Eikonal Expansion*

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A systematic study is made of high-energy scattering in nonrelativistic quantum mechanics. Approximate amplitudes are obtained for the scattering of a high-energy projectile off a single particle and off a compound system. Our results hold for all momentum transfers. Near the forward direction, they reduce to the well-known Glauber formulas, and for large momentum transfers, the leading terms are seen to correspond to the Mandelstam-cut diagrams.

I. INTRODUCTION

THE study of very-high-energy scattering in nonrelativistic quantum mechanics is interesting both because of its direct application to nuclear physics[1] and because one hopes that it will give some insight into the more complicated problem of relativistic high-energy scattering.[2] Since there has recently been a great deal of interest in both of these applications, it seems worthwhile to make a systematic study of high-energy potential scattering.

Several years ago, Glauber[3] presented a very elegant approximation method which is based on the fact that at very high energies most scattering occurs in the forward direction. Although Glauber's theory has been successfully applied to a wide range of problems, there are many interesting effects which occur at momentum transfers that are too large for it to be applicable. Our purpose here is to present approximation techniques which will be useful at all momentum transfers.

A start in this direction was made several years ago by Saxon and Schiff,[4] who presented a high-energy approximation to the two-praticle scattering amplitude which holds at all momentum transfers for a wide class of potentials. We shall discuss this problem, as well as the scattering of a high-energy projectile off a compound system.

In Sec. II we present the formal results which are necessary for the derivation of our approximations. In Sec. III we treat the two-body problem. For potentials which fall off no faster than a power in momentum space, we obtain the Saxon-Schiff approximation. We also present a much simplified form for their approximation, which at high energies is only somewhat less accurate. In the next sections we study the scattering of a high-energy projectile off a two-particle bound state. We obtain expressions for the elastic scattering in Sec. IV and breakup amplitudes in Sec. V, which are valid at all momentum transfers. In all cases, our results reduce to those of Glauber in the forward direction. Finally, in Sec. VI we summarize our results and comment on the trivial extension to the case of a projectile scattering off a multiparticle bound state. The connection between our results and Regge-cut theory is discussed briefly.

II. FORMAL CONSIDERATIONS

The basic approach to be used in our discussion of high-energy scattering is to develop approximate Green's functions which are accurate at high energies and which are simple enough so that one can solve the resulting approximate equations. We shall also be very interested in developing a convenient perturbation series for the corrections to the zeroth-order problem. First, we will consider the formal aspects of the theory; then, we will make a definite choice for the approximate Green's functions which corresponds to eikonal type of propagation.

The scattering matrix satisfies the familiar equation

$$T=V+VGT=V+TGV. \tag{1}$$

The Green's function will be separated into

$$G=G_j+R_j=G_j+G_jN_jG=G_j+GN_jG_j, \tag{2}$$

where

$$N_j=G_j^{-1}-G^{-1}. \tag{3}$$

The subscript j refers to the possible parameters in the approximate Green's function G_j, which hopefully can be chosen so as to improve the accuracy of the final answer. For example, in the high-energy limit, where the scattering is almost all in the forward direction, we will choose G_j to correspond to propagation in a definite direction. We shall always want to choose G_j to be simple enough so that when it is used in place of G in Eq. (1), the resulting approximate T matrix can be obtained in closed form.

The two versions of the equation for T, Eq. (1), will be solved using perhaps different choices for the approxi-

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[1] R. J. Glauber, in *High-Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland Publishing Co., Amsterdam, 1967), pp. 311–338.
[2] See, for example, T. T. Chou and C. N. Yang, Phys. Rev. **170**, 1591 (1968); R. C. Arnold, *ibid.* **153**, 1523 (1967); R. C. Arnold and M. L. Blackmon, *ibid.* **176**, 2082 (1968); S. Frautschi and B. Margolis, Nuovo Cimento **56A**, 1155 (1968); L. Durand, III, and R. Lipes, Phys. Rev. Letters **20**, 637 (1968).
[3] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by E. Brittin and L. G. Dunham (Wiley-Interscience, Inc., New York, 1959), Vol. I, p. 315.
[4] D. S. Saxon and L. I. Schiff, Nuovo Cimento **6**, 614 (1957). For an example in which the large-momentum-transfer corrections became important, see D. K. Ross, Phys. Rev. **173**, 1695 (1968).

mate Green's functions. Inserting (2) into (1), we get

$$T=T_i+TGN_iG_iT_i$$
$$=T_f+T_fG_fN_fGT\,, \quad (4)$$

where

$$T_i=V(1-G_iV)^{-1}\,, \quad (5)$$
$$T_f=(1-VG_f)^{-1}V\,,$$

and i and f are two possible values of j. Inserting each of the equations (4) into the other gives

$$T=T_j+T_j{}^1+T_fG_fN_f[G+GTG]N_iG_iT_i\,, \quad (6)$$

where $j=i$ or f, and

$$T_i{}^1=T_fG_fN_iG_iT_i\,, \quad (7)$$
$$T_f{}^1=T_fG_fN_fG_iT_i\,.$$

The sum of the first two terms on the right-hand side of Eq. (6) is equal for $j=i$ or f, as is easily proved by the use of Eqs. (3) and (5). This equivalence is required to preserve the symmetry of the T matrix. The quantity in the square bracket in last term in Eq. (6) is the exact Green's function g, in the presence of the potential V. The next step is to approximate g in a systematic manner so that the symmetry of T is preserved at any stage.

To that end, let us define

$$g=G+GTG=G+GVg=G+gVG\,, \quad (8)$$

and introduce the quantities

$$G=G_n+R_n\,,$$
$$N_n=G_n{}^{-1}-G^{-1}\,, \quad (9)$$
$$g_n=(1-G_nV)^{-1}G_n\,.$$

The Green's function can then be written in the symmetric form

$$g=g_n+g(N_n-N_ng_nN_n)g\,. \quad (10)$$

The T matrix now becomes

$$T=T_j+T_j{}^1+T_2$$
$$+T_fG_fN_fg(N_n-N_ng_nN_n)gN_iG_iT_i\,, \quad (11)$$

where

$$T_2=T_fG_fN_fg_nN_iG_iT_i\,. \quad (12)$$

Proceeding in the same manner, one can expand T in an explicitly symmetric form involving higher and higher powers of the "perturbation" N_j. One also has the freedom at each stage to choose the approximate Green's functions G_j in such a way as to minimize the corrections. The physical interpretation of this expansion follows from the fact that the structure $G_jN_jG=R_j$ propagates free waves in all ways except those included in G_j itself. These latter waves are included in earlier terms in the expansion of T.

III. TWO-PARTICLE SCATTERING

Our next problem is to find suitable approximations for the free Green's functions which are accurate at high energies. Let us start by considering the scattering of two nonrelativistic particles in their center-of-mass system. In momentum space, the free Green's function is given by

$$G=(k^2/2m-p^2/2m+i\epsilon)^{-1}\,. \quad (13)$$

m is the reduced mass, and the center-of-mass energy E is given by $E=k^2/2m$.

At high energies most of the scattering is in the forward direction, so it will be convenient to have approximate Green's functions which accurately describe propagation at energy E in a definite direction. To that end we expand the intermediate momentum $\mathbf{p}$ about the vector $\mathbf{k}_j$, where $k_j{}^2=k^2$. Neglecting terms of second order in $\mathbf{p}-\mathbf{k}_j$, we obtain the "eikonal Green's function"

$$G_j=m/(k^2-\mathbf{p}\cdot\mathbf{k}_j+i\epsilon)\,. \quad (14)$$

The perturbation is therefore

$$N_j=(\mathbf{p}-\mathbf{k}_j)^2/2m\,. \quad (15)$$

In coordinate space, the eikonal Green's function is most conveniently written in terms of the parametric integral:

$$G_j(\mathbf{r},\mathbf{r}')=-i\int_0^\infty dt\; e^{itk^2/m}\delta(\mathbf{r}-\mathbf{r}'-t\mathbf{k}_j/m)\,. \quad (16)$$

The full eikonal Green's function $g_j(\mathbf{r},\mathbf{r}')$ satisfies the differential equation

$$m^{-1}(k^2+i\mathbf{k}_j\cdot\boldsymbol{\nabla})g(\mathbf{r},\mathbf{r}')=V(\mathbf{r})g_j(\mathbf{r},\mathbf{r}')+\delta^3(\mathbf{r}-\mathbf{r}')\,, \quad (17)$$

the solution of which is easily seen to be

$$g_j(\mathbf{r},\mathbf{r}')=G_j(\mathbf{r},\mathbf{r}')e^{i[\chi_j(\mathbf{r})-\chi_j(\mathbf{r}')]}\,, \quad (18)$$

where

$$\chi_j(\mathbf{r})=-\int_0^\infty dt\; V(\mathbf{r}-\mathbf{v}_jt) \quad (19)$$

and

$$\mathbf{v}_j\equiv\mathbf{k}_j/m\,.$$

It is now a simple matter to construct the eikonal T matrix for scattering from k_i to k_f. Choosing $k_j=k_i$, we find

$$\langle\mathbf{k}_f|T_i|\mathbf{k}_i\rangle=\langle\mathbf{k}_f|V+Vg_iV|\mathbf{k}_i\rangle$$
$$=\int d^3r\; e^{-i\mathbf{k}_f\cdot\mathbf{r}}V(\mathbf{r})e^{i\mathbf{k}_i\cdot\mathbf{r}+i\delta_i(\mathbf{r})}\,, \quad (20)$$

while for $\mathbf{k}_j=\mathbf{k}_f$ we have

$$\langle\mathbf{k}_f|T_f|\mathbf{k}_i\rangle=\langle\mathbf{k}_f|V+Vg_fV|\mathbf{k}_i\rangle$$
$$=\int d^3r\; e^{-i\mathbf{k}_f\cdot\mathbf{r}+i\delta_f(\mathbf{r})}V(\mathbf{r})e^{i\mathbf{k}_i\cdot\mathbf{r}}\,, \quad (21)$$

where

$$\delta_i(\mathbf{r})=\chi_i(\mathbf{r})=-\int_0^\infty dt\; V(\mathbf{r}-\mathbf{v}_it)\,,$$
$$\delta_f(\mathbf{r})=\chi_{-f}(\mathbf{r})=-\int_0^\infty dt\; V(\mathbf{r}+\mathbf{v}_ft)\,. \quad (22)$$

In obtaining Eqs. (20) and (21), we have made use of the fact that

$$-i\int_0^\infty dt\, V(\mathbf{r}-\mathbf{v}_f t)e^{-i\chi_f(\mathbf{r}-\mathbf{v}_f t)}=1-e^{-i\chi_f(\mathbf{r})}, \quad (23)$$

which follows from an integration by parts. Equations (20) and (21) are only two alternative forms of the eikonal approximation of Molière and of Glauber.[5]

In order to obtain the correction terms T_i^1 and T_f^1 of Eq. (6), it is convenient to work out matrix elements in a mixed representation. We have, for example,

$$\langle\mathbf{r}|T_i|\mathbf{k}_i\rangle=V(r)e^{i\mathbf{k}_i\cdot\mathbf{r}+i\delta_i(\mathbf{r})}, \quad (24)$$

$$\langle\mathbf{r}|G_iT_i|\mathbf{k}_i\rangle=e^{i\mathbf{k}_i\cdot\mathbf{r}}(e^{i\delta_i(\mathbf{r})}-1), \quad (25)$$

and

$$\langle\mathbf{r}|N_iG_iT_i|\mathbf{k}_i\rangle=e^{i\mathbf{k}_i\cdot\mathbf{r}}(-\nabla^2/2m)(e^{i\delta_i(\mathbf{r})}-1). \quad (26)$$

Similar expressions hold for matrix elements of T_f. These results illustrate the main simplification of the eikonal T matrices, namely, the locality of Eqs. (24)–(26) in coordinate space.

The first correction to the eikonal T matrix can now be easily computed. We find

$$T_1=T_i+T_i^1=T_i+T_fG_fN_iG_iT_i$$
$$=\int d^3r\, e^{i\Delta\cdot\mathbf{r}}V(\mathbf{r})e^{i\delta_i(\mathbf{r})}$$
$$+\int d^3r\, e^{i\Delta\cdot\mathbf{r}}(e^{i\delta_f(\mathbf{r})}-1)[(-\nabla^2/2m)(e^{i\delta_i(\mathbf{r})}-1)], \quad (27)$$

or, alternatively,

$$T_1=T_f+T_f^1=T_f+T_fG_fN_fG_iT_i$$
$$=\int d^3r\, e^{i\Delta\cdot\mathbf{r}}V(\mathbf{r})e^{i\delta_f(\mathbf{r})}$$
$$+\int d^3r\, e^{i\Delta\cdot\mathbf{r}}[(-\nabla^2/2m)(e^{i\delta_f(\mathbf{r})}-1)](e^{i\delta_i(\mathbf{r})}-1), \quad (28)$$

where $\mathbf{\Delta}=\mathbf{k}_i-\mathbf{k}_f$. Equations (27) and (28) are recognized as the well-known Saxon-Schiff formulas.[4]

The physical interpretation of Eqs. (27) and (28) has been clearly stated in the paper by Saxon and Schiff.[4] However, it will be repeated here both to clarify the formula and to motivate the next step in our discussion. Let us concentrate on Eq. (27) and consider the case of large momentum transfer. As we shall show below, the correction terms T_i^1 and T_f^1 are negligible for small momentum transfers, and then T_1 reduces to the eikonal approximation.

The first term, T_i, corresponds to the particle traveling along the incident direction, accumulating a phase δ_i, but not suffering a large scattering until the last potential acts. It is then scattered into the final direction and propagates out of the interaction region without accumulating any more phase. The second term in Eq. (27) is interpreted in a similar way. The factor T_i corresponds to propagation along the incident direction accumulating a phase δ_i just as before. The factor $N_i=(\mathbf{p}-\mathbf{k}_i)^2/2m$ prevents propagation in the incident direction, so the large scattering must occur in the last potential in T_i. After this large scattering, the particle moves away from the interaction region in the final direction, accumulating the phase δ_f.

One easily sees that the second term is constructed in such a manner that double counting does not occur. For example, the factor N_i tends to make T_i^1 small in the forward direction, where the eikonal approximation is known to be very good.

One expects that T_1 will be an adequate representation of T at high energies and large momentum transfers, if the potential is such that a large-angle scattering event is due to the action of a single potential—in other words, if the probability of large-angle scattering occurring by scattering through momentum transfers of approximately $\frac{1}{2}\mathbf{\Delta}$ at two different potentials is much smaller than the probability of a single scattering with momentum transfer $\mathbf{\Delta}$. This requires that the Fourier transform of the potential not fall off too rapidly for large momentum transfers. This in turn implies that the potential is not too smooth in coordinate space.

For example, T_1 will certainly not reflect the large-momentum transfer behavior of T for the case of a Gaussian potential. However, for potentials such as the Yukawa or exponential potentials, which fall off like a power in momentum space, T_1 is expected to be an accurate approximation for a wide range of momentum transfers.

In order to clarify this point, let us estimate the magnitude of the terms contributing to T_1 and to $\bar{T}\equiv T-T_1$. We start by making the usual assumption that the potential is only appreciable in the region $|\mathbf{r}|\leq a$, and that it changes significantly only over a distance a, so that $|\nabla V(\mathbf{r})|\approx U/a$. Here U is some average value of the potential. We shall always assume that $ka\gg 1$ and that $k^2/2mU\gg 1$.

Let us start by considering the region where the eikonal approximation is known to hold: $\Delta\gtrsim 1/a$, i.e., $\theta\lesssim(ka)^{-1/2}$. We can write T_1 in the form [see Eq. (27)]

$$T_1=\int d^3r\, e^{i\Delta\cdot\mathbf{r}}e^{i\delta_i(\mathbf{r})}[V(\mathbf{r})+(e^{i\delta_f(\mathbf{r})}-1)(1/2m)$$
$$\times\nabla\delta_i(\mathbf{r})\cdot\nabla\delta_i(\mathbf{r})-i\nabla^2\delta_i(\mathbf{r})]. \quad (29)$$

The first term in Eq. (29) is the eikonal approximation. The second term is the Saxon-Schiff correction. For $\Delta\lesssim 1/a$, there is only one important length in the problem, a. It is therefore possible to write $|\nabla\delta_i|\approx Um/k$ and $\nabla^2\delta_i\approx Um/ka$. We then see that the ratio

[5] G. Molière, Z. Naturforsch 2, 133 (1947); R. J. Glauber, Phys. Rev. 91, 459 (1953).

of the Saxon-Schiff correction to the eikonal term is of order $1/ka$ or $2mU/k^2$, whichever is larger. By the same type of reasoning, one sees that the ratio of the remainder term $\bar{T}=T-T_1$ to the eikonal expression is at least as small.

The Saxon-Schiff correction T_i^1 becomes important for $\Delta>1/a$. In fact, after approximating ∇ by $\boldsymbol{\Delta}$ in Eq. (29), one is led to expect that T_i^1 becomes comparable to T_i for $\Delta^2\approx k/a$. However, $\bar{T}$ remains negligible for a wide range of potentials. This is most easily seen by going to momentum space, where the integrands do not oscillate rapidly even for large values of Δ. Defining

$$h_i(\mathbf{q})\equiv\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}e^{i\delta_i(\mathbf{r})}$$

and

$$h_f(\mathbf{q})\equiv\int d^3r\, e^{i\mathbf{q}\cdot\mathbf{r}}e^{i\delta_f(\mathbf{r})}, \tag{30}$$

we can write [see Eq. (6)]

$$T_i^1=\int\frac{d^3qd^3q'}{(2\pi)^6}h_f(\mathbf{q}+\boldsymbol{\Delta})\langle\mathbf{q}|VG_fN_iG_iV|\mathbf{q}'\rangle h_i(\mathbf{q}') \tag{31}$$

and

$$\bar{T}=\int\frac{d^3qd^3q'}{(2\pi)^6}h_f(\mathbf{q}+\boldsymbol{\Delta})\times\langle\mathbf{q}|VG_fN_fgN_iG_iV|\mathbf{q}'\rangle\cdot h_i(\mathbf{q}'). \tag{32}$$

Since h_i and h_f are not expected to oscillate rapidly, it is sufficient to compare the matrix elements in the integrands of Eqs. (31) and (32). Furthermore, since we are only interested in an order-of-magnitude estimate, we shall replace the full Green's function g in Eq. (32) by the free Green's function G. We must then compare

$$I_1=\langle\mathbf{q}|VG_fN_iG_iV|\mathbf{q}'\rangle=\int\frac{d^3q''}{(2\pi)^3}\frac{V(\mathbf{q}-\mathbf{q}'')(\mathbf{q}''^2/2m)V(\mathbf{q}''-\mathbf{q}')}{[(1/m)\mathbf{q}''\cdot\mathbf{k}_i][(1/m)(\mathbf{q}''+\boldsymbol{\Delta})\cdot\mathbf{k}_f]} \tag{33}$$

with

$$\bar{I}=\langle\mathbf{q}|VG_fN_fGN_iG_iV|\mathbf{q}'\rangle=\int\frac{d^3q''}{(2\pi)^3}V(\mathbf{q}-\mathbf{q}'')\frac{(\mathbf{q}''+\boldsymbol{\Delta})^2}{2m}\frac{\mathbf{q}''^2}{2m}V(\mathbf{q}''-\mathbf{q}')\Big/\{[(1/m)\mathbf{q}''\cdot\mathbf{k}_i][-(1/2m)(\mathbf{q}''^2+2\mathbf{q}''\cdot\mathbf{k}_i)]\times[(1/m)(\mathbf{q}''+\boldsymbol{\Delta})\cdot\mathbf{k}_f]\}. \tag{34}$$

Our assumption that the potential is only appreciable in the region $|\mathbf{r}|\leq a$ and changes significantly only over a distance a is equivalent to saying that, in momentum space, the important singularities of $V(\mathbf{q}^2)$ are not much more distant than $1/a$ from the real $|\mathbf{q}|$ axis. This allows us to make a simple estimate of $\bar{I}/I_1$. We find

$$\bar{I}/I_1\approx\max\left(1/ka;\frac{V(\Delta^2/4)\,V(\Delta^2/4)}{V(\Delta^2)\,V(0)}\right). \tag{35}$$

Since we have assumed that $1/ka\ll1$, the second term in Eq. (35) is the estimate of real interest.

The central assumption of Saxon and Schiff is that for $\Delta\gg1/a$, a single large-angle scattering is preferable to two intermediate ones, i.e., $V^2(\Delta^2/4)/V(\Delta^2)V(0)\ll1$. This will certainly be the case for potentials $V(\Delta^2)$, which for large values of Δ^2 go like Δ^{-2N}, $N\geq1$. For such potentials one can neglect $\bar{T}$ for all values of Δ^2. Recall that for $\Delta^2\lesssim1/a^2$, where $V^2(\Delta^2/4)/V(\Delta^2)V(0)\approx1$, both $\bar{T}$ and T_i^1 are negligible compared to T_i.

However, for potentials that fall off exponentially or faster in momentum space, $\bar{I}/I_1$ is not necessarily small, and one must take into account the multiple-scattering terms contained in $\bar{T}$. This is true, for example, for a Gaussian potential.

It is important to notice that the remainder term $\bar{T}$ is small for large values of Δ^2 only because we have expanded the amplitude symmetrically about both the initial and final moments. If, for example, we had expanded only about $\mathbf{k}_i$, so that the remainder term was proportional to N_i^2 instead of N_iN_f, then it would not have been negligible.

We can take advantage of the symmetry of our expansion to obtain a simplification of the Saxon-Schiff formula. Taking the average of Eqs. (27) and (28) and integrating by parts yields

$$T_1=\int d^3r\, e^{i\boldsymbol{\Delta}\cdot\mathbf{r}}[V(\mathbf{r})\tfrac{1}{2}(e^{i\delta_i}+e^{i\delta_f})+\tfrac{1}{2}(\Delta^2/2m)(e^{i\delta_f}-1)(e^{i\delta_i}-1)+(1/2m)\nabla(e^{i\delta_f}-1)\cdot\nabla(e^{i\delta_i}-1)]. \tag{36}$$

The same reasoning which led to Eq. (35) now shows that for $\Delta\gg1/a$, the last term in Eq. (36) can be neglected with an error of order $1/\Delta a$. As usual, for $\Delta\lesssim1/a$, all corrections to the eikonal approximation are negligible. We now have the much simplified result

$$T\cong\tfrac{1}{2}\int d^3r\, e^{i\boldsymbol{\Delta}\cdot\mathbf{r}}[V(\mathbf{r})(e^{i\delta_i}+e^{i\delta_f})+(\Delta^2/2m)(e^{i\delta_f}-1)(e^{i\delta_i}-1)]. \tag{37}$$

Although this expression is not quite as accurate as the Saxon-Schiff approximation (for the Yukawa potential, the error in the Saxon-Schiff formula is of order $1/\Delta^2a^2$ for $\Delta\gg1/a$), it should be considerably easier to evaluate.

For potentials that fall off faster than a power in momentum space, Eqs. (27), (28), and (37) are not expected to give a good approximation of the scattering amplitude, so it is necessary to take into account the multiple-scattering effects contained in $\bar{T}$. To illustrate the procedure, let us consider the double-scattering term T_2 defined in Eq. (12). From the previous physical interpretation of the expansion, it is clear that a large scattering must occur both in the initial term $(N_iG_iT_i)$ and in the final term $(T_fG_fN_f)$. We shall assume that each of these scatterings occurs with a momentum transfer of approximately $\frac{1}{2}\mathbf{\Delta}$. We therefore write

$$\mathbf{k}_n=\mathbf{k}_i-\tfrac{1}{2}\mathbf{\Delta}=m\mathbf{v}_n \tag{38}$$

and

$$G_n(\mathbf{r},\mathbf{r}')=-i\int_0^\infty dt\, e^{itk^2/m}\delta(\mathbf{r}-\mathbf{r}'-t\mathbf{v}_n). \tag{39}$$

It is a simple matter to write T_2 explicitly. One finds

$$T_2=-i\int d^3r\int_0^\infty dt\, e^{i\Delta\cdot\mathbf{r}}e^{it\Delta^2/4m}[(\nabla^2/2m)(e^{i\delta_f(\mathbf{r}+t\mathbf{v}_n)}-1)]$$
$$\times e^{i[\chi_n(\mathbf{r}+t\mathbf{v}_n)-\chi_n(\mathbf{r})]}[(\nabla^2/2m)(e^{i\delta_i(\mathbf{r})}-1)]$$
$$\cong -i\left(\frac{\Delta^2}{8m}\right)^2\int d^3r\int_0^\infty dt\, e^{i\Delta\cdot\mathbf{r}}e^{it\Delta^2/4m}(e^{i\delta_f(\mathbf{r}+t\mathbf{v}_n)}-1)$$
$$\times e^{i[\chi_n(\mathbf{r}+t\mathbf{v}_n)-\chi_n(\mathbf{r})]}(e^{i\delta_i(\mathbf{r})}-1). \tag{40}$$

Again, we have made an error of order $1/\Delta a$ in obtaining the simplified form of Eq. (40).

The interpretation of Eq. (40) is clear. The last term on the right corresponds to the incident particle moving along the initial direction, accumulating the phase δ_i and then undergoing a large scattering at the point $\mathbf{r}$. The center factor accounts for the phase acquired in going from the point $\mathbf{r}$ to the point $\mathbf{r}+\mathbf{v}_nt$. The next factor corresponds to a large scattering into the final direction at the point $\mathbf{r}+\mathbf{v}_nt$ and the propagation out of the interaction region. The parametric variable t integrates over the possible time spent traveling along $\hat{n}$ at velocity $\mathbf{v}_n$.

Near the forward direction, it is again simple to estimate the ratios of T_2 to T_i. One finds that for $\Delta\lesssim 1/a$, $T_2/T_i\ll 1$.

Following the method outlined in Sec. II, one can proceed to extract terms corresponding to any number of large-angle scatterings without double counting. How many, if any, of these terms are important for large momentum transfer depends of course on the rate of falloff of the potential.

IV. BOUND-STATE SCATTERING

We now turn to the problem of the scattering of a high-energy projectile from a compound system. Let us start by considering the elastic scattering of a particle from a two-body bound state. The masses of the particles will be denoted by m_j and their laboratory momenta by k_j, $j=1, 2, 3$. For simplicity, the particles will be assumed to be distinguishable. The projectile will be termed particle 3, with initial momentum $\mathbf{k}_i$ and final momentum $\mathbf{k}_f$. The bound state, composed of particles 1 and 2, is initially at rest and ends up with momentum $\mathbf{\Delta}=\mathbf{k}_i-\mathbf{k}_f$. We shall always assume that the potentials fall off like a power in momentum space.

If we introduce momenta and coordinate variables by

$$\begin{aligned}&\mathbf{K}\equiv\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3,\quad X\equiv(m_1\mathbf{r}_1+m_2\mathbf{r}_2)/(m_1+m_2),\\&\mathbf{q}\equiv(m_2\mathbf{k}_1-m_1\mathbf{k}_2)/(m_1+m_2),\quad \mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2,\\&\mathbf{p}\equiv\mathbf{k}_3,\quad \mathbf{R}\equiv\mathbf{r}_3-\mathbf{X},\end{aligned} \tag{41}$$

and define

$$M_{ij}\equiv m_i+m_j,\quad m_{ij}\equiv m_im_j/M_{ij}, \tag{42}$$

then the free Hamiltonian can be written in the form

$$H_0(\mathbf{p},\mathbf{q})=p^2/2m_3+q^2/2m_{12}+(\mathbf{K}-\mathbf{p})^2/2M_{12}. \tag{43}$$

The potential then becomes

$$V(\mathbf{R},\mathbf{r})=V_{12}(\mathbf{r})+V_{13}\left(\mathbf{R}-\frac{m_2}{M_{12}}\mathbf{r}\right)+V_{23}\left(\mathbf{R}+\frac{m_1}{M_{12}}\mathbf{r}\right)$$
$$\equiv V_{12}(\mathbf{r})+W(\mathbf{R},\mathbf{r}). \tag{44}$$

The variable $\mathbf{X}$ will be eliminated from the equations and ignored because its integration will always only produce the usual momentum-conservation δ function.

The wave function of the two-body bound state satisfies the equation

$$H_{12}\psi=[q^2/2m_{12}+V_{12}(\mathbf{r})]\psi(\mathbf{r})=-B\psi(\mathbf{r}). \tag{45}$$

Therefore, the total energy of the scattering system can be written

$$E=k_i^2/2m_3-B=k_f^2/2m_3+\Delta^2/2M_{12}-B. \tag{46}$$

The free three-particle Green's function is defined as

$$G_0(\mathbf{R},\mathbf{r})=\int\frac{d^3pd^3q}{(2\pi)^6}e^{i(\mathbf{p}\cdot\mathbf{R}+\mathbf{q}\cdot\mathbf{r})}[E-H_0(\mathbf{p},\mathbf{q})+i\epsilon]^{-1} \tag{47}$$

and satisfies the equation

$$[E-H_0(i^{-1}\nabla_R,i^{-1}\nabla_r)]G_0(\mathbf{R},\mathbf{r})=\delta(\mathbf{R})\delta(\mathbf{r}). \tag{48}$$

The total momentum $\mathbf{K}=\mathbf{k}_i=\mathbf{k}_f+\mathbf{\Delta}$ is, of course, to be treated as a fixed parameter. It is convenient to introduce the Green's function G_3, which includes only the potential acting between particles 1 and 2:

$$[E-H_0-V_{12}(\mathbf{r})]G_3(\mathbf{R},\mathbf{r})=\delta(\mathbf{R})\delta(\mathbf{r}). \tag{49}$$

Finally, the full three-particle Green's function g is defined by

$$[E-H_0-V(\mathbf{R},\mathbf{r})]g(\mathbf{R},\mathbf{r})=\delta(\mathbf{R})\delta(\mathbf{r}) \tag{50}$$

or, formally,

$$(1-G_3W)g=G_3. \tag{51}$$

The T matrix describing elastic scattering from the bound state is given by

$$T=W+WG_3T=W+TG_3W. \tag{52}$$

We are now in a position to apply the techniques developed in Sec. II and III. We wish to obtain an approximation for the Green's function G_3, which corresponds to straight-line propagation for the projectile. In the lowest-order approximation we shall assume that the Hamiltonian of the bound pair is negligible, compared with the energy of the projectile. We are thus led to consider the "eikonal Green's functions"

$$G_i=m_3/[(\mathbf{k}_i-\mathbf{p})\cdot\mathbf{k}_i+i\epsilon] \tag{53}$$

and

$$G_f=m_3/\{(\mathbf{k}_f-\mathbf{p})[\mathbf{k}_f-(m_3/M_{12})\mathbf{\Delta}]+i\epsilon\}. \tag{54}$$

The perturbations then become

$$N_i=G_i^{-1}-G_3^{-1}=(\mathbf{p}-\mathbf{k}_i)^2/2m_3+(\mathbf{p}-\mathbf{k}_i)^2/2M_{12}+H_{12}+B \tag{55}$$

and

$$N_f=G_f^{-1}-G_3^{-1}=(\mathbf{p}-\mathbf{k}_f)^2/2m_3+(\mathbf{p}-\mathbf{k}_f)^2/2M_{12}+H_{12}+B. \tag{56}$$

To clarify the interpretation of these choices, it is convenient to examine them in coordinate space. We have

$$G_i(\mathbf{R}-\mathbf{R}',\mathbf{r}-\mathbf{r}')=-i\int_0^\infty dt\,\delta(\mathbf{R}-\mathbf{R}'-\mathbf{v}_it)\delta(\mathbf{r}-\mathbf{r}')e^{it\mathbf{k}_i\cdot\mathbf{v}_i} \tag{57}$$

and

$$G_f(\mathbf{R}-\mathbf{R}',\mathbf{r}-\mathbf{r}')=-i\int_0^\infty dt\,\delta(\mathbf{R}-\mathbf{R}'-\mathbf{v}_ft)\delta(\mathbf{r}-\mathbf{r}')e^{it\mathbf{k}_f\cdot\mathbf{v}_f}, \tag{58}$$

where

$$\mathbf{v}_i=\mathbf{k}_i/m_3$$

and

$$\mathbf{v}_f=\mathbf{k}_f/m_3-\mathbf{\Delta}/M_{12}$$

are the initial and final velocities of the projectile relative to the bound system. The δ function in $\mathbf{r}-\mathbf{r}'$ means that in this approximation one is assuming that the projectile goes by so rapidly that the constituents of the bound state do not have a chance to move during the scattering process.

The full Green's function corresponding to G_i satisfies the differential equation

$$m_3^{-1}(k_i^2+i\mathbf{k}_i\cdot\mathbf{\Delta}_R)g_i(\mathbf{R}-\mathbf{R}',\mathbf{r}-\mathbf{r}')=W(\mathbf{R},r)g_i(\mathbf{R}-\mathbf{R}',\mathbf{r}-\mathbf{r}')+\delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{r}-\mathbf{r}'). \tag{59}$$

The solution of this equation is easily seen to be

$$g_i(\mathbf{R}-\mathbf{R}',\mathbf{r}-\mathbf{r}')=G_i(\mathbf{R}-\mathbf{R}',\mathbf{r}-\mathbf{r}')e^{i[\chi_i(\mathbf{R},r)-\chi_i(\mathbf{R}',r')]}, \tag{60}$$

where

$$\chi_i(\mathbf{R},\mathbf{r})=-\int_0^\infty dt\,W(\mathbf{R}-\mathbf{v}_it,\mathbf{r})=-\int_0^\infty dt[V_{13}(\mathbf{r}_3-\mathbf{r}_1-\mathbf{v}_it)+V_{23}(\mathbf{r}_3-\mathbf{r}_2-\mathbf{v}_it)]. \tag{61}$$

The eikonal T matrix is then easily computed to be

$$\langle\mathbf{k}_f,\Delta|T_i|\mathbf{k}_i,\mathbf{0}\rangle=\int d^3Rd^3r\,e^{i\Delta\cdot\mathbf{R}}|\psi(\mathbf{r})|^2W(\mathbf{R},\mathbf{r})e^{i\delta_i(\mathbf{R},\mathbf{r})}, \tag{62}$$

where $\delta_i=\chi_i$. This is of course the well-known Glauber formula.[3] The result for T_f is also easily obtained:

$$\langle\mathbf{k}_f,\mathbf{\Delta}|T_f|\mathbf{k}_i,\mathbf{0}\rangle=\int d^3Rd^3r\,e^{i\Delta\cdot\mathbf{R}}|\psi(\mathbf{r})|^2W(\mathbf{R},\mathbf{r})e^{i\delta_f(\mathbf{R},\mathbf{r})}, \tag{63}$$

where

$$\delta_f(\mathbf{R},\mathbf{r})=\chi_{-f}(R,r)=-\int_0^\infty dt\,W(\mathbf{R}+\mathbf{v}_ft,\mathbf{r})=\int_0^\infty dt[V_{13}(\mathbf{r}_3-\mathbf{r}_1+\mathbf{v}_ft)+V_{23}(\mathbf{r}_3-\mathbf{r}_2+\mathbf{v}_ft)]. \tag{64}$$

Error estimates can be made by using exactly the same procedures that were employed in Sec. III for the two-body problem. One finds that for $\Delta\lesssim 1/a$ the expressions given in Eqs. (62) and (63) are both good approximations to the exact scattering amplitude with errors of the order of $1/k_ia$. The real problem is to find an approximation for T which is accurate for large momentum transfers.

Let us start by considering the first correction term involving N_i. Proceeding in the usual way, we find

$$T_i^1=T_fG_fN_iG_iT_i=\int d^3Rd^3r\,e^{i\Delta\cdot\mathbf{R}}\psi(\mathbf{r})\times(e^{i\delta_f(\mathbf{R},\mathbf{r})}-1)[(-\nabla_R^2/2M_3)+H_{12}+B]\times\psi(\mathbf{r})(e^{i\delta_i(\mathbf{R},\mathbf{r})}-1), \tag{65}$$

where

$$M_3=m_3(m_1+m_2)/(m_1+m_2+m_3).$$

Making use of the equation for $\psi(\mathbf{r})$ [Eq. (45)] and carrying out an integration by parts with respect to $\mathbf{r}$ yields the more convenient form

$$T_i^1=\int d^3Rd^3r\,e^{i\Delta\cdot\mathbf{R}}|\psi(r)|^2\times[(e^{i\delta_f}-1)(-\nabla_R^2/2M_3)(e^{i\delta_i}-1)+(1/2m_{12})(\nabla_re^{i\delta_f})\cdot(\nabla_re^{i\delta_i})]. \tag{66}$$

It is again useful to symmetrize T_1 so that the analog of the Saxon-Schiff term for the three-particle case becomes

$$\tfrac{1}{2}(T_i^1+T_f^1)=\int d^3Rd^3r\,e^{i\Delta\cdot\mathbf{R}}|\psi(\mathbf{r})|^2\times[(\Delta^2/4M_3)(e^{i\delta_f}-1)(e^{i\delta_i}-1)+(1/2M_3)\nabla_Re^{i\delta_f}\cdot\nabla_Re^{i\delta_i}+(1/2m_{12})\nabla_re^{i\delta_f}\cdot\nabla_re^{i\delta_i}]. \tag{67}$$

If we use the techniques developed in Sec. III, it is easily seen that the second two terms in the square bracket in Eq. (67) can be neglected with an error of order $1/\Delta a$. Again we recall that unless $\Delta a \gg 1$, the entire Saxon-Schiff correction term becomes negligible.

Unfortunately, the correction term given in Eq. (62) does not fully explain large-angle elastic scattering. Our interpretation of T_i^1 is that the projectile enters the interaction region with momentum $\mathbf{k}_i$ and accumulates a phase δ_i. It undergoes a single large scattering off one of the constituents of the bound state and then builds up a phase δ_f while leaving the interaction region. However, since momentum is conserved in all intermediate states, after the large scattering the constituents of the bound state will have a relative momentum of order of magnitude Δ. Thus the probability that they will remain bound is small. In fact, if we make an order-of-magnitude estimate of T_i^1 similar to the one made in Sec. III, we find that T_i^1 is proportional to $V(\mathbf{\Delta}^2)\Psi(\mathbf{\Delta}^2)$, where $\Psi(q^2)$ is the Fourier transform of the bound-state wave function $\psi(\mathbf{r})$. Let us now consider the double-scattering term T_2 defined in Eq. (12). It contains new types of terms in which the projectile undergoes a large scattering off each of the constituents of the bound state, leaving them with a small relative momentum. The probability that particles 1 and 2 then remain bound is therefore large. The price that one pays of course, is, an extra power of $V(\Delta^2)$. In fact, a careful analysis shows that

$$T_2/T_i^1 \approx V(\Delta^2)/\Psi(\Delta^2). \tag{68}$$

In many cases of physical interest, the wave function falls off more rapidly than the potential. In this situation both the eikonal and the Saxon-Schiff correction terms are negligible compared with the double-scattering term at large momentum transfer. In any event, T_2 can certainly not be neglected.

In order to obtain an explicit expression for T_2, we must introduce eikonal Green's functions which accurately describe propagation when the relative momentum between particles 1 and 2 is large. For this purpose it is convenient to start by approximating the free Green's function G rather than G_3. Let us assume that in the intermediate state, $\mathbf{p}$ and $\mathbf{q}$ are close to the values $\mathbf{p}_n$ and $\mathbf{q}_n$. With expansion about these values and retention only of first-order terms, the eikonal Green's function becomes

$$G_n = [E - E_n + (\mathbf{p}_n - \mathbf{p})\cdot\mathbf{v}_n + (\mathbf{q}_n - \mathbf{q})\cdot\mathbf{u}_n + i\epsilon]^{-1}, \tag{69}$$

where

$$\begin{aligned} \mathbf{v}_n &= \mathbf{p}_n/m_3 - (\mathbf{k}_i - \mathbf{p}_n)/M_{12}, \\ \mathbf{u}_n &= \mathbf{q}_n/m_{12}, \end{aligned} \tag{70}$$

and

$$E_n = \frac{p_n^2}{2m_3} + \frac{(\mathbf{p}_n - \mathbf{k}_i)^2}{2M_{12}} + \frac{q_n^2}{2m_{12}}.$$

$\mathbf{v}_n$ is the velocity of the projectile relative to the center of mass of particles 1 and 2, and $\mathbf{u}_n$ is the relative velocity between these latter particles.

In coordinate space, G_n can be written in the form

$$G_n(\mathbf{R}-\mathbf{R}', \mathbf{r}-\mathbf{r}') = -i\int_0^\infty dt\, \delta(\mathbf{R}-\mathbf{R}'-\mathbf{v}_n t)\delta(\mathbf{r}-\mathbf{r}'-\mathbf{u}_n t) \times e^{it[E-E_n+\mathbf{p}_n\cdot\mathbf{v}_n+\mathbf{q}_n\cdot\mathbf{u}_n]}. \tag{71}$$

The main difference between G_n and G_i or G_f is the fact that the large values of $\mathbf{q}_n$ are taken into account by the δ function in $\mathbf{r}-\mathbf{r}'$. The spatial separation due to this relative velocity shows up in the integration over the parametric time t.

The rescattering corrections to G_n are easily included. The equation satisfied by the interacting eikonal Green's function is

$$G_n^{-1}g_n(\mathbf{R}-\mathbf{R}', \mathbf{r}-\mathbf{r}') = V(\mathbf{R},\mathbf{r})g_n(\mathbf{R}-\mathbf{R}', \mathbf{r}-\mathbf{r}') + \delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{r}-\mathbf{r}'). \tag{72}$$

The solution to this equation is

$$g_n(\mathbf{R}-\mathbf{R}', \mathbf{r}-\mathbf{r}') = G_n(\mathbf{R}-\mathbf{R}', \mathbf{r}-\mathbf{r}')e^{i[\varphi(\mathbf{R},\mathbf{r})-\varphi(\mathbf{R}',\mathbf{r}')]}, \tag{73}$$

where

$$\begin{aligned} \varphi(\mathbf{R},\mathbf{r}) &= -\int_0^\infty dt\, V(\mathbf{R}-\mathbf{v}_n t, \mathbf{r}-\mathbf{u}_n t) \\ &= -\int_0^\infty dt [V_{12}(\mathbf{r}_1-\mathbf{r}_2-(\mathbf{v}_1-\mathbf{v}_2)t) \\ &\quad + V_{13}(\mathbf{r}_3-\mathbf{r}_1-(\mathbf{v}_3-\mathbf{v}_1)t) \\ &\quad + V_{23}(\mathbf{r}_3-\mathbf{r}_2-(\mathbf{v}_3-\mathbf{v}_2)t)] \end{aligned} \tag{74}$$

and

$$\begin{aligned} \mathbf{v}_1-\mathbf{v}_2 &= \mathbf{u}_n, \\ \mathbf{v}_3-\mathbf{v}_2 &= \mathbf{v}_n - (m_1/M_{12})\mathbf{u}_n, \\ \mathbf{v}_3-\mathbf{v}_1 &= \mathbf{v}_n + (m_2/M_{12})\mathbf{u}_n. \end{aligned} \tag{75}$$

The major contribution of the double-scattering term T_2 comes from events in which the projectile makes a large scattering from both of the constituents of the bound state and leaves them with a small relative momentum. This means that the momentum transfer for the scattering off particle 1 must be $(m_1/M_{12})\mathbf{\Delta}$, while for the scattering off particle 2, it must be $(m_2/M_{12})\mathbf{\Delta}$. If particle 1 is struck first, we make use of the eikonal Green's function G_{n1} defined by Eq. (69) with the choice

$$\mathbf{p}_{n1} = \mathbf{k}_i - (m_1/M_{12})\mathbf{\Delta}, \quad \mathbf{q}_{n1} = (m_{12}/M_{12})\mathbf{\Delta}. \tag{76}$$

On the other hand, if particle 2 is struck first, the appropriate eikonal Green's function is G_{n2}, where

$$\mathbf{p}_{n2} = \mathbf{k}_i - (m_2/M_{12})\mathbf{\Delta}, \quad \mathbf{q}_{n2} = -(m_{12}/M_{12})\mathbf{\Delta}. \tag{77}$$

We can now write the double-scattering term in the form

$$T_2=T_fG_fN_fg_nN_iG_iT_i$$
$$\cong\langle\psi_f^-|V_{23}G_fN_fg_{n1}N_iG_iV_{13}|\psi_i^+\rangle$$
$$+\langle\psi_f^-|V_{13}G_fN_fg_{n2}N_iG_iV_{23}|\psi_i^+\rangle. \quad (78)$$

We have dropped the small terms which correspond to two large scatterings from the same potential. These terms are down by a factor of $\Psi(\Delta^2)$ from the terms included in Eq. (78). If we now write

$$T\cong\tfrac{1}{2}(T_i+T_f+T_i^1+T_f^1)+T_2, \quad (79)$$

then, by using the techniques of Sec. III, it is not too difficult to show that the error is of order $1/k_ia$ for all values of the momentum transfer.

The expression for T_2 given in Eq. (78) is rather cumbersome; however, it can be greatly simplified for large momentum transfers if one is willing to admit errors of order $1/\Delta a$. We again recall that all of the corrections to the Glauber approximation are small unless $\Delta a\gg1$. The first step in simplifying our expression for T_2 is to make use of the fact that all of the large scatterings occur at the potentials which have been explicitly displayed in Eq. (78). Therefore, the largest terms arise when all of the derivatives in N_i and N_f act on these potentials. In momentum space, we see that up to terms of order $1/\Delta a$,

$$T_2\cong(\Delta^2/2\mathfrak{M}_1)(\Delta^2/2\mathfrak{M}_2)(\langle\psi_f^-|V_{23}G_fg_{n1}G_iV_{13}|\psi_i^+\rangle$$
$$+\langle\psi_f^-|V_{13}G_fg_{n2}G_iV_{23}|\psi_i^+\rangle)$$
$$\cong(\Delta^2/2\mathfrak{M}_1)(\Delta^2/2\mathfrak{M}_2)(\langle\psi_f^-|V_{23}G_fg_{n1}G_iW|\psi_i^+\rangle$$
$$+\langle\psi_f^-|V_{23}G_fg_{n2}G_iW|\psi_i^+\rangle), \quad (80)$$

where

$$\frac{\Delta^2}{2\mathfrak{M}_1}=\frac{\Delta^2}{2M_3}\left(\frac{m_1}{M_{12}}\right)^2+\frac{\Delta^2}{2m_{12}}$$

and (81)

$$\frac{\Delta^2}{2\mathfrak{M}_2}=\frac{\Delta^2}{2M_3}\left(\frac{m_2}{M_{12}}\right)^2+\frac{\Delta^2}{2m_{12}}.$$

In the second form of Eq. (80), we have added terms corresponding to two large scatterings off the same particle. These terms are negligible, but their presence enables the integral over the parametric time in G_i to be carried out by making use of Eq. (23). However, having used this trick for the potentials adjacent to G_i, we cannot use it again for those adjacent to G_f, since we would then be introducing terms corresponding to large scatterings off two different particles. These terms are not small.

To get around this difficulty, we must make an additional approximation. The term $(\mathbf{q}_n-\mathbf{q})\cdot\mathbf{u}_n$ in G_n [see Eq. (69)] is dropped, which introduces a further error of order $1/\Delta a$. This new approximation takes into account only the fact that, even though the constituents of the bound state acquire a large relative velocity after the first major collision, they are not able to move very far in the time ($t_0\approx a/v_n$) that it takes the projectile to pass by.

If $m_1=m_2$, then in the present approximation $G_{n1}=G_{n2}$, so we can add the two terms contributing to T_2 and perform the integrals over the parametric times associated with G_f and G_i. If the masses of the bound particles are not equal, then, to the order to which we are presently working, we can write

$$G_{n1}\cong[E-E_n+(\mathbf{p}_{n1}-\mathbf{p})\cdot\mathbf{v}_n'+i\epsilon]^{-1}, \quad (82)$$
$$G_{n2}\cong[E-E_n+(\mathbf{p}_{n2}-\mathbf{p})\cdot\mathbf{v}_n'+i\epsilon]^{-1},$$

where

$$\mathbf{v}_n'=\mathbf{k}_i/m_3-\tfrac{1}{2}\mathbf{\Delta}/M_3. \quad (83)$$

$\mathbf{v}_n'$ is the value $\mathbf{v}_{ni}$ would take if $m_1=m_2$.

The use of Eq. (82) in our expression for T_2 [Eq. (80)] allows the integrals over the parametric times associated with G_f and G_i to be done for any value of the masses. To lowest order in $1/\Delta a$ we find

$$T_2\cong\frac{\Delta^2}{2\mathfrak{M}_1}\frac{\Delta^2}{2\mathfrak{M}_2}\int d^3rd^3R|\psi(\mathbf{r})|^2e^{i\Delta\cdot\mathbf{R}}$$
$$\times\int_0^\infty dt\, e^{i(E-E_n)t}(e^{i\delta_f(\mathbf{R},\mathbf{r})}-1)$$
$$\times e^{i[\delta_n(\mathbf{R},\mathbf{r})-\delta_n(\mathbf{R}-\mathbf{v}_n't,\mathbf{r})]}(e^{i\delta_i(\mathbf{R}-\mathbf{v}_n't,\mathbf{r})}), \quad (84)$$

where

$$\delta_n(R,r)=-\int_0^\infty dt\, W(\mathbf{R}-\mathbf{v}_n't,\mathbf{r}). \quad (85)$$

This final result is considerably simpler than the slightly more accurate expression given in Eq. (75). It is more difficult to calculate than T_1 only in that it involves one extra integral over the time between the two major collisions.

V. BREAKUP REACTIONS

It was difficult to obtain an accurate expression for elastic scattering at large momentum transfers because it is very unlikely that particles 1 and 2 actually remain bound. Therefore, it was necessary to take into account several small effects. On the other hand, the breakup reaction, which dominates the scattering at large momentum transfer, is much simpler to approximate.

We define the final state by ket$|\mathbf{k}_f,\mathbf{\Delta},\mathbf{q}_f\rangle$, where $\mathbf{k}_f$ is the final momentum of the projectile, and $\mathbf{\Delta}$ and $\mathbf{q}_f$ are the total and relative momenta of the 1-2 pair. In the eikonal approximation the breakup amplitude can be written

$$\langle\mathbf{k}_f\mathbf{\Delta}\mathbf{q}_f|T|\mathbf{k}_i,\mathbf{0}\rangle\cong\tfrac{1}{2}(T_f+T_i)$$
$$=\int d^3Rd^3r\, e^{i\Delta\cdot\mathbf{R}}\psi_{q_f}^-(\mathbf{r})^*\psi(r)W(\mathbf{R},\mathbf{r})$$
$$\times\tfrac{1}{2}(e^{i\delta_i(\mathbf{R},\mathbf{r})}+e^{i\delta_f(\mathbf{R},\mathbf{r})}). \quad (86)$$

Again, $\psi(\mathbf{r})$ is the wave function of the bound state and $\psi_{q_f}^-(\mathbf{r})$ is the exact wave function for particles 1 and

2 with relative momentum $\mathbf{q}_f$ and incoming-wave boundary conditions. δ_i and δ_f are defined in Eqs. (61) and (64). For $\Delta \lesssim 1/a$, Eq. (86) is accurate to order $1/ka$.

For large momentum transfers, the most likely occurrence is that the projectile undergoes a single large scattering off one of the constituents of the bound state. Thus the struck particle has a momentum of the order of Δ, while the other constituent of the bound state is left with a small momentum.

In analogy with Eq. (69), the free Green's function G is approximated by expanding $\mathbf{p}$ and $\mathbf{q}$ about the values $\mathbf{k}_f$ and $\mathbf{q}_f$. Keeping only the first-order terms, one obtains

$$G_\Delta = [(\mathbf{k}_f - \mathbf{p}) \cdot \mathbf{v}_f + (\mathbf{q}_f - \mathbf{q}) \cdot u_f + i\epsilon]^{-1}, \quad (87)$$

where

$$\mathbf{v}_f = \mathbf{k}_f/m_3 - \mathbf{\Delta}/M_{12}, \quad \mathbf{u}_f = \mathbf{q}_f/m_{12}. \quad (88)$$

The full Green's function g_Δ can then be read off from Eqs. (73) and (74).

We can now write

$$T \cong \tfrac{1}{2}(T_i + T_f) + \tfrac{1}{2}(T_i^1 + T_f^1), \quad (89)$$

where

$$T_i^1 = \langle \mathbf{k}_f \mathbf{\Delta} \mathbf{q}_f | V g_\Delta N_i G_i T_i | \mathbf{k}_i, \mathbf{0} \rangle, \quad T_f^1 = \langle \mathbf{k}_f \mathbf{\Delta} \mathbf{q}_f | V g_\Delta N_\Delta G_i T_i | \mathbf{k}_i, \mathbf{0} \rangle, \quad (90)$$

and

$$N_\Delta = \frac{(\mathbf{p}-\mathbf{k}_f)^2}{2m_3} + \frac{(\mathbf{p}-\mathbf{k}_f)^2}{2M_{12}} + \frac{(\mathbf{q}-\mathbf{q}_f)^2}{2m_{12}}. \quad (91)$$

In obtaining Eq. (89), we have utilized the fact that for large values of $\mathbf{q}_f$,

$$\psi_{q_f}^-(r)^* \cong e^{-i\mathbf{q}_f \cdot \mathbf{r} + i\delta_{12}(\mathbf{r})},$$

where

$$\delta_{12}(\mathbf{r}) = -\int_0^\infty dt\, V_{12}(\mathbf{r} + t\mathbf{u}_f).$$

Approximating N_i and N_Δ in the usual way then gives

$$T_i^1 + T_f^1 \cong \left(\frac{\Delta^2}{2M_3} + \frac{q_f^2}{2m_{12}}\right) \int d^3R d^3r\, e^{-i\mathbf{q}_f \cdot \mathbf{r}} e^{i\Delta \cdot \mathbf{R}} \psi(\mathbf{r}) \times (e^{i\varphi_\Delta(\mathbf{R},\mathbf{r})} - 1)(e^{i\delta_i(\mathbf{R},\mathbf{r})} - 1), \quad (92)$$

where

$$\varphi_\Delta(R,r) = -\int_0^\infty dt\, V(\mathbf{R} + \mathbf{v}_f t, \mathbf{r} + \mathbf{u}_f t). \quad (93)$$

The substitution of Eq. (92) into Eq. (89) then yields an expression for the breakup amplitude accurate to first order in $1/\Delta a$ provided $q_f \approx \Delta$. If $q_f \ll \Delta$, then one must include T_2 as in the case of elastic scattering. Of course, the probability that particles 1 and 2 end up with small relative momentum is itself very small for large values of Δ.

VI. CONCLUSION AND DISCUSSION

In many of the applications of the eikonal approximation to elementary-particle reactions, one attempts to explain structure in the scattering amplitude at rather large momentum transfers in terms of diffractive behavior in the forward direction. Unfortunately, this is precisely the region where one expects the eikonal approximation to break down. This is especially true if one is scattering off a bound state or a state which can be considered as a composite of other "elementary" particles, in which case terms involving more than one large scattering should dominate for large Δ.

To aid the reader, the main results of this paper, namely, improved and simplified eikonal formulas, will be repeated here. For two-particle scattering, the simplified Saxon-Schiff formula becomes

$$T = \tfrac{1}{2} \int d^3r\, e^{i\Delta \cdot \mathbf{r}} \left[V(\mathbf{r})(e^{i\delta_i} + e^{i\delta_f}) + \frac{\Delta^2}{2m}(e^{i\delta_f} - 1)(e^{i\delta_i} - 1) \right]. \quad (94)$$

The δ's are defined in Sec. III, and a detailed discussion of the errors is given there. If we use units in which the eikonal term is of order 1, then the simplified Saxon-Schiff correction [the second term in Eq. (94)] is of order $\Delta^2 a/k$. In obtaining Eq. (94), we have neglected terms of order $(1+\Delta a)/ka$, so that the error is largest for $\Delta^2 \approx k/a$, where it is of order $1/(ka)^{1/2}$.

For the elastic scattering of a high-energy projectile from a two-particle bound state, we find

$$\begin{aligned} \langle \mathbf{k}_f \mathbf{\Delta} | T | \mathbf{k}_i \mathbf{0} \rangle &\cong \int d^3r d^3R\, e^{i\Delta \cdot \mathbf{R}} |\psi(\mathbf{r})|^2 \Big[\tfrac{1}{2} W(\mathbf{R},\mathbf{r})(e^{i\delta_i(\mathbf{R},\mathbf{r})} + e^{i\delta_f(\mathbf{R},\mathbf{r})}) \\ &+ \tfrac{1}{2}(\Delta^2/2M_3)(e^{i\delta_f(\mathbf{R},\mathbf{r})} - 1)(e^{i\delta_i(\mathbf{R},\mathbf{r})} - 1) \\ &+ \frac{\Delta^2}{2\mathfrak{M}_1} \frac{\Delta^2}{2\mathfrak{M}_2} \int_0^\infty dt\, e^{i(E-E_n)t} (e^{i\delta_f(\mathbf{R},\mathbf{r})} - 1) \\ &\times e^{i[\delta_n(\mathbf{R},\mathbf{r}) - \delta_n(\mathbf{R} - \mathbf{v}_n' t, \mathbf{r})]} (e^{i\delta_i(\mathbf{R} - \mathbf{v}_n' t, \mathbf{r})} - 1) \Big]. \end{aligned} \quad (95)$$

The various quantities appearing in Eq. (95) are defined in Sec. IV. Again, if we use units in which the eikonal term is of order 1, then the second term in Eq. (95) is of order $\Delta^2 a/k$, and the third term, which corresponds to a large scattering off each constituent of the bound state is of order

$$\left(\frac{\Delta^2 a}{k}\right)^2 \frac{V^2(\tfrac{1}{4}\Delta^2)}{V(\Delta^2)\Psi(\Delta^2)}.$$

Once again, the error is of order $(1+\Delta a)/ka$.

Our expression for the breakup amplitude is

$$\langle \mathbf{k}_f \Delta \mathbf{q}_f | T | \mathbf{k}_i 0 \rangle \cong \tfrac{1}{2} \int d^3r d^3R\, e^{i\Delta\cdot\mathbf{R}} \psi_{q_f}^{-}(\mathbf{r})^* \psi(\mathbf{r})$$

$$\times W(\mathbf{R},\mathbf{r}) (e^{i\delta_i(\mathbf{R},\mathbf{r})} + e^{i\delta_f(\mathbf{R},\mathbf{r})})$$

$$+ \left(\frac{\Delta^2}{2M_3} + \frac{q_f^2}{2m_{12}}\right) \int d^3r d^3R\, e^{-i\mathbf{q}_f\cdot\mathbf{r}} e^{i\Delta\cdot\mathbf{R}} \psi(\mathbf{r})$$

$$\times (e^{i\varphi_\Delta(\mathbf{R},\mathbf{r})} - 1)(e^{i\delta_i(\mathbf{R},\mathbf{r})} - 1). \quad (96)$$

The various terms in Eq. (96) are defined in Sec. V. Again, the error is of order $(1+\Delta a)/ka$. In obtaining Eqs. (94)—(96), we have assumed that the potentials fall off no faster than a power in momentum space.[6]

The techniques which we have developed can be applied equally well to the scattering of a high-energy projectile from a multiparticle bound state. The amplitude for knocking a single particle out of the bound system is particularly simple; in fact, it is given by an expression which is virtually identical to Eq. (96). However, the amplitude for elastic scattering does become more difficult as the number of bound particles increases. Since the bound-state wave function ordinrily falls off much more rapidly in momentum space than the potential, if the number of bound particles is not too large, then for large values of Δ it will be most profitable for the projectile to scatter off each constituent of the bound state with a fairly large momentum transfer. As the number of bound particles increases, it eventually becomes more profitable for the projectile to undergo a single large scattering and to pay the price in the falloff of the wave function. Thus, our expression for the elastic scattering amplitude will simplify again when the number of bound particles becomes large.

As a final point of clarification and interpretation of our formulation of elastic bound-state scattering, it is interesting to recall that the term that dominates the large-momentum-transfer region corresponds to a graph in which each of the constituents scatters from the projectile through roughly one-half the momentum transfer. This type of diagram is exactly the nonrelativistic analog of the end piece of the Mandelstam diagram that produces cuts in the angular momentum plane.[7] Thus our theory predicts in a clear physical way the manner in which Regge cuts should dominate the large-momentum-transfer behavior of bound-state–bound-state elastic scattering. It should be stressed that this type of cut is quite different in its physical interpretation from the cutlike structure found in the simple eikonal pictures used in Ref. 2. In strong-interaction theory, where every particle must be considered to be at least partially a bound state, our model would lead one to expect that Regge cuts will dominate the large-momentum-transfer behavior because it is advantageous to let each particle in the bound state scatter and divide the momentum transfer rather than to allow one particle to scatter through the full momentum transfer and then be forced to absorb it in the bound-state form factor. There are indeed cuts in the Glauber double-scattering term, but in the large Δ region, where they might be expected to be important, the amplitude is actually dominated by T_2.

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[6] The physically interesting case of Gaussian potentials is more complicated and will be treated in detail by one of us (R. L. S.) in a forthcoming paper.

[7] S. Mandelstam, Nuovo Cimento 30, 1127 (1963); 30, 1148 (1963).