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[11]This approximation breaks down for low-frequency long-wavelength wave modes, whose period is long compared to whatever relaxation time is associated with the gas at the interface. One can introduce on the right-hand side of Eq. (5) a term proportional to $(\partial P/\partial N)_T \Delta N$ to test the importance of the approximation. The ensuing mathematical analysis becomes more complicated, but the qualitative results are unchanged, and the quantitative results presented below, for the velocity of third sound and the criterion for instability, are not changed very much even in the most extreme limit, where the chemical potential of the gas is taken to be constant everywhere. We therefore omit this complication, as did Atkins, in this simple model.

[12]In I it was asserted that thermally activated superflow could be expected to cease if $\bar{S}=0$. It may be seen from the equations of motion that in this model, that is indeed the case. When $\bar{S}=0$ the equations describe waves in N and v_s, with rapid damping of any temperature perturbation.

[13]It has been suggested (Rudnick, private communication; see also Ref. 3) that $\bar{S}<0$ might represent an unstable region for two fluid hydrodynamics. However, an examination of Eq. (17) shows that the film becomes stable in this region. It is an important result of the present formulation that $\bar{S}$ enters not only into heat flow in the film, in Eq. (6), but also into the two fluid hydrodynamics itself in Eq. (2). Thus if $\bar{S}$ were less than zero, superfluid would flow away from hot spots rather than toward them.

[14]This question is discussed briefly in Ref. 2.

[15]Along the locus of points $\bar{S}=\alpha N C_N/T S_g$ evaporative processes have no net effect and the wave propagates undamped. Thus in this region evaporative effects may be considered small, and the use of Eq. (27) is justified.

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[17a]*Note added in proof*. It has now been shown that if one assumes complete thermal and chemical potential equilibrium across the interface, third sound is always attenuated and the phase velocity is appreciably altered. [D. Bergmann, Report (unpublished)]. Bergmann's assumptions are most valid at low frequencies, whereas the present assumptions are most valid at high frequencies.

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Bose-Einstein Condensation for a Class of Wave Functions

L. Reatto
Istituto di Fisica, Gruppo Nazionale Struttura della Materia, Consiglio Nazionale delle Ricerche, Università di Milano, Milano, Italy
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It is verified that the class of wave functions used in variational calculations for ^{4}He has Bose-Einstein condensation. These trial wave functions have the form of a product of pair functions. Two cases are considered: In the first the correlations in the wave function are assumed to be short range and in the second the correlations include also the long-range part due to the zero-point motion of phonons. The one- and two-dimensional system is briefly considered.

The variational method seems to be the only method able to give, from first principles, quantitative information on the ground-state properties of liquid ^{4}He.[1] It is important to verify that the trial wave functions used in variational calculations have Bose-Einstein condensation. In fact it is generally accepted that the physical origin of the superfluid properties of bulk ^{4}He below the λ point lies in the phenomenon of Bose-Einstein (B. E.) condensation,[2] i. e., the largest eigenvalue of the one-particle reduced density matrix is an extensive quantity.[3]

The main purpose of this paper is to verify this property for the class of trial wave functions which has been used in the literature.[1]

A wave function which gives a good description of the short-range correlations present in the system has the form of a product of pair functions[4]:

$$\psi_u(\vec{r}_1, \ldots, \vec{r}_N) = Q_N^{-1/2}(u) \exp\left[-\frac{1}{2} \sum_{i<j}^{1\cdots N} u(\vec{r}_i - \vec{r}_j)\right], \tag{1}$$

where the correlation function $u(r)$, which contains the variational parameters, is doubly differentiable and has the properties

$$\lim_{r \to 0} u(r) = \infty, \tag{2}$$

$$|u(r)| < Ar^{-3-\epsilon}, \quad \text{for } r > b, \tag{3}$$

where A, b, ϵ are positive constants, and $u(r)$ is bounded below. Q_N is the normalization constant

$$Q_N(u) = \int d\vec{r}_1 \cdots d\vec{r}_N \exp\left[-\frac{1}{2} \sum_{i<j} u(\vec{r}_i - \vec{r}_j)\right]. \tag{4}$$

A better wave function which takes into account also the long-range correlations introduced by the zero-point motion of long-wavelength phonons is[5,6]

$$\psi_{u\chi}(\vec{r}_1, \ldots, \vec{r}_N) = Q_N^{-1/2}(u, \chi) \exp\left[-\frac{1}{2} \sum_{i<j}^{1\cdots N} u(\vec{r}_i - \vec{r}_j) + \chi'(\vec{r}_i - \vec{r}_j)\right], \tag{5}$$

where $$\chi'(r) = N^{-1} {\sum_{\vec{k}}}' (2mc/\hbar k)\, e^{i\vec{k}\cdot\vec{r}}, \tag{6}$$

c is the velocity of sound and the prime in the summation indicates that the $\vec{k} = 0$ term is omitted. We shall use also

$$\chi(r) = N^{-1} \sum_{\vec{k}} (2mc/\hbar k) e^{i\vec{k}\cdot\vec{r}} \to cm(\pi^2 n\hbar r^2)^{-1}, \quad \text{as } N \to \infty. \tag{7}$$

In (I) it was shown that the wave function (1) has B. E. condensation if $u(r) > 0$. In this note we demonstrate that the wave functions (1) and (5) have B. E. condensation under the additional condition that $u(r)$ has a hard core:

$$u(r) = \infty, \quad \text{for } r \leq a. \tag{8}$$

This is an unessential restriction in view of (2) and of the fact that we may choose the hard-core diameter a as small as we like.

If the system satisfies periodic boundary conditions and is spatially uniform, the eigenvalues of the one-particle density matrix coincide with the single-particle momentum distribution[3] N_k. For a system at rest, B. E. condensation shows up in the extensivity of N_0, the number of particles in the zero momentum state. In (I) it was shown that for a Bose system described by a ground-state wave function $\psi(\vec{r}_1 \cdots \vec{r}_N)$, the condensate $n_0 = N_0/V$ is given, in the infinite volume limit, by

$$n_0 = \lim_{V, N \to \infty} \frac{n}{V} \frac{\mathcal{Q}_{N+1}}{Q_N}, \tag{9}$$

where the number density, $n = N/V$, is kept constant, Q_N is the normalization constant, and $\mathcal{Q}_{N+1}$ is given by

$$\mathcal{Q}_{N+1} = Q_N \int d\vec{r}_1 \cdots d\vec{r}_{N+1}\, \psi(\vec{r}_1, \vec{r}_3, \vec{r}_4, \ldots, \vec{r}_{N+1}) \psi(\vec{r}_2, \vec{r}_3, \ldots, \vec{r}_{N+1}). \tag{10}$$

We consider first the case of the wave function (1). If $u(r)$ satisfies the conditions (3) and (8), there exists a positive constant ϕ such that

$$\sum_{i=1}^{t} u(\vec{r}_i - \vec{s}) \geq -\phi , \tag{11}$$

for all $t, \vec{s}, \vec{r}_1, \ldots, \vec{r}_t$, satisfying

$$\sum_{i<j\leq t} u(\vec{r}_i - \vec{r}_j) < \infty . \tag{12}$$

It is trivial to extend to this case Penrose's proof[7] which holds when $a = b$. Use of inequality (11) allows us to write

$$\mathcal{Q}_{N+1}(u) = \int d\vec{r}_1 \cdots d\vec{r}_{N+1} \exp\left\{ - \sum_{i<j}^{1\cdots N+1} u(\vec{r}_i - \vec{r}_j) + \frac{1}{2} \sum_{i=3}^{N+1} [u(r_1 - r_i) + u(r_2 - r_i)] + u(\vec{r}_1 - \vec{r}_2) \right\}$$

$$\geq e^{-\phi-\Delta} \int d\vec{r}_1 \cdots d\vec{r}_{N+1} \exp\left\{ - \sum_{i<j}^{1\cdots N+1} u(\vec{r}_i - \vec{r}_j) \right\} = e^{-\phi-\Delta} Q_{N+1}(u) , \tag{13}$$

where $\Delta = -\min[u(r)]$. Therefore the condensate satisfies the inequality

$$n_0 \geq [n^2/z(u)]\, e^{-\phi-\Delta} , \tag{14}$$

where
$$z(u) = \lim_{V,N\to\infty} [(N+1)Q_N(u)/Q_{N+1}(u)] . \tag{15}$$

The quantity z is formally equal to the activity of a system of classical particles interacting through the two-body potential $\varphi(r)$ and at a temperature $T_{\rm eff}$ such that $\varphi(r)/k_B T_{\rm eff} = u(r)$. If $u(r)$ satisfies the conditions (3) and (8), for this equivalent system the thermodynamic limit exists,[8] so that the activity z is finite if $n < n_c$, where n_c is the close-packing density for hard spheres of diameter a. We conclude that the wave function (1) has a condensate. If $u(r)$ is non-negative, $\phi = 0$ and $\Delta = 0$, and we recover the result of (I).

We consider now the wave function (5). We write the right-hand side of Eq. (9) in the following form:

$$\frac{n}{V} \frac{\mathcal{Q}_{N+1}(u,\chi)}{Q_N(u,\chi)} = \left[\frac{n}{V} \frac{Q_{N+1}(u,\chi)}{Q_N(u,\chi)} \right] \left[\frac{\mathcal{Q}_{N+1}(u,\chi)}{Q_{N+1}(u,\chi)} \right] . \tag{16}$$

If the two expressions inside the brackets of Eq. (16) have a finite limit in the infinite volume limit, we may consider independently these two expressions. For the quantity $\mathcal{Q}_{N+1}/Q_{N+1}$ use of inequality (11) gives

$$\frac{\mathcal{Q}_{N+1}(u,\chi)}{Q_{N+1}(u,\chi)} \geq e^{-\phi-\Delta} \frac{1}{Q_{N+1}(u,\chi)} \int d\vec{r}_1 \cdots d\vec{r}_{N+1} \exp\left\{ - \sum_{i<j}^{N+1} [u(\vec{r}_i - \vec{r}_j) + \chi'(r_i - r_j)] \right\}$$

$$\times \exp\left\{ \frac{1}{2} \sum_{i=3}^{N+1} [\chi'(\vec{r}_1 - \vec{r}_i) + \chi'(\vec{r}_2 - \vec{r}_i)] + \chi'(\vec{r}_1 - \vec{r}_2) \right\}$$

$$\equiv e^{-\phi-\Delta} \left\langle \exp\left\{ \frac{1}{2} \sum_{i=3}^{N+1} [\chi'(\vec{r}_1 - \vec{r}_i) + \chi'(\vec{r}_2 - \vec{r}_i)] + \chi'(\vec{r}_1 - \vec{r}_2) \right\} \right\rangle , \tag{17}$$

where the bracket $\langle \cdots \rangle$ indicates the configurational average

$$\langle F \rangle \equiv \frac{1}{Q_{N+1}(u,\chi)} \int d\vec{r}_1 \cdots d\vec{r}_{N+1} F \exp\left\{ - \sum_{i<j} [u(\vec{r}_i - \vec{r}_j) + \chi'(\vec{r}_i - \vec{r}_j)] \right\} . \tag{18}$$

Use of inequality[9]

$$\langle e^{f(x)}\rangle \geq \exp\langle f(x)\rangle \ , \tag{19}$$

which holds true if $\langle 1\rangle = 1$ and for non-negative weight function, gives for Eq. (17)

$$\lim_{V, N\to\infty} \frac{\mathcal{Q}_{N+1}(u,\chi)}{Q_{N+1}(u,\chi)} \geq e^{-\phi-\Delta} \exp\{n \int d\vec{r}\,\chi'(r)g(r)\} = e^{-\phi-\Delta}\exp\{n \int d\vec{r}\,\chi(r)[g(r)-1]\}, \tag{20}$$

where we have introduced the pair distribution function $g(r)$ relative to the wave function (5). By use of (16) and (20), we find that the condensate satisfies the inequality

$$n_0 \geq [n^2/z(u,\chi)]\, e^{-\phi-\Delta}\exp\{n \int d\vec{r}\,\chi(r)[g(r)-1]\}. \tag{21}$$

Here the quantity $z(u,\chi)$ is the activity of an equivalent classical fluid interacting through a two-body potential proportional to $u(r)+\chi'(r)$. The interaction in this equivalent fluid consists of a short-range part and of a r^{-2} long-range part screened by a uniform "neutralizing" background [this is due to the absence of the $\vec{k}=0$ term in expression (6)]. The existence of the thermodynamic limit for this fluid, and therefore the finiteness of $z(u,\chi)$, is strongly suggested by the similitude of this fluid with a one-component plasma in a uniform neutralizing background. For this plasma the existence of the thermodynamic limit has been always taken for granted,[10] and a first step toward a rigorous proof of it has been made with Dyson and Lenard's proof of the stability of a system of charged particles.[11] [*Note added in proof*: The existence of the thermodynamic limit for a system of charged particles has been proved by J. L. Lebowitz and E. H. Lieb (preprint).]

The activity z can be written[12] as

$$z = n\exp\{n \int_0^1 d\xi \int d\vec{r}[u(r)g(r;\xi)+\chi(r)(g(r;\xi)-1)]\}, \tag{22}$$

where $g(r;\xi)$ is the pair distribution function between two particles, one of which is coupled to the others by $\xi[\chi'(r)+u(r)]$ rather than by $\chi'(r)+u(r)$. From (21) and (22) it results that n_0 is finite if $g(r;\xi)\to 1$ as $r\to\infty$, and $g(r;\xi)-1$ approaches zero faster than r^{-1}. The first condition just tells us that in the system there is no long-range order in configurational space. With regard to the second condition, it is commonly accepted that $g(r)-1$ approaches zero at least as fast as the potential.[13] In fact in (I) it was shown that $g(r)-1\sim O(r^{-4})$ as $r\to\infty$. All this makes very plausible that the thermodynamic limit exists for a fluid with a r^{-2} "screened" interaction and, therefore, that the wave function (5) has a condensate.

Similar considerations may be repeated in the case of a two-dimensional Bose system for wave functions equal to (1) and (5). In this case in Eq. (6) the $\vec{k}$ vectors are two-dimensional so that $\chi'(r)\propto r^{-1}$. Therefore the same confidence we have in the existence of B.E. condensation for a three-dimensional system holds also for a two-dimensional system in its ground state.

In one dimension we expect that there is no condensation for the wave function (5).[5] Therefore we want to verify that in this case our inequality for the condensate is not important. In fact in one dimension $\chi'(r)$ has the form,[5] for a system of length L,

$$\chi'(r) = -(2mc/\pi n\hbar)(\ln 2\,|\sin\pi r/L|). \tag{23}$$

Our previous arguments do not apply due to the explicit and essential dependence of $\chi'(r)$ on L. We can obtain explicit results if we assume $u(r)\equiv 0$ in the wave function (5) and $2mc/\hbar n\pi = 2$ in Eq. (23). This wave function is the exact ground state for a system of point impenetrable bosons[14] and it has been proved it has no condensate.[15] Use of Dyson's results[16] for the partition function $Q_N = N!L^N$, and for the average

$$\left\langle \sum_{i<j}^{1,\ldots,N+1} \ln\left(2\left|\sin\pi\frac{\vec{r}_i-\vec{r}_j}{L}\right|\right)\right\rangle = \tfrac{1}{2}(N+1)[\ln(N+1)-1+\gamma], \quad \gamma = 0.577\cdots, \tag{24}$$

which appears in the inequality (19) applied to (17), gives

$$n_0 \geq [n/(N+1)]\, e^{2(1-\gamma)}, \tag{25}$$

which is an uninteresting bound on the condensate. In fact Lenard showed[15] that n_0 is bounded above by $N^{-1/2}$.

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Low-Temperature Ion Mobility in Interacting Fermi Liquids

Harvey Gould*
Department of Physics, The University of Michigan, Ann Arbor, Michigan 48104

and

Shang-keng Ma†
Department of Physics and Institute for Pure and Applied Physical Sciences, The University of California, San Diego, La Jolla, California 92038

(Received 24 February 1969)

The temperature dependence of the mobility μ of a heavy ion in a neutral interacting Fermi liquid is found to be of the form $1/\mu = R - ST^2 \ln 1/T + O(T^2)$ for low T where R and S are constants. The $T^2 \ln T$ term is a consequence of the Friedel density oscillations around the ion and would be absent for the noninteracting Fermi liquid investigated by other authors. The coefficients are calculated for the case of a large hard sphere. The pressure dependence of the coefficients and the predicted temperature dependence are in reasonable agreement with recent experimental data for negative ions in He^3.

I. INTRODUCTION

The low-field mobility of positive and negative ions in liquid He^3 for temperatures between 0.03 and 1°K and at various pressures has recently been measured by Anderson, Kuchnir, and Wheatley.[1] They observed that at temperatures near 0.03°K the mobility of the negative ion was a constant independent of temperature and for higher temperatures slowly increased with increasing