increasing temperature than previously found in multidomain material. This is not in the direction anticipated from the previous theoretical work of Suhl and Winter,[4,5] who predicted a more rapid decrease in walls than in domains.

A detailed interpretation of the present results and a discussion of NMR hyperfine field studies in ferromagnets are in progress, and will be published in a separate paper.

[4] H. Suhl, Bull. Am. Phys. Soc. 5, 175 (1960).
[5] J. M. Winter, Phys. Rev. 124, 452 (1961).

ACKNOWLEDGMENTS

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Three-Dimensional Magnetic Model with Classical Spins of High Dimensionality

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A magnetic model with ν-dimensional classical spins on an s-dimensional lattice has been analyzed by Stanley. He demonstrates that the term proportional to ν in the free energy is that given by a spherical-model assumption. Further terms are $O(\nu^0)$. We show, for a three-dimensional lattice, that even for large ν this next-order term makes a contribution which eventually dominates as the critical point is approached, and below the transition. A field-theoretic formulation of the Ising model ($\nu=1$) is modified to obtain series developments in ν^{-1} of the system's properties, but near the critical point these series may not be used directly. The difficulties near the critical point examined here go beyond the context of the model.

I. INTRODUCTION

RECENTLY, Stanley[1] has considered the properties of a system of ν-dimensional classical spins, situated at the sites of an s-dimensional lattice. In particular, he arrives at the conclusion that as the dimensionality ν of the spin vectors becomes infinite, the thermodynamic properties become those of the corresponding spherical model[2] for a fixed $T>T_c$. We shall demonstrate in this paper that for three-dimensional lattices his argument does not apply below the critical transition. Moreover, for any finite ν there exists a region above the transition temperature for which the spherical-model terms are not dominant. Thus the limit of approaching the critical point and passing to $\nu\rightarrow\infty$ may not be interchanged. The possibility that this may occur was recognized in Stanley's paper.

We begin with a rapid review of a formalism useful for consideration of this problem. It is next demonstrated that as the critical point is approached from above, the terms of $O(\nu)$ are dominated by terms of $O(\nu^0)$. A more complete analysis of the problem for large, but finite, ν may be obtained by a simple extension of recent work of the author.[3] In particular, one can obtain a Feynman-diagram theory that is ordered in $1/\nu$. The diagrammatic development cannot be directly utilized near the critical point. How to cure this difficulty remains one of the great mysteries of theoretical physics.

[1] H. E. Stanley, Phys. Rev. 176, 718 (1968); Phys. Rev. Letters 20, 589 (1968).
[2] T. H. Berlin and M. Kac, Phys. Rev. 86, 821 (1952).

II. FORMULATION

Consider the system described by the Hamiltonian[1]

$$H=-\tfrac{1}{2}J\sum_{i,j=1}^{N} v_{ij}\mathbf{S}_i\cdot\mathbf{S}_j. \tag{2.1}$$

The sum is over sites of a lattice, which in this paper we take as three-dimensional simple cubic. The ν-dimensional spins have magnitude $\nu^{1/2}$, i.e., $|\mathbf{S}_j|^2=\nu$. By increasing the magnitude of the spin vectors as ν is increased, the transition temperature is kept finite for $\nu\rightarrow\infty$. (One can return to a model where the ground-state energy is independent of ν by allowing J to be inversely proportional to ν.)

The normalized partition function is

$$Q=Z(\beta)/Z(0), \tag{2.2}$$

[3] E. Helfand, Phys. Rev. 180, 600 (1969). This paper will be referred to as I.

where

$$Z(\beta)=\int\cdots\int d\mathbf{S}_1\cdots d\mathbf{S}_N\prod_{j=1}^{N}\delta(\nu-|\mathbf{S}_j|^2)\times\exp[\beta\sum_{ij}v_{ij}\mathbf{S}_i\cdot\mathbf{S}_j], \quad (2.3)$$

$$\beta=J/2k_BT. \quad (2.4)$$

One can transform to an alternative form of the partition function by following a technique introduced by Montroll and Berlin[4,3,5] for the Ising model ($\nu=1$). Write the δ functions in the integral representation

$$\delta(\nu-|\mathbf{S}_j|^2)=(\beta/2\pi i)\int_{-i\infty}^{i\infty}dt_j\exp[\beta t_j(\nu-|\mathbf{S}_j|^2)]. \quad (2.5)$$

By shifting all t contours to the right by an amount

$$\gamma>\tilde{v}_0,\quad \tilde{v}_0\equiv\sum_j v_{ij}, \quad (2.6)$$

the order of $\mathbf{S}$ and t integrations may be interchanged.[5] The $\mathbf{S}$ integrals, possessing an integrand which is a multidimensional Gaussian form, can be performed. The result is

$$Z=(\beta/2\pi i)^N(\pi/\beta)^{N\nu/2}\int_{-i\infty+\gamma}^{i\infty+\gamma}d\{t\}\times\exp[\beta\nu\sum_j t_j]|\mathbf{T}-\mathbf{V}|^{-\nu/2}, \quad (2.7)$$

where $\{t\}$ is the set $t_1,\cdots,t_N$, $\mathbf{T}$ is a diagonal matrix with element t_i in the ii position, and $\mathbf{V}$ is a matrix composed of the potential v_{ij}.

In an investigation of the Ising problem, Helfand and Langer[5] were lead to a consideration of saddle-point approximation to the $\{t\}$ integral. Stanley has argued that for large-spin dimensionality, ν may serve as the large parameter necessary for the justification of that procedure. For temperatures above the critical, a stationary point of the integrand of Eq. (2.7) is t_j, a constant—let us call it z_s—determined by a condition which arises in the spherical model:

$$\beta=\tfrac{1}{2}(2\pi)^{-3}\int_{\mathrm{BZ}}d\mathbf{p}\,[z_s-\tilde{v}(\mathbf{p})]^{-1}. \quad (2.8)$$

Here $\tilde{v}(\mathbf{p})$ is the Fourier transform of the potential. Above the transition it is appropriate to use an integral over the Brillouin zone, rather than a summation, in the large-N limit.

The leading contribution to Z comes from inserting the saddle value for the t_j's. The first correction is determined in the standard manner by expanding the integrand of Eq. (2.7) as a Gaussian about the saddle point. The result is[1]

$$Z=(\beta/2\pi)^N(\pi/\beta)^{N\nu/2}(2\pi)^{N/2}|\mathbf{P}^0(z_s)|^{-1/2}\times\exp\left[N\nu\left(\beta z_s-\tfrac{1}{2}(2\pi)^{-3}\int d\mathbf{p}\ln[z_s-\tilde{v}(\mathbf{p})]\right)\right], \quad (2.9)$$

where $\mathbf{P}^0$ is a matrix with elements

$$P_{ij}^0(z)=-\tfrac{1}{2}\nu\partial^2\ln|\mathbf{T}-\mathbf{V}|/\partial t_i\partial t_j|_{\text{all } t\text{'s}=z}. \quad (2.10)$$

Stanley notes that the terms proportional to ν in $\ln Z$ are precisely those which would be given by the spherical model. The spherical-model critical point corresponds to the temperature β_c^{SM}, such that $z_s=\tilde{v}_0$. According to Eq. (2.8),

$$\beta_c^{\mathrm{SM}}=\tfrac{1}{2}(2\pi)^{-3}\int_{\mathrm{BZ}}d\mathbf{p}\,[\tilde{v}_0-\tilde{v}(\mathbf{p})]^{-1}. \quad (2.11)$$

We shall now demonstrate that in a region of temperatures above the critical, the contribution from the fluctuations about the saddle point may not be neglected.

III. ONSET OF CRITICAL ANOMALIES

To examine the onset of critical anomalies, which dominate the spherical-model terms, we should not make an *ad hoc* application of saddle-point procedures in the functional integral. A more complete formalism, presented in a recent paper by the present author,[3] may be applied with the minor modifications indicated in Sec. IV. However, the full theory is unnecessary for the demonstration of the critical breakdown. Therefore, we shall concentrate first on the vital aspects.

In Eq. (2.7), let us go to Fourier components of $\{t\}$ as integration variables:

$$\tilde{t}_p=N^{-1/2}\sum_j t_j\exp(i\mathbf{p}\cdot\mathbf{r}_j), \quad (3.1)$$

with $\mathbf{p}$ defined on the inverse lattice points of a Brillouin zone. It is useful to introduce new names:

$$z\equiv N^{-1}\sum_j t_j \equiv N^{-1/2}\tilde{t}_0, \qquad \tilde{\varphi}_p\equiv\tilde{t}_p,\quad p\neq0. \quad (3.2)$$

This enables us to use the variable φ_j to signify

$$\varphi_j=N^{-1/2}\sum_{p\neq0}\tilde{\varphi}_p\exp(-i\mathbf{p}\cdot\mathbf{r}_j) =t_j-z. \quad (3.3)$$

Then the functional integral for the partition function

[4] E. W. Montroll and T. H. Berlin, Commun. Pure Appl. Math. 4, 23 (1951).
[5] E. Helfand and J. S. Langer, Phys. Rev. 160, 434 (1967).

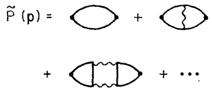


FIG. 1. Diagrammatic representation of terms in the P equation. All lines are renormalized.

may be written[6]

$$Z=(\beta/2\pi i)^N(\pi/\beta)^{N\nu/2}N^{1/2}\int_{-i\infty+\gamma}^{i\infty+\gamma}dz\exp(\nu N\beta z-\tfrac{1}{2}\nu\ln|z\mathbf{1}-\mathbf{V}|)\int d\{\tilde\varphi\}'|z\mathbf{1}-\mathbf{V}|^{\nu/2}/|z\mathbf{1}+\mathbf{\Phi}-\mathbf{V}|^{\nu/2},\quad(3.4)$$

where $\{\tilde\varphi\}'$ is the set of $\tilde\varphi_p$'s excluding $p=0$, and $\mathbf{\Phi}$ is a diagonal matrix with element φ_i in the ii position. The $\{\tilde\varphi\}'$ integrations are somewhat symbolic, and, as described in I, are best understood as integrals over sine and cosine transform variables, with contours $-i\infty$ to $+i\infty$.

As a close parallel to Stanley's procedure, described in Sec. II, let us evaluate the $\{\tilde\varphi\}'$ integral by expanding the integrand into a Gaussian in $\tilde\varphi_p$:

$$|z\mathbf{1}-\mathbf{V}|^{\nu/2}/|z\mathbf{1}+\mathbf{\Phi}-\mathbf{V}|^{\nu/2}\approx\exp[\tfrac{1}{2}\sum_{p\neq0}\tilde P^0(p)\tilde\varphi_p\tilde\varphi_{-p}].\quad(3.5)$$

For large N, $\tilde P^0(\mathbf{p})$ [Fourier transform with respect to $\mathbf{r}_i-\mathbf{r}_j$ of the $P_{ij}{}^0(z)$ matrix elements of Eq. (2.10)] is given by

$$\tilde P^0(\mathbf{p})=\tfrac{1}{2}\nu(2\pi)^{-3}\int d\mathbf{q}\,\tilde G^0(\mathbf{p}-\mathbf{q})\tilde G^0(\mathbf{q}).\quad(3.6)$$

$G_{ij}{}^0$ is the inverse (Green's function) of $z\mathbf{1}-\mathbf{V}$; hence,

$$\tilde G^0(\mathbf{p})=[z-\tilde v(\mathbf{p})]^{-1}.\quad(3.7)$$

The $\{\tilde\varphi\}'$ integral in Eq. (3.4) may be approximately evaluated in this way to yield [cf. Eq. (2.9)]

$$Z\propto\beta^{-N(\nu/2-1)}\int dz\exp\left(N\nu\beta z+\tfrac{1}{2}N\nu(2\pi)^{-3}\times\int d\mathbf{p}\ln\tilde G^0(\mathbf{p})-\tfrac{1}{2}N(2\pi)^{-3}\int d\mathbf{p}\ln\tilde P^0(\mathbf{p})\right).\quad(3.8)$$

A saddle-point integration on the z variable appears to be appropriate by virtue of N being large. The saddle point is determined by

$$\nu\beta-\tfrac{1}{2}\nu(2)\pi^{-3}\int d\mathbf{p}\,[z_s-\tilde v(\mathbf{p})]^{-1}-\tfrac{1}{2}(2\pi)^{-3}\int d\mathbf{p}\,(\partial\ln\tilde P^0/\partial z)_{z=z_s}=0.\quad(3.9)$$

[6] The question of a convergence factor which must be introduced to arrange the order of integration in this fashion has been discussed in I.

Well above the critical point, for large ν, the final term on the left-hand side (i.e., the contribution from the $\{\tilde\varphi\}'$ integration) may be neglected, and saddle condition (2.8) is recovered. It behooves us to exercise more care near the critical point.

In the spherical model the critical point occurs when $z_s=\tilde v_0$, so let us examine the region where

$$\epsilon\equiv z-\tilde v_0\quad(3.10)$$

is small. The second term of Eq. (3.9) may be expanded as

$$\tfrac{1}{2}(2\pi)^{-3}\int d\mathbf{p}\,[z-\tilde v(\mathbf{p})]^{-1}=\beta_c{}^{\mathrm{SM}}-(1/8\pi\sigma^3)\epsilon^{1/2}+\cdots.\quad(3.11)$$

This result is derived by noting that the $\epsilon^{1/2}$ term has its origins in the low-p regions of the integral (which is evident upon taking the z derivative) where it is appropriate to expand

$$\tilde v(\mathbf{p})=\tilde v_0-\sigma^2p^2+\cdots\quad(3.12)$$

[σ is a measure of the range of the forces when $\tilde v_0=O(1)$].

Next, the term

$$J=-\tfrac{1}{2}(2\pi)^{-3}\int_{\mathrm{BZ}}d\mathbf{p}\,\partial\ln\tilde P^0(\mathbf{p})/dz\quad(3.13)$$

of Eq. (3.9) will be examined. It helps to know how $\tilde P^0$, defined by Eq. (3.6), behaves for small p. Again this information can be extracted by approximating $\tilde v(\mathbf{p})$ by its quadratic expansion. The resulting integral for $\tilde P^0$ can be evaluated to leading order by extending the $\mathbf{q}$ integration to all space. The result is

$$\tilde P^0(p)\approx(\nu/8\pi\sigma^4p)\arctan(p\sigma/2\epsilon^{1/2}).\quad(3.14)$$

The above approximations require corrections which are negligible for $p\sigma$ and $\epsilon\ll1$. In the Appendix we find that significant contributions to J come from the $p\sigma\sim1$ region. Nevertheless, the nature of the small-ϵ behavior is determined as

$$J\sim B\epsilon^{-1/2},\quad(3.15)$$

with B a numerical constant. The saddle-point equation (3.9) may be written as

$$\nu\beta-\nu\beta_c{}^{\mathrm{SM}}+\nu(1/8\pi\sigma^3)\epsilon^{1/2}+\nu O(\epsilon)+B\epsilon^{-1/2}+O(1)=0.\quad(3.16)$$

One notes the following: The term $B\epsilon^{-1/2}$ becomes important when $\beta_c{}^{\mathrm{SM}}-\beta=O(\nu^{-1/2})$, and prior to z_s equalling $\tilde v_0$, the saddle z splits into a complex pair.

These features, however, are only a signal of the breakdown of the spherical-model approximation and the approximation wherein the $\{\tilde\varphi\}'$ integrand is expanded to a Gaussian. We believe that one cannot infer anything about the true critical behavior until a complete analysis has been performed. In this respect the Feynman- and Dyson-diagram theory of I may be of value. In Sec. IV, we briefly indicate how that theory is modified for the present problem, and how the terms are ordered in ν^{-1}.

It is to be observed that if the spherical-model approximation breaks down for small $z_s-\tilde{v}_0$, it is certainly inappropriate to apply that approximation below the transition. The subcritical state is described in the spherical model by a z_s which comes within $O(1/N)$ of $\tilde{v}_0$.

IV. DIAGRAM THEORY

The diagram theory of Ref. 3 ($\nu=1$) carries over nearly bodily to the general spin dimension problem. In view of the length of that treatment, it is inappropriate to reproduce it here, so that this section can best be read in conjunction with I.

The basic equation (I3.5) for $\mathcal{S}$ needs to be modified only by inserting a factor ν in the exponent. This is then incorporated in the diagram theory by changing the rule following Eq. (I3.15) so that with each closed loop of G bonds is associated a factor of $\frac{1}{2}\nu$ (instead of the original factor of $\frac{1}{2}$).

The resulting diagram theory can be expressed in a number of ways. Here we will mention only the two coupled integral equations for functions $\tilde{G}(\mathbf{p},z)$ and $\tilde{D}(\mathbf{p},z)$ [7]:

$$[\tilde{G}(\mathbf{p})]^{-1}=[\tilde{G}^0(\mathbf{p})]^{-1}+\tilde{M}(\mathbf{p})\,, \qquad (4.1)$$

$$\tilde{D}(\mathbf{p})=[\tilde{P}(\mathbf{p})]^{-1}. \qquad (4.2)$$

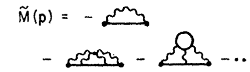
The "self-energy" functions $\tilde{M}$ and $\tilde{P}$ are given by an infinite series of terms involving the renormalized $\tilde{G}$ and $\tilde{D}$. In a standard manner the terms of these series are represented by diagrams, the first few of which are given in Figs. 1 and 2. In each diagram a heavy, solid line represents $\tilde{G}$ and a wavy line is $-\tilde{D}$. A momentum $\mathbf{p}$ is considered as being transmitted across the diagram. Momenta of the intermediate lines are to be integrated over a Brillouin zone, with conservation of momentum at the vertices. A factor of $\frac{1}{2}\nu$ is to be associated with each closed loop of G lines.

The equation for $\tilde{P}(\mathbf{p})$ involves at least one closed loop, so it is of $O(\nu)$ (cf. Fig. 1). The function $\tilde{D}(\mathbf{p})$ is equal to $1/\tilde{P}(\mathbf{p})$; hence $\tilde{D}(\mathbf{p})$ begins as $O(\nu^{-1})$. To get another closed loop (factor of ν) into a P diagram it is necessary to include two more D bonds (each a factor of ν^{-1}). This is illustrated by the third diagram of Fig. 1.

The "self-energy" $\tilde{M}(\mathbf{p})$ involves at least one D line and no loops (cf. Fig. 2). Loops are again accompanied by at least two more D lines, as in the third diagram, so there is no profit in introducing them.

[7] The reader is referred to Ref. 3 for the equation relating the spin-spin correlation function to a weighted z integral of $\tilde{G}(\mathbf{p},z)$ and the equation relating the energy-energy correlation to $\tilde{D}(\mathbf{p},z)$.

FIG. 2. Diagrammatic representation of terms in the M equation.



We conclude that the leading contribution to $\tilde{P}(\mathbf{p})$ is given by Eq. (3.6), and that $\tilde{G}$ is given by

$$[\tilde{G}(\mathbf{p})]^{-1}=[\tilde{G}^0(\mathbf{p})]^{-1} +(2\pi)^{-3}\int d\mathbf{q}\,[\tilde{P}^0(\mathbf{q})]^{-1}\tilde{G}^0(\mathbf{p}-\mathbf{q})+O(\nu^{-2})\,, \qquad (4.3)$$

where the second term on the right is implicitly $O(\nu^{-1})$.

The complete $\{\tilde{\varphi}\}'$ integral of Eq. (3.4) may be expressed in terms of renormalized $\tilde{G}$ and $\tilde{D}$ functions by a Luttinger-Ward[8,3] technique. The result is

$$Z\propto\beta^{-N(\nu/2-1)}\int_{-i\infty+\gamma}^{i\infty+\gamma}dz\,\exp\biggl(N\nu\beta z +\tfrac{1}{2}N\nu(2\pi)^{-3}\int d\mathbf{p}\,\ln\tilde{G}(\mathbf{p}) +\tfrac{1}{2}N(2\pi)^{-3}\int d\mathbf{p}\,\ln\tilde{D}(\mathbf{p})+\Omega^{\dagger}(z)\biggr)\,. \qquad (4.4)$$

$\Omega^{\dagger}(z)$ has a diagrammatic expansion which begins as in Fig. 3. The first term is apparently $O(\nu^0)$. We note, however, that to this order it involves

$$(2\pi)^{-3}\int d\mathbf{p}\,P^0(\mathbf{p})/P^0(\mathbf{p})=1\,.$$

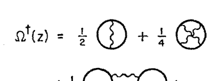
Thus its leading z-dependent contribution is $O(\nu^{-1})$. The second and third diagrams are $O(\nu^{-1})$, as are several diagrams similar to three involving crossings of the D lines.

We recognize the $O(\nu)$ terms in the integrand of Eq. (4.4) as the spherical-model terms. The contribution of $O(1)$ from the $\ln\tilde{G}$ term involves a diagram like the first of Fig. 3, and so is an immaterial constant. The only significant $O(1)$ contribution comes from lowest approximation to the $\ln\tilde{D}$ term, and is exactly the quantity we considered from the Gaussian integral approximation of Sec. III.

V. DISCUSSION OF DIFFICULTIES NEAR THE CRITICAL POINT

As indicated in Sec. III, the $1/\nu$ ordering just described is not useful near the critical point. In analogy

FIG. 3. Representation of $\Omega^{\dagger}$ in terms of diagrams with renormalized lines.



[8] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960).

to what has been shown by Vaks, Larkin, and Pikin[9] and by Thouless[10] in the context of related models, it is possible to demonstrate that part of the difficulty with the type of analysis used in Sec. II is that the expansions are carried out about the point $z_s=\tilde{v}_0$, rather than about a different critical value of z_s arising after (masslike) renormalization. These authors analyze the shifts, but in the context of a model where the Green's function remains of Ornstein-Zernike (OZ) form,

$$G^{-1}(p)=a+bp^2+\cdots. \tag{5.1}$$

However, a lack of self-consistency is to be noted. The OZ form implies that $D(p)\sim cp$. In turn, the second term on the right-hand side of Eq. (4.1) makes a $p^2\ln p$ contribution, which dominates the OZ term bp^2 of Eq. (5.1). If one tries to include the $p^2\ln p$ term in G, the difficulty compounds, leading to powers of $\ln p$. This type of hazard is characteristic of perturbation approaches to a field-theory infrared divergence.

It is possible, as in I, to draw general conclusions about the z integration from rigorous implications of a saddle-point integral and thus to point out another potential difficulty. For the moment we make assumption A:

The integrand of Eq. (4.4) has a single saddle point $z_s>\tilde{v}_0$ on the positive real axis, and the contour may be distorted to run through this saddle point.

Then it follows[3] that the energy per spin u is given by

$$u=-\tfrac{1}{2}J\nu[z_s-(\tfrac{1}{2}-\nu^{-1})\beta^{-1}]. \tag{5.2}$$

For $\nu\leq 2$ the requirement that $u>-\frac{1}{2}J\nu$ leads one to conclude that assumption A on z_s cannot be true. For $\nu>2$ the hypothesis may be true.

If assumption A is true, it also[3] follows that

$$\partial z_s/\partial\beta=-(\tfrac{1}{2}-\nu^{-1})\beta^{-2}+(1/\nu k_B\beta^2)c_H, \tag{5.3}$$

where c_H is the specific heat. For $\nu>2$ the saddle point moves to the left as temperature is decreased when $c_H<k(\frac{1}{2}\nu-1)$. If the specific heat diverges in this model at β_c corresponding to a $z_s=z_c$, and if assumption A is true at all temperatures above the critical, then $z_s(\beta)$ must have had the value z_c at some higher temperature β_1. But a concomitant to the divergence in the specific heat is a divergence of $\lim_{p\to 0}\tilde{D}(\mathbf{p},z_s)$.[3] Thus we are led to the contradiction of a specific-heat divergence at β_1. We conclude that it is necessary for assumption A to be invalid, at least at some higher temperature, to obtain a divergent specific heat at β_c.

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We thank Dr. H. E. Stanley for informing us, prior to publication, of his results, and for valuable conversations. We also have benefited from discussions with Dr. M. Fibich.

APPENDIX

We must evaluate the integral which gives rise to the critical anomaly in the saddle-point condition, Eq. (3.9):

$$J=-\tfrac{1}{2}(2\pi)^{-3}\int_{\mathrm{BZ}} d\mathbf{p}\;\partial\ln\tilde{P}^0(\mathbf{p})/dz. \tag{3.13}$$

Let us examine

$$\partial\tilde{P}^0/\partial z=-\nu(2\pi)^{-3}\int d\mathbf{q}\,[z-\tilde{v}(\mathbf{q})]^{-2}\times[z-\tilde{v}(\mathbf{p}-\mathbf{q})]^{-1}. \tag{A1}$$

From the neighborhood of $q=0$, we extract as the leading divergence

$$\partial\tilde{P}^0/\partial z\sim-(\nu/8\pi\sigma^3\epsilon^{1/2})[z-\tilde{v}(p)]^{-1},\quad \sigma p\gg\epsilon^{1/2} \tag{A2}$$

$$\sim-(\nu/32\pi\sigma^3\epsilon^{3/2}),\qquad \sigma p\ll\epsilon^{1/2}. \tag{A3}$$

In the J integral, the region with $\sigma p\ll\epsilon^{1/2}$ is only of volume $O(\epsilon^{3/2})$. $\tilde{P}^0$ goes like $\epsilon^{-1/2}$ in this region. Hence an estimate of the contribution to J is $O(\epsilon^{1/2})$. More careful analysis reveals an $\epsilon^{1/2}\ln\epsilon$ term.

The remainder of p space contributes a more important term. Inserting Eq. (A2), we find as the dominant contribution to J

$$J\sim B\epsilon^{-1/2}, \tag{A4}$$

$$B=(1/128\pi^4\sigma^3)\int_{\mathrm{BZ}} d\mathbf{p}\,[\nu^{-1}\tilde{P}^0(\mathbf{p},\,\epsilon=0)]^{-1}\times[\tilde{v}_0-\tilde{v}(\mathbf{p})]^{-1}. \tag{A5}$$

B is finite by virtue of the fact that for small p, one has $\tilde{P}^0\sim p^{-1}$.

[9] V. G. Vaks, A. I. Larkin, and S. A. Pikin, Zh. Eksperim. i Teor. Fiz. **51**, 761 (1966) [English transl.: Soviet Phys.—JETP 24, 240 (1967)].

[10] D. J. Thouless (to be published).