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Isospin Sum Rules and the Photodisintegration of the $A=3$ Nuclei

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Three sum rules for the electric dipole photodisintegration cross section of the $A=3$ nuclei are split into their final-state isospin components. The isospin doublet contribution to the three-nucleon breakup mode is estimated to be of the order of 10–20% and originates primarily in the high-energy region.

I. INTRODUCTION

ELECTRIC dipole sum rules allow for the calculation of the first few moments of the total photoabsorption cross section from the ground-state wave function and the two-nucleon potential. Since isospin is a good quantum number for low Z nuclei, it is interesting to split the photo cross section into its isospin components and analyze the relation between final isospin states and excitation or breakup modes. The first example of any such relation was found in the photodisintegration of the ^{3}H-^{3}He isospin doublet; where, by the use of isospin sum rules for a ground state with a fully symmetric space component, Barton[1] has shown that the two- and three-body photodisintegration channels are predominantly isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively.

In this paper, we derive isospin component sum rules. Although we restrict ourselves to the $A=3$ isospin doublet some of the results have a larger validity. In Sec. II, we show how the sum rules are written for each final-state isospin. In Sec. III, the general isospin wave functions for ^{3}H and ^{3}He are presented. The bremsstrahlung sum rule is calculated in Sec. IV and is related to the proton and neutron mean square radii. Its relation to the nonrelativistic Cabibbo-Radicati sum rule[2] is shown. The ordinary and energy-weighted isospin sum rules are derived for particular potentials in Secs. V and VI. Comparison with the data to extract the isospin purity of the breakup modes is made in Sec. VII. The Appendix gives the isospin algebra used in the derivation of the sum rules.

[1] G. Barton, Nucl. Phys. **A104**, 289 (1967).
[2] N. Cabibbo and L. A. Radicati, Phys. Letters **19**, 697 (1966).

II. DERIVATION OF THE SUM RULES

The electric dipole photodisintegration cross section is given by

$$\sigma=(4\pi^2/\hbar c)E_\gamma\,|\langle f\,|\,\mathfrak{D}\,|\,0\rangle|^2(df/dE),\qquad(1)$$

where $E_\gamma=E-E_0$ is the photon energy, $|\,0\rangle$ and $|\,f\rangle$ are, respectively, the initial and final states, df/dE is the density of final states, and $\mathfrak{D}$ is the electric dipole operator. In the long-wavelength approximation,

$$\mathfrak{D}=\tfrac{1}{2}e\sum_i(\hat{\epsilon}\cdot\mathbf{r}_i)\tau_3^{(i)},\qquad(2)$$

with $\hat{\epsilon}$ the photon polarization direction, $\mathbf{r}_i$ the position vector of the ith nucleon relative to the c.m., and $\tau_3^{(i)}$ the isospin component of the ith nucleon. The cross section is averaged over photon polarizations.

Sum rules[3] are derived by using closure over the complete set of final states. Thus, we find from

$$\sigma_n\equiv\int_0^\infty dE\,E_\gamma^n\sigma(E)=\frac{4\pi^2}{\hbar c}\int df(E-E_0)^{n+1}\times\langle 0\,|\,\mathfrak{D}\,|\,f\rangle\langle f\,|\,\mathfrak{D}\,|\,0\rangle\qquad(3)$$

that for $n=-1,0,1$

$$\sigma_{-1}=(4\pi^2/\hbar c)\langle 0\,|\,\mathfrak{D}^2\,|\,0\rangle,$$
$$\sigma_0=-(2\pi^2/\hbar c)\langle 0\,|[[\mathcal{H},\mathfrak{D}],\mathfrak{D}]|\,0\rangle,\qquad(4)$$
$$\sigma_1=-(4\pi^2/\hbar c)\langle 0\,|[\mathcal{H},\mathfrak{D}]^2\,|\,0\rangle,$$

[3] J. S. Levinger, *Nuclear Photo-Disintegration* (Oxford University Press, London, England, 1960).

where $\mathcal{H}$ is the nuclear Hamiltonian of which $|f\rangle$, E; $|0\rangle$, E_0 are eigenstates and eigenvalues.

If the isospin of the ground state is I_0, the electric dipole operator will produce final states $I=I_0-1$, I_0, I_0+1 (no $0\to 0$ or $I<I_{03}$) while conserving T_3, the third component of T: thus, $I_{03}=I_3=\frac{1}{2}(Z-N)$. The closure property applied to obtain the usual sum rules (4) may be separated into its final-state isospin components by means of the appropriate projection operators according to

$$\int df\,|f\rangle\langle f| = \sum_I P_I = 1. \quad (5)$$

A sum rule (4) may be now separated into its final-state isospin components

$$\sigma_{-1}(I) = (4\pi^2/\hbar c)\langle 0\,|\,\mathfrak{D}P_I\mathfrak{D}\,|\,0\rangle,$$
$$\sigma_0(I) = -(2\pi^2/\hbar c)\langle 0\,|\{[\mathcal{H},\mathfrak{D}]P_I\mathfrak{D}-\mathfrak{D}P_I[\mathcal{H},\mathfrak{D}]\}|\,0\rangle,$$
$$\sigma_1(I) = (4\pi^2/\hbar c)\langle 0\,|[\mathfrak{D},\mathcal{H}]P_I[\mathcal{H},\mathfrak{D}]|\,0\rangle. \quad (6)$$

III. GROUND-STATE WAVE FUNCTION

In ^{3}H and ^{3}He, the ground state is $I_0=\frac{1}{2}$. ($I_{03}=+\frac{1}{2}$ for ^{3}He, $I_{03}=-\frac{1}{2}$ for ^{3}H). For three nucleons there are two orthonormal isospin-$\frac{1}{2}$ eigenstates ζ', ζ'' constructed, respectively, by coupling nucleon 1 to the nucleon pair (23) in an isospin singlet and in an isospin triplet.[4] Consequently, ζ' is antisymmetric under interchange of nucleons 2 and 3, while ζ'' is symmetric. Explicitly, we have for ^{3}He

$$\zeta'=2^{-1/2}p(1)[p(2)n(3)-n(2)p(3)],$$
$$\zeta''=6^{-1/2}[2n(1)p(2)p(3)-p(1)p(2)n(3)-p(1)n(2)p(3)] \quad (7)$$

and similarly for ^{3}H.

The ground state of ^{3}H and ^{3}He has, in general, the form

$$|0\rangle=\Phi_0'\zeta''-\Phi_0''\zeta', \quad (8)$$

where Φ_0' and Φ_0'' are both space- and spin-dependent functions, Φ_0' being antisymmetric and Φ_0'' symmetric under exchange of the nucleons 2 and 3. The eigenstate (8) is then antisymmetric under any permutation. For a pure spin-$\frac{1}{2}$ state Verde[4] has shown the space- and spin-dependent components of the ground state have the general form

$$\Phi_0'=2^{-1/2}[(\psi_0^s+\psi_0'')\chi'+(\psi_0'-\psi_0^a)\chi''],$$
$$\Phi_0''=2^{-1/2}[(\psi_0^s-\psi_0'')\chi''+(\psi_0'+\psi_0^a)\chi'], \quad (9)$$

where χ', χ'' are the spin states ($S=\frac{1}{2}$) analogous to (7), and ψ_0^s, ψ_0^a, ψ_0', ψ_0'' are functions of the spatial coordinates. ψ_0^s is symmetric and ψ_0^a antisymmetric under exchange of any nucleon pair; ψ_0' is antisymmetric and ψ_0'' symmetric under exchange of the nucleon pair 23.

The ^{3}H and ^{3}He ground states are thought[5] to be approximately 92% fully symmetric ^2S state, 2% mixed symmetry $^2S'$ state and 6% ^4D state. If, as an initial approximation, the ground state is considered fully symmetric in the space coordinates, Eq. (8) becomes

$$|0\rangle=\psi_0^s 2^{-1/2}(\chi'\zeta''-\chi''\zeta'). \quad (10)$$

The electric dipole operator $\mathfrak{D}$ given by Eq. (2) acting on the ground state $I_0=\frac{1}{2}$ of the general form Eq. (8) will generate $I=\frac{1}{2}$ and $\frac{3}{2}$ states. The $I=\frac{3}{2}$ states ζ^s are symmetric in all nucleons. Explicitly the $I=\frac{3}{2}$, $I_3=+\frac{1}{2}$ state is

$$\zeta^s=3^{-1/2}[n(1)p(2)p(3)+p(1)n(2)p(3)+p(1)p(2)n(3)]. \quad (11)$$

Application of the operator $\mathfrak{D}$ to the ground state $|0\rangle$, using the relations in the Appendix, yields

$$\mathfrak{D}\,|\,0\rangle=-(e/2\sqrt{3})\hat{\epsilon}\cdot\{[(2/\sqrt{3})\boldsymbol{\varrho}\Phi_0'-\mathbf{r}\Phi_0'']\zeta''+[\mathbf{r}\Phi_0'+(2/\sqrt{3})\boldsymbol{\varrho}\Phi_0'']\zeta'+\sqrt{2}[(2/\sqrt{3})\boldsymbol{\varrho}\Phi_0'+\mathbf{r}\Phi_0'']\zeta^s\}. \quad (12)$$

In terms of the c.m. coordinates $\mathbf{r}$, $\boldsymbol{\varrho}$,

$$\mathbf{r}_1=\tfrac{2}{3}\boldsymbol{\varrho},$$
$$\mathbf{r}_2=\tfrac{1}{2}\mathbf{r}-\tfrac{1}{3}\boldsymbol{\varrho}, \quad (13)$$
$$\mathbf{r}_3=-\tfrac{1}{2}\mathbf{r}-\tfrac{1}{3}\boldsymbol{\varrho}.$$

From Eq. (12) we obtain by inspection the results of applying to $\mathfrak{D}\,|\,0\rangle$ the projection operators P_I

$$P_{1/2}\mathfrak{D}\,|\,0\rangle=-(e/2\sqrt{3})\hat{\epsilon}\cdot\{[(2/\sqrt{3})\boldsymbol{\varrho}\Phi_0'-\mathbf{r}\Phi_0'']\zeta''+[\mathbf{r}\Phi_0'+(2/\sqrt{3})\boldsymbol{\varrho}\Phi_0'']\zeta'\},$$
$$P_{3/2}\mathfrak{D}\,|\,0\rangle=-(e/\sqrt{6})\hat{\epsilon}\cdot[(2/\sqrt{3})\boldsymbol{\varrho}\Phi_0'+\mathbf{r}\Phi_0'']\zeta^s. \quad (14)$$

IV. $\int dE\sigma/E\gamma$ SUM RULES

According to the first of Eqs. (6) we obtain the $\sigma_{-1}(I)$ sum rules from the norm of the vectors given by Eqs. (14) and taking the polarization average. Thus,

$$\sigma_{-1}(\tfrac{1}{2})=\tfrac{1}{9}\pi^2\alpha\{\langle\Phi_0'\,|\,\tfrac{4}{3}\rho^2+r^2\,|\,\Phi_0'\rangle+\langle\Phi_0''\,|\,\tfrac{4}{3}\rho^2+r^2\,|\,\Phi_0''\rangle\}, \quad (15)$$
$$\sigma_{-1}(\tfrac{3}{2})=\tfrac{2}{9}\pi^2\alpha\{\langle\Phi_0'\,|\,\tfrac{4}{3}\rho^2\,|\,\Phi_0'\rangle+\langle\Phi_0''\,|\,r^2\,|\,\Phi_0''\rangle+(4/\sqrt{3})\langle\Phi_0'\,|(\boldsymbol{\varrho}\cdot\mathbf{r})|\,\Phi_0''\rangle\}, \quad (16)$$

where we have used the fact that $\langle(\hat{\epsilon}\cdot\mathbf{a})(\hat{\epsilon}\cdot\mathbf{b})\rangle_{\text{pol.av}}=\frac{1}{3}(\mathbf{a}\cdot\mathbf{b})$ for any vectors $\mathbf{a}$ and $\mathbf{b}$ and where $\alpha=e^2/\hbar c$.

The mean square matter radius R_m^2 is given by

$$R_m^2=\tfrac{1}{3}\langle 0\,|\sum_i r_i^2\,|\,0\rangle=\tfrac{1}{6}\langle 0\,|\,\tfrac{4}{3}\rho^2+r^2\,|\,0\rangle, \quad (17)$$

[4] M. Verde, *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. XXXIX; G. Derrick and J. M. Blatt, Nucl. Phys. **8**, 310 (1958).

[5] B. F. Gibson, Phys. Rev. **139**, B1153 (1965).

therefore, Eq. (15) can be written as

$$\sigma_{-1}(\tfrac{1}{2})=\tfrac{2}{3}\pi^2\alpha R_m^2. \tag{18}$$

To interpret the expression in brackets in the right-hand side of Eq. (16) we note that, for ^{3}He,

$$\langle 0|\sum_i r_i^2\tau^{(i)}|0\rangle=\tfrac{1}{3}\{\langle\Phi_0'|r^2|\Phi_0'\rangle+\langle\Phi_0''|\tfrac{4}{3}\rho^2|\Phi_0''\rangle-(4/\sqrt{3})\langle\Phi_0'|(\boldsymbol{\rho}\cdot\mathbf{r})|\Phi_0''\rangle\}$$

and, for both ^{3}He and ^{3}H,

$$\langle 0|\sum_i r_i^2\tau^{(i)}|0\rangle=2I_3R_V^2, \tag{19}$$

which defines the isovector radius R_V, hence,

$$\sigma_{-1}(\tfrac{3}{2})=\tfrac{4}{3}\pi^2\alpha(R_m^2-\tfrac{1}{2}R_V^2). \tag{20}$$

Equations (18) and (20) give the bremsstrahlung weighted integrated cross section in terms of the matter radius (17) and isovector radius (19) of the ground state. Addition of Eqs. (18) and (20) yields

$$\sigma_{-1}=\tfrac{2}{3}\pi^2\alpha(3R_m^2-R_V^2). \tag{21}$$

This reduces to the form of Davey and Valk[6] for their particular ground-state wave function.

Elimination of R_m^2 between Eqs. (18) and (20) yields the relation

$$2\sigma_{-1}(\tfrac{1}{2})-\sigma_{-1}(\tfrac{3}{2})=\tfrac{2}{3}\pi^2\alpha R_V^2, \tag{22}$$

which is the nonrelativistic form of the Cabibbo-Radicati sum rule.[2] For a fully symmetric ground state, of the form given by Eq. (10), $R_m^2=R_V^2=R^2$ and Eqs. (21) and (22) reduce to those derived in Ref. 1.

The matter and isovector radii are related to the proton and neutron mean square radii R_p^2, R_n^2. From the definitions

$$ZR_p^2=\langle 0|\sum_i r_i^2[\tfrac{1}{2}(1+\tau_3)]_i|0\rangle,$$
$$NR_n^2=\langle 0|\sum_i r_i^2[\tfrac{1}{2}(1-\tau_3)]_i|0\rangle, \tag{23}$$

and Eqs. (17) and (19), we have that

$$ZR_p^2=\tfrac{1}{2}AR_m^2+I_3R_V^2,$$
$$NR_n^2=\tfrac{1}{2}AR_m^2-I_3R_V^2. \tag{24}$$

For ^{3}He, ^{3}H, Eqs. (24) read

$$2R_p^2(^3\text{He})=2R_n^2(^3\text{H})=\tfrac{3}{2}R_m^2+\tfrac{1}{2}R_V^2,$$
$$R_n^2(^3\text{He})=R_p^2(^3\text{H})=\tfrac{3}{2}R_m^2-\tfrac{1}{2}R_V^2. \tag{25}$$

The equality of proton and neutron radii of the mirror nuclei is a result of isospin invariance.

From Eqs. (25) one gets the isovector radius

$$R_V^2=2R_p^2(^3\text{He})-R_n^2(^3\text{He})=2R_p^2(^3\text{He})-R_p^2(^3\text{H}),$$

which substituted in (22) yields

$$\tfrac{2}{3}\pi^2\alpha[2R_p^2(^3\text{He})-R_p^2(^3\text{H})]=2\sigma_{-1}(\tfrac{1}{2})-\sigma_{-1}(\tfrac{3}{2}). \tag{26}$$

[6] P. O. Davey and H. S. Valk, Phys. Letters **7**, 335 (1963).

Equation (26) was obtained by Gerasimov[7] and by Scheck and Schülke[8] by generalizing the nucleon photomeson production sum rule of Cabibbo and Radicati[2] to ^{3}H and ^{3}He. Here the result is obtained by using only isospin invariance and the long-wavelength approximation for the electric dipole operator.

V. $\int dE\sigma$ SUM RULES

To evaluate the second of Eqs. (6) we consider separately the kinetic energy and potential energy commutators with the dipole operator. We evaluate the contributions

$$\sigma_0(I)_{\text{kin}}=-(2\pi^2/\hbar c)\langle 0|\{[K,\mathfrak{D}]P_I\mathfrak{D}-\mathfrak{D}P_I[K,\mathfrak{D}]\}|0\rangle, \tag{27}$$

$$\sigma_0(I)_{\text{int}}=-(2\pi^2/\hbar c)\langle 0|\{[V,\mathfrak{D}]P_I\mathfrak{D}-\mathfrak{D}P_I[V,\mathfrak{D}]\}|0\rangle \tag{28}$$

with

$$K=(\hbar^2/M)(k^2+\tfrac{3}{4}p^2),$$
$$V=V_{12}+V_{23}+V_{31}, \tag{29}$$

separately. Then since $H=K+V$,

$$\sigma_0(I)=\sigma_0(I)_{\text{kin}}+\sigma_0(I)_{\text{int}}. \tag{30}$$

A. Kinetic Energy Term

k, **p** are $\hbar^{-1}$ times the momenta in the c.m. system canonically conjugate, respectively, to **r**, **ρ**. They are related to $\hbar$ times the momenta of the nucleons by

$$\mathbf{p}_1=\mathbf{p},$$
$$\mathbf{p}_2=\mathbf{k}-\tfrac{1}{2}\mathbf{p}, \tag{31}$$
$$\mathbf{p}_3=-\mathbf{k}-\tfrac{1}{2}\mathbf{p}.$$

By use of the commutation relations

$$[r_\alpha,k_\beta]=i\delta_{\alpha\beta},\quad [\rho_\alpha,p_\beta]=i\delta_{\alpha\beta},$$
$$[r_\alpha,p_\beta]=0,\quad [\rho_\alpha,k_\beta]=0,\quad (\alpha,\beta=x,y,z) \tag{32}$$

one finds that

$$[K,\mathfrak{D}]=-i(e\hbar^2/2M)\{(\hat{\epsilon}\cdot\mathbf{k})[\tau_3^{(2)}-\tau_3^{(3)}]+(\hat{\epsilon}\cdot\mathbf{p})[\tau_3^{(1)}-\tfrac{1}{2}(\tau_3^{(2)}+\tau_3^{(3)})]\}. \tag{33}$$

The result of the application of $[K,\mathfrak{D}]$ to the ground state $|0\rangle$ [Eq. (8)] can be found by application of the results in the Appendix. The $\sigma_0(I)_{\text{kin}}$ are evaluated using the $P_I\mathfrak{D}|0\rangle$ given by Eq. (14).

The kinetic energy contribution to the sum rule is then

$$\sigma_0(\tfrac{1}{2})_{\text{kin}}=\sigma_0(\tfrac{3}{2})_{\text{kin}}=\tfrac{2}{3}\pi^2\alpha(\hbar^2/M)=20\text{ MeV mb}. \tag{34}$$

[7] S. B. Gerasimov, Zh. Ehsperim. i Teor. Fiz. Pis'ma v Redaktsiyu **5**, 412 (1967) [English transl.: Soviet Phys.—JETP Letters **5**, 337 (1967)].

[8] F. Scheck and L. Schülke, Phys. Letters **25B**, 526 (1967).

The total contribution $\sigma_0(\frac{1}{2})_{\text{kin}}+\sigma_0(\frac{3}{2})_{\text{kin}}$ gives the Thomas-Reiche-Kuhn sum rule.[3]

B. Potential Energy Term

The potential energy contribution to the integrated cross section from

$$\sigma_0(I)_{\text{int}}=-(6\pi^2/\hbar c)\langle 0|\{[V_{23},\mathfrak{D}]P_I\mathfrak{D}-\mathfrak{D}P_I[V_{23},\mathfrak{D}]\}|0\rangle \quad (28')$$

depends on the form assumed for the nucleon-nucleon interaction. Since the ground state is predominantly fully symmetric, we will assume for simplicity that $|0\rangle$ is given by Eq. (10).

For a local potential of the form

$$V_{23}=V(r)[w+x\mathcal{P}_r+y\mathcal{P}_\tau+z\mathcal{P}_\sigma], \quad (35)$$

where $\mathcal{P}_r$, $\mathcal{P}_\tau$, $\mathcal{P}_\sigma$ are the space, isospin, and spin exchange operators of nucleons 2, 3, one obtains

$$[V_{23},\mathfrak{D}]=-\tfrac{1}{2}eV(r)(\hat{\epsilon}\cdot\mathbf{r})(\tau_3^{(2)}-\tau_3^{(3)})(x\mathcal{P}_r+y\mathcal{P}_\tau). \quad (36)$$

The result of the application of the commutator given by Eq. (36) to the ground state of the form in Eq. (10) is obtained by use of the formulas in the Appendix. The vectors $P_I\mathfrak{D}|0\rangle$ for the case of fully symmetric ground state are obtained from those given by Eq. (14) by the substitutions $\Phi_0'=2^{-1/2}\psi_0^s\chi'$, $\Phi_0''=2^{-1/2}\psi_0^s\chi''$. Then one can evaluate the right-hand side of Eq. (28') finding

$$\sigma_0(\tfrac{1}{2})_{\text{int}}=-\tfrac{2}{3}\pi^2\alpha x(\psi_0^s,r^2V(r)\psi_0^s),$$
$$\sigma_0(\tfrac{3}{2})_{\text{int}}=-\tfrac{2}{3}\pi^2\alpha(x-y)(\psi_0^s,r^2V(r)\psi_0^s). \quad (37)$$

The integrated cross sections for the two final isospin states are

$$\sigma_0(\tfrac{1}{2})=\tfrac{2}{3}\pi^2\alpha(\hbar^2/M)[1-x(M/\hbar^2)(\psi_0^s,r^2V(r)\psi_0^s)],$$
$$\sigma_0(\tfrac{3}{2})=\tfrac{2}{3}\pi^2\alpha(\hbar^2/M)[1-(x-y)(M/\hbar^2)(\psi_0^s,r^2V(r)\psi_0^s)], \quad (38)$$

when the nuclear interaction is assumed to be given by Eq. (35).

The sum $\sigma_0(\frac{1}{2})+\sigma_0(\frac{3}{2})$ agrees with the expression derived by Verde.[9]

A nonlocal separable potential[10,11] that has been used[12,13] in calculations of the photodisintegration of ^{3}H and ^{3}He is the s-wave spin-dependent form

$$\langle\mathbf{k}|V^{(s)}|\mathbf{k}'\rangle=-(\lambda_s/M)g_s^*(k)g_s(k') \quad (39)$$

with $g_s(k)=(k^2+\beta_s^2)^{-1}$, in the momentum representation. The strength λ_s and range β_s parameters are different in the spin singlet and spin triplet states ($s=0,1$) of the nucleon pair.

For a symmetric ground state, the contribution to the sum rule given by Eq. (28') can again be reduced to a matrix element involving only the spatial part of the ground state. One proceeds in the way described for the previous case and finds

$$\sigma_0(\tfrac{1}{2})_{\text{int}}=-\tfrac{1}{6}\pi^2\alpha(\psi_0^s,\tfrac{1}{2}[V^{(0)}+V^{(1)}]r^2\psi_0^s),$$
$$\sigma_0(\tfrac{3}{2})_{\text{int}}=-\tfrac{1}{6}\pi^2\alpha(\psi_0^s,V^{(1)}r^2\psi_0^s). \quad (40)$$

VI. $\int dEE_\gamma\sigma$ SUM RULES

These sum rules are given by the last of Eqs. (6). We shall assume in this case that the ground state is fully symmetric, i.e., that $|0\rangle$ is given by Eq. (10), and we shall also assume that the nuclear interaction has the form given in Eq. (35). From the reality properties of the vectors $[K,\mathfrak{D}]|0\rangle$ and $[V,\mathfrak{D}]|0\rangle$, one obtains

$$\langle 0|[\mathfrak{D},\mathcal{H}]P_I[\mathcal{H},\mathfrak{D}]|0\rangle=\langle 0|[\mathfrak{D},K]P_I[K,\mathfrak{D}]|0\rangle+9\langle 0|[\mathfrak{D},V_{23}]P_I[V_{23},\mathfrak{D}]|0\rangle. \quad (41)$$

The commutators $[K,\mathfrak{D}]$ and $[V_{23},\mathfrak{D}]$ have been evaluated in Eqs. (33) and (36). Application of these results to the ground state of Eq. (10) with use of the formulas in the Appendix and separation of the $I=\frac{1}{2},\frac{3}{2}$ components gives the results

$$\langle 0|[\mathfrak{D},K]P_{1/2}[K,\mathfrak{D}]|0\rangle=\langle 0|[\mathfrak{D},K]P_{3/2}[K,\mathfrak{D}]|0\rangle=(e^2\hbar^2/9M)(\psi_0^s,K\psi_0^s) \quad (42)$$

and

$$\langle 0|[\mathfrak{D},V_{23}]P_{1/2}[V_{23},\mathfrak{D}]|0\rangle=\tfrac{1}{9}e^2(x^2+y^2)\times(\psi_0^s,[rV(r)]^2\psi_0^s),$$
$$\langle 0|[\mathfrak{D},V_{23}]P_{3/2}[V_{23},\mathfrak{D}]|0\rangle=\tfrac{1}{9}e^2(x-y)^2\times(\psi_0^s,[rV(r)]^2\psi_0^s). \quad (43)$$

The sum rules are then

$$\sigma_1(\tfrac{1}{2})=(4/9)\pi^2\alpha(\hbar^2/M)\times(\psi_0^s,\{K+9(x^2+y^2)(M/\hbar^2)[rV(r)]^2\}\psi_0^s),$$
$$\sigma_1(\tfrac{3}{2})=(4/9)\pi^2\alpha(\hbar^2/M)\times(\psi_0^s,\{K+9(x-y)^2(M/\hbar^2)[rV(r)]^2\}\psi_0^s). \quad (44)$$

VII. COMPARISON WITH THE DATA

The cloud-chamber experiment of Gorbunov *et al.*[14] gives the following sums for ^{3}He with uncertainties due to counting statistics and intensity measurement

[9] M. Verde, Nuovo Cimento **8**, 152 (1951).
[10] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[11] A. G. Sitenko and V. F. Kharchenko, Nucl. Phys. **49**, 15 (1963).
[12] I. M. Barbour and A. C. Phillips, Phys. Rev. Letters **19**, 1388 (1967).
[13] J. S. O'Connell and F. Prats, Phys. Letters **26B**, 197 (1968).
[14] V. N. Fetisov, A. N. Gorbunov, and A. T. Varfolomeev, Nucl. Phys. **71**, 305 (1965). The experimental cross section includes, of course, all multipoles. Gorbunov *et al.* estimate that $(11\pm4)\%$ may be electric quadrupole and should be subtracted. However, S. B. Gerasimov [Phys. Letters **13**, 240 (1964)] shows that for σ_0 retardation in electric dipole transitions is cancelled by the higher multipoles.

uncertainties:

$$\sigma_{-1}(2)\equiv\int_0^{170\text{ MeV}} dE\sigma(\text{2-body})/E\gamma=1.34\pm0.05\text{ mb},$$

$$\sigma_{-1}(3)\equiv\int_0^{170\text{ MeV}} dE\sigma(\text{3-body})/E\gamma=1.42\pm0.07\text{ mb},$$

$$\sigma_{0}(2)\equiv\int_0^{170\text{ MeV}} dE\sigma(\text{2-body})=26.5\pm1.3\text{ MeV mb},$$

$$\sigma_{0}(3)\equiv\int_0^{170\text{ MeV}} dE\sigma(\text{3-body})=43.6\pm2.7\text{ MeV mb}. \quad (45)$$

The charge radii measured by electron scattering[15] with uncertainties due to counting statistics are

$$\langle R_{\text{CH}}^2(^3\text{He})\rangle^{1/2}=1.87\pm0.05\text{ fm},$$
$$\langle R_{\text{CH}}^2(^3\text{H})\rangle^{1/2}=1.70\pm0.05\text{ fm},$$

from which we deduce the like (L) and odd (O) nucleon radii[16]

$$R_{\text{L}}^2=R_p^2(^3\text{He})=R_n^2(^3\text{H})=R_{\text{CH}}^2(^3\text{He})-R_{\text{CH}}^2(^1\text{H})=2.86\text{ fm}^2,$$

$$R_{\text{O}}^2=R_n^2(^3\text{He})=R_p^2(^3\text{H})=R_{\text{CH}}^2(^3\text{H})-R_{\text{CH}}^2(^1\text{H})=2.25\text{ fm}^2,$$

where the proton charge radius was taken as 0.80 fm.

The matter and isovector radii are given by

$$R_m^2=\tfrac{1}{3}(2R_{\text{L}}^2+R_{\text{O}}^2)=2.66\text{ fm}^2,$$
$$R_{\text{V}}^2=2R_{\text{L}}^2-R_{\text{O}}^2=3.47\text{ fm}^2.$$

Using Eq. (21), the total photonuclear sum for ^{3}He or ^{3}H calculated from the charge radii is $\sigma_{-1}=2.16\pm0.06$ mb. This is to be compared with the measured value[14] for ^{3}He of $\sigma_{-1}=2.76\pm0.18$ mb. There is some evidence from other measurements[17] of the ^{3}He cross sections that the two-body breakup cross section of Ref. 14 is 10–15% too large and that the three-body breakup cross section is 30% too large. These corrections would reduce the total sum to the value calculated from the charge radii.

In the following, we shall accept the values of the partial sums given in Eq. (45) to estimate the isospin purity of three-body breakup. The results do not change appreciably if the indicated reductions in the experimental sums are made.

The two-body breakup of ^{3}He into a deuteron plus proton is a $I=\frac{1}{2}$ channel, since the deuteron is isospin zero. The three-body breakup, however, can be both $I=\frac{1}{2}$ and $\frac{3}{2}$. It has been argued by several authors[1,7,8] that the near equality $\sigma_{-1}(2)\simeq\sigma_{-1}(3)$ implies $\sigma_{-1}(3)$ is almost all $I=\frac{3}{2}$ final state. Using Eq. (22) and the experimental value for the total sum $\sigma_{-1}(\frac{1}{2})+\sigma_{-1}(\frac{3}{2})=2.76$ mb, we obtain

$$\sigma_{-1}(\tfrac{1}{2})=1.48\text{ mb},\qquad \sigma_{-1}(\tfrac{3}{2})=1.28\text{ mb}.$$

The fraction f_{-1} of $\sigma_{-1}(\frac{1}{2})$ that goes into three-body breakup is given by

$$\sigma_{-1}(3)=\sigma_{-1}(\tfrac{3}{2})+f_{-1}\sigma_{-1}(\tfrac{1}{2})$$

from which we compute $f_{-1}=0.095$. This confirms the relative purity of at least the low-energy part of the cross sections.

The isospin purity of the σ_0 sum rules can be checked (1) if we assume some set of exchange coefficients (x and y for the local central potential or λ_0 and λ_1 for the separable potential), and (2) if we choose a value of the matrix element $\langle r^2V(r)\rangle$ to give the experimental σ_0(total).

For the local central potential we use, as an example, the Rosenfeld mixture $x=0.93$, $y=0.26$.[18] To get the experimental total sum, we require

$$-(M/\hbar^2)(\psi_0^s, r^2V(r)\psi_0^s)=0.936,$$

a value somewhat larger than those obtained by explicit calculations of this matrix element 0.55,[19] 0.72,[20] 0.724.[21] The isospin sums are then

$$\sigma_0(\tfrac{1}{2})=37.4\text{ MeV mb},\qquad \sigma_0(\tfrac{3}{2})=32.6\text{ MeV mb}.$$

The fraction f_0 of $\sigma_0(\frac{1}{2})$ that goes into three-body breakup is given by

$$\sigma_0(3)=\sigma_0(\tfrac{3}{2})+f_0\sigma_0(\tfrac{1}{2})$$

from which we compute $f_0=0.29$.

The same type of computation for the separable potential using the Yamaguchi[11] parameters $\lambda_0=0.292$ fm^{-3}, $\lambda_1=0.415$ fm^{-3}, $\beta_0=\beta_1=1.45$ fm^{-1} yields $f_0=0.21$.

The fact that $f_0>f_{-1}$ implies that the $I=\frac{1}{2}$ contribution to three-body breakup comes mainly in the high-energy part of the cross section.

In conclusion, we have shown how the σ_{-1} sums for the different isospin states can be related to measured charge radii and how the σ_0 isospin sums are related to matrix elements of the space and isospin exchange potentials. A recent calculation[22] of three-body sum rules has been made with the Hamada-Johnston potential and a variational wave function. This calculation shows how tensor forces and ground-state ^4D states affect the σ_0 sum.

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APPENDIX

Using the convention $\tau_3^{(i)}p(i)=p(i)$, $\tau_3^{(i)}n(i)=-n(i)$, the third component of the nucleon isospin operator applied to the isospin functions of ^{3}He given by Eqs. (7) and (11) yields

$$\tau_3^{(1)}\zeta'=\zeta',$$
$$\tau_3^{(2)}\zeta'=-(1/\sqrt{3})\zeta''+(\sqrt{\tfrac{2}{3}})\zeta^s,$$
$$\tau_3^{(3)}\zeta'=+(1/\sqrt{3})\zeta''-(\sqrt{\tfrac{2}{3}})\zeta^s,$$
$$\tau_3^{(1)}\zeta''=-\tfrac{1}{3}\zeta''-\tfrac{2}{3}\sqrt{2}\zeta^s,$$
$$\tau_3^{(2)}\zeta''=-(1/\sqrt{3})\zeta'+\tfrac{2}{3}\zeta''+\tfrac{1}{3}\sqrt{2}\zeta^s,$$
$$\tau_3^{(3)}\zeta''=+(1/\sqrt{3})\zeta'+\tfrac{2}{3}\zeta''+\tfrac{1}{3}\sqrt{2}\zeta^s.$$

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Independent-Particle-Model Energy-Level Formula*

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The Morse function is assumed to approximate a nonlocal or velocity-dependent nucleon-nuclear potential. Analytic neutron and proton wave functions and eigenvalue formulas are obtained for all states of all nuclei, using a modified version of an analytic perturbation method due to Pekeris. The eigenvalue formula is in approximate agreement with the following experimental data: (1) the last-particle binding energies of neutrons and protons, (2) the neutron and proton magic numbers, (3) the positions of the S-wave-size resonances in total-neutron-cross-section data, and (4) the recent experimental work on deeply bound inner particle states by Amaldi *et al.* The eigenvalue formula gives results that are consistent with recent Hartree-Fock calculations.

I. INTRODUCTION

PREVIOUSLY, Green, Darewych, and Berezdivin[1] (GDB) noted the possibility of using the Morse potential to arrive at analytic eigenvalues and analytic wave functions for single-particle states in velocity-dependent nuclear potentials. Their results compared favorably with the numerical solutions of Wyatt, Wills, and Green.[2] The procedure of GDB involved fitting the effective energy-dependent potential together with the centrifugal potential by an approximating Morse potential.[3] In the present work, the numerical fitting procedure used in the earlier study is replaced by a modified version of a technique used by Pekeris.[4] Thus, one arrives at explicit formulas for the effective state-dependent Morse-function parameters and for the eigenvalues. This should greatly enhance the utility of the approximate method.

II. VELOCITY-DEPENDENT POTENTIALS

In this work, velocity-dependent potentials of the form

$$V(\mathbf{r},\mathbf{p})=-V_0\xi_0(\mathbf{r})-(\delta\hbar^2/8m)\times[\nabla^2\xi(\mathbf{r})+2\nabla\cdot\xi(\mathbf{r})\nabla+\xi(\mathbf{r})\nabla^2] \quad (1)$$

are considered. Here, $\xi_0(r)$ is the static form factor, which may differ from $\xi(r)$, the velocity-dependent form factor. If use is made of the wave function $R(r)Y_l^m(\theta,\phi)$ and the definitions

$$x=r/a,\quad \epsilon_w^2=-W/E_0,\quad R(x)=G(x)/x,$$
$$E_0=\hbar^2/2ma^2,\quad \epsilon_0^2=V_0/E_0,\quad X(x)=(1+\delta\xi)^{1/2}G(x); \quad (2)$$

it is possible to transform the usual Schrödinger equation for such potentials into the form

$$X''-v(x,\epsilon_w)X-[\epsilon_w^2/(1+\delta)]X=0, \quad (3)$$

where $v(x,\epsilon_w)$ is the energy-dependent potential

$$v(x,\epsilon_w)=-\frac{\epsilon_0^2\xi_0}{1+\delta\xi}+\frac{1}{4}\frac{\delta\xi''}{1+\delta\xi}-\frac{1}{4}\frac{\delta^2\xi'^2}{(1+\delta\xi)^2}+\frac{\delta\xi'}{2x(1+\delta\xi)}+\frac{l(l+1)}{x^2}+\frac{\delta(1-\xi)\epsilon_w^2}{(1+\delta\xi)(1+\delta)}. \quad (4)$$

In this paper, an analytic method is developed for the approximate solution of Eq. (3) in certain cases of interest in nuclear physics. Form factors ξ_0 and ξ are assumed, for which the first four terms of Eq. (4) become a Morse function. Then all remaining terms are treated by the quadratic perturbation method.

III. MATHEMATICAL PROPERTIES OF THE MORSE POTENTIAL

The form of the Morse function used in this calculation is[3]

$$v_m(x)=\alpha_0^2\{\exp[-2(x-x_0)/d]-2\exp[-(x-x_0)/d]\}. \quad (5)$$

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