Generalizations of the Veneziano and Virasoro Models*

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(Received 28 February 1969)

It is suggested that Virasoro's fixed poles in the j plane of the Veneziano amplitude should be regarded as Gribov-Pomeranchuk poles. The third Veneziano term can probably have other effects associated with a third double-spectral function, such as cuts in the angular-momentum plane. The alternative to the Veneziano formula proposed by Virasoro may therefore not conflict with unitarity. A generalized formula is proposed which contains the Veneziano and Virasoro amplitudes as special cases. The new formula has a double-integral representation which is similar to the beta-function representation for the Veneziano formula. We propose another generalization of the Veneziano formula, in which the signature degeneracy can be broken by an arbitrary amount.

1. INTRODUCTION

An alternative to the Veneziano[1] formula for an amplitude with linear trajectories has been proposed by Virasoro[2] and, independently, by Rubinstein.[3] The formula has many features in common with that of Veneziano, and the leading term alone has no redundant trajectories. One essential difference between the two formulas is that the Regge residues in the Virasoro model have poles when the trajectories pass through the nonsense wrong-signature integers. The question then arises whether these poles are in conflict with unitarity.

It has been pointed out by Virasoro[4] that the Veneziano model has poles in the angular-momentum plane at the negative wrong-signature integers. Such poles arise from the third Veneziano term; in other words, the poles in the j plane corresponding to the t channel will come from the su Veneziano term. The two models differ in that the fixed poles in the j plane do not combine multiplicatively with the moving poles in the model of Veneziano, whereas they do in that of Virasoro. In either case, it is necessary to examine a possible conflict with unitarity.

The first obvious question is whether the poles in the Veneziano model should be regarded as analogous to the Gribov-Pomeranchuk poles or to the poles in potential scattering which begin to move as soon as the coupling is turned on. The difference between the two types of pole is that the residue at the first is independent of t, whereas the residue at the second is an analytic function of t, with a left-hand cut which originates from the third double-spectral function. If the residue is independent of t, the pole occurs in the inhomogeneous term of the N/D equations but not in the kernel, while the Gribov-Pomeranchuk pole with its t-dependent residue occurs in the kernel as well. On solving the equations in the elastic-unitarity approximation, one then finds that the potential-theory pole moves, whereas the Gribov-Pomeranchuk pole becomes an essential singularity.

The residue at the pole in the Veneziano model has been found by Virasoro to involve a factor 2^{-t}. A pole with a t-dependent residue would be expected to have consequences similar to those of the Gribov-Pomeranchuk poles, not to those of the potential-theory poles. The essential singularity at $t=-\infty$ replaces the left-hand cut, and the third Veneziano term will be similar to the third double-spectral function in its effects.

If Gribov-Pomeranchuk poles can occur in elastic amplitudes, one would expect Regge cuts to occur in amplitudes with multiparticle intermediate states. The Veneziano formula has been generalized to production processes by Bardakci and Ruegg[5] and, independently, by Virasoro.[6] The amplitudes have terms corresponding to uncrossed and crossed loops, and it is natural to suppose that terms of the latter type would combine to produce Regge cuts in the elastic amplitude.

Considerations such as these will have an effect on the relation between the importance of the cuts and the widths of the resonances. One may make the argument that the discontinuities across the Regge cuts will not be large in a system with narrow resonances, since multiparticle intermediate states, as opposed to quasi-two-particle intermediate states, will not be important except at high energies. If the Veneziano terms had no effects of the type usually attributed to the double-spectral functions, the three-particle intermediate states would not produce Regge cuts in the lowest order in which they appeared. If the third Veneziano term is analogous to a third double-spectral function, the three-particle intermediate states will always produce cuts. One may possibly hope that their effects are not too large, since they result from relatively distant singularities in the s-t plane.

Once one allows the presence of Regge cuts, one avoids the contradiction between the Gribov-Pomeranchuk poles and the unitarity condition. One then has no obvious reason for excluding poles in the

* Research supported by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Grant No. AF-AFOSR-68-1471.

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968); also, M. Suzuki (private communication).
[2] M. A. Virasoro, Phys. Rev. **177**, 2309 (1969).
[3] See Ref. 7.
[4] M. A. Virasoro (private communication).
[5] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968).
[6] M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969).

Regge residues when the trajectories pass through negative wrong-signature integers, so that the Virasoro amplitude should be considered as a possible alternative to that of Veneziano. Having two distinct models, similar in their general form, for an amplitude with linear Regge trajectories, we are inclined to wonder whether they may be particular cases of a more general model. It is the purpose of the first part of the paper to find such a model. We shall show that the Virasoro amplitude has a representation which is similar to the beta-function representation for the Veneziano amplitude, but which involves a double integral instead of a single integral. The representation admits of an obvious generalization, and our final formula depends on three parameters: ν_1, ν_2, and ν_3. For a particular value of these parameters, the double integral becomes a single integral, and our formula then reduces to the Veneziano formula.

The Virasoro model has the characteristic that the nth highest trajectory has a definite signature $(-1)^n$, even if all three channels are different. By adding nonleading terms to the Veneziano formula or to the generalized formula, we can obtain an amplitude with this property, but neither formula automatically possesses it. The Virasoro formula is thus the special case of the generalized formula in which the leading term has trajectories with signature $(-1)^n$ only, whereas the Veneziano formula is the special case where the Regge residues have no negative wrong-signature poles. If we require a spectral function to be absent in one of the channels, or if we wish to avoid resonances with exotic quantum numbers, we cannot allow poles in the Regge residues, and we are forced to adopt the Veneziano formula.

In Sec. 2 we write the double-integral representation for the Virasoro formula and generalize it. We examine the properties of the generalized amplitude, and we show that it has poles in the physical region which correspond to the expected particles without ancestors. In Sec. 3 we show that the amplitude does have a Regge asymptotic behavior.

Our aim in the second part of the paper is to examine a generalization of a completely different type. We have pointed out that the Veneziano amplitude, as well as our generalized amplitude, possesses "signature degeneracy" if all three channels are different. The leading trajectory has a definite signature, but all other trajectories contain components with both signatures. We can get rid of the trajectories with signature $-(-1)^n$ by adding nonleading terms or by fixing the ν's at the Virasoro value. It may still appear as though the breaking of the signature degeneracy is a rather special case; the generalization of the Veneziano formula which we propose in Sec. 4 will show that it is by no means a special case. We obtain an amplitude in which the trajectories of signature $(-1)^n$ are lowered by an amount ϵ from their positions as given by the Veneziano formula, while the trajectories of signature $-(-1)^n$ are raised by an amount ϵ. The quantity ϵ is arbitrary and, if it is zero, our formula reduces to the Veneziano formula. It is thus possible to break the signature degeneracy continuously. If, therefore, the dynamics suggest that signature degeneracy is the exceptional rather than the normal case, the Veneziano formula provides no evidence to the contrary.

We shall be able to give a similar generalization of our double-integral formula, provided the three parameters ν are equal. For unequal values of the ν's, we have not found an amplitude in which the signature degeneracy is broken, and we shall give reasons for believing that it may not be possible to do so.

2. GENERALIZED FORMULA FOR AMPLITUDE WITH LINEAR REGGE TRAJECTORIES

We show in Appendix A that the amplitude proposed by Virasoro has the following integral representation:

$$\frac{\Gamma(-\frac{1}{2}S)\Gamma(-\frac{1}{2}T)\Gamma(-\frac{1}{2}U)}{\Gamma(-\frac{1}{2}S-\frac{1}{2}T)\Gamma(-\frac{1}{2}S-\frac{1}{2}U)\Gamma(-\frac{1}{2}T-\frac{1}{2}U)} =\pi^{-1/2}2^{-D-1}[\Gamma(-\tfrac{1}{2}-\tfrac{1}{2}D)]^{-1}\int dxdy \times\left(\frac{(1-x)(1-y)(x+y-1)}{x^2y^2(2-x-y)^2}\right)^{-\frac{1}{2}D-\frac{3}{2}} \times x^{-S-2}y^{-T-2}(2-x-y)^{-U-2}, \quad (2.1)$$

where

$$S=\alpha_s(s)\,, \quad (2.2a)$$

$$T=\alpha_t(t)\,, \quad (2.2b)$$

$$U=\alpha_u(u)\,, \quad (2.2c)$$

$$D=S+T+U\,. \quad (2.2d)$$

The range of integration is the triangle

$$x<1\,, \quad y<1\,, \quad x+y>1\,. \quad (2.3)$$

Equation (2.1) obviously suggests that we should examine the following possible formula for an amplitude with linear trajectories:

$$A(s,t)=\int dxdy \times\left[\frac{1-x}{y(2-x-y)}\right]^{\nu_1}\left[\frac{1-y}{x(2-x-y)}\right]^{\nu_2}\left[\frac{x+y-1}{xy}\right]^{\nu_3} \times x^{-S-2}y^{-T-2}(2-x-y)^{-U-2}, \quad (2.4)$$

where the range of integration is again given by (2.3). We shall show that (2.4) has all the desirable properties of the original amplitudes proposed by Veneziano and Virasoro. In the present section we examine the general properties of the amplitude and the residues at the single-particle poles. We examine the asymptotic behavior in Sec. 3.

We begin by remarking that the crucial feature of the formula is the range of integration (2.3). In fact, we can write a more general formula as follows:

$$A'(s,t)=\int dxdy\, f(x,y)x^{-S'-2}y^{-T'-2}(2-x-y)^{-U'-2},$$

where f is any function which can be expanded in a powers series in x and y within the triangle, but which may have power branch points along the sides. If the branch-point factors $(1-x)^{\nu_1}$, $(1-y)^{\nu_2}$, and $(x+y-1)^{\nu_3}$ are separated from f, and the remaining function is expanded in a Taylor series, the successive terms will give the amplitude (2.4), together with nonleading terms of a similar form. The factors of x, y, and $2-x-y$ in the denominators of the first three factors of (2.4) could have been absorbed in the factors x^{-S-2}, y^{-T-2}, and $(2-x-y)^{-U-2}$. We have written them explicitly in order that the functions S, T, and U should be equal to the α's of the leading trajectories in the three channels.

We also remark that the three integration variables x, y, and $2-x-y$ occur symmetrically in (2.4); they are associated with the variables s, t, and u, respectively. One of the variables x, y, and $2-x-y$ is equal to zero at each vertex of the triangle, but they are all greater than zero within the triangle or along the sides. The three variables $1-x$, $1-y$, and $x+y-1$ also occur symmetrically in the formula. One of these variables is equal to zero along each of the sides, but they are all greater than zero within the triangle.

Now let us examine the poles of the amplitude. We must show that there is a pole at $T=0$ whose residue is constant, a pole at $T=1$ whose residue is a linear function of S, and so on. There should be similar poles in the s and u channels; apart from them, the amplitude should be analytic.

We shall begin by assuming that ν_1, ν_2, and ν_3 are all greater than -1. As long as S, T, and U are all negative, the integral is uniformly convergent, so that A is an analytic function of S and T. If the variable T now approaches zero (from the negative side), the factor y^{-T-2} will cause the integral to diverge at $y=0$. When y becomes small and x remains within the range of integration (2.3), the variables x and $x+y$ will both approach 1. The three factors $(1-x)/[y(2-x-y)]$, $(1-y)/[x(2-x-y)]$, and $(x+y-1)/xy$ in (2.4) will all remain finite. In fact, the x integral of the product of all factors of (2.4), except the factor y^{-T-2}, will be an analytic function of y at $y=0$, or

$$\int_{1-y}^{1} dx\left[\frac{1-x}{y(2-x-y)}\right]^{\nu_1}\left[\frac{1-y}{x(2-x-y)}\right]^{\nu_2}\left[\frac{x+y-1}{xy}\right]^{\nu_3}\times x^{-S-2}y^{-T-2}(2-x-y)^{-U-2}=ya(y,S,U)y^{-T-2}. \quad (2.5)$$

The factor y on the right-hand side of (2.5) comes from the length of the range of the x integral. We observe further that the function a is independent of S and U at $y=0$, since the factors x^{-S-2} and $(2-x-y)^{-U-2}$ in (2.4) and (2.5) then become unity. Hence, if we integrate (2.5) with respect to y, the integral will contain a pole in T and $T=0$, and the residue at this pole will be independent of S.

When the variable T is increased above zero, Eq. (2.4) must be defined by analytic continuation. If $0<T<1$, we can evaluate the y integration of (2.5) by parts:

$$\int dy\, y^{-T-1}a(y,S,U)=\frac{1}{T}\int dy\, y^{-S}\frac{\partial}{\partial y}a(y,S,U). \quad (2.6)$$

Our definition by analytic continuation enables us to discard the surface terms in (2.6). The derivative $(\partial/\partial y)a(y,S,U)$ will contain explicit factors linear in S or U, which arise from differentiating the last two factors on the left-hand side of (2.5) with respect to y. Thus, as T approaches 1, we can conclude as before that there will be a pole in T, but the residue will now be a linear function of S or U. By repeated integration by parts, we can show that there will be a pole at $T=n$, and that the residue will be a polynomial of the nth degree in S or U. There will be similar poles in the s or u channels, but no other singularities in the st plane, except at infinity.

The Veneziano formula is the special case of (2.4) where one or more of the ν's is equal to -1. The integral (2.4) actually diverges in this case, but if we multiply the integral by $\nu+1$ and then take the limit, we obtain a finite contribution from a side of the triangle which is precisely the same as the Veneziano formula. Thus, if ν_3 approaches -1, for instance, the factor $[(x+y-1)/xy]$ in (2.4) causes the integral to diverge along the side of the triangle $x+y-1=0$. On taking the limit, we easily see that

$$\lim_{\nu_3\to-1}(\nu_3+1)\left[\frac{1-x}{y(2-x-y)}\right]^{\nu_1}\left[\frac{1-y}{x(2-x-y)}\right]^{\nu_2}\left[\frac{x+y-1}{xy}\right]^{\nu_3}\times x^{-S-2}y^{-T-2}(2-x-y)^{-U-2}=\int^1 dx\, x^{-S-1}(1-x)^{-T-1}. \quad (2.7)$$

The right-hand side of (2.5) is simply the Veneziano formula with an st term alone. By summing such formulas, or by letting ν_1, ν_2, and ν_3 tend to -1 with suitably adjusted values of the ratio of ν_1+1, ν_2+1, and ν_3+1, we can obtain a Veneziano formula with arbitrary ratios among the three terms.

We have thus demonstrated that the Veneziano and Virasoro formulas are both particular cases of (2.4).

An obvious question one may ask is whether the arbitrariness in the choice of ν_1, ν_2, and ν_3 in (2.4) represents an additional degree of freedom besides those associated with the addition of nonleading terms.[7]

[7] G. Altarelli and H. R. Rubinstein, Phys. Rev. **178**, 2165 (1969).

We prefer to regard this choice, not as an additional degree of freedom, but as a method of performing divergent summations over the nonleading terms. If we are given two expressions of the form (2.4), with different values of ν_1, ν_2, and ν_3, one expression may be obtained from the other by power-series expansion of the integrand; higher terms in the expansion represent nonleading terms in A. As long as both sets of values of ν_1, ν_2, and ν_3 are greater than -1, and as long as we are working in the region for which S, T, and U are less than -1, the sum will converge. When we continue into the regions where S, T, and U have general values, the sum will no longer necessarily converge, so that Eq. (2.4) may be regarded as a means of analytically continuing the sum into such regions. The limiting formula with one of the ν's equal to -1 cannot be treated in this way, but Eq. (2.4) may then be regarded as a means of passing to the limit.

It might be thought that one could generalize (2.4) still further by allowing the function f of (2.5) to have singularities along lines other than $1-x=0$, $1-y=0$, and $x+y-1=0$. In the cases we have investigated, it turned out either that the function could be expanded in a convergent series of nonleading terms of the form (2.4) or that the amplitude $A(s,t)$ increased exponentially when one of the variables approached infinity. Such a possible exponential increase can be investigated either by using the method of Sec. 3 or by finding the residues at the poles. The number of necessary integrations by parts increases with T and, if the expansion of $f(x,y)$ as a power series in y does not converge sufficiently fast, the final result will increase exponentially with the number of integrations by parts.

3. ASYMPTOTIC BEHAVIOR OF SCATTERING AMPLITUDE

We now show that the amplitude defined by (2.4) does have Regge asymptotic behavior. We shall investigate the amplitude when T is fixed and S or U becomes infinite.

It is first necessary to make a change of variables in order that only one factor in the integrand should depend exponentially on S or U. We therefore define

$$\xi=1-y\,, \tag{3.1a}$$

$$\eta=(2-x-y)/x\,. \tag{3.1b}$$

The integral (2.4) then becomes

$$\begin{aligned} A(s,t)=&\int_0^1 d\xi\int_\xi^{\xi^{-1}} d\eta(\eta-\xi)^{\nu_1}\xi^{\nu_2}(1-\eta\xi)^{\nu_3} \\ &\times(1+\xi)^{T-D-\nu_1-2\nu_2-\nu_3-3}(1-\xi)^{-T-\nu_1-\nu_3-2} \\ &\times(1+\eta)^{D-T+2\nu_2+2}\eta^{S+T-D-\nu_1-\nu_2-2} \\ \equiv&\int_0^1 d\xi\int_\xi^{\xi^{-1}} d\eta\,\mathfrak{F}(\xi,\eta)\,. \end{aligned} \tag{3.2}$$

It will be convenient to divide (3.2) into two terms, each involving one limit of the η integration. This can be done by a suitable deformation of the contour of integration in the η plane. The integrand has four branch points, at $\eta=-1$, 0, ξ, and ξ^{-1}. We shall draw all cuts extending along the real axis to the right of their branch points. Equation (3.2) can then be rewritten

$$\begin{aligned} A(s,t)=\int_0^1 d\xi\Bigg[&\frac{-e^{-i\pi(S+\nu_2)}}{2i\sin\pi(S+\nu_2)}\int_C d\eta \\ &-e^{-i\pi\nu_1}\frac{\sin\pi(S-\nu_1+\nu_2)}{\sin\pi(S+\nu_2)}\int_{0+i\epsilon}^{\xi+i\epsilon} d\eta \\ &-e^{i\pi\nu_3}\frac{\sin\pi(S+\nu_2+\nu_3)}{\sin\pi(S+\nu_2)}\int_{\xi^{-1}+i\epsilon}^{\infty+i\epsilon} d\eta\Bigg]\mathfrak{F}(\xi,\eta)\,. \end{aligned} \tag{3.3}$$

The contour C goes from ∞ to 0 below the real axis, and then from 0 to ∞ above the real axis. The $+i\epsilon$'s on the limits of the other two integrations indicate that the contour is to be taken above the real axis.

We can now deform the contour C to go from 0 to -1 above the real axis and then back to 0 below the real axis. If we do so, we may rewrite the factor η^S as $|\eta|^S e^{i\pi S}$, and we notice that it decreases exponentially as S approaches infinity in any complex direction within the upper half-plane. The integral over the contour C in (3.3) will therefore decrease exponentially in this limit. The factors in front of the other two integrals will approach unity like $1-e^{-\mathrm{Im}S}$, so that we may write

$$A(s,t)\approx-\int_0^1 d\xi\left[\int_{0+i\epsilon}^{\xi+i\epsilon} d\eta+\int_{\xi^{-1}+i\epsilon}^{\infty+i\epsilon} d\eta\right]\mathfrak{F}(\xi,\eta)\,. \tag{3.4}$$

We shall first examine the second term of (3.4), which is the more difficult. We shall make another change of variable in order that the limits of the η integration should not depend on ξ. A convenient choice is

$$\xi=\xi'(1-\eta')+\eta'\,, \tag{3.5a}$$

$$\eta=\eta'^{-1}\,. \tag{3.5b}$$

The second term of (3.4) then becomes

$$\begin{aligned} A^{(2)}(s,t)\approx&-e^{-i\pi\nu_3}\int_0^1 d\xi'\int_{0-i\epsilon}^{1-i\epsilon} d\eta' \\ &\times(1+\eta'-\eta'\xi')^{\nu_1}[\eta'+\xi'(1-\eta')]^{\nu_2}\xi'^{\nu_3} \\ &\times[1+\eta'+\xi'(1-\eta')]^{T-D-\nu_1-2\nu_2-\nu_3-3} \\ &\times(1-\xi')^{-T-2-\nu_1-\nu_3}(1+\eta')^{-T+D+2\nu_2+2} \\ &\times(1-\eta')^{-T-1}\eta'^{-S-\nu_2-\nu_3-2}\,. \end{aligned} \tag{3.6}$$

We next interchange the limits of integration, and we imagine that the ξ' integration has been performed. The result will have branch points when η' is -1, 0, or 1. If we are interested in the limit $S\to+\infty$, we should like to change the path of the η' integration to run between the limits -1 and ∞, so that the factor $\eta'^{-S-\nu_2-\nu_3-2}$ will decrease with S. It is not difficult to show that such a change is permissible, i.e., that

$$A^{(2)}(s,t)=e^{-i\pi\nu_3}\int_{1-i\epsilon}^{\infty-i\epsilon}d\eta'\int_0^1 d\xi' \times(1+\eta'-\eta'\xi')^{\nu_1}[\eta'+\xi'(1-\eta')]^{\nu_2}\xi'^{\nu_3} \times[1+\eta'+\xi'(1-\eta')]^{T-D-\nu_1-2\nu_2-\nu_3-3} \times(1-\xi')^{-T-2-\nu_1-\nu_3}(1+\eta')^{-T+D+2\nu_2+2} \times(1-\eta')^{-T-1}\eta'^{-S-\nu_2-\nu_3-2}, \quad (3.7)$$

apart from terms with factors $\sin\pi S$ in the denominator, which we may neglect as before.

The last two factors of (3.7) give us the familiar beta-function integral, while the remaining factors may all be expanded in powers of $1-\eta'$. We then change the order of integration once more, after which we can easily perform the integrals. The leading term will behave like $(-S)^T$, while terms with higher powers of $1-\eta'$ in the integrand will behave like $(-S)^{T-\eta}$. It follows that the term in question does have Regge asymptotic behavior as S approaches infinity.

It is important that we carry out the steps in the order outlined: first to imagine the ξ' integration performed, then to show that the path of the η' integration can be deformed, and finally to expand the integrand in powers of $1-\eta'$. If we were to expand the integrand and then deform the path of integration term by term, we could miss exponentially increasing terms. Once we have shown that the path of the η integration can be deformed if the steps are carried out in the correct order, we may expand the integrand and perform both integrations term by term.

The first term of (3.4) may be handled in a similar way, except that no deformation of the contour of integration is necessary. We obtain integrals of the form $\int_0^1 d\eta'(1-\eta')^{T-1}\eta'^{S+T-D-\nu_1-\nu_2-2}$, together with nonleading terms. Again the amplitude has a Regge asymptotic behavior as S becomes infinite.

We shall evaluate explicitly the leading term in the Regge asymptotic expansion, and we shall be able to observe the nonsense wrong-signature poles directly. The leading term in $A^{(2)}$ is obtained by putting $\eta=1$ in all factors of (3.7) except the last two. Thus

$$A^{(2)}(s,t) \approx -e^{-i\pi\nu_3}2^{-\nu_1-\nu_3-1}\int_0^1 d\xi'\xi'^{\nu_3}(1-\xi')^{-T-2-\nu_1-\nu_3}(2-\xi')^{\nu_1} \times\int_{1-i\epsilon}^{\infty-i\epsilon}d\eta'(1-\eta')^{-T-1}\eta'^{-S-\nu_2-\nu_3-2} + \text{nonleading terms}$$
$$=-2^{-\nu_3-1}e^{-i\pi(T+\nu_3)}\Gamma(-T)\frac{\Gamma(\nu_3+1)\Gamma(-T-\nu_1-\nu_3-1)}{\Gamma(-T-\nu_1)} \times F(-\nu_1,\nu_3+1,-T-\nu_1,\tfrac{1}{2}) + \text{nonleading terms.} \quad (3.8)$$

On evaluating the leading term of $A^{(1)}$ in a similar way, we find that

$$A^{(1)}(s,t) \approx -2^{-\nu_1-1}e^{i\pi\nu_1}\Gamma(-T)\frac{\Gamma(\nu_1+1)\Gamma(-T-\nu_1-\nu_3-1)}{\Gamma(-T-\nu_3)} \times F(-\nu_3,\nu_1+1,-T-\nu_3,\tfrac{1}{2}) + \text{nonleading terms.} \quad (3.9)$$

The expressions (3.8) and (3.9) do not correspond to the right- and left-hand cuts in the s plane, because of the phase factors $e^{-i\pi\nu_3}$ and $e^{i\pi\nu_1}$. We can easily recombine them into the contributions from the two cuts by making use of the identity

$$f_1(t)e^{i\pi\nu_1}+f_2(t)e^{-i\pi(\nu_3+t)}=g_1(t)+g_2(t)e^{-i\pi t}, \quad (3.10a)$$

where

$$g_1(t)=(\sin\pi t)^{-1} \times[f_1(t)\sin\pi(\nu_1+t)-f_2(t)\sin\pi\nu_3], \quad (3.10b)$$

$$g_2(t)=(\sin\pi t)^{-1} \times[f_2(t)\sin\pi(\nu_3+t)-f_1(t)\sin\pi\nu_1]. \quad (3.10c)$$

On substituting (3.8) and (3.9) in (3.10), and making use of linear transformations of the hypergeometric function, we find that

$$A^{(1)}(s,t)+A^{(2)}(s,t)\approx g_1+g_2e^{-i\pi T}, \quad (3.11a)$$

where

$$g_1=\frac{\pi}{2\sin\pi T}\frac{2^{-\nu_1}\Gamma(\nu_1+1)}{\Gamma(T+\nu_1+2)} \times F(-\nu_3,\nu_1+1,T+\nu_1+2,\tfrac{1}{2}), \quad (3.11b)$$

$$g_2=\frac{\pi}{2\sin\pi T}\frac{2^{-\nu_3}\Gamma(\nu_3+1)}{\Gamma(T+\nu_3+2)} \times F(-\nu_1,\nu_3+1,T+\nu_3+2,\tfrac{1}{2}). \quad (3.11c)$$

The expressions for g_1 and g_2 contain factors $\sin\pi T$, which give the expected poles at the positive integers as well as poles at the negative integers. By making use of further identities satisfied by the hypergeometric functions, we can show that the poles of g_1+g_2 at the negative even integers, or the poles of g_1-g_2 at the negative odd integers, cancel. We therefore remain with poles at the negative wrong-signature integers. If $\nu_1=\nu_3$, the hypergeometric functions reduce to Legendre functions of zero, which can be evaluated explicitly. When ν_1 or ν_3 approach -1, the functions $\Gamma(\nu_1+1)g_1$ and $\Gamma(\nu_3+1)g_2$ remain finite. There will no longer be poles at the negative integers, since they will be canceled by the poles of the Γ functions in the denominators of (3.11). We thus confirm that the nonsense wrong-signature poles disappear when our formula reduces to that of Veneziano.

4. BREAKING OF SIGNATURE DEGENERACY

In an amplitude where the s and u channels are not identical, and where the intercepts of the linear functions α_s and α_u are therefore different, we would expect trajectories of both signatures in the t channel. In the Veneziano formula, or in the generalized VV formula where the three ν's do not have the Virasoro value, the trajectories of positive and negative signature coincide. We distinguish such "signature degeneracy" from the stronger condition of "exchange degeneracy," which requires, in addition, that the residues associated with the two trajectories be the same in some channel. In this section our object is to investigate whether signature degeneracy is an essential feature of the model, or whether it can be broken by a suitable generalization. We shall find that the latter is the case for a wide class of amplitudes.

It is certainly possible to find *particular* amplitudes which do not have signature degeneracy. One such amplitude is the following[8]:

$$A(s,t)=\int_0^1 dx\, x^{-S-1}(1-x)^{-T-1}[1-x(1-x)]^{\delta}, \quad (4.1a)$$

where

$$\delta=D+1. \quad (4.1b)$$

By suitably modifying the reasoning of Ref. 8, one can show that the nth trajectory in the amplitude (4.1a) has a definite signature of $(-1)^n$, and that there is therefore no signature trajectory. We emphasize that the definition (4.1b) is not the same as the definition actually used in Ref. 8, namely,

$$\delta=\tfrac{1}{2}(4a\mu^2+3b+1), \quad (4.2a)$$

unless $\alpha_s=\alpha_u$. Equation (4.1b) is the condition for the trajectories to have definite signature $(-1)^n$; Eq. (4.2a) is the condition for alternate trajectories to disappear. Only if $\alpha_s=\alpha_u$ are these two features equivalent. If the intercepts b_s and b_t associated with the s and t channels are not the same, Eq. (4.2a) for the t channel becomes

$$\delta=\tfrac{1}{2}(4a\mu^2+b_t+2b_s+1). \quad (4.2b)$$

Another special case of an amplitude without signature degeneracy is the Virasoro amplitude, where the nth trajectory again has signature $(-1)^n$. In the general amplitude (2.4), with the three ν's equal to one another but not to the Virasoro value, it is again possible to eliminate the trajectories of signature $-(-1)^n$ by adding nonleading terms, as we shall show below.

Our aim now is to examine an *arbitrary* amplitude of the Veneziano or the generalized VV form, where the ν's and the coefficients of the nonleading terms are unrestricted. We wish to show that it is possible to make a slight change of the amplitude so as to break the signature degeneracy. The sequence of trajectories with signature $(-1)^n$ (where n denotes the distance of the trajectory below the leading trajectory) is to be moved by a small distance relative to the sequence with signature $-(-1)^n$. In other words, the poles in T at the even integers are to be moved slightly, relative to the poles at the odd integers.

We begin by examining the Veneziano formula and by attempting to break the signature degeneracy in the t channel alone. It will be convenient to write the formula as

$$A(s,t)=\int_0^1 dy\, f_s(y)y^{-T-1}(1-y)^{-S-1} +\int_0^1 dy\, f_u(y)y^{-T-1}(1-y)^{-U-1}. \quad (4.3)$$

Equation (4.3) could of course be generalized by adding a third term or by allowing f_s and f_u to be polynomials in S and T, but such generalizations are not relevant to our present problem. Our first step is to find a condition on the function f which is equivalent to the condition that there be only trajectories of signature $(-1)^n$ in the t channel. We can carry out the analysis either by examining the asymptotic behavior or by examining the t-channel poles. We shall use the latter method, since it will be the easier when applied to the generalized VV formula.

It is thus necessary to examine a function

$$G(T)=\int_0^1 dx\, g(T,y)y^{-T-1} \quad (4.4)$$

and to find the condition for the poles at odd integral values of T to disappear. The residue of the pole at $T=N$ is given by the formula

$$r_N=-\frac{1}{T(T-1)\cdots(T-N+1)}\left[\frac{\partial^N}{\partial y^N}g(N,y)\right]_{y=0}. \quad (4.5)$$

We therefore require the Nth derivative of $g(T,y)$ to vanish at $y=0$, $T=N$, whenever N is an odd integer. A condition for this to happen is given by the following theorem: *In order for the Nth derivative of $g(T,y)$ to vanish at $T=0$, $y=N$, whenever N is a positive odd integer, it is sufficient for g to have the property*

$$y^{-T-1}g(T,y)\to(-y)^{-T-1}(1-y)^2 g(T,y) \quad \text{when } y\to -y/(1-y). \quad (4.6)$$

Equation (4.6) is not a necessary condition for our requirement if $g(T,y)$ is a general function, but it is probably necessary when applied to the type of functions with which we are dealing. In any case, all we require for our subsequent reasoning is that (4.6) be a sufficient condition.

Let us now apply (4.6) to (4.3). In order to remove one of the redundant variables S or U, we rewrite (4.3)

[8] S. Mandelstam, Phys. Rev. Letters 21, 1724 (1968).

in the form

$$A(S,T)=\int_0^1 dy\, f_s(y)y^{-T-1}(1-y)^{\frac{1}{2}(T-D)-1}(1-y)^{-\frac{1}{2}(S-U)}$$
$$+\int_0^1 dx\, f_u(y)y^{-T-1}(1-y)^{\frac{1}{2}(T-D)-1}(1-y)^{\frac{1}{2}(S-U)}. \quad (4.7)$$

Under the transformation $y\to y/(1-y)$, the expression $1-y$ becomes $(1-y)^{-1}$, so that the last factors of the two terms of (4.7) transform into one another. We therefore require that

$$y^{-T-1}f_s(y)(1-y)^{\frac{1}{2}(T-D)-1}\to(-y)^{-T-1}(1-y)^2 \times f_u(y)(1-y)^{\frac{1}{2}(T-D)-1}, \quad (4.8)$$

or

$$f_s(y)\to f_u(y)(1-y)^{-D-1},\quad y\to -y/(1-y). \quad (4.9)$$

In our present limited problem, where we are attempting to break signature degeneracy in the t channel alone, we shall be able to obtain a solution of (4.9) with $f_s=f_u$. The required property is then

$$f_s(y)\to f_s(y)(1-y)^{-D-1},\quad y\to -y/(1-y). \quad (4.10)$$

One might attempt to solve the problem by breaking the Veneziano formula into two parts, each of which had trajectories of one signature sequence only. One could then change the position of the trajectories in one of the sequences and recombine the formulas. If one did so, one would find that new trajectories had been introduced into the s or u channels. It is our aim to break the signature degeneracy in the t channel without introducing new trajectories in the other two channels, and the solution obtained by this method is therefore not satisfactory.

We must therefore choose the function f_s in such a way that the amplitude $A(s,t)$ has trajectories of both signature sequences, but with the first sequence at $\alpha=T-\epsilon-n$, the second at $\alpha=T+\epsilon-n$. At the same time, the positions of the trajectories in the s and u channels must remain unchanged. This can be achieved if we require f_s to have the following properties:

(i) $$f_s(y)=y^{\epsilon}h_1(y)+y^{-\epsilon+1}h_2(y), \quad (4.11a)$$

where h_1 and h_2 are analytic functions of y at $y=0$.

(ii) The two terms of (4.11a) must separately satisfy (4.10), i.e.,

$$y^{\epsilon}h_1(y)\to(-y)^{\epsilon}h_1(y)(1-y)^{-D-1},$$
$$y^{-\epsilon+1}h_2(y)\to(-y)^{-\epsilon+1}h_2(y)(1-y)^{-D-1},$$
$$y\to -y/(1-y). \quad (4.11b)$$

(iii) The function $f_s(y)$ must be an analytic function of y at $y=1$.

(iv) $$f_s(y)=1\quad \text{if}\quad \epsilon=0. \quad (4.11c)$$

Property (i) implies that the amplitude A has two sets of trajectories, one at $\alpha=T-\epsilon+n$ $(n=0,1,\cdots)$ and the other at $\alpha=T+\epsilon-n$ $(n=1,2,\cdots)$. Property (ii) ensures that the first set of trajectories has particles only when $T-\epsilon$ is an even integer, the second only when $T+\epsilon-1$ is an even integer, i.e., when $T+\epsilon$ is an odd integer. The phase factors in (4.11b) correspond to the phase factors $(-1)^{-T-1}$ in (4.6) and (4.8), since the effective value of T is shifted by $-\epsilon$ and $\epsilon-1$, respectively. Property (iii) is required in order that the positions of the trajectories in the s and u channels be unaltered by the addition of the factor f_s; the positions of these trajectories are governed by the analytic properties of the integrand at $y=1$. Finally, if Eq. (4.11c) is satisfied, our new amplitude will coincide with the original amplitude at $\epsilon=0$. We can thus start with the ordinary Veneziano amplitude and gradually break the signature degeneracy.

In order to obtain an idea of the type of function which will satisfy our requirements, we observe that the two terms of (4.11a) will not separately be analytic at $y=1$. If they were, we would be able to divide the amplitude into two parts, each with only one sequence of trajectories in the t channel, and we would not introduce new trajectories in the s and u channels by doing so. We have already remarked that this is not possible. Our function will therefore be the sum of two terms, each proportional to a different nonintegral power of y at $y=0$. At $y=1$, the sum will be analytic, but the individual terms will not. The simplest functions with such properties are the hypergeometric functions. The particular class of hypergeometric functions which satisfy (4.11b), in addition to (4.11a) and (4.11c), turns out to be the associated Legendre functions.

The following function, in fact, satisfies all the properties (4.11):

$$f_s(y)=\Gamma(-D)(1-y)^{\frac{1}{2}(D+1)}P_{-\epsilon}{}^{D+1}[(2-y)/y]. \quad (4.12)$$

At $y=1$, the argument $(2-y)/y$ is equal to 1, and the function $P_{-\epsilon}{}^{D+1}$ has simple analytic properties; it behaves like $(1-y)^{-\frac{1}{2}(D+1)}$. Thus property (iii) is satisfied. At $y=0$, the argument $(2-y)/y$ is infinite, and we can express $P_{-\epsilon}{}^{D+1}$ as the sum of two Q functions, each of which has simple analytic properties:

$$P_{-\epsilon}{}^{D+1}\left(\frac{2-y}{y}\right)=-\frac{e^{-i\pi D}\sin\pi(D+\epsilon-1)}{\pi\cos\pi\epsilon}Q_{\epsilon}{}^{D+1}\left(\frac{2-y}{y}\right)$$
$$-\frac{e^{-i\pi D}\sin\pi(D-\epsilon+1)}{\pi\cos\pi\epsilon}Q_{-\epsilon}{}^{D+1}\left(\frac{2-y}{y}\right). \quad (4.13)$$

The property (4.11a) is an immediate consequence of the behavior of the Q functions when their argument is infinite, while the property (4.11b) follows from the relation

$$Q_{\epsilon-1}{}^{D+1}(-z)=(-1)^{\epsilon}Q_{\epsilon-1}{}^{D+1}(z) \quad (4.14)$$

and the fact that the argument $(2-y)/y$ changes sign under the transformation $y\to -y/(1-y)$. When $\epsilon=0$, Eq. (4.12) does reduce to the right-hand side of (4.11c), so that all our requirements are met.

We can now extend our problem and attempt to break the signature degeneracy in all three channels simultaneously. In place of (4.3) we require a Veneziano formula with all three terms:

$$A(s,t)=\int_0^1 dy\, f_{st}(y)y^{-T-1}(1-y)^{-S-1} + \int_0^1 dy\, f_{tu}(y)y^{-T-1}(1-y)^{-U-1} + \int_0^1 dy\, f_{su}(y)y^{-U-1}(1-y)^{-S-1}. \quad (4.15)$$

The condition (4.9) must be supplemented by two further conditions, so that the requirements are

$$f_{st}(y)\to f_{tu}(y)(1-y)^{-D-1}, \quad (4.16a)$$

$$f_{st}(1-y)\to f_{su}(1-y)(1-y)^{-D-1}, \quad (4.16b)$$

$$f_{tu}(1-y)\to f_{su}(y)(1-y)^{-D-1},\ \ y\to -y/(1-y). \quad (4.16c)$$

If we wish to break the signature degeneracy by amounts ϵ_1, ϵ_2, and ϵ_3 in the t, s, and u channels, respectively, conditions (4.11) become generalized as follows:

$$\begin{aligned}
\text{(i)}\quad f_{st}(y)&=y^{\epsilon_1}h_1(y)+y^{-\epsilon_1+1}h_2(y), & y&=0\\
&=(1-y)^{\epsilon_2}h_3(1-y)+(1-y)^{-\epsilon_2+1}h_4(1-y), & y&=1\\
f_{tu}(y)&=y^{\epsilon_1}h_5(y)+y^{-\epsilon_1+1}h_6(y), & y&=0\\
&=(1-y)^{\epsilon_3}h_7(1-y)+(1-y)^{-\epsilon_3+1}h_8(1-y), & y&=1\\
f_{su}(y)&=y^{\epsilon_3}h_9(y)+y^{-\epsilon_3+1}h_{10}(y), & y&=0\\
&=(1-y)^{\epsilon_2}h_{11}(1-y)+(1-y)^{-\epsilon_2+1}h_{12}(y), & y&=1
\end{aligned} \quad (4.17a)$$

where the h's are analytic when their arguments are equal to zero.

$$\begin{aligned}
\text{(ii)}\quad y^{\epsilon_1}h_1(y)&\to(-y)^{\epsilon_1}h_5(y)(1-y)^{-D-1},\\
y^{-\epsilon_1+1}h_2(y)&\to(-y)^{-\epsilon_1+1}h_6(y)(1-y)^{-D-1},\\
y^{\epsilon_2}h_3(y)&\to(-y)^{\epsilon_2}h_{11}(y)(1-y)^{-D-1},\\
y^{-\epsilon_2+1}h_4(y)&\to(-y)^{-\epsilon_2+1}h_{12}(y)(1-y)^{-D-1},\\
y^{\epsilon_3}h_7(y)&\to(-y)^{\epsilon_3}h_9(y)(1-y)^{-D-1},\\
y^{-\epsilon_3+1}h_8(y)&\to(-y)^{-\epsilon_3+1}h_{10}(y)(1-y)^{-D-1},
\end{aligned} \quad (4.17b)$$

when $y\to -y/(1-y)$.

$$\text{(iii)}\quad f_{st}(y)=f_{tu}(y)=f_{su}(y)=1,\quad \epsilon=0. \quad (4.17c)$$

It is possible to find functions satisfying (4.17). They are given in Appendix B, as special cases of the functions which will be used to solve the problem for the generalized VV representation. The simplest functions we have been able to find are double integrals over products of Legendre functions.

We now turn to the problem of breaking the signature degeneracy in the generalized VV representation. We start from Eq. (2.5), and, if we eliminate one of the redundant variables S or U as before, the equation reads

$$A(s,t)=\int dydx\, f(x,y)x^{\frac{1}{2}(T-D)-2}x^{-\frac{1}{2}(S-U)}y^{-T-2} \times(2-x-y)^{\frac{1}{2}(T-D)-2}(2-x-y)^{\frac{1}{2}(S-U)}. \quad (4.18)$$

The condition for trajectories of only one signature sequence to exist is that the x integral in (4.18) should satisfy (4.6). We wish to replace this condition by a condition on the integrand in (4.18). One might try demanding that the integrand be unchanged under the transformation $y\to -y/(1-y)$. It turns out that this is not a convenient transformation. The range of the x integration given by (2.3) becomes changed, and the factors $x+y-1$ and $2-x-y$ do not transform in a simple way. We can overcome both of these difficulties by considering the behavior of the integrand of (4.18) under the transformation

$$y\to -y/(1-y),\quad x\to x/(1-y). \quad (4.19a)$$

The linear combinations of x and y of interest transform as follows:

$$\begin{aligned}
2-x-y&\to(2-x-y)/(1-y),\\
1-y&\to 1/(1-y),\\
1-x&\to -(x+y-1)/(1-y),\\
x+y-1&\to -(1-x)/(1-y).
\end{aligned} \quad (4.19b)$$

The range of integration $1-y<x<1$ becomes $1<x<1-y$, and the product $x^{-\frac{1}{2}(S-U)}(2-x-y)^{\frac{1}{2}(S-U)}$ remains unchanged. We can therefore replace (4.6) by the following condition on the integrand of (4.18):

$$dxf(x,y)x^{\frac{1}{2}(T-D)-2}y^{-T-2}(2-x-y)^{\frac{1}{2}(T-D)-2} \to(1-y)^2dxf(x,y)x^{\frac{1}{2}(T-D)-2} \times(-y)^{-T-2}(2-x-y)^{\frac{1}{2}(T-D)-2},$$
$$y\to -y/(1-x),\quad x\to x/(1-y),$$

or

$$f(x,y)\to(1-y)^{-D-3}f(x,y),\quad y\to -y/(1-y),\quad x\to x/(1-y). \quad (4.20)$$

We shall now insert the factors

$$\{(1-x)/[y(2-x-y)]\}^{\nu_1},$$
$$\{(1-y)/[x(2-x-y)]\}^{\nu_2},$$

and

$$[(x+y-1)/xy]^{\nu_3}$$

explicitly into the function f, but we shall restrict ourselves to the case where $\nu_1=\nu_2=\nu_3$. In other words, we modify (2.4) to read

$$A(s,t)=\int dxdy\, g(x,y)\left[\frac{(1-x)(1-y)(x+y-1)}{x^2y^2(2-x-y)^2}\right]^\nu \times x^{-S-2}y^{-T-2}(2-x-y)^{-U-2}. \quad (4.21)$$

The relation between f and g is simply

$$f(x,y)=g(x,y) \times\{(1-x)(1-y)(x+y-1)/[x^2y^2(2-x-y)^2]\}^\nu.$$

Equation (4.20) is equivalent to the condition

$$g(x,y)\to(1-y)^{-2\nu-D-3}g(x,y)\,,\quad y\to -y/(1-y)\,,\quad x\to x/(1-y)\,. \tag{4.22}$$

The first obvious consequence of (4.23) is that the transformation property is satisfied with $g=1$, provided

$$\nu=-\tfrac{1}{2}(D+3)\,. \tag{4.23}$$

Equation (4.23) is the same for all three channels. The Virasoro value of ν therefore does correspond to the condition that there exist trajectories of only one signature sequence in all three channels.

If (4.23) is not satisfied, we can add nonleading terms in such a way as to eliminate the trajectories of signature $-(-1)^n$. We simply choose the function g as follows:

$$g(x,y)=[1-x(1-x)-y(1-y)-(2-x-y)(x+y-1)]^{\frac{1}{2}(2\nu+D+3)}\,. \tag{4.24}$$

The condition (4.22) can then be verified at once. Equation (4.21), with g defined by (4.24), is the analog of Eq. (4.1) for the Veneziano formula.

We now attempt to choose the function $g(x,y)$ in (4.21) so as gradually to break the signature degeneracy. As in the case of the Veneziano formula, we should like to break the signature degeneracy in all three channels t, s, and u by an amount ϵ_1, ϵ_2, and ϵ_3, respectively. The transformations analogous to (4.19) for the s and u channels will be

$$s\text{ channel: } x\to -x/(1-x),\quad y\to y/(1-x)\,; \tag{4.25}$$

$$u\text{ channel: } x\to x/(x+y-1),\ y\to y/(x+y-1)\,. \tag{4.26}$$

The three transformations (4.19), (4.25), and (4.26), together with the identity transformation, form a group, so that it is reasonable that the combination of all our conditions will not be too restrictive.

The required properties of the function $g(x,y)$ will now be a straightforward generalization of the properties which we imposed in the Veneziano model. At $y=0$, the function g should consist of two terms which behave like y^{ϵ_1} and $y^{-\epsilon_1+1}$, respectively. The t-channel trajectories will thereby be split by an amount $2\epsilon_1$. Each of the terms must satisfy (4.22), so that the trajectories in each series will alternate in signature as before. The function g should have similar properties at $x=0$ and at $2-x-y=0$. The required properties are therefore as follows:

$$\begin{aligned}
&\text{(i)}\quad \text{(a)}\quad g(x,y)=y^{\epsilon_1}h_1(x,y)+y^{-\epsilon_1+1}h_2(x,y)\,, && y=0\\
&\phantom{\text{(i)}}\quad \text{(b)}\quad =x^{\epsilon_2}h_3(x,y)+x^{-\epsilon_2+1}h_4(x,y)\,, && x=0\\
&\phantom{\text{(i)}}\quad \text{(c)}\quad =(2-x-y)^{\epsilon_3}h_5(x,y)+(2-x-y)^{-\epsilon_3+1}h_6(x,y)\,, && 2-x-y=0
\end{aligned} \tag{4.27a}$$

where h_1 and h_2 are analytic at $y=0$, h_3 and h_4 at $x=0$, and h_5 and h_6 at $2-x-y=0$.

$$\begin{aligned}
&\text{(ii)}\quad \text{(a)}\quad y^{\epsilon_1}h_1(x,y)\to(-y)^{\epsilon_1}(1-y)^{-2\nu-D-3}h_1(x,y)\,, && \Big|\ y\to -y/(1-y)\\
&\qquad\qquad y^{-\epsilon_1+1}h_2(x,y)\to(-y)^{-\epsilon_1+1}(1-y)^{-2\nu-D-3}h_2(x,y)\,, && \Big|\ x\to x/(1-y)\\
&\qquad \text{(b)}\quad x^{\epsilon_2}h_3(x,y)\to(-x)^{\epsilon_2}(1-x)^{-2\nu-D-3}h_3(x,y)\,, && \Big|\ x\to -x/(1-x)\\
&\qquad\qquad x^{-\epsilon_2+1}h_4(x,y)\to(-x)^{-\epsilon_2+1}(1-x)^{-2\nu-D-3}h_4(x,y)\,, && \Big|\ y\to y/(1-x)\\
&\qquad \text{(c)}\quad (2-x-y)^{\epsilon_3}h_5(x,y)\to(-2+x+y)^{\epsilon_3}(x+y-1)^{-2\nu-D-3}h_5(x,y)\,, && \Big|\ x\to x/(x+y-1)\\
&\qquad\qquad (2-x-y)^{-\epsilon_3+1}h_6(x,y)\to(-2+x+y)^{-\epsilon_3+1}(x+y-1)^{-2\nu-D-3}h_6(x,y)\,, && \Big|\ y\to y/(x+y-1)\,.
\end{aligned} \tag{4.27b}$$

$$\text{(iii)}\qquad g(x,y)=1\quad\text{when } \epsilon_1=\epsilon_2=\epsilon_3=0\,. \tag{4.27c}$$

It is possible to find functions g satisfying (4.27), and we have given an example in Appendix B. The example is a double integral over products of Legendre functions. We therefore can break the signature degeneracy of the amplitude (2.4), provided that $\nu_1=\nu_2=\nu_3$.

For the generalized VV formula with all ν's different, we have not succeeded in breaking the signature degeneracy in one channel without introducing new trajectories in the others. Though we have no proof that the signature degeneracy cannot be broken in this general case, there are indications that it cannot be done. Let us investigate the behavior of the residue at the pole associated with the particles on the highest trajectory in the t channel, as t becomes large while s remains fixed. We can do so by examining the function g_1 given by (3.11b), and we notice that it contains a factor $[\Gamma(T+\nu_1+2)]^{-1}$, which may be written $T^{-\nu_1}[\Gamma(T+2)]^{-1}$ when T is large. The function g_2 is associated with the u-channel cuts and will only contribute an oscillatory term, which we ignore. If, on the other hand, we examine the residues as t becomes large while u remains fixed, g_2 will be the relevant amplitude, and it will contain a factor $T^{-\nu_3}[\Gamma(T+2)]^{-1}$. The residues associated with the lower trajectories will also have factors $T^{-\nu_1}$ and $T^{-\nu_3}$. An attempt to break the signature degeneracy in the t channel would involve constructing the two functions g_1+g_2 and g_1-g_2, displacing them by an amount ϵ and $-\epsilon$, respectively, and recombining them. The residues associated with the individual trajectories would then contain a factor which behaves like a $T^{-\nu_1}+bT^{-\nu_3}$ as t becomes large at fixed s. One would therefore expect a new trajectory in the s channel, situated at a distance $\nu_1-\nu_3$ above the original trajectory. This is precisely what one finds when one attempts to break the signature degeneracy by the methods explained in this section.

5. CONCLUDING REMARKS

From the purely kinematical point of view, the generalizations of the Veneziano formula proposed by Virasoro and extended in the present paper appear to be as attractive as the original formula itself. All particles lie on linear Regge trajectories, there are no ancestors, and the scattering amplitude is meromorphic in the j plane. Nevertheless, the Veneziano formula has one feature which possibly makes it more attractive on both dynamical and experimental grounds: It admits exchange degeneracy. We have already defined exchange degeneracy to mean that the Regge residues, as well as the positions of the trajectories, are independent of the signature for certain processes and in certain channels. If t is the energy of the channel in question, an exchange-degenerate amplitude will have an s cut but no u cut (or vice versa), so that exchange degeneracy is equivalent to the absence of resonances with certain quantum numbers. Any formula of the type considered in this paper, other than the Veneziano formula itself, necessarily has resonances in all three channels and is therefore in conflict with exchange degeneracy.

Exchange degeneracy appears experimentally to be fairly well satisfied, in agreement with the fact that low-lying resonances with certain quantum numbers have not been found. The quark model also possesses exchange degeneracy, as do certain bootstrap models. For all these reasons, one may feel inclined to prefer the Veneziano model to the more general amplitudes with poles in the Regge residues.

Exchange degeneracy is certainly not satisfied exactly in nature. If it is meaningful to use an approximation where exchange degeneracy is broken, but where the resonances are still narrow, one may have to replace the Veneziano formula by its generalization. The Pomeranchuk trajectory, if it exists, does not appear to be exchange-degenerate with another trajectory, and there is no reason for believing that the lower trajectories associated with it are exchange-degenerate. Thus, if it is possible to incorporate this trajectory in a narrow-resonance amplitude, the treatment may involve the generalized VV formula.

Another point of difference between the Veneziano formula and the generalized formula is the possibility of eliminating alternate trajectories.[9] This is a different question from that of eliminating trajectories with signature $-(-1)^n$; the two problems are the same only when the two crossed channels are identical. For the Veneziano formula the two problems can be solved along similar lines. In both cases one uses an amplitude of the form (4.1a) but, to eliminate the trajectories with signature $-(-1)^n$, one uses Eq. (4.1b) for δ, whereas, to eliminate the alternate trajectories, one uses (4.2). One cannot eliminate the alternate trajectories from all channels simultaneously, since Eq. (4.2b) is not symmetric in the s and t channels unless $b_s=b_t$. One can eliminate them in all channels which carry no quantum numbers that do not commute with C, and the requirements of conspiracy theory, in any case, prevent us from eliminating them in the other channels.

We have shown in Sec. 4 that the trajectories with signature $-(-1)^n$ can be eliminated from the generalized VV formula, provided $\nu_1=\nu_2=\nu_3$. As far as we can see, one cannot eliminate the alternate trajectories from this formula. Furthermore, one cannot eliminate the alternate trajectories and, at the same time, eliminate the trajectories of signature $-(-1)^n$ or break the signature degeneracy, even with the Veneziano formula.

Since exchange degeneracy appears to be fairly well satisfied in nature, one would not in general attempt to eliminate trajectories of signature $-(-1)^n$, although one might attempt to do so in certain cases. One may well attempt to eliminate the alternate trajectories, especially if one believes that a harmonic-oscillator quark model corresponds in some way to nature. If this were done, one would probably not be able to use the generalized VV formula. If, on the other hand, one did not attempt to eliminate the alternate trajectories, one could conceive of an approximation where exchange degeneracy was broken and where exotic resonances appeared. In such an approximation one might use the generalized VV formula and, at the same time, break the signature degeneracy by a small amount.

It is almost certainly possible to generalize our formula (2.4) to production processes. For the elastic amplitudes treated in this paper, the integration region of the generalized formula is a triangle, while the integration regions of the three terms of the Veneziano formula are the sides of the triangle. The integration region in the generalized formula for production amplitudes will be a solid in a multidimensional space, and the boundaries or corners of the solid will be the integration regions associated with the different terms in the formula due to Bardakci and Ruegg and to Virasoro.[10]

ACKNOWLEDGMENT

I would like to thank M. A. Virasoro for communicating to me his results on fixed poles in the angular-momentum plane.

[9] The alternate trajectories of a system where C is a good quantum number are not the odd daughters required by conspiracy theory. The charge-conjugation quantum number of an odd-daughter trajectory is opposite to that of the parent.

[10] M. A. Virasoro (private communication) has found a function which is probably the appropriate generalization to the five-point amplitude.

APPENDIX A: DERIVATION OF INTEGRAL REPRESENTATION FOR VIRASORO FORMULA

In this Appendix we shall prove Eq. (2.1). From the formula $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ it follows that

$$A(s,t)\equiv\frac{\Gamma(-\frac{1}{2}S)\Gamma(-\frac{1}{2}T)\Gamma(-\frac{1}{2}U)}{\Gamma(-\frac{1}{2}S-\frac{1}{2}T)\Gamma(-\frac{1}{2}S-\frac{1}{2}U)\Gamma(-\frac{1}{2}T-\frac{1}{2}U)}$$
$$=-\frac{\sin\frac{1}{2}\pi(S+T)}{\pi\Gamma(-1-\frac{1}{2}D)}\times\frac{\Gamma(-\frac{1}{2}T)\Gamma(-\frac{1}{2}U)}{\Gamma(-\frac{1}{2}T-\frac{1}{2}U)}\frac{\Gamma(-\frac{1}{2}S)\Gamma(-\frac{1}{2}U)}{\Gamma(-\frac{1}{2}S-\frac{1}{2}U)}\times\frac{\Gamma(1+\frac{1}{2}S+\frac{1}{2}T)\Gamma(-1-\frac{1}{2}S-\frac{1}{2}T-\frac{1}{2}U)}{\Gamma(-\frac{1}{2}U)}. \quad \text{(A1)}$$

By using integral representations for each of the three beta functions in (A1), we may write

$$A(s,t)=-\frac{\sin\frac{1}{2}\pi(S+T)}{\pi\Gamma(-1-\frac{1}{2}D)}\int_1^\infty dx\int_1^\infty dy\int_0^\infty dz\times x^{\frac{1}{2}(T+U)}(x-1)^{-\frac{1}{2}U-1}y^{\frac{1}{2}(S+U)}\times(y-1)^{-\frac{1}{2}U-1}z^{\frac{1}{2}(S+T)}(z+1)^{\frac{1}{2}U}$$
$$=-\frac{\sin\frac{1}{2}\pi(S+T)}{\pi\Gamma(-1-\frac{1}{2}D)}\int_1^\infty dx\int_1^\infty dy\int_0^\infty dz\times\left[x\left(\frac{z+1}{(x-1)(y-1)}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}(T+U)}\times\left[y\left(\frac{z+1}{(x-1)(y-1)}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}(S+U)}\times\left[z\left(\frac{(x-1)(y-1)}{z+1}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}(S+T)}\times[(x-1)(y-1)^{-1}]. \quad \text{(A2)}$$

The form of Eq. (A2) suggests that we define new variables:

$$x'=x\left(\frac{z+1}{(x-1)(y-1)}\right)^{\frac{1}{2}}, \quad \text{(A3a)}$$

$$y'=y\left(\frac{z+1}{(x-1)(y-1)}\right)^{\frac{1}{2}}, \quad \text{(A3b)}$$

$$z'=-z\left(\frac{(x-1)(y-1)}{z+1}\right)^{\frac{1}{2}}. \quad \text{(A3c)}$$

From (A3) it follows that

$$W\equiv\left(\frac{1+z}{(x-1)(y-1)}\right)^{\frac{1}{2}}=\tfrac{1}{2}[x'+y'+z'-(x'^2+y'^2+z'^2-2x'y'-2x'z'-2y'z'+4)^{\frac{1}{2}}], \quad \text{(A4a)}$$

$$\left|\frac{\partial(x,y,z)}{\partial(x',y',z')}\right|=2(z+1)W^{-2}(x'^2+y'^2+z'^2-2x'y'-2x'z'-2y'z'+4)^{-\frac{1}{2}}. \quad \text{(A4b)}$$

On making this change of variables in (A2) and dropping the primes, we find that

$$A(s,t)=-\frac{2\sin\frac{1}{2}\pi(S+T)}{\pi\Gamma(-1-\frac{1}{2}D)}\int_0^\infty dx\int_{1/x}^\infty dy\int_{-\infty}^0 dz\times x^{\frac{1}{2}(T+U)}y^{\frac{1}{2}(S+U)}(-z)^{\frac{1}{2}(S+T)}\times\{x^2+y^2+z^2-2xy-2xz-2yz+4\}^{-\frac{1}{2}}. \quad \text{(A5)}$$

We can deform the path of the z integration so as to remove the factor $\sin\frac{1}{2}\pi(S+T)$. On doing so, we shall obtain a formula which is symmetric in S, T, and U. The integrand of (A5) contains three singularities in the z plane. The first is at $z=0$; the other two are at the points $z=x+y\pm2(xy-1)^{1/2}$, which are both on the real axis. We can replace the integral along the negative axis of z by an integration along a contour surrounding the negative real axis, provided we divide by the factor $2i\exp[-\frac{1}{2}i\pi(S+T)]\sin\frac{1}{2}\pi(S+T)$. We can then deform the contour of integration so as to surround the two branch points at $z=x+y\pm2(xy-1)^{1/2}$. The integral thus becomes

$$A(s,t)=\frac{2}{\pi}[\Gamma(-1-\tfrac{1}{2}D)]^{-1}\int dxdydz\times x^{\frac{1}{2}(T+U)}y^{\frac{1}{2}(S+U)}z^{\frac{1}{2}(S+T)}\times\{-x^2-y^2-z^2+2xy+2xz+2yz-4\}^{-\frac{1}{2}}, \quad \text{(A6)}$$

where the range of integration is $x>0$, $y>0$, $z>0$, $-x^2-y^2-z^2+2xy+2xz+2yz-4>0$.

We next attempt to simplify the factor within the curly brackets. The factor can be rewritten as

$$\{(x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}}-z^{\frac{1}{2}})\times(x^{\frac{1}{2}}-y^{\frac{1}{2}}+z^{\frac{1}{2}})(-x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}})-4\}^{-\frac{1}{2}}.$$

We therefore make the obvious change of variables

$$x'=-x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}},\quad y'=x^{\frac{1}{2}}-y^{\frac{1}{2}}+z^{\frac{1}{2}},\quad z'=x^{\frac{1}{2}}+y^{\frac{1}{2}}-z^{\frac{1}{2}}, \quad \text{(A7)}$$

and (A6) becomes

$$A(s,t)=\frac{2^{-2D-1}}{\pi}[\Gamma(-1-\tfrac{1}{2}D)]^{-1}\int dxdydz\times(y+z)^{T+U+1}(x+z)^{S+U+1}(x+y)^{S+T+1}\times\{xyz(x+y+z)-4\}^{-\frac{1}{2}}. \quad \text{(A8)}$$

We can get rid of the factor xyz in the first term within the curly bracket by making the change of variables

$$x=x'(x'y'z')^{-1/4},\quad y=y'(x'y'z')^{-1/4},\quad z=z'(x'y'z')^{-1/4},$$

$$\left|\frac{\partial(x,y,z)}{\partial(x',y',z')}\right|=\tfrac{1}{4}(x'y'z')^{-3/4}. \quad \text{(A9)}$$

The integral then reads

$$A(s,t)=\frac{2^{-2D-3}}{\pi}[\Gamma(-1-\tfrac{1}{2}D)]^{-1}\int dxdydz \times(xyz)^{-\frac{1}{2}D-\frac{3}{2}}(y+z)^{T+U+1}(x+z)^{S+U+1} \times(x+y)^{S+T+1}(x+y+z-4)^{-\frac{1}{2}}, \quad \text{(A10)}$$

the range of integration being $y+z>1$, $x+z>1$, $x+y>1$, $x+y+z-4>1$.

We can reduce (A10) from a triple to a double integral by integrating over the variable $x+y+z$, while the ratios x/y and x/z are kept constant. We therefore make a final change of variables:

$$x=w(1-x'),\quad y=w(1-y'),\quad z=w(x'+y'-1),$$

$$\left|\frac{\partial(x,y,z)}{\partial(x',y',w)}\right|=w^2. \quad \text{(A11)}$$

On substituting (A11) in (A10), the integral becomes

$$A(s,t)=\frac{2^{-2D-3}}{\pi}[\Gamma(1-\tfrac{1}{2}D)]^{-1}\int_0^1 dx\int_{1-x}^1 dy \times\left[\frac{(1-x)(1-y)(x+y-1)}{x^2y^2(2-x-y)^2}\right]^{-\frac{1}{2}D-\frac{3}{2}} \times x^{-S-2}y^{-T-2}(2-x-y)^{-U-2} \times\int_4^\infty dw\, w^{\frac{1}{2}(D+1)}(w-4)^{-\frac{1}{2}}. \quad \text{(A12)}$$

The w integral can be performed as a beta function, so that we obtain finally

$$A(s,t)=\pi^{-\frac{1}{2}}2^{-D-1}[\Gamma(\tfrac{1}{2}(-1-D))]^{-1}\int_0^1 dx\int_{1-x}^1 dy \times\left[\frac{(1-x)(1-y)(x+y-1)}{x^2y^2(2-x-y)^2}\right]^{-\frac{1}{2}D-\frac{3}{2}} \times x^{-S-2}y^{-T-2}(2-x-y)^{-U-2}. \quad \text{(A13)}$$

APPENDIX B: EXPLICIT EXAMPLES OF FUNCTIONS THAT BREAK SIGNATURE DEGENERACY

We wish to write an integral representation for a function $g(x,y)$ which has the properties (4.27). Since we were able to satisfy (4.11) by taking a hypergeometric function of a single variable, an obvious first trial would be to investigate hypergeometric functions of two variables.[11] There do indeed exist hypergeometric functions of two variables which have simple transformation properties under precisely the group of transformations (4.19), (4.25), and (4.26). Furthermore, there exist other hypergeometric functions which can be expressed as a sum of the functions in question, just as the P's can be expressed as a sum of two Q's.

The hypergeometric functions of two variables are not exactly the functions we require. They contain too few variables to satisfy all the conditions (4.27); a rough examination indicates that they might possibly be used to break signature degeneracy in two channels, but not in all three. Moreover, the standard hypergeometric functions which have simple transformation properties under (4.19), (4.25), and (4.26) are defined in the region $x<1$, $y<1$, $x+y<1$ rather than in the region $x<1$, $y<1$, $x+y>1$. Nevertheless, the integral representation for the hypergeometric function does give us an indication of the type of function for which we are looking. A hypergeometric function of two variables can be expressed as a single integral of a product of two hypergeometric functions of one variable. The function which solves our problem is a double integral of a product of three hypergeometric functions of one variable. As in the case of the Veneziano formula, the particular hypergeometric functions which we require are the associated Legendre functions.

Guided by such considerations, we have found the following function, which solves our problem:

$$g(x,y)=\left[\frac{\Gamma(1-\Delta)}{\Gamma(1+\Delta)}\right]^3\Gamma(3+3\Delta) \times\int_0^1 dv\int_0^{1-v} dw[vw(1-v-w)]^\Delta \times\left[\left(1-\frac{x}{v}\right)\left(1-\frac{y}{w}\right)\left(1-\frac{2-x-y}{1-v-w}\right)\right]^{\Delta/2} \times P_{-\epsilon_1}{}^\Delta\left(\frac{2w-y}{y}\right)P_{-\epsilon_2}{}^\Delta\left(\frac{2v-x}{x}\right) \times P_{-\epsilon_3}{}^\Delta\left(\frac{2(1-v-w)-(2-x-y)}{2-x-y}\right), \quad \text{(B1a)}$$

where

$$\Delta=\tfrac{1}{3}(2\nu+D+1). \quad \text{(B1b)}$$

In order to show that (B1) has the properties (4.27) at $y=0$, we express the function $P_{\epsilon_1}{}^\Delta((2w-y)/y)$ as the sum of two Q's. Before doing so, however, we have to make a change in the path of the w integration. The integrand of (B1) is analytic at $w=y$, the singularities

[11] *Higher Transcendental Functions*, edited by A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 222.

of the factors $(1-y/w)^{\Delta/2}$ and $P_{\epsilon_1}{}^{\Delta}((2w-y)/y)$ just canceling one another. The individual terms obtained by expressing P as a sum of two Q's will no longer be analytic at $w=y$. We shall replace the w integration in (B1) by an integration over a Pochhammer contour,[12] one loop of which encloses the branch points at $w=0$ and $w=y$, and the other the branch point at $w=1-v$. On making this replacement, we must include an extra factor $-e^{\frac{3}{2}i\pi\Delta}/(4\sin\pi\Delta\sin\frac{1}{2}\pi\Delta)$.

The factor $P_{\epsilon_1}{}^{\Delta}((2w-y)/y)$ is now expressed as a sum of two Q's as follows:

$$P_{-\epsilon_1}{}^{\Delta}\left(\frac{2w-y}{y}\right)=\frac{e^{-i\pi\Delta}\sin\pi(\Delta+\epsilon)}{\pi\cos\pi\epsilon}Q_{\epsilon_1-1}{}^{\Delta}\left(\frac{2w-y}{y}\right)+\frac{e^{-i\pi\Delta}\sin\pi(\Delta-\epsilon)}{\pi\cos\pi\epsilon}Q_{-\epsilon_1}{}^{\Delta}\left(\frac{2w-y}{y}\right). \quad \text{(B2)}$$

Since the two Q functions behave like y^{ϵ_1} and $y^{-\epsilon_1+1}$ at $y=0$, we conclude that property i(a) of (4.27) is satisfied. To prove property ii(a), we make the transformation (4.19) and, at the same time, we define new variables of integration in (B1):

$$w'=\frac{w-y}{1-y},\quad v'=\frac{v}{1-y},\quad 1-v'-w'=\frac{1-v-w}{1-y}. \quad \text{(B3)}$$

The argument of the two Q functions in (B2) then changes sign, while the arguments of the last two P functions of (B1) remain unchanged. Thus, apart from factors $(-1)^{\epsilon_1}$ and $(-1)^{-\epsilon_1+1}$, the P and Q functions remain unchanged under the transformations (4.19), (B3). The other factors of (B1), including the differentials dv and dw, transform into themselves together with a factor $(1-y)^{-3\Delta-2}$. The Pochhammer contour transforms into itself. Property ii(a) of (4.27) is therefore satisfied.

In the same way, by expressing the other two P's as the sum of two Q's, we can show that properties i(b), ii(b), i(c), and ii(c) are satisfied. When the ϵ's are zero, the three P's in (B1a) become equal to

$$[\Gamma(1-\Delta)]^{-1}(1-y/w)^{-\Delta/2},\quad [\Gamma(1-\Delta)]^{-1}(1-x/v)^{-\Delta/2},$$

and

$$[\Gamma(1-\Delta)]^{-1}\left(1-\frac{2-x-y}{1-v-w}\right)^{-\Delta/2}.$$

The variables x and y then disappear from (B1a) and, on performing the integration over v and w, we find that property (iii) of (4.27) is satisfied.

Having obtained explicit functions for the integrand of the generalized VV formula which break the signature degeneracy, we can obtain the corresponding functions for the Veneziano formula as a special case. We wish to break the signature degeneracy in all three channels simultaneously; the integrand which breaks the degeneracy in one channel has already been written down. The Veneziano formula is the limiting case of the generalized formula when $\nu=-1$, and the integrand corresponding to the *st*, *tu*, or *su* Veneziano terms is obtained by taking the integrand of the generalized formula along the edges $x+y=1$, $x=1$, and $y=1$ of the triangle. With the integration variables defined by (4.15), the three functions f_{st}, f_{tu}, and f_{su} will then be

$$\begin{aligned} f_{st}(y)&=g(1-y,y)\,,\\ f_{tu}(y)&=g(1,y)\,,\\ f_{su}(y)&=g(1-y,1)\,. \end{aligned} \quad \text{(B4)}$$

The function g is again given by (B1a), with $\Delta=\frac{1}{3}(D-1)$. By using reasoning similar to that of the last few paragraphs, one can check directly that the functions defined by (B4) do have properties (4.17).

[12] For a definition of the Pochhammer contour, see Ref. 11, p. 41.