Dispersion Sum Rules for the Nonstrange Baryon Spectrum*

P. H. FRAMPTON AND B. HAMPRECHT
The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637
(Received 10 March 1969)

A study is made of the saturation of superconvergence relations by the known N (isospin-$\frac{1}{2}$) and Δ (isospin-$\frac{3}{2}$) nonstrange baryon resonances. Account is taken of the shape of the strong-interaction form factors as well as of the magnitude of the coupling constants. At a $\pi\Delta(1236)$ vertex, there are two independent form factors (spin component $\frac{1}{2}$ and $\frac{3}{2}$), and at a πN (nucleon) vertex, only one. Making a physically plausible assumption about the smoothness of these form-factor functions, a detailed study is made of sum rules in $\pi\Delta \to \pi\Delta$ and $\pi N \to \pi\Delta$ scattering. Two especially interesting relations occur in the elastic process $\pi\Delta \to \pi\Delta$, one relation involving only the N spectrum, the other only the Δ spectrum. The sum rule for the N spectrum is not saturated by the set of states which we insert, and this leads us to conjecture that there exists a spin-$\frac{3}{2}$ $\pi\Delta$ resonance which is at quite low mass and which is not reported in the πN phase-shift analysis. A possible fit to a larger set of sum rules is then considered; the fit is consistent with linear Regge trajectories and provides a starting point for a detailed check when better partial-width data are available.

I. INTRODUCTION

IN this paper we are concerned with the saturation of superconvergence relations obeyed by certain amplitudes for the scattering of pions off nonstrange baryons.[1] The resonance approximation, which we wish to test, tries to approximate the amplitudes by putting the set of known resonances into the s channel (the direct channel with squared c.m. energy s) as single-particle intermediate states. A meaningful comparison with experiment can only be expected if the couplings of the external baryons to a pion and to most of the s-channel resonances are known at least approximately. This consideration limits us to the nucleon (N) and the $\Delta(1236)$ resonance as initial and final baryons. None of the amplitudes in elastic πN scattering is superconvergent, without an assumption relevant to the exactly $SU(3)$ symmetric case,[2] because of the low-spin and isospin. Thus, the only processes of interest here are elastic $\pi\Delta$ scattering and the inelastic reaction $\pi N \to \pi\Delta$.

If $A(\nu,t)$ is a superconvergent amplitude, i.e., $A(\nu,t) = o(1/\nu)$ as $\nu \to \infty$ at fixed t (t = crossed energy variable, $\nu = \Sigma - 2s - t$, where Σ = sum of external masses), then

$$\int_{-\infty}^{\infty} \mathrm{Im}A(\nu,t)d\nu = 0 \tag{1.1}$$

is a typical superconvergence sum rule. Such a relation may be considered for each value of t or else all derivatives with respect to t may be written down at $t=0$. In either case we have an infinity of sum rules. Not all of these can be expected to be saturated by low-lying resonances in the s-channel. In particular, at $t=0$ derivative sum rules are not very suitable because the lowest contributing spin rises as the number of times the sum rule has been differentiated goes up, such that a high-derivative sum rule contains no information on resonances of low spin.

Sum rules for elastic $\pi\Delta$ scattering and for the reaction $\pi N \to \pi\Delta$ have previously been considered by Jones and Scadron[3] and by Frampton.[4] The former included only the nucleon and the $\Delta(1236)$ intermediate states and were concerned mainly with the connection to the higher symmetry groups. It is not surprising that in this approximation only a subset of the nonderivative sum rules at $t=0$ are consistent with one another in both reactions. In Ref. (4) it was shown subsequently that all nonderivative sum rules for elastic $\pi\Delta$ scattering at $t=0$ could be saturated in the resonance approximation. Two independent couplings, however, were allowed for each particle of spin $\frac{3}{2}$ and $\frac{5}{2}$. But it is well known that these couplings satisfy an angular momentum constraint at threshold. Since the low-lying resonances enter with form factors which are evaluated near threshold, arbitrary couplings may mean rapidly varying form factors. This is in fact the case in the solutions of Ref. 4.

Our question is, therefore, whether the now larger number of proposed resonances allows a saturation which is compatible with the assumption of "smooth" form factors. Or, in other words, we add to the assumptions of analyticity, Regge asymptotic behavior, and the single-particle approximation a further assumption about smoothness of form factors and investigate whether the known resonances may provide a solution to this extended problem.

It turns out that the known resonances do not suffice to saturate the set of nonderivative sum rules considered here. We can, however, say something about the nature of the missing contributions. A resonance of mass 1.55 GeV with $J^P = \frac{3}{2}^+$ and isospin $T=\frac{1}{2}$, and another one at 1.3 GeV with $J^P = \frac{1}{2}^-$ and $T=\frac{3}{2}$, would saturate the sum rules. These resonances fit on linear Regge trajectories, but although their existence is not

* This work supported in part by the U. S. Atomic Energy Commission.

[1] Some of the results of the work described here are given in P. H. Frampton and B. Hamprecht, Phys. Letters 28B, 664 (1969).

[2] B. Sakita and K. C. Wali, Phys. Rev. Letters 18, 29 (1967); G. Altarelli, F. Buccella, and R. Gatto, Phys. Letters 24B, 57 (1967); P. Babu, F. J. Gilman, and M. Suzuki, *ibid.* 24B, 65 (1967).

[3] H. F. Jones and M. D. Scadron, Nuovo Cimento 48A, 545 (1967); 52A, 62 (1967).

[4] P. H. Frampton, Nucl. Phys. B2, 518 (1967).

ruled out experimentally because they might couple only very weakly to the πN channel, they need be nothing else but a suitable parametrization of the background which contributes to the sum rules.

In Sec. II., we parametrize the form factors and state our smoothness assumptions. The choice of sum rules which we make is explained in Sec. III. In Sec. IV. the saturation of the sum rules is discussed. Two appendices are added, which contain the kinematics for elastic $\pi\Delta$ scattering and for the reaction $\pi N \rightarrow \pi\Delta$, respectively.

II. FORM FACTORS

Consider a baryon of mass M_i which may decay into a baryon of mass M_f and a (real or virtual) pion. Let p be the magnitude of the 3-momentum of each particle in the final state when the initial particle decays at rest. The quantity

$$x=\sinh^{-1}(p/M_f) \tag{2.1}$$

will be called the "rapidity" of the final particle of mass M_f. The vertex function may then be parametrized in the following way:

$$V_\lambda{}^{j'j\pm}=gC_\lambda{}^{j'j\pm}x^{|j'-j|+n}[1-\alpha_\lambda{}^{jj'\pm}x^2+O(x^4)], \tag{2.2}$$

where λ is the helicity of the final baryon; j' and j are the initial and final spins; n takes the values $n=0$ or $n=1$ according to whether the transition is of normal (+) or abnormal (−) parity type;[5] g is an over-all coupling constant; the $C_\lambda{}^{j'j\pm}$ are coefficients whose dependence on λ is fixed by rotational symmetry. We normalize the $C_\lambda{}^{j'j\pm}$ such that $C_{1/2}{}^{j'j\pm}=1$, and some of their values are stated in Table I.

We now introduce the *assumption of smoothness* by which we shall mean that the term $O(x^4)$ causes the vertex function to vanish as x tends to infinity, but is small compared to the term proportional to x^2 as long as $x<0.5$. Further, the assumption will mean that the "sizes" α_λ are essentially those given by an $O(3,1)$ dynamical group model for pseudoscalar form factors[6] and do not differ from them by an order of magnitude or more. This assumption seems to be justified because the $O(3,1)$ for pseudoscalar form factors is very successful in fitting the baryon decays.[7]

If in a decay the rapidity x becomes very large the final particle tends to be transversely polarized, i.e., its spin tends to be aligned along the direction of motion so that large values of $|\lambda|$ are favored. To the extent that this tendency shows up already for relatively small values of the rapidity x, the α_λ in (2.2) would satisfy the inequality

$$\alpha_\lambda>\alpha_\mu \quad \text{if} \quad |\lambda|<|\mu|. \tag{2.3}$$

[5] M. D. Scadron, Phys. Rev. **165**, 1640 (1967).

[6] A. O. Barut and H. Kleinert, Phys. Rev. Letters **18**, 754 (1967); H. Kleinert, Fortschr. Physik **16**, 1 (1968).

[7] B. Hamprecht and H. Kleinert, Fortschr. Physik **16**, 635 (1968).

TABLE I. Some values of the coefficients $C_\lambda{}^{j'j\pm}$ occurring in Eq. 2.2.

$C_{1/2}{}^{j'j\pm}=1$	$j=\frac{3}{2}$	$j=\frac{5}{2}$
$C_{3/2}{}^{\frac{3}{2}j+}$	1	$\sqrt{\frac{2}{3}}$
$C_{3/2}{}^{\frac{3}{2}j-}$	3	$\sqrt{6}$

We note that the α_λ of the $O(3,1)$ group model always satisfy these relations. In fact it will turn out that without the validity of the spin-alignment relation, Eq. (2.3), the sum rules would be still harder to satisfy.

Consider now a pole diagram with a resonance of spin J and mass M_R in the s channel. Then the s-channel helicity amplitudes for the process can be written

$$\operatorname{Im} f_{c0,a0}{}^{s,J}(s,t) = V(M_R\rightarrow M_f)d_{ac}{}^J(\theta_s)V(M_i\rightarrow M_R), \tag{2.4}$$

where θ_s is the s-channel c.m. scattering angle and the vertex functions are taken from Eq. (2.2).

Equation (2.4) does not have all the correct kinematic singularities in s; in fact it is not MacDowell-symmetric[8] in $\sqrt{s}$. We checked, however, that in the region of $\sqrt{s}$, which is relevant here, the difference between the form of Eq. (2.4) and the more complicated expression required by MacDowell symmetry is negligible. Hence, we shall for convenience, always work with Eq. (2.4).

III. CHOICE OF SUM RULES

Any scattering process with spin can be described in terms of invariant amplitudes, s-channel helicity amplitudes or t-channel helicity amplitudes. These three sets of amplitudes are related by linear transformations which are functions of s and t. Regge exchange in the t channel determines the asymptotic behavior of the amplitudes at fixed (small) t and $\nu\rightarrow\infty$. Some of the amplitudes are superconvergent. The set of all nonderivative and derivative sum rules at a certain value of t which can be written down for invariant amplitudes is equivalent to the set derived from t-channel helicity amplitudes. But if only *nonderivative* sum rules are considered, then, due to singularities in t of the matrix transforming invariant amplitudes into helicity amplitudes, a different number of sum rules may be obtained from the two procedures at some values of t. To clarify this point, let A_i be a set of invariant amplitudes and $f_i{}^t$ a set of t-channel helicity amplitudes. Then a relation exists which is of the form

$$f_i{}^t=\sum_j M_{ij}(s,t)A_j. \tag{3.1}$$

Suppose now that the $f_i{}^t$ satisfy a constraint at $t=t_0$ (it is well known that such constraints exist at certain

[8] W. E. A. Davies, Nuovo Cimento **53A**, 828 (1968).

points[9]). The A_j will be independent here and M_{ij} is therefore singular. The simplest form that such a singularity can take, which also was found to be the one that actually occurs in the present case, comes about when $M_{ij}(s,t_0)=0$ for some j ($j=k$, say) and all i. This means that A_k is not needed to express the helicity amplitudes at $t=t_0$. Only the derivatives of these helicity amplitudes at $t=t_0$ would involve A_k. Therefore, a nonderivative sum rule for A_k is independent of all nonderivative sum rules for helicity amplitudes.

Our procedure will be to consider all nonderivative sum rules for t-channel helicity amplitudes at a fixed value of t. We would like to fix t in the neighborhood of $t=0$. If $t=t_0$ is a point at which the helicity amplitudes satisfy a constraint then it would not be wise to choose t to be near but different from t_0 because at $t=t_0+\epsilon$ the helicity amplitudes would still satisfy a constraint up to terms of order ϵ and the invariant amplitude A_k which drops out at $t=t_0$ will enter into the sum rules at $t=t_0+\epsilon$ with coefficients of order ϵ. The considerations guide us to fix t such that we get the maximum number of constraints between helicity amplitudes. These points are $t=0$ for elastic $\pi\Delta$ scattering and the pseudo-threshold value $t=(M_\Delta-M_N)^2$ for the reaction $\pi N\rightarrow\pi\Delta$.

In this way we shall find seven sum rules for elastic $\pi\Delta$ scattering, compared to eight which can be written down for invariant amplitudes. For the reaction $\pi N\rightarrow\pi\Delta$ we will have only two sum rules, compared to four for invariant amplitudes.

The kinematic details are developed in Appendices A and B.

IV. SATURATION OF SUM RULES

In this section the contributions of the various s-channel resonances to the sum rules for elastic $\pi\Delta$ scattering ("elastic sum rules") and to the sum rules for the reaction $\pi N\rightarrow\pi\Delta$ ("inelastic sum rules") are evaluated numerically. For the elastic sum rules we find the relevant formulas in Appendix A. Equations (A22–A28) give the seven sum rules in terms of invariant amplitudes. Equations (A19–A21) express invariant amplitudes in terms of s-channel helicity amplitudes at $t=0$. Single-particle contributions to the latter are defined in Eqs. (A29–A32). In order to do numerical

TABLE II. Elastic sum rules, $\pi\Delta\rightarrow\pi\Delta$. Here numbers 1–24 list the contributions of known resonances to the seven elastic sum rules These contributions have to be multiplied by the modulus squared of coupling constants. Bounds on these coupling constants are given in the column labelled $g_{\min}^2$ and $g_{\max}^2$. The next column shows the coupling constants which give the best saturation of all the sum rules; in the last column the corresponding partial widths for $R\rightarrow\Delta(1236)\pi$ are listed, except for particle No. 1 where of course we take $\Delta(1236)\rightarrow N\pi$ instead. Numbers 25 and 26 list the contributions of hypothetical particles described in the text. At the end the positive (S^+) and negative (S^-) contributions are summed up separately and their discrepancy is found. The relative discrepancy is determined as $100(S^+-S^-)/\max(S^+,S^-)$.

No.	Particle Mass	j^P	I $A_1(T_s=\frac{1}{2})$	II $A_1(T_s=\frac{3}{2})$	III $\nu A_1(T_t=1)$	IV $\nu^2 A_1(T_t=2)$	V $A_2(T_t=1)$	VI $\nu A_2(T_t=2)$	VII $A_3(T_t=2)$	$g_{\min}^2$	$g_{\max}^2$	g_{opt}^2	Γ (MeV)
1	0.939	$\frac{1}{2}^+$	0.078		0.104	0.139	0.341	0.453	0.738	44	45.5	44	118
2	1.47	$\frac{1}{2}^+$	0.026		−0.0321	0.039	0.14	−0.17	−0.38	0	16	0	0
3	1.518	$\frac{3}{2}^-$	−0.71		1.08	−1.63	−0.47	0.71	−1.08	0.2	0.37	0.37	49
4	1.55	$\frac{1}{2}^-$	0.084		−0.14	0.25	−0.053	0.090	−0.013	0	30	30	8.7
5	1.68	$\frac{5}{2}^-$	0.061		−0.16	0.40	0.11	0.27	−0.065	0	30	3	9
6	1.69	$\frac{5}{2}^+$	0.037		−0.10	0.25	−0.030	0.079	0.25	0	6	4.5	34
7	1.71	$\frac{1}{2}^-$	0.049		−0.13	0.37	−0.046	0.13	−0.02	0	36	24	43
8	1.73	$\frac{3}{2}^-$	−0.23		0.66	−0.91	−0.074	0.22	−0.80	0	2	2	225
9	1.75	$\frac{1}{2}^+$	0.0087		−0.026	0.080	0.052	−0.16	−0.15	0	12	0	0
10	1.86	$\frac{3}{2}^+$	−0.63		2.42	−9.32	0.53	−2.03	−2.24	0	2	2	244
11	1.98	$7/2^+$	0.0082		−0.039	0.18	−0.025	0.12	−0.011	0	80	80	194
12	2.06	$\frac{3}{2}^-$	−0.031		0.17	−0.89	0.0069	−0.037	−0.028	0	4	0	0
13	2.20	$7/2^-$	0.0009		−0.0057	0.039	−0.0021	0.014	0.033	0	50	50	168
14	2.65	$11/2^-$	0.00002		−0.0003	0.0027	0.0001	−0.0012	0.014	0	120	0	0
15	3.03	$15/2^-$	−0.00002		0.0004	−0.0055	0.0003	−0.0050	0.015	0	340	0	0
16	1.236	$\frac{3}{2}^+$		−11.09	−0.342	0.0264	−1.746	0.135	3.55	···	···	4.0	···
17	1.64	$\frac{1}{2}^-$		0.062	−0.11	−0.52	−0.040	−0.18	0.029	0	121	50	47
18	1.69	$\frac{3}{2}^+$		−1.61	3.35	17.45	0.75	3.89	4.41	0	3	2.5	189
19	1.69	$\frac{3}{2}^-$		−0.29	0.60	3.15	−0.090	−0.47	1.39	0	2	2	242
20	1.91	$\frac{5}{2}^+$		0.011	−0.036	−0.31	−0.017	−0.14	−0.16	0	30	30	340
21	1.93	$\frac{1}{2}^+$		0.0040	−0.014	−0.12	−0.020	0.17	0.13	0	9	0	0
22	1.95	$7/2^+$		0.0092	−0.033	−0.30	−0.020	−0.18	0.013	30	60	60	122
23	1.95	$\frac{5}{2}^-$		−0.013	0.046	0.42	−0.022	−0.19	0.29	0	17	17	249
24	2.42	$11/2^+$		0.0018	−0.013	−0.22	−0.0068	−0.12	−0.0060	0	60	60	95
25	1.55	$\frac{3}{2}^+$	−3.26		5.58	−9.55	1.17	−2.00	−2.90	···	···	2.1	70
26	1.30	$\frac{1}{2}^-$		0.177	−0.040	−0.022	−0.018	−0.010	−0.0086	···	···	255	···
Positive contributions			7.0	−49.2	+35.4	+114	21.5	51	+34				
Negative contributions			−7.0	−49.2	−37.7	−97	−23.5	−45.7	−42				
Discrepancy			0 (0%)	0 (0%)	−2.3 (6%)	+17 (15%)	−2.0 (9%)	−5.3 (10%)	−8 (19%)				

[9] H. F. Jones, Nuovo Cimento **50A**, 814 (1967); G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. **46**, 239 (1968).

calculations we need a more specific expression for the vertex functions than Eq. (2.2). As a first trial we shall represent the bracket in Eq. (2.2) by the function $\exp(-\alpha_\lambda{}^{j'j\pm}x^2)$, where the $\alpha_\lambda{}^{j'j\pm}$ are taken from an $O(3,1)$ dynamical group model.[6] If no saturation is obtained in this way, we shall consider to what extent the vertex functions can be modified within the limits set by the assumption of smoothness described in Sec. II above.

The inelastic sum rules are given in Eqs. (B19) and (B20). Single-particle contributions are defined in Eqs. (B3)–(B6), and they are worked out explicitly for the particular cases of $J^P=\frac{1}{2}^+$, $\frac{3}{2}^+$ intermediate states a $t=(M_\Delta-M_N)^2$ in Eqs. (B10), (B16), respectively. In Tables II and III the contributions of the known resonances to all nine sum rules are listed. The known resonances (excluding those above 3 GeV in mass) are numbered 1–24. In Table III the particles which couple only weakly in either vertex of the inelastic reaction are omitted. Generally some bounds on the decay rates of resonances are known experimentally[10] and are listed as bounds on the absolute squares of coupling constants in Table II. An advantage of the elastic sum rules is that the sign of each contribution is fixed, since only the moduli of the coupling constants enter.

As expected from general considerations (we mean from the available convergence factor in ν extra to what is needed for validity of the sum rule), the convergence of the two nonmoment sum rules for A_1 in Table II is very strong. The sum rules for νA_1 and A_2 can also be expected to have negligible contributions from states above 2 GeV. The remaining three sum rules for νA_2, $\nu^2 A_1$, and A_5 might be expected to have appreciable contributions from higher mass states. The two sum rules in Table III are seen to converge quite well.

The sum rule for $A_1(T_s=\frac{1}{2})$, Eq. (A14), is of especial interest because it should be saturated by $T=\frac{1}{2}$ resonances alone. The most interesting feature of this sum rule is the large size of the nucleon contribution and a glance at the other contributions to the sum rule tells us that however we adjust the coupling constants within their limits we cannot cancel the large positive contribution of the nucleon. The rapid convergence here confines our attention to the low-mass region where the contributions to lowest order in (k/M_Δ), given in Table IV, are expected to provide us with a reasonable picture of the situation. From Table IV we infer that the spin alignment relations, Eq. (2.3), work in our favor, in that the spin-$\frac{3}{2}$ contributions are negative. The $N^*(1518)$ is, however, too small by an order of magnitude to cancel the nucleon contribution and an increase by an order of magnitude in α_1, which would take care of this, is incompatible with the assumptions made about the form factors in Sec. II. The particles numbered 8 and 10 in Table II provide appreciable contributions of the right sign in this sum rule if we attribute a large fraction of their inelastic widths to decay into $\pi\Delta$. But even taking them as large as allowed by the upper bounds on the coupling constants and giving to all other contributions the smallest possible values, we fail to balance the nucleon contribution by 100%. We see in this fact an indication of the failure of the known isospin-$\frac{1}{2}$ resonances to approximate the amplitude in Eq. (A14). In order to discuss the nature of the missing contribution, we first observe that significant terms can arise only from the low-mass region because the amplitude A_1 converges so fast. In this region the expressions of Table IV indicate how the various spin-parity channels contribute to the sum rules. It is seen that saturation of the sum rule requires the missing contribution to be of spin $\frac{3}{2}$ (rather than spin $\frac{1}{2}$ or $\frac{5}{2}$). Consideration also of the other elastic sum rules leads us to consider a $J^P=\frac{3}{2}^+$ state at 1.55 GeV. Whether this is a true isospin-$\frac{1}{2}$ resonance or just a parametrization of background cannot be decided on the basis of the sum rules. If it were a reasonance it could lie on the same Regge trajectory as the $J^P=\frac{7}{2}^+$ $N^*(1980)$; the partial decay width into $\pi\Delta$ would be $\Gamma_{\pi\Delta}\sim 70$ MeV, whereas $\Gamma_{\pi N}$ might be very small, explaining why the resonance is not seen in the pion-nucleon phase-shift analyses. The contributions of this hypothetical $N^*(1550)$ are listed in Table II as number 25, with its coupling to $\pi\Delta$ fixed in such a way that the first sum rule, for $A_1(T_s=\frac{1}{2})$, is satisfied identically.

TABLE III. Inelastic sum rules, $\pi N \rightarrow \pi\Delta$. This table gives the contributions to the two inelastic sum rules, *omitting* all particles with zero contribution by virtue of the solution. The last column lists the relative signs of the g_N and g_Δ that give the best saturation.

No.	Particle Mass	j^P	I $A(T_t=1)$	II $\nu A(T_t=2)$	$\lvert g_\Delta\rvert$	$\lvert g_N\rvert$	$\mathrm{sgn}(g_\Delta/g_N)$
1	0.939	$\frac{1}{2}^+$	0.324	−0.192	6.64	4.75	+
3	1.518	$\frac{3}{2}^-$	−0.059	−0.13	0.61	4.87	−
4	1.55	$\frac{1}{2}^-$	0.042	0.10	5.5	1.04	+
5	1.68	$\frac{5}{2}^-$	0.0004	0.001	1.73	5.04	+
6	1.69	$\frac{5}{2}^+$	0.0049	0.016	2.12	8.15	+
7	1.71	$\frac{1}{2}^-$	0.018	0.064	4.9	4.01	+
8	1.73	$\frac{3}{2}^-$	−0.018	−0.065	1.41	3	−
10	1.86	$\frac{3}{2}^+$	0.011	0.050	1.41	3.27	+
11	1.98	$7/2^+$	−0.00007	−0.0004	8.95	3.92	−
13	2.20	$7/2^-$	−0.0011	−0.0079	7.07	5.12	−
16	1.236	$\frac{3}{2}^+$	0.71	−0.099	2.0	6.64	−
17	1.64	$\frac{1}{2}^-$	0.084	−0.051	7.07	1.57	−
18	1.69	$\frac{3}{2}^+$	0.055	−0.037	1.58	2.17	−
19	1.69	$\frac{3}{2}^-$	−0.072	0.048	1.41	2.46	+
20	1.91	$\frac{5}{2}^+$	0.011	−0.011	5.5	4.60	−
22	1.95	$7/2^+$	−0.0004	0.0004	7.75	7.05	+
23	1.95	$\frac{5}{2}^-$	−0.0046	0.0048	4.12	3.78	+
24	2.42	$11/2^+$	−0.0005	0.0007	7.75	4.35	+
Positive contributions			11.2	+5.75			
Negative contributions			−10.1	−5.95			
Discrepancy			0.9 (8%)	−0.2 (3%)			

[10] N. Barash-Schmidt *et al.*, University of California Radiation Laboratory Report No. 8030 Pt. 1., 1968 (unpublished). A. Donnachie, *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968*, (CERN, Geneva, 1968), p. 139.

TABLE IV. Contributions from spins $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ to the $\pi\Delta \to \pi\Delta$ invariant amplitudes. Here the contributions are listed to lowest order in (k/M_Δ). The relation to the spin alignment relations, Eq. (2.3), is discussed in Sec. IV. M_R is the mass of the intermediate state.

j^P	A_1	A_2	A_5
$\frac{1}{2}^+$	$\frac{3}{32M_R^3}g^2$	$\frac{3M_\Delta g^2}{8M_R^2}\left(1+\frac{M_\pi}{2M_\Delta}\right)$	$-\frac{3M_\Delta}{4M_R^2}g^2$
$\frac{1}{2}^-$	$\frac{3}{8M_R^3}g^2$	$-\frac{3M_\pi}{4M_R^2}g^2$	0
$\frac{3}{2}^+$	$-\frac{3g^2}{4M_R^3}[1+3(\alpha_{1/2}-\alpha_{3/2})]$	$-\frac{3M_\Delta g^2}{2M_R^2}\left[2\left(1+\frac{M_\pi}{2M_\Delta}\right)-\frac{3M_\pi}{M_\Delta}(\alpha_{1/2}-\alpha_{3/2})\right]$	$\frac{3}{M_R^3}(2M_\Delta-3M_R)$
$\frac{3}{2}^-$	$-\frac{3g^2}{16M_R^3}(\alpha_{1/2}-\alpha_{3/2})$	$-\frac{3M_\Delta}{4M_R^3}g^2(\alpha_{1/2}-\alpha_{3/2})\left(1+\frac{M_\pi}{2M_\Delta}\right)$	$-\frac{3}{4M_R^2}[M_R+2(M_R-M_\Delta)(\alpha_{1/2}-\alpha_{3/2})]$
$\frac{5}{2}^+$	$\frac{g^2}{32M_R^3}[1+8(\alpha_{1/2}-\alpha_{3/2})]$	$-\frac{M_\Delta}{8M_R^3}g^2\left[1+\frac{M_\pi}{2M_\Delta}+\frac{4M_\pi}{M_\Delta}(\alpha_{1/2}-\alpha_{3/2})\right]$	0
$\frac{5}{2}^-$	$\frac{9g^2}{8M_R^2}$	$-\frac{9M_\pi}{4M_R^3}g^2$	0

A second observation, looking at Tables II and III, is that the sum rules in Table III require a value for $g_{\Delta\Delta\pi}$ which is incompatible with the second sum rule of Table II [that for $A_1(T_s=\frac{3}{2})$]. This discrepancy is too large to be resolved by a modification of the form factors within the restrictions imposed by the smoothness assumptions of Sec. II. Therefore, we again try to establish the nature of the missing contribution. The simplest solution turns out to be a $T=\frac{3}{2}$, $J^P=\frac{1}{2}^-$ enhancement a 1.3 GeV. Again we have no means to decide the question, whether this is a true resonance or a parametrization of background. If it were a resonance it could fit on the Regge trajectory of the $\Delta(1950)$ $\frac{5}{2}^-$ state; its partial width into πN could be small but it would decay strongly into $N\pi\pi$ through $\pi\Delta(1236)$ as a (virtual) intermediate state. Contributions of such a hypothetical $\Delta(1300)$ have been listed under number 26 in Table II. It is now possible to find a reasonable set of coupling constants which saturate all nine sum rules and which are compatible with presently known information about decay rates, and the results are noted in Tables II and III.

Finally in this section we mention that our values of

$$g_{NN\pi}^2=22.5 \quad \text{and} \quad g_{\Delta\Delta\pi}^2=4$$

correspond in the conventional notation to

$$G_{NN\pi}^2/4\pi=15 \quad \text{and} \quad G_{\Delta\Delta\pi}^2/4\pi=43.2\,,$$

with $G_{NN\pi}$ and $G_{\Delta\Delta\pi}$ defined in

$$\mathcal{H}_{\text{int}}=G_{NN\pi}\bar{\psi}_n\gamma^5\psi_n\phi_{\pi^0}+G_{\Delta\Delta\pi}\bar{\psi}_{\Delta^{++}}\gamma^5\psi_{\Delta^{++}}\phi_{\pi^0}$$

V. CONCLUDING REMARKS

Our calculations differ from other attempts to saturate superconvergence sum rules in that we make additional considerations about the shape of the strong-interaction form factors. In the discussion of the nature of the form factors in Sec. II, we have mentioned our assumptions of smoothness and the spin-alignment relations, Eq. (2.3).

The elastic sum rules are sensitive not only to the coupling constants, but also to the shapes of the form factors, and in particular to spin alignment—unlike sum rules for processes with external particles of lower spins. The reason can be very clearly seen from Eqs. (A19) and (A20) of Appendix A. The combination of s-channel helicity amplitudes and hence of the strong-interaction form factors which enters the sum rules has to cancel a factor k^4 in the denominator. For an intermediate state (like $J^P=\frac{3}{2}^-$) which can couple in an s wave, the result will depend therefore on higher-order terms in the form factors. This can also be seen in the formulas of Table IV.

Both the smoothness assumption and the particular aspect of spin alignment seem physically reasonable. Additional support for the smoothness assumption lies in the good agreement with the experiment of the fit to the baryon decays of the $O(3,1)$ dynamical group model for the form factors.[7] The fact that the sum rules here are more difficult to saturate without the spin alignment relationships may be taken as evidence for *their* validity.

The two elastic sum rules, Eqs. (A14) and (A15), are particularly suited to the separate study of the N (isospin $\frac{1}{2}$) and Δ (isospin $\frac{3}{2}$) resonance spectra. The results for these two sum rules indicate that the N resonances so far proposed cannot saturate Eq. (A14), whereas Eq. (A15) *by itself* does not lead to any difficulty. Only when it is considered simultaneously with Eqs. (B19) and (B20) do inconsistencies show up. These failures of saturation can be resolved either by a drastic departure from our assumptions about the form factors or by inclusion of additional contributions which, if resonances, fit into the Regge classification scheme or, if not, may be suitable parametrizations of the background.

The inelastic sum rule, Eq. (B19), is approximately saturated by the nucleon and $\Delta(1236)$ alone, which give the dominant contributions. This supports the work of Jones and Scadron,[3] who saturate a sum rule related to our Eq. (B19) with nucleon and $\Delta(1236)$ alone. In the elastic sum rules we find that the higher resonances are important, giving large contribution, so that in the philosophy of Scadron and Jones the symmetry-breaking contribution is large.

Finally we emphasize that more experimental data on the $\pi\Delta(1236)$ partial widths of the nonstrange baryons will provide, through the sum rules discussed here, a method of checking the resonance spectrum found by pion-nucleon phase-shift analysis.

ACKNOWLEDGMENT

We would like to thank Dr. J. J. G. Scanio for helpful discussions.

APPENDIX A

The Process $\pi\Delta \to \pi\Delta$

In this Appendix the superconvergence sum rules for elastic $\pi\Delta$ scattering at $t=0$ are developed. The t-channel reaction is taken as $\pi\pi \to \Delta\bar{\Delta}$. The asymptotic behavior of the t-channel helicity amplitudes $f_{ab,00}{}^t(\nu,T_t)$ at $t=0$ is given by Regge theory as

$$\bar{f}_{ab,00}{}^t(\nu,T_t) \sim \nu^{\alpha(T_t)-|a-b|} \quad \text{as} \quad \nu \to \infty\,,$$

where $\alpha(T_t)$ is the intercept at $t=0$ of the leading exchanged trajectory with isospin T_t. Here we have defined in the usual way[11]

$$\bar{f}_{ab,00}{}^t(\nu,T_t) = (\tfrac{1}{2}\sin\theta_t)^{-|a-b|} f_{ab,00}{}^t(\nu,T_t)\,,$$

where θ_t is the t channel c.m. scattering angle.

We assume $\alpha(T_t=0)=1$, $\alpha(T_t=1)\simeq 0.5$, and $\alpha(T_t=2)<0$ and then find the following superconvergence relations

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t(\nu,\, T_t=0)d\nu=0\,, \tag{A1}$$

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t(\nu,\, T_t=2)d\nu=0\,, \tag{A2}$$

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t(\nu,\, T_t=1)\nu d\nu=0\,, \tag{A3}$$

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t(\nu,\, T_t=2)\nu^2 d\nu=0\,, \tag{A4}$$

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{1}{2},00}{}^t(\nu,\, T_t=1)d\nu=0\,, \tag{A5}$$

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{1}{2},00}{}^t(\nu,\, T_t=2)d\nu=0\,, \tag{A6}$$

$$\int_{-0}^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{1}{2},00}{}^t(\nu,\, T_t=2)d\nu=0\,, \tag{A7}$$

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{1}{2}-\frac{1}{2},00}{}^t(\nu,\, T_t=2)d\nu=0\,. \tag{A8}$$

Using the Trueman–Wick crossing matrix [12] for the helicity amplitudes, the sum rules (A1–A8) may be written in terms of s-channel helicity amplitudes for which the contributions of the s-channel resonances are known through Eq. (2.4). At $t=0$ we find

$$\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t=\frac{3M_\Delta{}^3M_\pi{}^3}{k^4s^{3/2}}\left[\frac{E_\Delta}{M_\Delta}(\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s)+\frac{2}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{1}{2}0,-\frac{1}{2}0}{}^s\right]\sqrt{-t}+O(t^{3/2})\,, \tag{A9}$$

$$\bar{f}_{\frac{3}{2}-\frac{1}{2},00}{}^t=\frac{\sqrt{3}M_\Delta{}^2M_\pi{}^2}{k^2s}(\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s)+O(t)\,, \tag{A10}$$

$$\bar{f}_{\frac{3}{2}\frac{1}{2},00}{}^t=\frac{\sqrt{3}M_\Delta M_\pi}{4k^2s^{1/2}}\left[3\frac{E_\Delta}{M_\Delta}(\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s)+\frac{2}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{1}{2}0,-\frac{1}{2}0}{}^s-4\left(\frac{1}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s-\frac{E_\Delta}{M_\Delta}\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s\right)\right]\times\sqrt{-t}+O(t^{3/2})\,, \tag{A11}$$

$$\bar{f}_{\frac{1}{2}-\frac{1}{2},00}{}^t=\frac{M_\Delta M_\pi}{4k^2s^{1/2}}\left[-5\frac{E_\Delta}{M_\Delta}(\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s)-\frac{2}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s+\bar{f}_{\frac{1}{2}0,-\frac{1}{2}0}{}^s+8\left(\frac{1}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s-\frac{E_\Delta}{M_\Delta}\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s\right)\right]\times\sqrt{-t}+O(t^{3/2})\,. \tag{A12}$$

Here E_Δ and k are the energy and magnitude of 3-momentum respectively for the $\Delta(1236)$ external particle in the s-channel c.m. frame. Again the notation is the usual one:

$$\bar{f}_{a0,b0}{}^s=(\cos\tfrac{1}{2}\theta_s)^{-|a+b|}(\sin\tfrac{1}{2}\theta_s)^{-|a-b|}f_{a0,b0}{}^s$$

It is easy to see that the $\bar{f}^t$ are not all independent at $t=0$. In fact we find the constraint

$$8\sqrt{3}M_\Delta{}^2M_\pi{}^2\bar{f}_{\frac{3}{2}\frac{1}{2},00}{}^t+12M_\Delta{}^2M_\pi{}^2\bar{f}_{\frac{1}{2}-\frac{1}{2},00}{}^t-k^2s\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t=0 \tag{A13}$$

[11] T. L. Trueman Phys. Rev. Letters **17**, 1198 (1966).

[12] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

Thus, the sum rule, Eq. (A8), is not independent of the others and may be dropped.

The factor $(\sqrt{-t})$ in Eqs. (A9), (A11), and (A12) must of course be divided out before the sum rule at $t=0$ is evaluated.[13]

The isospin crossing matrix for this case is[14]

$$X_{ts}=\begin{pmatrix} -\frac{1}{3}\sqrt{3} & -\frac{2}{3}\sqrt{3} & -\sqrt{3} \\ -\frac{1}{6}\sqrt{10} & -\frac{2}{5}\sqrt{10} & \frac{3}{10}\sqrt{10} \\ -\frac{1}{6}\sqrt{6} & (4/15)\sqrt{6} & -\frac{1}{10}\sqrt{6} \end{pmatrix}$$

Assuming the absence of isospin-$\frac{5}{2}$ contributions in the s channel and taking advantage of the possibility to rescale the amplitudes in the sum rules, we use the following prescriptions

$$A(T_t=0)\rightarrow A(T_s=\tfrac{1}{2})+2A(T_s=\tfrac{3}{2}),$$
$$A(T_t=1)\rightarrow A(T_s=\tfrac{1}{2})+\tfrac{4}{5}A(T_s=\tfrac{3}{2}),$$
$$A(T_t=2)\rightarrow A(T_s=\tfrac{1}{2})-(8/5)A(T_s=\tfrac{3}{2}).$$

In particular the sum rules, Eqs. (A1) and (A2) become linear combinations of the two sum rules

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t(\nu,T_s=\tfrac{1}{2})d\nu=0, \quad \text{(A14)}$$

$$\int_0^\infty \mathrm{Im}\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t(\nu,T_s=\tfrac{3}{2})d\nu=0. \quad \text{(A15)}$$

We shall replace Eqs. (A1) and (A2) by Eqs. (A14) and (A15) since the latter two equations have the distinct advantage of testing separately the N (isospin $\frac{1}{2}$) and Δ (isospin $\frac{3}{2}$) resonance spectra.

For completeness we shall also exhibit the relation between invariant amplitudes and helicity amplitudes at $t=0$. To define invariant amplitudes we introduce the T-matrix development

$$T(\nu,t)=\bar{u}_\beta(q')M^{\beta\alpha}u_\alpha(q)$$

and

$$M^{\beta\alpha}=(A_1\boldsymbol{P}+A_2)P^\beta P^\alpha+(A_3\boldsymbol{P}+A_4)Q^\beta Q^\alpha+(A_5\boldsymbol{P}+A_6)g^{\beta\alpha},$$

where $P=p+p'$, $Q=q+q'$ for s-channel momenta $\Delta(q)+\pi(p)\rightarrow\Delta(q')+\pi(p')$. The quantities $u_\alpha(q)$ and $u_\beta(q')$ are Rarita-Schwinger spinors. With this definition A_1, A_2, A_3, and A_5 are superconvergent. We find at $t=0$,

$$\bar{f}_{\frac{3}{2}-\frac{3}{2},00}{}^t=16M_\pi{}^2(\sqrt{-t})A_1, \quad \text{(A16)}$$

$$\bar{f}_{\frac{3}{2}-\frac{1}{2},00}{}^t=\frac{8M_\pi{}^2}{\sqrt{3}M_\Delta}(M_\Delta A_2-\tfrac{1}{2}\nu A_1), \quad \text{(A17)}$$

$$\bar{f}_{\frac{1}{2}\frac{1}{2},00}{}^t=\frac{M_\pi}{\sqrt{3}M_\Delta}\left[-2A_s-\frac{4}{M_\Delta{}^2}\left(\frac{\nu^2}{16}+M_N{}^2M_\pi{}^2\right)A_1+\frac{\nu}{M_\Delta}A_2\right]\sqrt{-t}. \quad \text{(A18)}$$

These relations do not involve A_3, which is a reflection of the fact that the t-channel helicity amplitudes satisfy a constraint at $t=0$. A sum rule involving A_3 corresponds in the helicity formalism to a derivative sum rule at $t=0$, and will not be considered further here.

Using Eqs. (A16), (A17), and (A18) we may now also express A_1, A_2, and A_5 in terms of s-channel helicity amplitudes:

$$A_1=\frac{3M_\Delta{}^3}{16k^4s^{3/2}}\left[E_\Delta(\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s)+M_\Delta\left(\frac{2}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{1}{2}0,-\frac{1}{2}0}{}^s\right)\right], \quad \text{(A19)}$$

$$A_2=-\frac{3M_\Delta{}^3}{8k^4s^{3/2}}\left[M_\Delta E_\pi(\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s)+(k^2+E_\pi E_\Delta)\left(\frac{2}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{1}{2}0,-\frac{1}{2}0}{}^s\right)\right], \quad \text{(A20)}$$

$$A_5=\frac{3M_\Delta}{4k^2s}\left[-M_\Delta{}^2(\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s-\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s)+2(\sqrt{s})\left(\frac{M_\Delta}{\sqrt{3}}\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s-E_\Delta\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^s\right)\right]. \quad \text{(A21)}$$

In terms of A_1, A_2, and A_5 we have the following sum rules:

$$\int_0^\infty \mathrm{Im}A_1(\nu,T_s=\tfrac{1}{2})d\nu=0, \quad \text{(A22)}$$

$$\int_0^\infty \mathrm{Im}A_1(\nu,T_s=\tfrac{3}{2})d\nu=0, \quad \text{(A23)}$$

$$\int_0^\infty \mathrm{Im}A_1(\nu,T_t=1)\nu d\nu=\int_0^\infty \mathrm{Im}[A_1(T_s=\tfrac{1}{2})+\tfrac{4}{5}A_1(T_s=\tfrac{3}{2})]\nu d\nu=0, \quad \text{(A24)}$$

$$\int_0^\infty \mathrm{Im}A_1(\nu,T_t=2)\nu^2 d\nu=\int_0^\infty \mathrm{Im}[A_1(T_s=\tfrac{1}{2})-(8/5)A_1(T_s=\tfrac{3}{2})]\nu^2 d\nu=0, \quad \text{(A25)}$$

[13] Because of the factor $\sqrt{(-t)}$ in the amplitudes of Eqs. (A9), (A11), and (A12), they lead to the "Class II" sum rules of F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968). But we see no reason why they should be on a different footing from "Class I" sum rules as resulting from Eq. (A10). One might argue that as $\sqrt{(-t)}$ tends to zero the relevant integral also tends to zero, so that the sum rule seems trivially satisfied in this limit. The absolute magnitude of the integral is, however, quite meaningless. It is the ratio of this value to the sum of all contributions of the same sign which indicates how well the sum rule is satisfied. This ratio is independent of the factor $\sqrt{(-t)}$, and we shall simply divide out this kinematic branch point where it occurs and make no distinction between the "Class I" and "Class II" sum rules.

[14] P. Carruthers and J. P. Krisch, Ann. Phys. (N. Y). **33**, 1 (1965).

$$\int_0^\infty \mathrm{Im}A_2(\nu, T_t=1)d\nu = \int_0^\infty \mathrm{Im}[A_2(T_s=\tfrac{1}{2}) + \tfrac{4}{5}A_2(T_s=\tfrac{3}{2})]d\nu = 0\,, \quad \text{(A26)}$$

$$\int_0^\infty \mathrm{Im}A_2(\nu, T_t=2)\nu d\nu = \int_0^\infty \mathrm{Im}[A_2(T_s=\tfrac{1}{2}) - (8/5)A_2(T_s=\tfrac{3}{2})]\nu d\nu\,, \quad \text{(A27)}$$

$$\int_0^\infty \mathrm{Im}A_5(\nu, T_t=2)d\nu = \int_0^\infty \mathrm{Im}[A_5(T_s=\tfrac{1}{2}) - (8/5)A_5(T_s=\tfrac{3}{2})]d\nu\,. \quad \text{(A28)}$$

When saturating these sum rules, the following two procedures are equivalent:

(i) ν runs from 0 to ∞, but s-channel and u-channel resonances are included.

(ii) ν runs from $-\infty$ to $+\infty$ and only s-channel resonances are included.

We follow the second procedure.

The contribution of a particle of mass M_R, spin j, and parity P to the imaginary part of the s-channel helicity amplitudes is given by

$$\mathrm{Im}f_{a0,b0}{}^{s,j} = \bar{V}_b(\tfrac{3}{2}^+, j^P, x)d_{ba}{}^j(\theta_s)V_a(j^P, \tfrac{3}{2}^+, x)\delta(\nu-\nu_R)\,,$$

where

$$\nu_R = 2(M_\Delta{}^2 + M_\pi{}^2 - M_R{}^2)$$

A simple calculation gives for the $\bar{f}^s$ at $t=0$, the following contributions from a particle of spin j and normality $\pm$ (the imaginary part and $\delta(\nu-\nu_R)$ being understood):

$$\bar{f}_{\frac{3}{2}0,\frac{3}{2}0}{}^{s,j} = |V_{3/2}{}^{j\pm}(x)|^2\,, \quad \text{(A29)}$$

$$\bar{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^{s,j} = |V_{1/2}{}^{j\pm}(x)|^2\,, \quad \text{(A30)}$$

$$\bar{f}_{\frac{1}{2}0,-\frac{1}{2}0}{}^{s,j} = \pm\frac{2j+1}{2}|V_{1/2}{}^{j\pm}(x)|^2\,, \quad \text{(A31)}$$

$$\bar{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^{s,j} = [(j+\tfrac{3}{2})(j-\tfrac{1}{2})]^{1/2} \times V_{1/2}{}^{j\pm}(x)V_{3/2}{}^{j\pm}(x) \geq 0\,. \quad \text{(A32)}$$

These expressions are used in evaluating the sum rules. The results are given in Tables II and IV and discussed in Sec. IV.

APPENDIX B

The Process $\pi N \to \pi\Delta$

Let the s-channel process here be $\pi N \to \pi\Delta$ and the t-channel process $\pi\pi \to \bar{N}\Delta$. The t-channel helicity amplitudes at the pseudothreshold $t=(M_\Delta - M_N)^2$ satisfy two constraints, such that there exists only one independent superconvergent amplitude at this point, which we take to be

$$A = (M_N/M_\Delta)^{1/2}\{4M_N M_\Delta[(M_\Delta - M_N)^2 - 4M_\pi{}^2]\}^{-1} \bar{f}_{\frac{1}{2}-\frac{3}{2},00}{}^t$$
$$= \mathcal{S}_{\pi N}{}^{-1/2}\mathcal{S}_{\pi\Delta}{}^{-3/2}[\sqrt{3}(f_{\frac{1}{2}0,\frac{1}{2}0}{}^s - if_{-\frac{1}{2}0,\frac{1}{2}0}{}^s) - f_{-\frac{3}{2}0,\frac{1}{2}0}{}^s + if_{\frac{3}{2}0,\frac{1}{2}0}{}^s]\,, \quad \text{(B1)}$$

where

$$\mathcal{S}_{\pi N} = \{[s-(M_N+M_\pi)^2][s-(M_N-M_\pi)^2]\}^{1/2}\,,$$
$$\mathcal{S}_{\pi\Delta} = \{[s-(M_\Delta+M_\pi)^2][s-(M_\Delta-M_\pi)^2]\}^{1/2}\,, \quad \text{(B2)}$$

and

$$f_{b0,a0}{}^{s,j\pm} = \bar{V}_a{}^{\frac{3}{2}j\pm}d_{ab}{}^j(\theta_s)V_b{}^{j\pm\frac{1}{2}}\,. \quad \text{(B3)}$$

Since at $t=(M_\Delta - M_N)^2$ the s-channel process is unphysical $\cos\theta_s > 1$ and $\sin\theta_s$ is purely imaginary. Therefore, it is convenient to introduce the function

$$\tilde{d}_{ab}{}^j(\theta_s) = i^{a-b}d_{ab}{}^j(\theta_s) \quad \text{(B4)}$$

which is purely real. Then also

$$\tilde{f}_{b0,a0}{}^{s,j\pm} = i^{a-b}f_{b0,a0}{}^{s,j\pm} \quad \text{(B5)}$$

is purely real, so that also A is real:

$$A = \mathcal{S}_{\pi N}{}^{-1/2}\mathcal{S}_{\pi\Delta}{}^{-3/2}[\sqrt{3}(\tilde{f}_{\frac{1}{2}0,\frac{1}{2}0}{}^s + \tilde{f}_{-\frac{1}{2}0,\frac{1}{2}0}{}^s) + \tilde{f}_{-\frac{3}{2}0,\frac{1}{2}0}{}^s + \tilde{f}_{\frac{3}{2}0,\frac{1}{2}0}{}^s]\,. \quad \text{(B6)}$$

For s above threshold $\mathcal{S}_{\pi N}$, $\mathcal{S}_{\pi\Delta}$ and $\cos\theta_s$ are all real and can consistently be chosen positive. But for s below threshold, these quantities can become imaginary, so that in A several complex numbers have to cancel. This will only happen if the signs of all square roots have been chosen consistently. The easiest way to achieve this is to write out the $\tilde{f}^{s,j}$ explicitly as functions of $\mathcal{S}_{\pi N}$, $\mathcal{S}_{\pi\Delta}$, and θ_s and then simplify the resulting expression to the point where all relevant sign ambiguities have been eliminated. We do this here for particles with $j^P = \frac{1}{2}^+, \frac{3}{2}^+$, because the nucleon and the $\Delta(1236)$ contributions are the only ones below threshold.

For a $\frac{1}{2}^+$ intermediate state of mass M_R, we find

$$\mathrm{Im}f_{\pm\frac{3}{2}0,\frac{1}{2}0}{}^s = 0\,,$$
$$\mathrm{Im}f_{\pm\frac{1}{2}0,\frac{1}{2}0}{}^s = V_{\frac{1}{2}}{}^{\frac{1}{2}\frac{1}{2}-}(x_1)d_{\frac{1}{2}\pm\frac{1}{2}}{}^{\frac{1}{2}}(\theta_s)V_{\frac{1}{2}}{}^{\frac{1}{2}\frac{3}{2}}(x_2)\delta(\nu_R-\nu)\,, \quad \text{(B7)}$$

where

$$x_1 = \frac{\mathcal{S}_{\pi N}}{2M_N M_R}\,, \quad x_2 = \frac{\mathcal{S}_{\pi\Delta}}{2M_\Delta M_R}\,,$$
$$\nu_R = 2(M_R{}^2 - M_\pi{}^2 - M_N M_\Delta)\,.$$

Splitting off the threshold factors of the V-functions, such that

$$V_{\frac{1}{2}}{}^{\frac{1}{2}\frac{1}{2}-}(x_1) = \mathcal{S}_{\pi N}B_1{}^{1-}(x_1)\,, \quad V_{\frac{1}{2}}{}^{\frac{1}{2}\frac{3}{2}+}(x_2) = \mathcal{S}_{\pi\Delta}B_3{}^{1+}(x_2)\,, \quad \text{(B8)}$$

we obtain for the function A, defined in Eq. (B1):

$$\mathrm{Im}A^{\frac{1}{2}+} = \sqrt{3}B_1{}^{1-}(x_1)B_3{}^{1+}(x_2)\mathcal{S}_{\pi N}{}^{\frac{1}{2}}\mathcal{S}_{\pi\Delta}{}^{-\frac{1}{2}} \times(\cos\theta_s + i\sin\theta_s)^{1/2}\delta(\nu_R-\nu) \quad \text{(B9)}$$

Inserting the well-known expressions for $\cos\theta_s$ and $\sin\theta_s$, after some algebra we arrive at

$$\mathrm{Im}A^{1/2+}=\sqrt{3}B_1^{1-}(x_1)B_1^{1+}(x_2)\times\left[\frac{(M_R+M_N-M_\pi)(M_R+M_N+M_\pi)}{(M_R+M_\Delta-M_\pi)(M_R+M_\Delta+M_\pi)}\right]^{1/2}\times\delta(\nu_R-\nu)\,. \quad \text{(B10)}$$

For intermediate states of positive mass ($M_R>0$), the $\frac{1}{2}^+$ contribution to A will have no kinematical poles or zeros.[15]

The $\frac{3}{2}^+$ contribution is treated similarly. In this case

$$\begin{aligned}\mathrm{Im}f_{\pm\frac12 0,\frac12 0}{}^s&=\pm\overline{V_{\frac12}{}^{\frac12\frac32+}}(x_1)d_{\frac12\pm\frac12}{}^{\frac32}(\theta_s)V_{\frac12}{}^{\frac32\frac32-}(x_2)\delta(\nu_R-\nu)\\ \mathrm{Im}f_{\pm\frac32 0,\frac12 0}{}^s&=\pm\overline{V_{\frac12}{}^{\frac12\frac32+}}(x_1)d_{\frac12\pm\frac32}{}^{\frac32}(\theta_s)V_{\frac32}{}^{\frac32\frac32-}(x_2)\delta(\nu_R-\nu)\end{aligned} \quad \text{(B11)}$$

Factoring off the threshold factors,

$$V_{\frac12}{}^{\frac12\frac32+}(x_1)=\mathcal{S}_{\pi N}B_1{}^{3+}(x_1),\quad V_{j/2}{}^{\frac32\frac32-}(x_2)\mathcal{S}_{\pi\Delta}B_{3j}{}^{3-}(x_2)\,, \quad \text{(B12)}$$

we obtain, again after some algebra,

$$\begin{aligned}\mathrm{Im}A^{\frac32+}=&\,\mathcal{S}_{\pi N}{}^{1/2}\mathcal{S}_{\pi\Delta}{}^{1/2}B_1{}^{3+}(x_1)\{\sqrt{3}B_{31}{}^{3-}(x_2)\\ &\times[\cos\tfrac12\theta_s(3\cos\tfrac12\theta_s-2)+i\sin\tfrac12\theta_s(3\sin\tfrac12\theta_s-2)]\\ &+\tfrac12\sqrt{3}B_{33}{}^{3-}(x_2)\sin\theta_s[\sin\tfrac12\theta_s+i\,\mathrm{cox}\tfrac12\theta_s]\}\delta(\nu_R-\nu)\,.\end{aligned} \quad \text{(B13)}$$

Note that for small x

$$B_{33}{}^{3-}(x)=3B_{31}{}^{3-}(x)+O(x^2)\,.$$

Therefore,

$$D(x)=(B_{33}{}^{3-}(x)-3B_{31}{}^{3-}(x))/x^2 \quad \text{(B14)}$$

is a regular function at $x=0$. Equation (B13) may be rewritten as

$$\begin{aligned}\frac{\mathrm{Im}A^{3/2+}}{\delta(\nu_R-\nu)}=&\,\mathcal{S}_{\pi N}{}^{1/2}\mathcal{S}_{\pi\Delta}{}^{-1/2}\sqrt{3}B_1{}^{3+}(x_1)B_{31}{}^{3-}(x_2)\\ &\times(\cos\theta_s+i\sin\theta_s)^{\frac12}+\frac{\sqrt{3}}{8M_NM_\Delta M_R{}^2}\mathcal{S}_{\pi N}{}^{1/2}\mathcal{S}_{\pi\Delta}{}^{3/2}\\ &\times B_1{}^{3+}(x_1)D(x_2)i\sin\theta_s(\cos\theta_s-i\sin\theta_s)^{\frac12}\end{aligned} \quad \text{(B15)}$$

[15] With the correct Mac.Dowell symmetric expressions for the pole diagram, mentioned after Eq. (2.4), the poles and zeros would disappear for negative M_R also.

and, again after some algebra,

$$\begin{aligned}\mathrm{Im}A^{3/2+}=&\,\sqrt{3}B_1{}^{3+}(x_1)\\ &\times\left\{B_{31}{}^{3-}(x_2)\left[\frac{(M_R+M_N-M_\pi)(M_R+M_N+M_\pi)}{(M_R+M_\Delta-M_\pi)(M_R+M_\Delta+M_\pi)}\right]^{\frac12}\right.\\ &+\frac{D(x_2)\nu(M_\Delta-M_N)}{8M_NM_\Delta M_R}\\ &\left.\times\left[\frac{(M_R+M_\Delta-M_\pi)(M_R+M_\Delta+M_\pi)}{(M_R+M_N-M_\pi)(M_R+M_N+M_\pi)}\right]^{\frac12}\right\}\delta(\nu_R-\nu)\,,\end{aligned} \quad \text{(B16)}$$

where

$$\nu=2(s-M_\pi{}^2-M_NM_\Delta)\,.$$

$\mathrm{Im}A$ satisfies two superconvergence relations

$$\int_0^\infty \mathrm{Im}A(T_t=1)d\nu=0 \quad \text{(B17)}$$

and

$$\int_0^\infty \mathrm{Im}A(T_t=2)\nu d\nu=0\,. \quad \text{(B18)}$$

The isospin crossing matrix for this case is

$$X_{ts}=\frac13\begin{pmatrix}1 & \sqrt{10}\\ \sqrt{3} & -\frac15\sqrt{30}\end{pmatrix}$$

so that the sum rules become

$$\int_0^\infty \mathrm{Im}[A(T_s=\tfrac12)+(\sqrt{10})A(T_s=\tfrac32)]d\nu=0\,, \quad \text{(B19)}$$

$$\int_0^\infty \mathrm{Im}[A(T_s=\tfrac12)-(\sqrt{\tfrac25})A(T_s=\tfrac32)]\nu d\nu=0\,. \quad \text{(B20)}$$

The contributions to these $\pi N\to\pi\Delta$ sum rules are given in Table III and discussed in Sec. IV.