with recent Hartree-Fock calculations. The first calculations [Bassichis, Kerman, and Svenne (BKS)] have been corrected for deformations.[10] The second calculation [McCarthy and Köhler (MK)] has used a two-body interaction derived from meson field theory.[11] As can be seen, the agreement is quite good. Recent advances[12–14] have enabled Hartree-Fock calculations to be made for quite large nuclei, and in Table VI Hartree-Fock calculations for the neutron levels of Zr^{90} and Pb^{208} [Tarbutton and Davies (TD)] are compared

TABLE V. Comparison of the authors' model (MG) with Hartree-Fock calculations (BKS) (Ref. 10) and (MK) (Ref. 11) for neutrons states.

State	O^{16} BKS	O^{16} MK	O^{16} MG	Ca^{40} BKS	Ca^{40} MG
$1S_{1/2}$	−52	−61.0	−65		
$1P_{3/2}$	−26.59	−33.2	−38	−62.74	−64
$1P_{1/2}$	−17.90	−25.1	−28	−50.88	−58
$1D_{5/2}$				−35.15	−35
$1D_{3/2}$				−19.92	−20

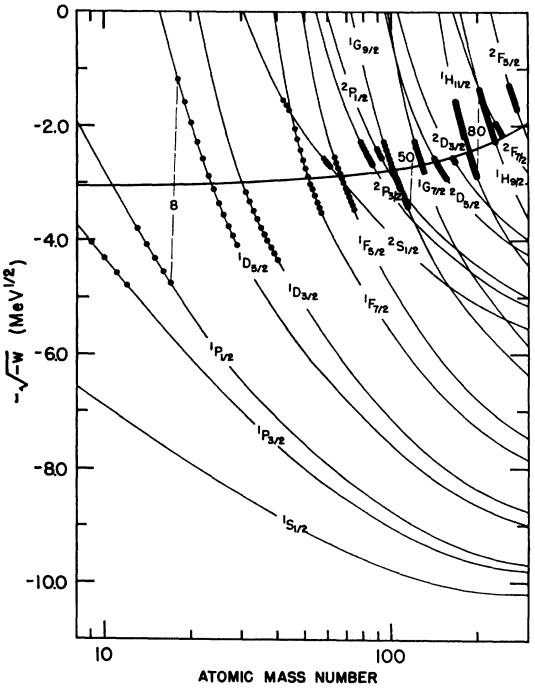

FIG. 5. The square root of proton single-particle energies along the line of β stability. Magic numbers and last-particle binding energies are indicated.

[9] U. Amaldi, invited paper to the fifty-second Congresso della Società Italiana di Fisica, 1966 (unpublished).

[10] W. H. Bassichis, A. K. Kerman, and J. P. Svenne, Phys. Rev. **160**, 746 (1967).

[11] R. J. McCarthy and H. S. Köhler, Phys. Rev. Letters **20**, 671 (1968).

[12] R. M. Tarbutton and K. T. R. Davies, Nucl. Phys. **A120**, 1 (1968).

[13] M. Baranger, in *Proceedings of the International Nuclear Physics Conference, Gatlinburg, Tenn., 1966*, edited by R. Becker *et al.* (Academic Press Inc., New York, 1967).

[14] C. W. Nestor, K. T. R. Davies, S. J. Kreiger, and M. Baranger, Nucl. Phys. **A113**, 14 (1968).

with the authors' model.[12] The agreement is reasonable, except for the $2S_{1/2}$ and $3S_{1/2}$ levels, which are much too high. This is due, of course, to the strong repulsive core of the Morse function. Note that the agreement is still reasonable for the $l>0$ states for higher principal quantum numbers.

A major improvement for the S states could be had for particular nuclei or a group of nuclei by subtracting a quadratic perturbation term which cancels part of the repulsive core of the Morse function. For example, one could replace $l(l+1)$ in the centrifugal perturbation term by $l(l+1)-\lambda$ and readjust the basic Morse parameters. This would correspond approximately to subtracting the term λ/x^2 from the Morse potential. The parameter λ would establish a control over the strength of the repulsive core of the Morse potential.

TABLE VI. Comparison of the authors' model (MG) with Hartree-Fock calculations (TD) (Ref. 12) for neutron states of heavy nuclei.

State	Zr^{90} TD	Zr^{90} MG	Pb^{208} TD	Pb^{208} MG
$1S_{1/2}$	−90.8	−96.0	−106.2	−105.34
$1P_{3/2}$	−68.0	−82.0	−86.05	−96.10
$1P_{1/2}$	−67.3	−77.2	−84.85	−93.035
$1D_{5/2}$	−46.3	−58.7	−66.95	−79.39
$1D_{3/2}$	−44.7	−51.6	−65.64	−74.49
$2S_{1/2}$	−41.4	−22.66	−63.94	−37.68
$1F_{7/2}$	−26.2	−33.75	−48.88	−58.98
$1F_{5/2}$	−23.1	−25.79	−47.54	−52.89
$2P_{3/2}$	−20.2	−17.52	−44.23	−33.38
$2P_{1/2}$	−18.7	−15.49	−43.35	−31.75
$1G_{9/2}$	−8.4	−11.73	−32.11	−38.65
$1G_{7/2}$	−3.2	−8.43	−30.34	−32.07
$2D_{5/2}$			−26.58	−25.60
$2D_{3/2}$			−24.75	−23.05
$3S_{1/2}$			−23.97	−8.83
$1H_{11/2}$			−16.88	−20.34
$1H_{9/2}$			−14.18	−13.96
$2F_{7/2}$			−11.27	−15.81
$2F_{5/2}$			−8.50	−12.85
$3P_{3/2}$			−8.18	−6.80
$3P_{1/2}$			−6.89	−6.13
$1I_{13/2}$			−3.30	−4.64

VII. CONCLUSIONS

An analytic wave function and an eigenvalue formula have been obtained which represent neutron and proton single-particle states in all nuclei. Effects due to the spin-orbit interaction, symmetry effect, Coulomb interaction, and velocity dependence are included. The model agrees approximately with data on last-particle separation energies and magic numbers. Fair agreement with experimental data on single-particle energies is obtained. With the exception of some S states, fair agreement with recent Hartree-Fock calculations is obtained. The agreement could be improved by local variations of the model parameters.

Some justification for a repulsive core in the effective n-N interaction arising from velocity-dependent effects has been given. Even should such a core exist, it is probable that the core of the Morse function is unrealistically strong. Still it seems that only S-state energy levels are harmed by this.

The wave function Eq. (9) is analytic; however, it is not the true radial wave function. The true radial wave function is given by

$$G(x)=(1+\delta\xi)^{-1/2}e^{-z/2}z^{b/2}F(z). \tag{25}$$

Since ξ is undetermined, the true radial wave function is analytic only in the static limit ($\delta=0$). The analyticity of the static Morse single-particle wave functions suggests that they may form a convenient basis set from which to do Hartree-Fock calculations. The energy levels in the Morse potential are closer to the Hartree-Fock results than are those of the harmonic-oscillator potential. Difficulties arising from the unrealistic behavior of the harmonic-oscillator potential at the nuclear surface probably would not occur with the Morse wave functions. On the other hand, one would be almost certain to encounter difficulties with the S states with the Morse wave functions. The Morse potential has a finite number of discrete levels, which may or may not be an advantage over the infinite number of harmonic-oscillator levels. It will probably be necessary to attempt Hartree-Fock calculations with the Morse basis before the question may be resolved.