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Statistical Properties of Waves in a Random Medium

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Averages of a number of properties of waves or quantum particles in random media are determined. Problems are formulated in terms of functional integrals. Techniques of the theory of Markov processes are employed to express the averaged properties in terms of the solutions of Fokker-Planck-Kolmogorov equations. Explicitly, the average density of states and Green's function of a particle in a one-dimensional (white noise) random medium are reexamined. A three-dimensional model system with a spherically symmetric, random potential $V(|\vec{r}|)$ is also considered. This model is relevant to several physical problems. Finally, statistics of the reflectivity of a slab with random, complex dielectric constant $\epsilon(x)$, are determined. A discussion is included of mathematical aspects of the type of functional integral involved in the wave-random-medium problem.

I. INTRODUCTION

The interaction of a wave or quantum particle with a random medium is a problem that arises in a variety of physical situations. Examples are electrons in extrinsic semiconductors[1,2]; scattering of microwaves from an inhomogeneous plasma[3,4]; and, after a mathematical transformation, the polymer excluded volume problem.[5] In these systems one may be concerned with the density of states in the medium; problems of transmission through the substance; the use of the material to guide the wave; or scattering properties of the wave as a probe of the medium.

A more difficult class of problems arises when the wave can influence the state of the medium. Examples include the many-body problem, where the waves are the particle or quasiparticles of which the matter is composed, and quantum field theory, where interactions occur with the virtual particles of the vacuum. Functional integral formulations of these systems – quantum fluids[6,7]; the Ising model[8]; quantum field theory[9] – bring out clearly the relation to the simpler problems arising in media of fixed properties.

Perturbative treatments are of value for many purposes. The small parameter may be the covariance, the inverse correlation range, or the concentration of the inhomogeneities. To provide guidelines to the breakdown of perturbation theory it is interesting to study exactly soluble systems. Several years ago Frisch and Lloyd[10] determined the average density of states in a one-dimensional random medium with a Poisson distribution of δ function scatterers. Halperin[11] presented a theory for the average Green's function and the density of states of a particle in a stochastic potential $V(x)$ characterized by white noise statistics. We present what, in our opinion, is a slightly more direct derivation of these results. We also consider a functional integral formulation of the density-of-states-problem in order to develop a greater understanding of this class of functional integrals.

Cases exist where the restriction to one dimension eliminates interesting physical phenomena, such as phase transitions. For this reason we have examined a model three-dimensional system in which the potential $V(r)$ is constant on shells, but is a random function of the radial variable. We present only a preliminary analysis of this system. An indication of the relation of the model to real physical situations is included.

The methods developed here find immediate application to a problem posed by Hochstim and Martens.[12] These authors performed Monte Carlo calculations of the average reflection intensity from a slab of dielectric characterized by a complex dielectric constant which is a stochastic function of x. In Sec. V we present an equation for the probability distribution of the intensity and phase of reflections from such a random medium.

In general terms, we will consider the following problem. Some property of a wave is a functional of the stochastic element of the medium. We wish to calculate the average of this property over an ensemble of media. For example, the Green's function $\mathcal{G}[x, x', E \mid V(\cdot)]$ of a particle in a one-dimensional random medium is a functional of the potential $V(x)$. We are interested in the average Green's function

$$G(x,x',E) \equiv \langle \mathcal{G} \rangle = \int \delta V(\cdot)\, \mathcal{F}[V(\cdot)]\, \mathcal{G}[x,x',E \mid V(\cdot)] \,. \tag{1.1}$$

This integral is a functional integral over all possible manifestations of the potential $V(x)$, with a probability density $\mathcal{F}[V(\cdot)]$ associated with each manifestation.[13] When V is characterized by certain types of statistics, the logarithmic derivative of the Green's function $Z(\dot{x})$ is a Markovian variable. Z is related to V by a Ricatti equation, which has the form of a Langevin stochastic equation. (Such a relationship has been noted by a number of authors[10,11] and generalized by Lax.[14] This is essentially the basis for previous discussions.) A Fokker-Planck-Kolmogorov equation can then be written for the requisite probability distribution of the variable Z.[14] The Markovian variable Z diffuses and drifts as a function of the x variable.

We believe that the functional integral formulation helps to clarify the problem. At the same time the solutions serve to illustrate techniques for the evaluation of functional integrals. The latter is especially significant in view of the interest, previously alluded to, in functional integral techniques for a variety of problems.[15]

II. STATISTICAL PROPERTIES OF THE MEDIUM

Our prototype problem will be to determine the average properties of a quantum particle in a one-dimensional medium with a stochastic potential $V(x)$. In this paper we shall concern ourselves exclusively with white-noise statistics for the potential. Thus the probability distribution function $\mathcal{F}[V(\cdot)]$ is Gaussian, with mean zero

$$\langle V(x)\rangle = 0, \qquad (2.1)$$

and with covariance

$$\langle V(x)V(x')\rangle = D\delta(x-x')\;; \qquad (2.2)$$

i.e.,

$$\mathcal{F}[V(\cdot)] = \mathfrak{N}\exp[-(1/2D)\int_0^L dx\,V^2(x)]. \qquad (2.3)$$

Throughout this paper we use the convention that $\mathfrak{N}$ is a normalization constant, the exact nature of which should be clear from the context. We shall limit discussion to real functions $V(x)$ until Sec. V, where the simple extension to a complex stochastic process will be made.

III. QUANTUM PARTICLE IN A ONE-DIMENSIONAL RANDOM MEDIUM

A. Formulation

The properties of a quantum particle are effectively characterized by the Green's function $\mathcal{G}(x, x', E)$, given by

$$\left[E+\frac{d^2}{dx^2}-V(x)\right]\mathcal{G}(x,x',E)=\delta(x-x'), \qquad (3.1)$$

where $\hbar = 2m = 1$ and where boundary conditions are to be specified below. We shall be interested in the average Green's function $\langle\mathcal{G}\rangle$ defined by Eq. (1.1). In particular, the average density of states is given by

$$\rho(E) = -\pi^{-1}\mathrm{Im}\langle\mathcal{G}(x,x,E)\rangle, \qquad (3.2)$$

and the average spectral density is given by

$$A(k,E) = -\frac{1}{\pi}\,\mathrm{Im}\int dx\,e^{ik(x-x')}\langle\mathcal{G}(x,x',E)\rangle, \qquad (3.3)$$

where E in the Green's function is considered to approach the real axis from the upper half plane.

$\mathcal{G}(x, x, E\,|\,V)$ is a complicated functional of V and the direct calculation of the average is impractical. We therefore introduce a new random variable in a manner familiar from WKB calculations. The Green's function is written in the form

$$\mathcal{G}(x,x',E) = -[Z_l(x')+Z_r(x')]^{-1}\times\exp\{-\int_{x'}^{x} Z_r(\xi)d\xi\}, \quad x>x'; \qquad (3.4)$$

$$\mathcal{G}(x,x',E) = -[Z_l(x')+Z_r(x')]^{-1}\times\exp\{-\int_{x}^{x'} Z_l(\xi)d\xi\}, \quad x<x'. \qquad (3.5)$$

The prefactor is determined by the condition of continuity of $\mathcal{G}$ and the unit discontinuity of $\partial\mathcal{G}/\partial x$ when x crosses x'. We shall find it convenient to define the variable

$$\begin{aligned} t &= x, && \text{when used in an } l \text{ equation,}\\ &= L-x, && \text{when used in an } r \text{ equation.}\end{aligned}$$

Since $\mathcal{G}$ satisfies the wave equation, the Z's satisfy a Ricatti equation

$$\frac{dZ_s(x)}{dt} = -Z_s^2(x) - E + V(x)\,, \quad s=r,l\,. \qquad (3.6)$$

This equation can also be recognized as a Langevin equation linking stochastic process Z_s and V. Since white-noise V is statistically uncorrelated from point to point, and the term dZ_s/dt introduces correlation only between $Z_s(x)$ and $Z_s(x+dt)$, it is clear that the Z_s process is Markovian. The distance variable (measured from right to left for discussions of Z_r) has replaced the time variable that occurs in most applications of Markov theory.

If the box is very long we shall find that the bulk properties are independent of the boundary conditions, so at this point it is sufficient to specify the boundary conditions as being

$$Z_l(0)=z_l{}^0, \tag{3.7}$$

$$Z_r(L)=z_r{}^0. \tag{3.8}$$

B. Density of States

Consider first the average density of states

$$\rho(E)=\frac{1}{\pi}\operatorname{Im}\mathfrak{N}\int_{z_l{}^0,z_r{}^0}\delta V(\cdot)\times\exp[-(1/2D)\int_0^L d\xi V^2(\xi)]\times[Z_l(x)+Z_r(x)]^{-1}. \tag{3.9}$$

[The subscripts on the integral remind us of the constraints on the function $V(\xi)$.] This may also be written in the form

$$\rho(E)=\frac{1}{\pi}\operatorname{Im}\int dz_r dz_l P(z_l,x\,|\,z_l{}^0,0)\times(z_r+z_l)^{-1}P(z_r,L-x\,|\,z_r{}^0,0), \tag{3.10}$$

where

$$P(z,t\,|\,z^0,0)=\mathfrak{N}\int_{z^0,z}\delta V(\cdot)\times\exp[-(1/2D)\int_0^t d\tau\, V^2(\tau)]. \tag{3.11}$$

The latter functional integral is to be carried out under the constraint that $Z(t)=z$, where $Z(t)$ is the solution of the Langevin equation (3.6) with boundary condition $Z(0)=z^0$. Note that t rather than x has been introduced as the independent variable of Z. We may also say that $P(z,t\,|\,z^0,0)$ is the probability density that a stochastic process Z, which develops in t according to the Langevin equation (3.6), will go from z^0 at $t=0$ to z at $t=t$. In Sec. VI we delve more deeply into the equivalence of these two perspectives, but we first wish to point out that the latter point of view enables us to use standard results of Markov theory, [14] to write an equation for P. The probability density of a process which satisfies the Langevin equation (3.6) is given by the solution of the Fokker-Planck-Kolmogorov equation

$$\frac{\partial P(z,t\,|\,z^0,0)}{\partial t}=\left[\frac{D}{2}\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial z}(z^2+E)\right]\times P(z,t\,|\,z^0,0), \tag{3.12}$$

with initial condition

$$P(z,0\,|\,z^0,0)=\delta(z-z^0), \tag{3.13}$$

and the normalization condition

$$\int dz P(z,t\,|\,z^0,0)=1. \tag{3.14}$$

The density of states can now be evaluated by Eq. (3.2). An integration of the imaginary part of the Ricatti equation (3.6) reveals that if E has a small, positive imaginary part then z has a negative imaginary part. Thus, in the limit of the imaginary parts approaching zero,

$$\operatorname{Im}(z_l+z_r)^{-1}=\pi\delta(z_l+z_r). \tag{3.15}$$

When x is far from the walls, $0\ll x\ll L$, P becomes independent of t and the boundary condition. Thus one has

$$\rho(E)=\int_{-\infty}^{\infty}dz\,P(z)P(-z), \tag{3.16}$$

where $P(z)$ satisfies

$$\tfrac{1}{2}Dd^2P/dz^2+d[(z^2+E)P]/dz=0. \tag{3.17}$$

The normalized solution is

$$P(z)=C\exp[-(\tfrac{2}{3}z^3+2Ez)/D]\times\int_{-\infty}^{z}d\zeta\exp[(\tfrac{2}{3}\zeta^3+2E\zeta)/D], \tag{3.18}$$

$$C^{-1}=(\pi D/2)^{1/2}\int_0^\infty d\eta\,\eta^{-1/2}\times\exp\{-[(\eta^3/6)+2E\eta]/D\}. \tag{3.19}$$

This leads to a density of states

$$\rho(E)=(\pi D/2)^{1/2}C^2\int_0^\infty d\eta\,\eta^{1/2}\times\exp\{-[(\eta^3/6)+2E\eta]/D\}, \tag{3.20}$$

as given by Frisch and Lloyd. [10]

For large negative E the integrals can be evaluated by saddle-point techniques and the density of states approaches

$$\rho(E)\sim(4/\pi D)|E|\exp[-(8/3D)|E|^{3/2}]. \tag{3.21}$$

This asymptotic formula has been obtained by Halperin and Lax, [2] and by Zittartz and Langer. [16] The nonanalytic dependence on D (the variance of the potential fluctuations) accounts for the failure of perturbation theories which predict a sharp cutoff of the density of states rather than the extended tail that is manifested by Eq. (3.21).

C. Green's Function

It is nearly as simple a matter to develop equations for the average Green's function. The functional integral formulation proves extremely useful.

Consider first the evaluation of $G(x,x')=\langle \mathcal{G}(x,x')\rangle$ for $x<x'$. It is given by

$$G(x,x') = -\mathfrak{N}\int \delta V(\cdot)\exp\left[-\frac{1}{2D}\int_0^L d\xi V^2(\xi)\right]$$

$$\times[Z_l(x')+Z_r(x')]^{-1}\exp[-\int_x^{x'} d\xi\, Z_l(\xi)]\,. \tag{3.22}$$

In this case it is convenient to divide the range of ξ into three intervals: 0 to x, x to x', and x' to L. The values of $Z_l(x)=z_l$, $Z_l(x')=z_l'$, and $Z_r(x')=z_r'$ are at first held fixed and then integrated over. The average Green's function can be written in the form analogous to Eq. (3.10):

$$G(x,x') = -\int dz_l\, dz_l'\, dz_r' P(z_l) \times \frac{Q(z_l', x'|z_l, x)}{z_l'+z_r'} P(z_r), \quad x<x'. \tag{3.23}$$

We have already assumed that x and x' are far from the ends of the box, so that we may employ the asymptotic function $P(z)$ rather than $P(z,x|z^0,0)$. The function Q, which represents the functional integral for the intermediate region x to x', is given by

$$Q(z',x'|z,x) = \mathfrak{N}\int_{z,z'}\delta V(\cdot)$$

$$\times\exp\left[-\frac{1}{2D}\int_x^{x'} d\xi\, V^2(\xi)\right]$$

$$\times\exp[-\int_x^{x'} d\xi\, Z_l(\xi)]\,. \tag{3.24}$$

This equation for Q bears strong resemblance to the Kac functional integrals.[15,17] Indeed, similar techniques are utilized for its evaluation in Sec. VI. At present it is of interest to continue to employ a Langevin-equation approach for its evaluation.

Define a stochastic variable $Y(\xi)$, related to the stochastic variable $Z(\xi)$, by

$$Y(\xi) = \int_x^\xi d\xi' Z(\xi')\,, \tag{3.25}$$

or, in differential form, by

$$dY/d\xi = Z(\xi)\,. \tag{3.26}$$

This is coupled to the Ricatti equation

$$dZ/d\xi = -Z^2(\xi) - E + V(\xi)\,, \tag{3.27}$$

and the two may be viewed as a simultaneous set of Langevin equations. Consider the functional integral

$$P^{(2)}(z',y',x'|z,y,x) = \mathfrak{N}\int_{\substack{z,z'\\ y,y'}}\delta V(\cdot)$$

$$\times\exp[-(1/2D)\int_x^{x'} d\xi\, V^2(\xi)]\,, \tag{3.28}$$

where the value of Z and Y are fixed at the end points, and Z and Y envolve according to the Langevin equations (3.26)–(3.27). $P^{(2)}$ is given by the Fokker-Planck-Kolmogorov equation

$$\frac{\partial P^{(2)}}{\partial x'} = \left[\frac{D}{2}\frac{\partial^2}{\partial z'^2} + \frac{\partial}{\partial z'}(z'^2+E) - \frac{\partial}{\partial y'}z'\right]P^{(2)}, \tag{3.29}$$

with initial condition

$$P^{(2)}(z',y',x|z,y,x) = \delta(z'-z)\delta(y'-y)\,. \tag{3.30}$$

We are interested in

$$Q(z',x'|z,x) = \int_{-\infty}^{\infty} dy' P^{(2)}(z',y',x'|z,0,x)e^{-y'}, \tag{3.31}$$

which satisfies

$$\frac{\partial Q}{\partial x'} = \left[\frac{D}{2}\frac{\partial^2}{\partial z'^2} + \frac{\partial}{\partial z'}(z'^2+E) - z'\right]Q, \tag{3.32}$$

with $Q(z',x|z,x) = \delta(z'-z)\,.$ (3.33)

Equations (3.23), (3.17), and (3.32) constitute the solution to the problem of determining the average Green's function for $x<x'$. One could repeat the derivation employing the form of $\mathcal{G}$ appropriate to $x>x'$, but it is simpler to use the fact that the symmetry of a Green's function, with respect to interchange of x and x', is retained upon averaging. This, together with Eq. (3.23), leads us to conclude that $G(x,x')$ is a function of $|x-x'|$.

The spectral density may be obtained from these results in two steps. To take the imaginary part of G, we use Eq. (3.23) and integrate over dz_r' using the $\delta(z_l'+z_l')$. The result is

$$-\pi^{-1}\,\mathrm{Im}G(x,x') = \int dz\, dz' P(z) \times Q(z',x'|z,x)P(-z'), \quad x<x' \tag{3.34}$$

with x and x' interchanged for $x>x'$.

In order to take the Fourier transform introduce the functions

$$Q_l(z', x'-x|z) = Q(z', x'|z, x)\theta(x'-x) , \quad (3.35)$$

$$Q_r(z', x'-x|z) = Q_l(z', x-x'|z), \quad (3.36)$$

where θ is the unit step function. Q_l satisfies

$$\left[\frac{\partial}{\partial x'} - \frac{D}{2}\frac{\partial^2}{\partial z'^2} - \frac{\partial}{\partial z'}(z'^2+E)+z'\right] Q_l(z', x'-x|z) = \delta(x'-x)\delta(z'-z). \quad (3.37)$$

Then

$$-\pi^{-1}\,\mathrm{Im}G(x, x') = \int dz\, dz' P(z) \times [Q_l(z', x'-x|z) + Q_r(z', x'-x|z)] P(-z') . \quad (3.38)$$

When we take the Fourier transform with respect to $x - x'$ we obtain

$$A(k, E) = 2\int dz dz' P(z') \mathrm{Re}[\tilde{Q}_l(z', k|z)] P(-z), \quad (3.39)$$

where $\tilde{Q}_l$ satisfies

$$\left[-ik - \frac{D}{2}\frac{\partial^2}{\partial z'^2} - \frac{\partial}{\partial z'}(z'^2+E)+z'\right] \times \tilde{Q}_l(z', k|z) = \delta(z'-z) . \quad (3.40)$$

These equations are equivalent to those given by Halperin[11] for the average Green's function.

IV. A THREE-DIMENSIONAL MODEL

A variety of differences exist in our qualitative picture of waves in a three-dimensional (as opposed to a one-dimensional) system. A wave may be scattered in an infinity of directions. Obstacles may be circumvented. Diffraction effects abound. (A manifestation of the importance of dimensionality is the absence of phase transitions in ordinary one-dimensional systems.) The question arises as to what differences are manifest in the various problems involving the interactions of waves and random media.

In order to probe these differences, we feel it is worthwhile to study a model three-dimensional system, which captures some, though surely not all, of the above features. We shall consider the average properties of a particle in a random medium, where the potential $V(r)$ is a function of radial distance only. The key to the tractability of the one-dimensional system was the reduction to a Markov process. Our model has been created to maintain this feature. But, as we shall show later, our model also represents the leading term in a systematic treatment of the complete three-dimensional problem.

Let us study how to average the Green's function given by

$$[E + \nabla^2 - V(r)]\, \mathcal{G}(\vec{r}, \vec{r}', E) = \delta(\vec{r} - \vec{r}') . \quad (4.1)$$

It is convenient to make a spherical harmonic expansion. Define the z axis as lying along $\vec{r}'$. Then

$$\mathcal{G}(\vec{r}, \vec{r}', E) = (1/4\pi) \sum_{l=0} (2l+1)\, \mathcal{G}_l(r, r', E) P_l(\cos\theta) , \quad (4.2)$$

where θ is the angle between $\vec{r}$ and $\vec{r}'$ and $\mathcal{G}_l$ satisfies

$$\left\{E + \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} - V(r)\right\} \mathcal{G}_l[r, r', E|V(\cdot)] = \frac{1}{r'^2}\delta(r-r') . \quad (4.3)$$

Our desideratum is an average of this solution over a weighting:

$$\mathfrak{F}[V(\cdot)] = \mathfrak{N}\exp[-(\tfrac{1}{2}D)\int_0^\infty dr\, 4\pi r^2 V^2(r)] . \quad (4.4)$$

This form of the measure allows one to make closest contact with physical problems, as shall be seen subsequently. The insertion of r^2 in the weight, results in an attenuation of the fluctuations at large distances from the origin.

The solution of this model problem closely follows the one-dimensional procedure, if it is assumed that the order of l summation and potential averaging may be interchanged. Write the Green's function's

l component in the form

$$\mathcal{G}_l(r,r') = -\{[Z_i(r') + Z_e(r')]r' + (2l+1)\}^{-1}(r^l/r'^{l+1})\exp\{-\int_r^{r'} Z_i(\rho)d\rho\}, \quad r < r'; \tag{4.5}$$

$$\mathcal{G}_l(r,r') = -\{[Z_i(r') + Z_e(r')]r' + (2l+1)\}^{-1}(r'^l/r^{l+1})\exp\{-\int_{r'}^{r} Z_e(\rho)d\rho\}, \quad r > r'. \tag{4.6}$$

The behavior of the field-free-interior solution at small r and of the exterior solution at large r has been explicitly included so that the singularities of the field-free problem do not cloud the issue.

The Ricatti equations satisfied by the Z's are

$$dZ_i/dr = -Z_i^2 - 2(l+1)r^{-1}Z_i - E + V(r), \qquad r < r', \tag{4.7}$$

$$-dZ_e/dr = -Z_e^2 - 2lr^{-1}Z_e - E + V(r), \qquad r > r'. \tag{4.8}$$

For the density of states, interest centers on the average value $G(\vec{r},\vec{r},E) \equiv \langle \mathcal{G}(\vec{r},\vec{r},E)\rangle$ given by

$$G(\vec{r},\vec{r},E) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)G_l(r,r,E),$$

$$G_l(r,r,E) = -r^{-1}\mathfrak{N}\int \delta V(\cdot)\exp[-(\tfrac{1}{2}D)\int_0^\infty d\rho\, 4\pi\rho^2 V^2(\rho)]\{[Z_i(r)+Z_e(r)]r + (2l+1)\}^{-1}. \tag{4.9}$$

We have assumed that it is legitimate to interchange the order of l summation and functional integration. In analogy to Eq. (3.23), we may write

$$G_l(r,r,E) = -r^{-1}\int dz_i dz_e P_i^{\,l}(z_i, r|z_i^0, 0)[(z_i+z_e)r + (2l+1)]^{-1}P_e^{\,l}(z_e, r|z_e^0, \infty). \tag{4.10}$$

The P's satisfy Fokker-Planck-Kolmogorov equations, which may be derived[14] from the Langevin equations (4.7) and (4.8),

$$\frac{\partial P_i^{\,l}(z,r|z^0,r^0)}{\partial r} = \left[\frac{D}{8\pi r^2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}\left(z^2 + \frac{2(l+1)z}{r} + E\right)\right]P_i^{\,l}, \tag{4.11}$$

$$-\frac{\partial P_e^{\,l}(z,r,z^0,r^0)}{\partial r} = \left[\frac{D}{8\pi r^2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}\left(z^2 + \frac{2lz}{r} + E\right)\right]P_e^{\,l}, \tag{4.12}$$

with the boundary condition

$$P_s^{\,l}(z,r^0|z^0,r^0) = \delta(z - z^0), \quad s = i \text{ or } e. \tag{4.13}$$

In close analogy to the one-dimensional problem, the functions $P_i^{\,l}(z,r|z^0,0)$ and $P_e^{\,l}(z,r|z^0,\infty)$ are independent of z^0.

The density of states for this system is given by

$$\rho(E) = -\frac{1}{\pi V}\int d\vec{r}\,\mathrm{Im}G(\vec{r},\vec{r},E)$$
$$= \frac{1}{V}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dr\int_{-\infty}^{\infty}dz\, P_i^{\,l}(z,r|z_i^0,0)P_e^{\,l}[-z-(2l+1)/r, r|z_e^0,\infty]. \tag{4.14}$$

It is to be expected that the effect of the random potential will be negligible far from the origin. Hence the density of states is expected to be that of a free particle, plus the interesting terms of $O(1/V)$ arising from the stochastic features centered about the origin. If there is a dilute concentration of such centers, we can obtain the term, in the density of states proportional to this concentration, by multiplying the $O(1/V)$ term by the number of centers.

To approximate the density of states in a homogeneously stochastic medium we must adapt a different point of view. Consider a medium with potential $V(\vec{r})$, which is Gaussianly distributed according to the white noise probability distribution

$$\mathcal{F}[V(\cdot)] = \mathfrak{N} \exp[-(1/2D) \int d\vec{r}\, V^2(\vec{r})] \; . \tag{4.15}$$

The $V(\vec{r})$ may always be resolved into spherical harmonics about an arbitrary center,

$$V(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_{lm}(r)\, Y_{lm}(\Omega) \, , \tag{4.16}$$

so that also

$$\mathcal{F}[V(\cdot)] = \mathfrak{N} \exp[-(1/2D) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int dr\, 4\pi r^2 |V_{lm}(r)|^2] \; . \tag{4.17}$$

It is rigorous to write the average Green's function as

$$\langle \mathcal{G}[\vec{r},\vec{r}\,'\,|\,V(\cdot)] \rangle = \mathfrak{N} \int \delta V_{00}(\cdot) \exp[-(1/2D) \int dr\, 4\pi r^2 V_{00}^{\,2}(r)] \langle \mathcal{G}[\vec{r},\vec{r}\,'\,|\,V(\cdot)] \rangle^{00} ; \tag{4.18}$$

$$\langle \mathcal{G}[\vec{r},\vec{r}\,'\,|\,V(\cdot)] \rangle^{00} = \mathfrak{N} \int \prod_{l=1}^{\infty} \prod_{m=-l}^{l} \delta V_{lm}(\cdot) \exp[-(1/2D) \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \int dr\, 4\pi r^2 |V_{lm}(r)|^2]\, \mathcal{G}[\vec{r},\vec{r}\,'\,|\,V(\cdot)] \; . \tag{4.19}$$

The superscript 00 on $\langle \cdots \rangle^{00}$ means average over all components, except for the V_{00} component of the potential. For certain problems, of which, we believe, the density of states at large negative E is an example, it may be useful to approximate $\langle \mathcal{G}[\vec{r},\vec{r}\,'\,|\,V(\cdot)] \rangle^{00}$ by $\mathcal{G}[\vec{r},\vec{r}\,'\,|\,V_{00}(\cdot)]$, the Green's function in the potential $V_{00}(r)$ [cf. Eq. (4.1)]. This would lead precisely to the calculation we have just performed. Corrections would be determined by performing the integral in Eq. (4.18), by a modification of the usual perturbation procedure such that $\mathcal{G}[\vec{r},\vec{r}\,'\,|\,V_{00}(\cdot)]$ is the zeroth order Green's function. It may be possible to use the general procedures of this section to average correction terms over V_{00}.

An arbitrariness remains as to where to place the origin for the spherical harmonic expansion of $V(\vec{r})$. A choice of origin far from the $\vec{r}$ and $\vec{r}\,'$ of $\langle \mathcal{G}(\vec{r},\vec{r}\,'\,|\,V) \rangle$ would clearly be unwise. If one is going to proceed completely with the perturbation theory, the arbitrariness of the origin makes no difference in principle, except with respect to convergence. For, given $\vec{r}$ and $\vec{r}\,'$, there should be a choice for the location of the origin $\vec{r}_0$ which optimizes convergence. Then Eq. (4.14) for the density of states is replaced by the approximation

$$\rho(E) = -\pi^{-1} \operatorname{Im} G(\vec{r} - \vec{r}_0, \vec{r} - \vec{r}_0) \, , \tag{4.20}$$

and $\rho(E)$ clearly has $O(1)$ corrections from the field-free case. To a given order in some small parameter, like $1/|E|$, the answer for $\rho(E)$ may be independent of the precise choice of origin.

At this time we will not explicitly apply the equations, developed here, to any particular three-dimensional problem. We also feel that the details of the calculation of $G(\vec{r},\vec{r}\,')$ for $\vec{r}\,' \neq \vec{r}$ need not be presented, as they are easily inferred from Sec. III. Work is in progress on further aspects of this three-dimensional model.

V. STATISTICS OF THE REFLECTION FROM A ONE-DIMENSIONAL RANDOM MEDIUM

Consider any electromagnetic plane wave polarized in the y direction:

$$\vec{\mathcal{E}}(x,\tau) = \mathcal{E}(x)\, e^{i\omega\tau}\, \vec{e}_y \, , \tag{5.1}$$

where τ is the time.

The quantity $\mathcal{E}(x)$ satisfies the wave equation

$$[d^2\mathcal{E}(x)/dx^2] + k^2 \epsilon(x) \mathcal{E}(x) = 0 \, , \tag{5.2}$$

with $k = \omega/c$. The complex dielectric constant $\epsilon(x) = \epsilon_1(x) + i\epsilon_2(x)$ is unity, except in a region from 0 to L. There, it is stochastic, characterized by white noise statistics; i.e., $\epsilon_1(x)$ and $\epsilon_2(x)$ have independent Gaussian distributions with mean

$$\langle \epsilon_m(x) \rangle = \bar{\epsilon}_m(x), \quad 0 < x < L, \quad m = 1, 2, \tag{5.3}$$

and covariance

$$\langle \Delta\epsilon_m(x) \Delta\epsilon_n(x) \rangle = D_m \delta(x - x')\delta_{mn} \, , \quad 0 < x < L \, , \tag{5.4}$$

$$\Delta\epsilon_m(x)=\epsilon_m(x)-\bar{\epsilon}_m(x). \qquad (5.5)$$

(The extension to a cross correlation between real and imaginary parts would be trivial.) The probability density for a particular realization is

$$\mathfrak{F}[\epsilon(\cdot)]=\mathfrak{N}\exp\left(-\frac{1}{2}\sum_{m=1}^{2}\frac{1}{D_m}\int_0^L dx[\Delta\epsilon_m(x)]^2\right). \qquad (5.6)$$

Returning to the wave equation (5.2), we shall use as boundary conditions:

(i) A wave of unit intensity is incoming from the right,

(ii) the wave to the left of the slab is outgoing. Write the solution in the form

$$\begin{aligned}\mathcal{E}(x)&=e^{-ik(x-L)}+Re^{ik(x-L)}, & x\geq L,\\ &=A\exp[-\textstyle\int_0^x Z(\xi)d\xi], & 0\leq x\leq L \qquad (5.7)\\ &=Te^{-ikx} & x\leq 0.\end{aligned}$$

R is the complex reflection coefficient, containing information about the intensity and phase of the reflected wave. Continuity of the logarithmic derivative of $\mathcal{E}(x)$ at the ends of the slab leads to the relations

$$Z(0)=ik, \qquad (5.8)$$

$$Z(L)=ik(1-R)/(1+R); \qquad (5.9)$$

the latter being equivalent to

$$R=[ik-Z(L)]/[ik+Z(L)]. \qquad (5.10)$$

The physical constraint $|R|\leq 1$ implies $-\infty\leq Z_1(L)\leq\infty$, $0\leq Z_2(L)\leq\infty$, where $Z_1(L)$ and $Z_2(L)$ are the real and imaginary parts of $Z(L)$. Observe that a knowledge of the statistics of the pair $[Z_1(L), Z_2(L)]$, gives us all the information about the statistics of $[R_1, R_2]$.

As in Sec. III, Z satisfies a (complex) Ricatti-Langevin equation, whose real and imaginary parts are

$$dZ_1/dx=Z_1^2-Z_2^2+k^2\bar{\epsilon}_1(x)+k^2\Delta\epsilon_1(x), \qquad (5.11)$$

$$dZ_2/dx=2Z_1Z_2+k^2\bar{\epsilon}_2(x)+k^2\Delta\epsilon_2(x). \qquad (5.12)$$

From Eq. (5.10) we see that the basic statistical question is the following: For a Markovian process that evolves according to the Langevin equations (5.11–5.12), what is the probability density $P(z_1,z_2,L)$ that $[Z_1(L),Z_2(L)]=[z_1,z_2]$, given that $[Z_1(0),Z_2(0)]=[0,k]$. The answer is that P satisfies the Fokker-Planck-Kolmogorov equation

$$\begin{aligned}\frac{\partial P}{\partial x}&=\frac{D_1k^4}{2}\frac{\partial^2P}{\partial z_1^2}+\frac{D_2k^4}{2}\frac{\partial^2P}{\partial z_2^2}\\ &-\frac{\partial}{\partial z_1}[z_1^2-z_2^2+k^2\bar{\epsilon}_1(x)]P\\ &-\frac{\partial}{\partial z_2}[2z_1z_2+k^2\bar{\epsilon}_2(x)]P; \qquad (5.13)\end{aligned}$$

with the "initial" condition

$$P(z_1,z_2,0)=\delta(z_1)\delta(z_2-k), \qquad (5.14)$$

and normalization

$$\int_{-\infty}^{\infty}dz_1\int_0^{\infty}dz_2P(z_1,z_2,x)=1. \qquad (5.15)$$

Once $P(z_1,z_2,L)$ is determined, the probability density of R_1 and R_2, $\hat{P}(R_1,R_2)$ follows from Eq. (5.9). This leads to the relation

$$\hat{P}(R_1,R_2)=4k^2|1+R|^{-4}\times P\left(\mathrm{Re}\left\{ik\frac{1+R}{1-R}\right\},\ \mathrm{Im}\left\{ik\frac{1+R}{1-R}\right\}\right), \qquad (5.16)$$

where the prefactor is the Jacobian of the transformation from $Z(L)$ to R variables.

One can calculate averages of functions of R, such as the average reflected intensity

$$\langle|R|^2\rangle=\iint_{|R|\leq 1}dR_1dR_2(R_1^2+R_2^2)\hat{P}(R_1,R_2). \qquad (5.17)$$

It may be more convenient to carry out the integration in the z variables

$$\langle|R|^2\rangle=\int_{-\infty}^{\infty}dz_1\int_0^{\infty}dz_2\left|\frac{ik-z_1-iz_2}{ik+z_1+iz_2}\right|^2P(z_1,z_2,L). \qquad (5.18)$$

VI. THE FUNCTIONAL INTEGRAL

We shall now probe more deeply, into the type of functional integral which arises in problems of wave and random medium interaction: specifically the one-dimensional density of states calculation of Sec. III.

A. Change of Variables from V to Z

Let us return to the functional integral

$$I = \mathfrak{N}\int \delta V(\cdot)\exp[-(1/2D)\int_0^L d\xi\, V^2(\xi)][Z_l(x)+Z_r(x)]^{-1}, \tag{6.1}$$

with Z given by the Ricatti-Langevin equation (3.6). A treatment of the integral I evolves from a more basic definition, wherein the V and Z functions are considered to be piecewise continuous over intervals of width Δ.[18] The functional integral may then be regarded as a limit, as $\Delta \to 0$ of the multiple integral

$$I = \mathfrak{N}\int dV_1 \cdots dV_N \exp\Big[-(\Delta/2D)\sum_{i=1}^{N} V_i^2\Big][Z_{l,n}+Z_{r,n+1}]^{-1}, \tag{6.2}$$

with $N = L/\Delta$ and $n = x/\Delta$. There is some arbitrariness in writing a discretized form of the Ricatti equation (3.6) which links V and Z. We shall find it convenient to use

$$(Z_{l,i} - Z_{l,i-1})/\Delta = -Z^2_{l,i-1} - E + V_i, \tag{6.3}$$

$$(Z_{r,i} - Z_{r,i+1})/\Delta = -Z^2_{r,i+1} - E + V_i, \tag{6.4}$$

with $Z_{l,0} = z_l{}^0$ and $Z_{r,N+1} = z_r{}^0$. The simplicity that ensues from this particular form of discretization motivates the choice. In Part D of this section, we demonstrate the manner in which different discretizations yield the same answer, but in a more cumbersome fashion.

We shall perform the integration by a change of variables from V_i to $Z_{l,i}$ for $1 \le i \le n$, and V_i to $Z_{r,i}$ for $n+1 \le i \le N$. As a result of the use of a different index for Z and V, on the right hand side of Eqs. (6.3 - 6.4), one finds a simple Jacobian:

$$\partial(V_1 \cdots V_N)/\partial(Z_{l,1} \cdots Z_{r,N}) = \Delta^{-N}. \tag{6.5}$$

The contours of integration for the Z's may be determined by considering that the integrals are done in the order $Z_{l,n}$ to $Z_{l,1}$ and $Z_{r,n+1}$ to $Z_{r,N}$. Since V_i takes values from $-\infty$ to $+\infty$, so do the Z's.

The values of Z are given at the ends of the box, and the $Z_{l,n}$ and $Z_{r,n+1}$ integrations must be reserved for last (since these variables appear in the integrand). We therefore rearrange the variables to the order

$$Z_{l,1}, Z_{l,2}, \ldots, Z_{l,n-1}, Z_{r,N}, \ldots, Z_{r,n+2}, Z_{l,n}, Z_{r,n+1}.$$

This progression of the i of $Z_{l,i}$ from left to right, and of $Z_{r,i}$ from right to left, is analogous to the use of the variable $t = x$ for Z_l and $t = L - x$ for Z_r, with t regarded as developing in a positive sense.

Now that we have settled the questions of the Jacobian, the contours of integration, and the order of integration, we may return to a continuous formulation of the functional integral, and change to the Z's as variables of integration. It is natural to divide the range of the ξ variable, which labels $Z(\xi)$, into $0 < \xi < x$ and $x < \xi < L$. Then we may write

$$I = \mathfrak{N}\int \frac{dZ_l(x)\,dZ_r(x)}{Z_l(x)+Z_r(x)} \int_{z_l{}^0,\, Z_l(x)} \delta Z_l \exp\{-(1/2D)\int_0^x d\xi\,[dZ_l(\xi)/d\xi + Z_l{}^2(\xi) + E]^2\}$$

$$\times \int_{z_r{}^0,\, Z_r(x)} \delta Z_r \exp\{-(1/2D)\int_x^L d\xi\,[dZ_r(\xi)/d\xi - Z_r{}^2(\xi) - E]^2\}. \tag{6.6}$$

This form is closely related to Eq. (3.10), except that the change of variables from V to Z has been made. Thus we are concerned with the calculation of

$$P(z,t\,|\,z^0,0) = \mathfrak{N}\int_{z^0,\, z} \delta Z(\cdot)\exp\{-(1/2D)\int_0^t d\tau\,[dZ(\tau)/d\tau]^2\}$$

$$\times \exp\{-(1/D)\int_0^t d\tau\, a[Z(\tau)][dZ(\tau)/d\tau] - (1/2D)\int_0^t d\tau\, a^2[Z(\tau)]\}, \tag{6.7}$$

$$a(Z)=Z^2+E \ . \tag{6.8}$$

This is a type of Wiener integral[15]; the first term of the integrand is the usual Wiener measure,

$$d_W\mu_Z=\mathfrak{N}\delta Z(\cdot)\exp\{-(1/2D)\int_0^t d\tau[dZ/d\tau]^2\} \ . \tag{6.9}$$

It is not quite the Kac[15, 17] integral because of the term

$$K=\int_0^t d\tau a[Z(\tau)](dZ/d\tau) \ . \tag{6.10}$$

We shall consider two ways of handling this Wiener integral. Either we transform it to a Kac integral (which one knows how to do), or we return to basics and make the appropriate modifications in the Darling-Siegert[19] derivation of the Kac integral formula. Both approaches are instructive and will be outlined.

B. Transformation to a Kac Integral

The integral K of Eq. (6.10) can be performed in a probabilistic sense by a procedure due to Itô.[20, 14] To understand the Itô integral, note that if dZ/dt were a continuous function, the integral K would trivially be given by

$$\int_{Z(0)}^{Z(t)} a(y)\,dy \equiv A[Z(t)]-A[Z(0)] \ , \tag{6.11}$$

where A is the indefinite integral

$$A(z)=\int^z a(y)dy \ . \tag{6.12}$$

However, the functions $Z(\tau)$, which enter the integrand of K, generally have everywhere a discontinuous derivative, according to the Wiener measure. Consider application of a Taylor-like expansion to

$$A(Z_n)-A(Z_0)\equiv\sum_{i=1}^n[A(Z_i)-A(Z_{i-1})]=\sum_{i=1}^n a(Z_i)(Z_i-Z_{i-1})+\tfrac{1}{2}\sum_{i=1}^n a'(Z_i)(Z_i-Z_{i-1})^2+R \ , \tag{6.13}$$

where R is such as to render the final form an identity. The first term on the right is a discretized form of K, where $dt\to\Delta$ (cf. Part A this section). For analytic functions $Z(\tau)$ the second term would be of $O(\Delta^2)$, but, with Wiener measure (6.9), the average value of $(Z_i-Z_{i-1})^2$ is $D\,\Delta$. Itô shows that the deviations from this average, as well as the term R, lead to contributions which vanish with probability unity, as $\Delta\to 0$. Thus we may write the Itô integral

$$K=A[Z(t)]-A[Z(0)]-\tfrac{1}{2}D\int_0^t a'[Z(\tau)]d\tau \ . \tag{6.14}$$

With this result, Eq. (6.7) for P is proportional to a Kac functional integral

$$P(z,t\,|\,z^0,0)=\exp\{-D^{-1}[A(z)-A(z_0)]\}L(z,t\,|\,z^0,0), \tag{6.15}$$

$$L(z,t\,|\,z^0,0)=\int_{z,z^0} d_W\mu_Z\exp\{-(1/2D)\int d\tau(a^2[Z(\tau)]+Da'[Z(\tau)]\} \ . \tag{6.16}$$

The Kac integral L is given by

$$\frac{\partial L}{\partial x}=\frac{D}{2}\frac{\partial^2 L}{\partial z^2}-\frac{1}{2}[D^{-1}a^2(z)+a'(z)]L, \tag{6.17}$$

with the boundary condition

$$L(z,0\,|\,z^0,0)=\delta(z-z^0) \ . \tag{6.18}$$

It follows simply that $P(z,t\,|\,z_0,0)$ satisfies the Fokker-Planck-Kolmogorov equation (3.12).

C. Extension of the Darling-Siegert Technique

Darling and Siegert[19] have presented a derivation of the differential equation – for evaluating the Kac integral – which is easily modified for Eq. (6.7). Again the derivation requires a return to a discretization of the t variable.

Define the quantity

$$I_n = \exp\left(-D^{-1}\sum_{i=1}^{n}\left[\tfrac{1}{2}a^2(Z_{i-1})\Delta + a(Z_{i-1})(Z_i - Z_{i-1})\right]\right). \tag{6.19}$$

We can write

$$P(z,t\,|\,z^0,0) = \lim_{\Delta\to 0}\langle I_n\rangle_{0n}\ , \tag{6.20}$$

where the angular brackets denote the discretized Wiener integral

$$\langle F\rangle_{mn} = \mathfrak{N}\int dZ_{m+1}\cdots dZ_{n-1}\exp\{-\tfrac{1}{2}D^{-1}\Delta\sum_{i=m+1}^{n}[(Z_i - Z_{i-1})/\Delta]^2\}F\ , \tag{6.21}$$

with conditions on the end points of Z_i; i.e., Z_m and Z_n are specified.

We shall also need the fact that

$$\langle 1\rangle_{mn} \equiv P^0(Z_n, t\,|\,Z_m, 0) = (2\pi Dt)^{-1/2}\exp\{-(Z_n - Z_m)^2/2Dt\} \tag{6.22}$$

is independent of Δ.

Note that by expansion

$$(I_n - I_{n-1}) = I_{n-1}[-\tfrac{1}{2}\Delta D^{-1}a^2(Z_{n-1}) - D^{-1}a(Z_{n-1})(Z_n - Z_{n-1}) + \tfrac{1}{2}D^{-2}a^2(Z_{n-1})(Z_n - Z_{n-1})^2 + \cdots]\ , \tag{6.23}$$

where $I_0 \equiv 1$. In the identity

$$I_n = I_0 + \sum_{m=1}^{n}(I_m - I_{m-1})\ , \tag{6.24}$$

apply the $\langle\cdots\rangle_{0n}$ operation. With the aid of Eq. (6.23) one obtains

$$\begin{aligned}\langle I_n\rangle_{0n} = \langle 1\rangle_{0n} &+ \sum_{m=1}^{n}\int dZ_{m-1}dZ_m\langle I_{m-1}\rangle_{0,m-1}\\ &\times[-\tfrac{1}{2}\Delta D^{-1}a^2(Z_{m-1}) - D^{-1}a(Z_{m-1})(Z_m - Z_{m-1}) + \tfrac{1}{2}D^{-2}a^2(Z_{m-1})(Z_m - Z_{m-1})^2]\\ &\times(2\pi D\Delta)^{-1/2}\exp\{-(Z_m - Z_{m-1})^2/2D\Delta\}\langle 1\rangle_{mn} + 0(\Delta),\end{aligned} \tag{6.25}$$

where the corrections are due to the end terms of the summation. One may perform the Z_{m-1} integration by Taylor series expansion of $\langle I_{m-1}\rangle_{0,m-1}$ and $a(Z_{m-1})$, to change the variables from Z_{m-1} and $(m-1)$ to Z_m and m. The contribution that does not vanish as $\Delta\to 0$ comes from the middle term of the bracket. One obtains

$$\langle I_n\rangle_{0n} = \langle 1\rangle_{0n} + \Delta\sum_{m=1}^{n}\int dZ_m\left\{\frac{\partial}{\partial Z_m}[a(Z_m)\langle I_m\rangle_{0m}]\right\}\langle 1\rangle_{mn} + \cdots\ . \tag{6.26}$$

The important feature to note is that terms in $(Z_m - Z_{m-1})^2$ contribute a term proportional to Δ rather than Δ^2. In the limit $\Delta\to 0$ Eq. (6.26) becomes

$$P(z,t\,|\,z^0,0) = P^0(z,t\,|\,z^0,0) + \int_0^t dt'\int_{-\infty}^{\infty}dz'P^0(z,t\,|\,z',t')\frac{\partial}{\partial z'}[a(z')P(z',t'\,|\,z^0,0)]. \tag{6.27}$$

This is the integral equation equivalent to the Fokker-Planck-Kolmogorov equation (3.12).

D. A Different Discretization

In this section we shall examine the additional elements that contrive to give the same results when a different discretized version of the Langevin equation is employed. Consider replacing $a(Z_{l,i-1})$ by $a(Z_{l,i})$ in Eq. (6.3):

$$(Z_{l,i}-Z_{l,i-1})/\Delta=-a(Z_{l,i})+V_i\,, \tag{6.28}$$

the Jacobian for the transformation from V to Z would be

$$\partial(V_1\cdots V_n)/\partial(Z_1\cdots Z_n)=\Delta^{-n}\prod_{i=1}^{n}[1+\Delta a'(Z_i)]\sim\Delta^{-n}\exp\left[\Delta\sum_{i=1}^{n}a'(Z_i)\right]. \tag{6.29}$$

Thus we expect that $P(z,t|z^0,0)$ is also given by

$$P(z,t|z^0,0)=\lim_{\Delta\to 0}\langle J_n\rangle_{0n}\,,$$

$$J_n=\exp\{-\sum_{i=1}^{n}[D^{-1}a^2(Z_i)\Delta+D^{-1}a(Z_i)(Z_i-Z_{i-1})-a'(Z_i)\Delta]\}\,. \tag{6.30}$$

By a discussion parallel to that applied to I_n, we find

$$\langle J_n\rangle_{0n}=\langle 1\rangle_{0n}+\sum_{m=1}^{n}\int dZ_{m-1}dZ_m\langle J_{m-1}\rangle_{0,m-1}$$
$$\times[-\tfrac{1}{2}\Delta D^{-1}a^2(Z_m)-D^{-1}a(Z_m)(Z_m-Z_{m-1})+\Delta a'(Z_m)+\tfrac{1}{2}D^{-2}a^2(Z_m)(Z_m-Z_{m-1})^2]$$
$$\times(2\pi D\Delta)^{-1/2}\exp\{-(Z_m-Z_{m-1})^2/2D\Delta\}\langle 1\rangle_{mn}+0(\Delta)\,. \tag{6.31}$$

Now we change the variables of $\langle J_{m-1}\rangle_{0,m-1}$ to Z_m and m by Taylor expansion, and integrate over Z_{m-1}. The result

$$\langle J_n\rangle_{0n}=\langle 1\rangle_{0n}+\Delta\sum_{m=1}^{n}\int dZ_m[a(Z_m)(\partial/\partial Z_m)\langle J_m\rangle_{0m}+a'(Z_m)\langle J_m\rangle_{0m}]\langle 1\rangle_{mn}+\cdots\,, \tag{6.32}$$

is identical with Eq. (6.26).

VII. CONCLUDING REMARKS

We have presented some problems involving the interaction of waves with random media. The Fokker-Planck-Kolmogorov equations, in which the solutions are embodied, are probably best solved in a fashion that is tailored to the specific information one desires to extract.

At present our research is directed at further analysis of the three-dimensional model. We also are investigating, with closely related techniques, the statistical properties of solutions of Burger's one-dimensional hydrodynamics model, wherein the stochastic element is the initial state.

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