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The author wishes to thank Professor G. V. Chester for the helpful correspondence.

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PHYSICAL REVIEW VOLUME 183, NUMBER 1 5 JULY 1969

Low-Temperature Ion Mobility in Interacting Fermi Liquids

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(Received 24 February 1969)

The temperature dependence of the mobility μ of a heavy ion in a neutral interacting Fermi liquid is found to be of the form $1/\mu = R - ST^2 \ln 1/T + O(T^2)$ for low T where R and S are constants. The $T^2 \ln T$ term is a consequence of the Friedel density oscillations around the ion and would be absent for the noninteracting Fermi liquid investigated by other authors. The coefficients are calculated for the case of a large hard sphere. The pressure dependence of the coefficients and the predicted temperature dependence are in reasonable agreement with recent experimental data for negative ions in He^3.

I. INTRODUCTION

The low-field mobility of positive and negative ions in liquid He^3 for temperatures between 0.03 and 1°K and at various pressures has recently been measured by Anderson, Kuchnir, and Wheatley.[1] They observed that at temperatures near 0.03°K the mobility of the negative ion was a constant independent of temperature and for higher temperatures slowly increased with increasing temperature. The measurements of the positive-ion mobility were not conclusive due to the probable presence of impurities in colloidal suspension and will only be briefly discussed. We have reported a theoretical interpretation of the negative-ion data based on a model of a hard sphere moving through a weakly interacting Fermi liquid.[2] In this paper we give a more complete discussion and derivation of our results.

The model for the negative-ion structure is an electron inside a bubble created in the liquid due to the strong, short-range repulsion between the electron and a helium atom and the large zero-point energy of the electron.[3] The observed pressure dependence of the mobility is directly connected to the dependence of the radius of the electron bubble on pressure.

Previous theoretical analyses of the ion mobility have been based on the Boltzmann equation for the motion of an ion moving through a degenerate Fermi liquid of noninteracting particles.[4–7] The collision integral is difficult to calculate analytically in general and has only been considered in the two limiting cases where the thermal momentum of the ion is much less or much greater than the Fermi momentum of the fluid particles. The former case which corresponds to temperatures such that $T \ll m\epsilon_F/M$ was considered by Clark,[4] Abe and Aizu,[5] and Schappert[6] who found that

$$1/\mu \sim (MT)^2 . \qquad (1.1)$$

T is the absolute temperature, k_F is the Fermi momentum, $\epsilon_F \equiv k_F{}^2/2m$ is the Fermi energy,[8] M is the effective mass of the ion, and m is the mass of the fluid particles. The mass of the negative ion was estimated in Ref. 1 to be approximately 390 m at a pressure of 0.26 atm. Thus for (1.1) to hold $T \ll 0.012°$K which is below the temperatures attained. In the latter limit ($m\epsilon_F/M \ll T \ll \epsilon_F$) Davis and Dagonnier[7] show that

$$1/\mu \sim k_F{}^4(1 + cT^2) , \qquad (1.2)$$

where c is a positive constant.[9] The observed temperature independence and pressure dependence of the negative-ion mobility below ~ 0.07°K are in qualitative agreement with (1.2), but at higher temperatures (1.2) predicts a decrease rather than the observed increase in the mobility.

The effective mass of the positive ion is only about 40 m so that the experimental data of Ref. 1 covers the temperature range between the two limiting cases. Thus for $T < 0.12°$K, μ would be expected to increase with decreasing temperature as was observed qualitatively in Ref. 1.

In this paper we show that it is necessary to consider the interaction between fermions to account for the observed temperature dependence of the mobility. We first derive in Sec. II a general expression for the inverse mobility of a heavy ion. To demonstrate the qualitative features we assume in Sec. III that the ion-fermion interaction can be treated in Born approximation and apply our general expression to noninteracting fermions and recover the results of Davis and Dagonnier. We then take into account the interaction between fermions to first order and find a $T^2 \ln T$ term for $1/\mu$. In Sec. IV we repeat the calculation of Sec. III treating the ion-fermion interaction exactly. Finally in Sec. V we compare our theoretical calculations with the experimental data of Ref. 1 for negative ions in He^3 and find qualitative agreement.

II. BASIC FORMULA

In the presence of a uniform, constant electric field the ion will acquire a constant drift velocity v through the liquid. In the frame of reference in which the ion is at rest the Hamiltonian for the system to first order in v is

$$H' = H - vP ,$$
$$H = H_0 + \int d^3x\, \rho(x)\, U(x) , \qquad (2.1)$$

where P is the total momentum of the liquid along the direction of v, ρ is the density of the liquid, $U(x)$ is the potential due to the ion, and H_0 is the Hamiltonian of the liquid at rest in the absence of the ion. The mobility μ is related to the power dissipated by the ion by the definition

$$\text{power} = v^2/\mu . \qquad (2.2)$$

If we treat the vP term in (2.1) as a perturbation and apply Fermi's "golden rule" we find

$$\text{power} = 2\pi v^2 \sum_{i,f} W_i |P_{fi}|^2 \times \delta(E_f - E_i)(E_f - E_i)$$
$$= \frac{v^2}{2} \lim_{\omega \to 0} \omega \int dt\, e^{i\omega t} \langle [P(t), P] \rangle , \qquad (2.3)$$

where E_n and W_n are the energy and statistical weight respectively of the eigenstate n of H. The bracket $\langle \cdots \rangle$ represents the average over the equilibrium ensemble describing the liquid in the presence of a fixed ion, and $P(t)$ is the momentum operator in the Heisenberg representation $e^{iHt} P e^{-iHt}$.

We integrate (2.3) by parts twice to obtain the more familiar expression

$$1/\mu = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt\, e^{i\omega t} \langle [F(t), F] \rangle \ , \qquad (2.4)$$

where $F = dP/dt$

$$= \int d^3x\, \rho(x) \frac{\partial U(x)}{\partial z} \qquad (2.5)$$

is the force between the ion and the fluid along the direction of flow (taken to be the z axis). The calculation of the mobility has now been reduced to finding the density response function in the presence of a static field $U(x)$.

In the derivation of (2.4) the recoil of the impurity was not taken into account and (2.4) is applicable only to sufficiently massive ions. We have also assumed that the term vP could be treated by perturbation theory which implies that the drift velocity of the ion is much less than the average velocity of the fluid particles.[10]

III. BORN APPROXIMATION

The presence of the field $U(x)$ makes the fluid nonuniform and hence the calculation very difficult. If we treat the ion-fermion interaction in Born approximation, we avoid the complications of nonuniformity and (2.4) reduces to

$$1/\mu = -\frac{1}{3}\int \frac{d^3q}{(2\pi)^2} U(q)^2 q^2 \operatorname{Im}\chi'(q, 0) \ , \qquad (3.1)$$

$$\chi(q, \omega) = -i\int d^3x\, dt\, e^{-i\vec{q}\cdot\vec{x}} e^{i\omega t} \times \langle [\rho(x, t), \rho(0, 0)] \rangle\, \theta(t) \ , \qquad (3.2)$$

$$\chi'(q, 0) = \left.\frac{\partial\chi(q, \omega)}{\partial\omega}\right|_{\omega = 0} ,$$

where χ is the density response function for the uniform system, i.e., that without the impurity. $U(q)$ is the Fourier transform of $U(r)$, and $\theta(t)$ is the usual step function. If we neglect the interaction between the fermions and calculate the contribution from Fig. 1(a) we obtain

$$\chi_0(q, \omega) = 2\int \frac{d^3q}{(2\pi)^3} \frac{f_{p+q} - f_p}{\epsilon_{p+q} - \epsilon_p - \omega} \ , \qquad (3.3)$$

so that

$$\operatorname{Im}\chi_0'(q, 0) = -\frac{m^2}{2\pi q}\left[\exp\{\beta(q^2/8m - \epsilon_F) + 1\}\right]^{-1} , \qquad (3.4)$$

where $f(\epsilon_p) = [\exp(\epsilon_p/T) + 1]^{-1}$,

and ϵ_p is the particle energy $p^2/2m$ measured from the Fermi energy. The factor of 2 has been

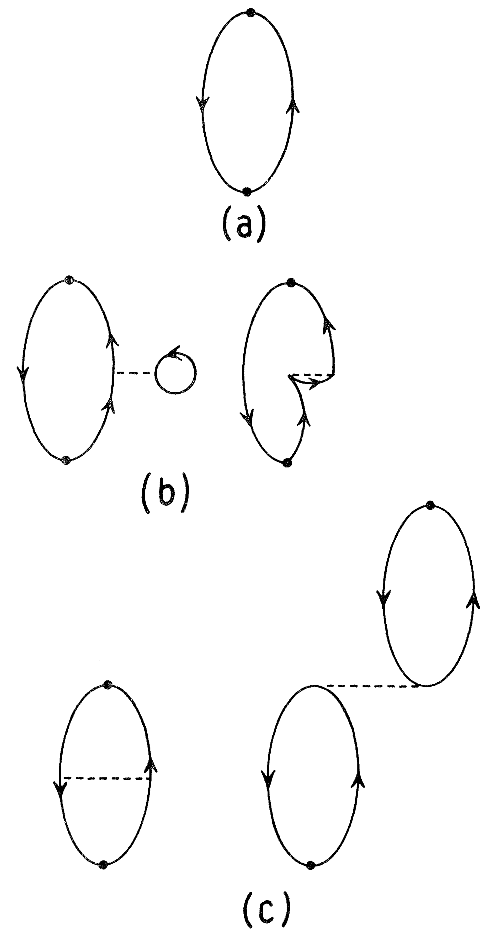

FIG. 1. Zero- and first-order diagrams for the density correlation function χ. The self-energy corrections to the hole line in (b) are not shown.

introduced in (3.3) to account for the spin of the fermions. To further simplify the calculation we represent the ion-fermion interaction by a pseudopotential $U(q) = 4\pi a/m$ where a is some length characterizing the ion ($k_F a \ll 1$). Then substituting (3.4) into (3.1) and performing the integral over q for $T \ll \epsilon_F$, we obtain the result

$$1/\mu = (16a^2k_F^4/3\pi[1 + \tfrac{1}{3}\pi^2(T/T_F)^2] \ . \qquad (3.5)$$

which agrees with that of Davis and Dagonnier. As was mentioned in the introduction, (3.5) predicts that the mobility should decrease with increasing temperature which is opposite to that observed experimentally. We are thus led to consider the interaction between the fermions.

Equation (3.3) indicates that at low temperatures only those particles near the Fermi level contribute to $1/\mu$. We can consider a normal interacting Fermi liquid to be a gas of weakly interacting quasiparticles of effective mass m^* so long as only particles near the Fermi level are involved. Thus a study of a weakly interacting system to which we shall devote the rest of this paper will lead to some qualitative conclusions for a strongly interacting system also.

The effect of a weak fermion-fermion interaction may be viewed in the following way. The particles see the self-consistent field due to all the other particles plus the ion. Because of the presence of the ion, the self-consistent field has a long-range oscillating tail of wave number $2k_F$ as a function of the distance from the ion.[11] This tail is due to long-range density oscillations, commonly referred to as Friedel density oscillations, and will be shown to lead to the presence of a $T^2\ln T$ term in $1/\mu$. The presence of a logarithmic term should not be surprising since they seem to occur often when long-range effects are present.

The first-order corrections to χ_0 are shown in Figs. 1(b) and 1(c). The corresponding contribution to $1/\mu$ can easily be shown to be

$$\Delta\left(\frac{1}{\mu}\right)=-\frac{1}{3}\int\frac{d^3q}{(2\pi)^3}\,q^2U(q)^2 \times \mathrm{Im}\chi_0'(q,0)\,V\chi_0(q,0)\,, \tag{3.6}$$

where we have assumed the effective quasiparticle interaction to be a constant V, i.e., a point interaction. The angular integrals in (3.3) for χ_0 can easily be performed so that we can write χ_0 in the form

$$\chi_0(q,0)=-\frac{m}{q\pi^2}\int_0^\infty p\,dp f_p \ln\left|\frac{q+2p}{q-2p}\right|. \tag{3.7}$$

The dominant contribution to the integral in (3.6) comes from energies near the Fermi level so we rewrite (3.6) as

$$\Delta(1/\mu)=\int d\epsilon\int d\epsilon' f(\epsilon)f(\epsilon')I(\epsilon,\epsilon')\ln\left|\epsilon-\epsilon'\right|\,, \tag{3.8}$$

where

$$\epsilon=p^2/2m-\epsilon_F,\quad \epsilon'=q^2/8m-\epsilon_F\,,$$

and $I(\epsilon,\epsilon')$ is a slowly varying function near $\epsilon=\epsilon'=0$ (remember that we are measuring energies from ϵ_F). Let us separate $f(\epsilon)$ into a zero-temperature part and a temperature-dependent part:

$$f(\epsilon)=\theta(-\epsilon)+h(\epsilon/T)\,,$$
$$h(\epsilon/T)=\mathrm{sgn}\epsilon\,[\exp|\epsilon|/T+1]\,. \tag{3.9}$$

The temperature dependence is contained in the antisymmetric function $h(\epsilon/T)$. Substituting (3.9) into (3.8) we find two temperature-dependent terms

$$\int d\epsilon\,d\epsilon' h(\epsilon/T)h(\epsilon'/T)I(\epsilon,\epsilon')\ln|\epsilon-\epsilon'|$$
$$\approx T^2 I(0,0)\int dx\,dx' h(x)h(x')\ln T|x-x'| \tag{3.10}$$
$$=O(T^2)$$

and

$$\int d\epsilon\,d\epsilon' h(\epsilon/T)\theta(-\epsilon')\times[I(\epsilon,\epsilon')+I(\epsilon',\epsilon)]\ln|\epsilon-\epsilon'|$$
$$\approx-\partial I(0,0)\int d\epsilon\,h(\epsilon/T)[\epsilon\ln\epsilon+O(\epsilon)]$$
$$=-\tfrac{1}{3}\pi^2 I(0,0)\,T^2\ln T+O(T^2)\,. \tag{3.11}$$

The $\epsilon\ln\epsilon$ term in (3.11) reflects the effect of the Friedel density oscillation. Combining (3.5), (3.6), (3.8), and (3.11) and again interpreting $U(q)$ as a pseudopotential we obtain the result

$$\frac{1}{\mu}=\frac{16}{3\pi}a^2k_F^4\left[1+\frac{1}{6}mk_FV\left(\frac{T}{\epsilon_F}\right)^2\ln\frac{\epsilon_F}{T}\right]+O(T^2). \tag{3.12}$$

Equation (3.12) is based on the Born approximation for the ion-fermion interaction which is not a realistic approximation for an ion in liquid He^3. In the following section we obtain a general expression for the coefficient of the $T^2\ln T$ term and apply it to the case of a hard sphere of large radius ($k_Fa\gg1$). The nonmathematically inclined reader can skip to (4.45). The main result will be that the coefficient of the $T^2\ln T$ term is proportional to a^3 instead of a^2.

IV. GENERAL TREATMENT

We follow the same program as in the previous section, namely we first consider a noninteracting Fermi liquid and then calculate the first-order corrections. Let $\phi_n(x)$ be the normalized wave functions satisfying the one-particle Schrödinger equation

$$[-\nabla^2/2m^*+U(x)]\,\phi_n(x)=\epsilon_n\phi_n(x)\,. \tag{4.1}$$

Then $$\rho(x)=\sum_{m,n}\phi_m^*(x)\phi_n(x)a_m^+a_n\,, \tag{4.2}$$

where a_n^+ creates a fermion in the state n. It follows from (2.4) that the contribution to $1/\mu$ from Fig. 1(a) is

$$\frac{1}{\mu}=2\pi\sum_{n,m}(f_n-f_m)\delta'(\epsilon_n-\epsilon_m)\,|F_{nm}|^2, \quad (4.3)$$

where $$F_{nm}=\int d^3x\,\phi_n^*(x)\,\frac{\partial U(x)}{\partial z}\,\phi_m(x)\,. \quad (4.4)$$

We shall assume $U(x)$ to be spherically symmetric. In spherical coordinates we have

$$\phi_{klm}=(\sqrt{2}/r)\,y_l(k,r)\,Y_{lm}(\hat{r})\,,$$
$$\sum_{klm}\rightarrow\frac{1}{\pi}\int dk\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\,, \quad (4.5)$$

and y_l satisfies the equation

$$\left(\frac{d^2}{dr^2}+k^2-2m^*U(r)-\frac{l(l+1)}{r^2}\right)y_l=0\,, \quad (4.6)$$

and the asymptotic condition

$$\lim_{r\to\infty}y_l=\sin(kr+\delta_l-\tfrac{1}{2}\pi l). \quad (4.7)$$

In the above δ_l is the lth phase shift for ion-fermion scattering and Y_{lm} is the spherical harmonic.[12] Substituting (4.5) into (4.3) and using the fact that

$$\int d\Omega\cos\theta\,Y^*_{l-1m}(\hat{r})\,Y_{lm}(\hat{r})=\left(\frac{(l-m)(l+m)}{(2l+1)(2l-1)}\right)^{1/2}\,,$$

we obtain

$$1/\mu=-\int d\epsilon_k f'(\epsilon_k)R(\epsilon_k)\,, \quad (4.8)$$

where

$$R(\epsilon_k)=\frac{16m^{*2}}{3\pi k^2}\sum_{l=0}^{\infty}(l+1)\,|A_l(k)|^2\,, \quad (4.9)$$

$$A_l(k)=\int dr\,y_l(k,r)\,y_{l+1}(k,r)\,\frac{\partial U(r)}{\partial r}\,. \quad (4.10)$$

It can be shown that if U is a hard-core interaction, A_l can be written in the more familiar form[13]

$$A_l(k)=(k^2/2m^*)\sin(\delta_l-\delta_{l+1})\,, \quad (4.11)$$

but we have not been able to obtain (4.11) in general.

For temperatures much lower than the Fermi energy we can use arguments similar to those leading to (3.12), expand $R(\epsilon)$ in powers of ϵ, and obtain from (4.8)

$$1/\mu=R(0)+(\pi^2/6)\,T^2R''(0)+O(T^4)\,. \quad (4.12)$$

Equation (4.9) shows that R is always positive, but the sign of $R''(0)$ depends on the nature of $U(r)$.

The first-order corrections to the density response function are shown in Figs. 1(b) and 1(c). After some algebra it can be shown that the corresponding contribution to $1/\mu$ is

$$\Delta\frac{1}{\mu}=2\pi\sum_{nm}f_n\delta'(\epsilon_n-\epsilon_m)\times\mathrm{Re}(F_{mn}\Delta F_{nm}+F_{nm}\Delta F_{mn})\,, \quad (4.13)$$

$$\Delta F_{nm}=\Delta F_{nm}^{(b)}+\Delta F_{nm}^{(c)}\,,$$

where

$$\Delta F_{nm}^{(b)}=2\sum_{n'm'}\frac{f_{m'}F_{n'm}}{\epsilon_n-\epsilon_{n'}}\times(V_{nm'n'm'}-V_{nm'm'n'})\,, \quad (4.14)$$

$$\Delta F_{nm}^{(c)}=2\sum_{n'm'}\frac{f_{m'}F_{n'm'}}{\epsilon_{m'}-\epsilon_{n'}}\times(V_{nm'mn'}-V_{m'nmn'})\,, \quad (4.15)$$

and

$$V_{nm'n'm}=\int d^3x\,d^3x'\,\phi_n^*(x)\,\phi_{m'}^*(x')\times V(x-x')\,\phi_{n'}(x)\,\phi_m(x')\,. \quad (4.16)$$

The quasiparticle interaction has now been assumed to depend on the interparticle distance. Let us concentrate on the calculation of $\Delta F^{(b)}$. The calculation of $\Delta F^{(c)}$ is similar and is given in the Appendix. We write

$$\rho(x',x)=\sum_{m'}f_{m'}\,\phi_{m'}^*(x')\,\phi_{m'}(x)\,, \quad (4.17)$$

$$\rho(x)=2\rho(x,x)\,,$$

$$G(x,x';\epsilon)=\sum_{n'}\frac{\phi_{n'}(x)\,\phi_{n'}^*(x')}{\epsilon-\epsilon_{n'}}\,, \quad (4.18)$$

$$\Delta\phi_n(r) = \int d^3x' G(r, x', \epsilon_n)\,\phi_n(x') \times \int d^3x V(x'-x)\rho(x) \,, \quad (4.19)$$

$$\Delta\phi_n^{\text{ex}}(r) = -\int d^3x' G(r, x', \epsilon_n) \times \int d^3x\, V(x'-x)\,\rho(x', x)\,\phi_n(x). \quad (4.20)$$

Using the above equations we can write (4.14) for $\Delta F^{(b)}$ in the more transparent form

$$\Delta F_{nm}^{(b)} = 2\int d^3r [\Delta\phi_n(r) + \Delta\phi_n^{\text{ex}}(r)] \times \frac{\partial U(r)}{\partial z}\,\phi_m(r) \,. \quad (4.21)$$

Using spherical coordinates as before, (4.13) can be written as

$$\Delta(1/\mu) = -\int d\epsilon_k f'(\epsilon_k)\,\Delta R(\epsilon_k, T) \,, \quad (4.22)$$

$$\Delta R = \frac{16m^{*2}}{3\pi k^2} \sum_{l=0}^{\infty} (l+1) 2\text{Re}A_l(k)\,\Delta A_l(k) \,, \quad (4.23)$$

where ΔA_l is defined by

$$\Delta F_{l+1m,\,lm} + \Delta F_{lm,\,l+1m} = 4[(l+m+1)(l-m+1)/(2l+1)(2l+3)]^{1/2}\Delta A_l \,. \quad (4.24)$$

The temperature dependence of ΔR comes from the factors of $f_{m'}$ in (4.14) and (4.15) so that (4.22) is an integral over two Fermi distribution functions. If we separate the leading temperature-dependent part from the zero-temperature part part as was done in Sec. III, we obtain from (4.22)

$$\Delta(1/\mu) = \Delta R(0,0) - 2\int_0^{\infty} f'(\epsilon) \times [\Delta R(\epsilon, 0) - \Delta R'(-\epsilon, 0)]\,d\epsilon \,. \quad (4.25)$$

We now must evaluate ΔA_l at $T = 0$, so our first task is to calculate the wave-function correction $\Delta\phi$ given by (4.19). Since the fermion density $\rho(x)$ is spherically symmetric, the effective potential

$$\Delta U = \int d^3x' V(x - x')\,\rho(x') \quad (4.26)$$

must be also. Thus $\Delta\phi_{klm}$ is of the form

$$\Delta\phi_{klm} = (\sqrt{2}/r)\,\Delta y_l(r)\,Y_{lm}(\hat{r}) \,. \quad (4.27)$$

The contribution of Δy_l to ΔA_l can be found from (4.10) or (4.24) and is given by

$$\Delta A_l = \int dr (\Delta y_l y_{l+1} + y_l \Delta y_{l+1}) \frac{\partial U}{\partial r} \,. \quad (4.28)$$

The Green's function (4.18) is written in the form (suppressing the energy variable)

$$G(\vec{r}, \vec{r}') = -\frac{2m}{rr'} \times \sum_{lm} Y_{lm}(\hat{r})\,Y^*_{lm}(\hat{r}')\,G_l(r, r'), \quad (4.29)$$

where $G_l(r, r')$ is obtained from the radial Schrödinger equation with a δ-function source:

$$\left(-\frac{d^2}{dr^2} - k^2 - 2m^*U(r) + \frac{l(l+1)}{r^2}\right) \times G_l(r, r') = \delta(r - r') \,. \quad (4.30)$$

It can easily be shown that the solution to (4.30) is

$$G_l(r, r') = z_l(r_>)\,y_l(r_<)/k \,, \quad (4.31)$$

where $r_<$ is the lesser of r and r', and $r_>$ is the greater of r and r', and $z_l(r)$ satisfies (4.6) together with the asymptotic condition

$$\lim_{r\to\infty} z_l(r) = \cos(kr + \delta_l - \pi l/2) \,. \quad (4.32)$$

By combining (4.19), (4.27), (4.29) and (4.31), we write $\Delta y_l(r)$ in the form

$$\Delta y_l(r) = -(2m/k)\int dr' y_l(r_<) \times z_l(r_>)\,y_l(r')\,\Delta U(r') \,. \quad (4.33)$$

The $T^2 \ln T$ term depends only on the long-range behavior of y_l and z_l. Substituting (4.33) into (4.28), using the asymptotic forms of y_l and z_l, (4.7) and (4.32), respectively, and the definition of A_l, (4.10), we find

$$\Delta A_l = -A_l \sin(\delta_l - \delta_{l+1})(2m^*/k) \times \int dr\,\Delta U(r)\cos(2kr - l\pi + \delta_l + \delta_{l+1}) \,. \quad (4.34)$$

The form of the long-range density oscillation is well known[11]:

$$\Delta\rho(r) = -(2\pi^2 r^3)^{-1}\sum_l (2l+1) \times \sin\delta_l \cos(2k_F r + \delta_l - l\pi)$$

$$= -\frac{k_F}{2\pi^2 r^3}\,\mathrm{Re}f(k_F,\pi)\exp(2ik_F r)\,, \quad (4.35)$$

where $f(k,\theta)$ is the fermion-impurity scattering amplitude

$$f(k,\theta)=k^{-1}\sum_l (2l+1)\sin\delta_l\,\exp(2i\delta_l)P_l(\cos\theta).$$

Combining (4.24) and (4.33), we see that the effective quasiparticle potential is

$$\Delta U(r) = -(k_F/2\pi^2 r^3)\,V(2k_F)$$
$$\times\,\mathrm{Re}f(k_F,\pi)\exp(2ik_F r)\,, \quad (4.36)$$

where $V(k)$ is the Fourier transform of $V(x)$. Then substituting (4.36) into (4.34), we obtain

$$\Delta A_l = A_l\sin(\delta_l-\delta_{l+1})[m^*k_F V(2k_F)/4\pi^2]$$
$$\times\,\mathrm{Re}\left\{k_F f(k_F,\pi)(-1)^l\exp[-i(\delta_l+\delta_{l+1})]\right.$$
$$\left.\times\left[\left(\frac{\epsilon}{\epsilon_F}\right)^2\ln\epsilon-\frac{i\epsilon/\epsilon_F}{k_F a}+\frac{1}{(k_F a)^2}+O(\epsilon^2)\right]\right\}. \quad (4.37)$$

We have repeatedly used the asymptotic form of the wave functions valid for $r \gg k_F^{-1}$ and for r outside the range of the impurity potential. The parameter a in (4.37) is the lower cutoff of the r integral. The $\epsilon^2\ln\epsilon$ term in (4.37) depends only on the asymptotic behavior of the wave functions and leads to a $T^2\ln T$ term for $1/\mu$. Odd powers of ϵ will have no effect according to (4.25) and will be ignored. Equation (4.37) makes sense only if the impurity is of large size.

The contribution of $\Delta\phi^{ex}$ to ΔA_l is found in the Appendix and can be taken into account by simply replacing $V(2k_F)$ by $V(2k_F)-\frac{1}{2}V(0)$ in (4.37). The contribution of Fig. 1(c) to ΔA_l is found to be [see (A7) through (A14)]

$$\Delta A_l^{(c)} = -\{m^*k_F[V(2k_F)-\tfrac{1}{2}V(0)]/4\pi^2\}$$
$$\times\,\mathrm{Re}\Big((-1)^l\exp[-i(\delta_l+\delta_{l+1})]$$
$$\times\sum_{l'}(-1)^{l'}(l'+1)A_{l'}\exp[i(\delta_{l'}+\delta_{l'+1})]\Big)$$
$$\times[(\epsilon/\epsilon_F)^2\ln\epsilon+1/(k_F a)^2+O(\epsilon^2)]\,. \quad (4.38)$$

Now by combining (4.23), (4.37), and (4.38) and rearranging terms, we obtain

$$\Delta R(\epsilon,0)=\frac{3B}{2\pi^2}\left[\frac{1}{(k_F a)^2}+\left(\frac{\epsilon}{\epsilon_F}\right)^2\ln\frac{\epsilon_F}{\epsilon}+O(\epsilon^2)\right], \quad (4.39)$$

where the quantity B is

$$B=k_F^{\,2}(4/9\pi)m^*k_F[V(2k_F)-\tfrac{1}{2}V(0)]$$
$$\times\,\epsilon_F^{-2}\sum_{l,l'}(l+1)(l'+1)$$
$$\times\exp[i(\delta_l+\delta_{l+1}-\delta_{l'}-\delta_{l'+1})]$$
$$\times[A_lA_{l'}-A_l^2\sin(\delta_l-\delta_{l+1})\sin(\delta_{l'}-\delta_{l'+1})]\,. \quad (4.40)$$

Then substituting (4.39) into (4.25) and performing the integral over ϵ, we obtain finally

$$\Delta\left(\frac{1}{\mu}\right)=B\left[\tfrac{3}{2}(\pi k_F a)^2+\left(\frac{T}{\epsilon_F}\right)^2\ln\left(\frac{\epsilon_F}{T}\right)+O(T^2)\right]. \quad (4.41)$$

Equation (4.41) is the general expression for the first-order correction to the inverse mobility.

To evaluate the quantities $R(0)$ and E explicitly, a simple model for the ion-fermion interaction $U(r)$ is needed. We choose a hard-sphere interaction here since it has only one parameter, the radius a, and it has some resemblence to the electron bubble in liquid helium. As is well known the phase shifts are given by[12]

$$\tan\delta_l=-j_l(ka)/n_l(ka)\,. \quad (4.42)$$

For a large sphere such that $k_F a \gg 1$ we approximate δ_l by

$$\delta_l=-ka+\tfrac{1}{2}\pi(l+1)+ka[1-(1-x^2)^{1/2}-x\sin^{-1}x]\,,$$
$$\text{for } l+\tfrac{1}{2}<ka\,,\ =0,\ \text{for } l+\tfrac{1}{2}>ka\,, \quad (4.43)$$

where $x=(l+\frac{1}{2})/ka$. Equation (4.43) can be obtained from the asymptotic form of the Bessel functions or from the WKB approximation.[12] Thus from (4.9) and (4.11) we have

$$A_l=-(k^2/2m)(1-x^2)^{1/2}\theta(1-x)$$

and $$R(0)=(3\pi)^{-1}k_F^{\,2}(k_F a)^2\,. \quad (4.44)$$

In evaluating the double sum in (4.40) we note that the exponential factor is a rapidly oscillating function due to the large phase shifts except for terms with $l=l'$. Thus combining (4.12), (4.40) (4.41), and (4.44) and evaluating B keeping only terms with $l=l'$, we arrive finally at the result

$$1/\mu=(3\pi)^{-1}k_F^{\,2}(k_F a)^2$$

$$\times [1 + \tfrac{8}{315} m^* k_F [V(2k_F) - \tfrac{1}{2} V(0)] k_F a \times (T/\epsilon_F)^2 \ln(\epsilon_F/T)] + O(T^2). \quad (4.45)$$

Since we have assumed that $k_F a \gg 1$, we have neglected the temperature-independent correction term in (4.41). Note that the coefficient of the $T^2 \ln T$ term is proportional to a^3.

V. COMPARISON WITH EXPERIMENT

In order to compare our results with the experimental measurements of Ref. 1, we need to make some assumptions regarding the nature of the interaction of the electron bubble and the helium atom. A complete calculation of this interaction has never been done and is certainly beyond the scope of this paper. We make the reasonable approximation that the electron bubble can be treated as a hard sphere of effective radius a with the additional term

$$U_{\text{pol}}(r) = -\tfrac{1}{2} \alpha e^2 / r^4$$

due to the polarization of the liquid around the ion. It can be shown that for a large sphere ($k_F a \gg 1$) the effect of the polarization interaction is to multiply the constant term in (4.45) by the factor

$$[1 - (\alpha e^2 / 4a^4) \epsilon_F^{-1}]^2$$

and to leave the coefficient of the $T^2 \ln T$ term unchanged.

The experimental values of $1/\mu$ at 0.03°K (where the $T^2 \ln T$ term is unimportant) are given in Table I as a function of pressure. The values of the parameter a that give the best agreement with the mobility data at low temperatures can be inferred from the first term of (4.45) together with the polarization correction and are also presented in Table I.[14–16]

Since the radius of the negative ion in liquid He^3 has not been measured, we perform a theoretical calculation, similar to the one commonly used for He^4,[17] based on a simple bubble model of an electron in a potential well. The total energy involved in the formation of the bubble is

$$E = E_0 + \tfrac{4}{3} \pi P a^3 + 4\pi a^2 \gamma - \tfrac{1}{2} \alpha e^2 / a^4 , \quad (5.1)$$

where the right-hand side is the sum of the electron's zero-point energy in a square well of depth V_0, the work done against the external pressure P, the work done against the surface tension γ, and the polarization interaction energy with the particles outside the bubble respectively. To find the equilibrium radius we set $\partial E / \partial a = 0$. V_0 will be estimated by the simple form

$$V_0 = (\hbar^2 / 2m)\, 4\pi \rho a_t ,$$

where $a_t = 1.1 \hbar^2 / m e^2$ is the low-energy electron-helium scattering length,[18] m is the free-electron mass, and ρ is the liquid density of He^3. We take $\gamma = 0.15$ dyn/cm, independent of T for low T, and to scale with the density.[19] The calculated values of a, which will be referred to as a_{BM} are shown in column six of Table I. The major sources of error in the values of a_{BM} are due to the uncertainty in the values of γ and V_0. The differences between the values of the parameter a and a_{BM} are of the same order of magnitude as has been found in He^4.[17] Using the values of a_{BM}, we can then proceed to calculate the theoretical mobilities. As noted in Ref. 1 there is reasonable agreement between experiment and theory especially in regard to the pressure dependence of the mobility (see column four of Table I).

We now turn to higher temperatures where the $T^2 \ln T$ term becomes important. The quantity $V(2k_F) - \tfrac{1}{2} V(0)$ appearing in (4.45) is as yet unknown and can only be estimated very roughly. Since a realistic interaction between the fermions generally includes a hard core, $V(q)$ must be interpreted as the T matrix for two quasiparticles at the Fermi surface. For liquid He^3 at intermediate pressures the hard-core radius $d \sim 2.5$ Å

TABLE I. Pressure dependence of mobility and radius of electron bubble. (Pressures in atmospheres, radii in angstroms, mobilities in cm^2/V sec, and S in V sec/cm^2 °K^2.) The values of k_F were calculated using the values of the liquid density given in Ref. 14, and ϵ_F was computed using the values of the fermion effective mass given in Ref. 15. The static polarizability α was taken to be 0.2 Å^3 Ref. 16. μ_{theor} was calculated using the first term of Eq. (4.45) with the polarization correction and the values of a_{BM}. The parameter a was determined using the first Eq. (4.45) with the polarization correction and the values of μ_{expt}. a_{BM} was computed according to the bubble model, Eq. (5.1).

Pressure	μ_{expt} $T = 0.03$°K	μ_{theor}	$\mu_{\text{expt}}/\mu_{\text{theor}}$	a	a_{BM}	S (Fig. 2)	S/a^3	S/a_{BM}^3	S/a^2
0.32	0.011	0.009	1.2	18.8	20.1	258	0.039	0.032	0.73
7.5	0.019	0.016	1.2	14.0	13.8	110	0.040	0.042	0.56
27.9	0.025	0.021	1.2	12.3	10.7	82	0.044	0.067	0.54

and $k_F \sim 0.8$ Å^{-1}, so that $k_F d \sim 2$ and several particle waves must be included in estimating V. The hard-core contribution was found by keeping five particle waves to be

$$[V(2k_F) - \tfrac{1}{2}V(0)]_{\text{hard core}} \approx -(1.3)4\pi/m^* k_F . \tag{5.2}$$

The contribution of the $l=4$ term is less than five percent of the total. The attractive part of the interaction certainly will make (5.2) more negative. A rough estimate shows that it is of comparable magnitude, and we write

$$[V(2k_F) - \tfrac{1}{2}V(0)]_{\text{total}} = -(4\pi/m^* k_F)b, \tag{5.3}$$

where $b \sim O(1)$ and is positive.

If we substitute (5.3) into (4.45), we see that the mobility will increase with temperature for low temperatures. In Fig. 2, $1/\mu$ is plotted against $T^2 \ln(3/T)$. It is seen that the predicted temperature dependence is in qualitative agreement with experiment. It should be noted that because of the experimental error, it is difficult to distinguish experimentally between a T^2 and a $T^2 \ln T$ dependence.

Another test of the theory is to compare the observed pressure dependence of the coefficient S of the $T^2 \ln T$ term with that predicted by (4.45). In Table I we show the experimental values of S taken from the slope of the straight lines in Fig. 2. Since the pressure dependence of S is dominated by a^3, we include the ratios of S/a^3 as well as S/a^2 for comparison. If we use the values of the parameter a determined from the mobility data at lower temperatures, we see that for the two lower pressures S/a^3 is approximately a constant as is predicted by the theory and our results are consistent. The discrepancy at high pressure is probably due to the pressure dependence of the effective quasiparticle parameters. However, the dependence of S on a^2 cannot be ruled out by the experimental data.

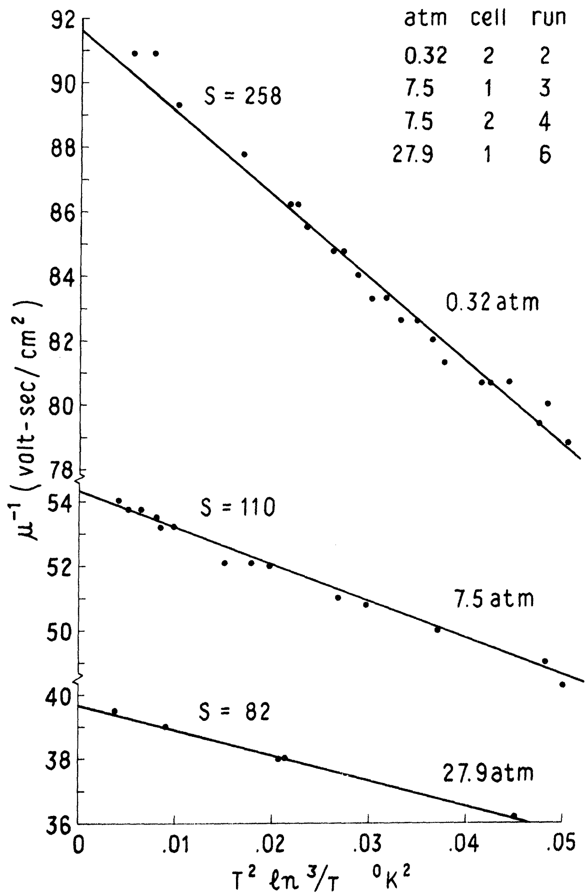

FIG. 2. Temperature and pressure dependence of the mobility of negative ions in liquid He^3. The experimental data are taken from Appendix 3 of Kuchnir's thesis (Ref. 1). The straight lines represent the best fit through the experimental points and S is the slope of the lines. Note units of $1/\mu$.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor T. M. Sanders, Professor J. C. Wheatley, and Dr. R. C. Clark for numerous helpful discussions. One of us (H. G.) would like to thank the Physics Department of The University of California, San Diego, for its hospitality during part of this work.

APPENDIX

In Sec. IV we evaluated the contribution to ΔA_l from the "direct term" in Fig. 1(b), i.e. that given by (4.19). We shall now evaluate the remaining terms. We begin by considering the "exchange term" in Fig. 1(c), i.e., that given by $\Delta\phi^{\text{ex}}$, (4.20). The expression for $\rho(r', r)$ for large r and r' is

$$\rho(r', r) = -(k_F/4\pi^2 r^3)\,\mathrm{Re}\exp[ik_F(r+r')]f(k_F, \pi). \tag{A1}$$

Equation (A1) is similar to (4.35) except that the coefficient in front differs by $\frac{1}{2}$ because only one of the spin states is involved. Equation (4.19) can be written in spherical coordinates in the form

$$[\Delta\phi_{lm}^{\rm ex}(r'')]^* = -(\sqrt{2}/r'')\int d^3r\,d^3r'\,V(\vec{r}-\vec{r}\,')\,\rho(\vec{r}\,',\vec{r})$$

$$\times Y^*_{lm}(\hat{r})\,y_l(r)\sum_{l'm'}\left(\frac{-2m}{rr'}\,G_l(r',r'')\,Y_{l'm'}(\hat{r}')\,Y^*_{l'm'}(\hat{r}'')\right). \qquad \text{(A2)}$$

For r, r' much larger than the range of the interaction V

$$\hat{r}\approx\hat{r}', \qquad r\approx r'+\hat{r}'\cdot(\vec{r}-\vec{r}\,')\,. \qquad \text{(A3)}$$

Substituting (A1) and (A3) into (A2) and integrating over the solid angle $\hat{r}$, we obtain

$$[\Delta\phi_{lm}^{\rm ex}(r'')]^* = (\sqrt{2}/r'')\,Y^*_{lm}(\hat{r}'')\int dr'2mG_l(r',r'')\,{\rm Re}(2i)^{-1}\{V(k+k_F)\exp\{i(kr'+\delta_l-\pi l/2)\}$$

$$-V(k-k_F)\exp[-i(kr'+\delta_l-\pi l/2)]\}(-k_F/4\pi^2r'^3)f(k_F,\pi)\exp(2ik_Fr')\,. \qquad \text{(A4)}$$

Using (4.31) for G_l and the asymptotic form of z_l, (4.32), we can write

$$[\Delta\phi_{lm}^{\rm ex}(r'')]^* = \frac{\sqrt{2}}{r''}\,y_l(r'')\,Y^*_{lm}(\hat{r}'')\,\frac{mk_F{}^2V(0)}{16\pi^2}\,{\rm Re}(-i)f(k_F,\pi)\exp(-2i\delta_l)\left(\frac{\epsilon}{\epsilon_F}\right)^2\ln\epsilon+O(\epsilon^2). \qquad \text{(A5)}$$

Using (4.21), (4.24), and (A5) we find

$$\Delta A_l = -A_l\sin(\delta_l-\delta_{l+1})\,\frac{mk_F{}^2}{8\pi^2}\,V(0)\,{\rm Re}f(k_F,\pi)(-1)^l\exp[-i(\delta_l+\delta_{l+1})]\left(\frac{\epsilon}{\epsilon_F}\right)^2\ln\epsilon+O(\epsilon^2)\,, \qquad \text{(A6)}$$

where we have used the identity

$$\exp(-2i\delta_l)-\exp(-2i\delta_{l+1}) = -2i\sin(\delta_l-\delta_{l+1})\exp[-2i(\delta_l+\delta_{l+1})]\,.$$

We see that (A6) may be obtained from (4.3) by replacing $V(2k_F)$ by $-V(0)/2$.
We proceed to evaluate the first term of (4.15). This term may be written as

$$\Delta F_{nm}^{(c)} = \int d^3r'\,V_{nm}(r')\sum\nolimits_{m'}\phi^*_{m'}(r')X_{m'}(r')f_{m'}\,, \qquad \text{(A7)}$$

where $$V_{nm}(r') = \int d^3r\,V(r'-r)\,\phi^*_n(r)\,\phi_m(r) \qquad \text{(A8)}$$

and $$X_{m'}(r') = \int d^3r\,G(r',r)\,\phi_{m'}(r)\,\partial U(r)/\partial z\,. \qquad \text{(A9)}$$

The function G results from the sum over n'. Converting to spherical coordinates and using (4.29) and (4.31) we obtain in the same manner as before

$$X_{l'm'}(r') = -(2^{3/2}m/kr')\{Y_{l'+1,m}(\hat{r}')z_{l'+1}(r')A_{l'}[(l'+m'+1)(l'-m'+1)/(2l'+1)(2l'+3)]^{1/2}$$

$$+Y_{l'-1,m}(\hat{r}')z_{l'-1}A_{l'-1}[(l'+m')(l'-m')/(2l'+1)(2l'-1)]^{1/2}\}\,. \qquad \text{(A10)}$$

Consider now Eq. (A9) for V_{nm}. We are only interested in the matrix element ΔF with the magnetic quantum number m unchanged and with the total angular momentum l changed by 1 [see (4.24)]. We are thus led to write (A8) in the form

$$V_{lm,l+1m}(\vec{r}\,') = 4\pi\sum_{LM}\int r^2dr\,d\Omega\,V_L(r,r')\,Y_{LM}(\hat{r})\,Y^*_{LM}(\hat{r}')\,\frac{2}{r^2}\,y_l(r)\,y_{l+1}(r)\,Y^*_{lm}(\hat{r})\,Y_{l+1m}(\hat{r})$$

$$= 3\cos\theta'[(l+m+1)(l-m+1)/(2l+1)(2l+3)]^{1/2}\int dr\,y_l(r)\,y_{l+1}(r)\,V_L(r,r')\,. \qquad \text{(A11)}$$

We have used the expansion

$$V(\vec{r}-\vec{r}')=4\pi\sum_{LM}V_L(r,r')Y_{LM}(\hat{r})Y^*_{LM}(\hat{r}),\quad V_L(r,r')=\frac{1}{4\pi}\int d\Omega V(\vec{r}-\vec{r}')P_l(\hat{r}\cdot\hat{r}'). \tag{A12}$$

Substituting (A10) and (A11) into (A7) we obtain

$$\Delta F^{(c)}_{l+1m,\,lm}=\frac{8}{\pi}\int dkf(\epsilon_k)\int dr'\sum_{l'}[(l'+1)A_{l'}z_{l'+1}(r')+l'A_{l'-1}z_{l'-1}(r')]$$
$$\times y_{l'}(r')\int dry_l(r)y_{l+1}(r)V_1(r,r')[l+m+1)(l-m+1)/(2l+1)(2l+3)]^{1/2} \tag{A13}$$

Then using the asymptotic forms of z_l and y_l, we finally arrive at

$$\Delta A_l=-[mk_F V(2k_F)/4\pi^2]\,\mathrm{Re}\sum_{l'}(-1)^{l+l'}(l'+1)A_{l'}\exp[i(\delta_{l'}+\delta_{l'+1}-\delta_l-\delta_{l+1})](\epsilon/\epsilon_F)^2\ln\epsilon\,. \tag{A14}$$

The evaluation of the last term in (4.15) is similar. The result is the same as (A14) except that $V(2k_F)$ is replaced by $-V(0)/2$.

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*Research supported by the National Science Foundation.

†Research supported in part by the Advanced Research Projects Agency of the Department of Defense and was monitored by the U. S. Army Research Office, Durham, under Contract No. DA-31-124-ARO-D-257, and in part by the U. S. Atomic Energy Commission.

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