Mellin transformation with respect to $|k^2|$ then yields

$$\bar{M}_L(\zeta)=\int_0^\infty d|k^2|\,(|k^2|/m^2)^{-\zeta}M_L(k^2)$$
$$=2m^{2\zeta}\pi(1-\zeta)\csc\pi\zeta\int_0^\infty d\mathbf{q}_\perp{}^2(\mathbf{q}_\perp{}^2+\lambda^2)^{-1}\int_0^1 dA\int_0^1 dx\,x(1-x)[A(1-A)]^\zeta[x(1-x)\mathbf{q}_\perp{}^2+m^2]^{-\zeta}$$
$$=2\pi(1-\zeta)\csc\pi\zeta\,[\Gamma(1+\zeta)]^2[\Gamma(2+2\zeta)]^{-1}\zeta^{-1}\int_0^1 dx\,x(1-x)F(\zeta,1;1+\zeta;1-x(1-x)\lambda^2/m^2).\quad\text{(C3)}$$

Let ζ be small and positive so that (B5) can be used. Then

$$\bar{M}_L(\zeta)\sim2\zeta^{-2}(1-3\zeta)\int_0^1 dx\,x(1-x)\{1-\zeta\ln[x(1-x)\lambda^2/m^2]\}\sim\tfrac{1}{3}\zeta^{-2}\{1-\zeta[\ln(\lambda^2/m^2)+\tfrac{4}{3}]\}.\quad\text{(C4)}$$

Accordingly, the desired answer is that, for large $|k^2|$ but fixed nonzero values of m and λ,

$$M_L(k^2)\sim\tfrac{1}{3}|k^2|^{-1}[\ln(|k^2|/\lambda^2)-\tfrac{4}{3}].\quad\text{(C5)}$$

Note again that m does not appear in (C5).

PHYSICAL REVIEW VOLUME 183, NUMBER 5 25 JULY 1969

Coupled Dynamical Equations Incorporating Unitarity and Crossing. I. $\pi\pi$ and πK Scattering*

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(Received 17 February 1969)

An equation proposed by Zimmermann, which takes account of the normal thresholds in s, t, and u channels, is generalized to two-body reactions involving particles of any spin or isospin, and extended to include a number of coupled processes in each channel. In this form, the resulting equations, besides satisfying the requirements of low-energy unitarity and crossing symmetry, take some account of the effect of all reactions in each channel, as well as the detailed energy-dependent effects of one or two low-mass processes. The coupled $\pi\pi$ and πK system is considered, and approximate solutions are found which give the ρ and K^* resonance parameters within 15% of the accepted experimental values, as well as a κ resonance in the $I=\frac{1}{2}$ s-wave πK partial wave. The $\pi\pi$ s-wave phase shifts in this case are found to be very negative, decreasing rapidly to $-\pi$; but with the coupling of the $\pi\pi\rightarrow\pi\omega$ reaction to this system; a solution is obtained giving an $I=0$, $\pi\pi$ s-wave which, at low energies, is in agreement with some recent phenomenological analyses.

I. INTRODUCTION

THE development of S-matrix theory based upon the requirements of Lorentz invariance, unitarity, analyticity, and crossing has been a very powerful tool in understanding the strong interactions of fundamental particles. In principle, it is able to predict the scattering amplitude for any hadron reaction, and it has been suggested that the simultaneous determination of these amplitudes, coupled via the unitarity and crossing conditions, could be made with only the input of an over-all scale. Such complete solutions, however, are clearly impossible to calculate, since they involve the solution of an infinite number of coupled channels. Nevertheless, it has been hoped that approximate low-energy solutions could be found which at least reflect the physical situation on the assumption that the effects of the higher-mass processes are either negligible or can be inserted phenomenologically.

Most of the work in this direction has concentrated on the solution of dispersion relations which stress the cut-plane analyticity of the scattering amplitudes, with the corresponding discontinuities given by the unitarity conditions. Unfortunately, whether one uses the full Mandelstam representation[1] or the multichannel N/D partial-wave equations of Bjorken,[2] the

* Supported by Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Grant No. AF-AFOSR-30-67.

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

computational difficulties necessitate the truncation of the unitarity equations to include only a small number of inelastic channels, if any. Only in the single-channel N/D equations[3] can the effect of an infinite number of inelastic channels be included, at least phenomenologically, but here one has the added difficulty of possible Castillejo-Dalitz-Dyson (CDD) poles.[4] Furthermore, in any partial-wave dispersion relation, the requirement of crossing can only be approximately fulfilled, since only a small number of partial waves can be considered in the crossed partial-wave expansion, and this fails to converge outside a certain range.

Because of these difficulties, it would seem advantageous to develop an alternative method of calculating scattering amplitudes which, while still based on the requirements of Lorentz invariance, unitarity, analyticity, and crossing, does not have the above drawbacks of dispersion relations. In this paper, therefore, a system of equations first proposed by Zimmermann[5] is generalized to describe two-body amplitudes for particles of any spin and isospin. In this form they can be extended to include any number of coupled inelastic two-body processes, and thus should provide an excellent computational tool for hadron scattering. These integral equations have several very desirable features. First, they are crossing-symmetric and take account of elastic and some inelastic thresholds in all three energy channels. Also, all particles are on their mass shells and, unlike dispersion relations, the integrals extend only over a finite range. Furthermore, while it is clearly impractical to consider explicitly more than one or two coupled channels, the equations are capable of taking some account of the effect of *all* such channels, at least in the low-energy region.

Its ability to include inelastic effects as well as crossing suggests that this system of equations should be a very satisfactory model for studying many different and apparently complex processes. To demonstrate the possible potential of the model, we shall concentrate in this paper on the simultaneous solution of the coupled $\pi\pi$ and πK reactions. This system is chosen not only because of the spinless nature of the particles, which makes the algebra somewhat simpler, but also because there is mounting evidence to suggest that "self-consistent" $\pi\pi$ dispersion relations do not agree with experimental analyses. For instance, recent phenomenological studies[6] indicate, besides the existence of the ρ and f^0 resonances, that the $I=0$, s-wave $\pi\pi$ phase shift is large and positive near 700 MeV, if not in fact resonant. On the other hand, the self-consistent s- and p-wave calculation of Chew, Mandelstam, and Noyes[7] produced no appreciable p-wave scattering, but a very large and negative $I=0$, s-wave phase shift δ_0^0. This s-wave-dominant solution has also been found in other studies[8] and is apparently the prevailing solution of elastic partial-wave dispersion relations. We should mention that the so-called ρ bootstrap is "successful" only because the large, negative s-wave phase shift which is invariably produced at the same time is overlooked, supposedly annihilated by some unestimated short-range effects.[8] Even so, the resulting "ρ" meson has too low a mass and an extremely large width.

One of the main characteristics of these calculations is the assumption of purely elastic unitarity. The argument is that the first inelastic threshold, the four-pion threshold at $s=16\mu^2$ (where μ is the pion mass), is thought to be "relatively distant" from the two-pion threshold at $s=4\mu^2$. However, this is less than the corresponding distance in πN scattering, for instance, where inelastic effects are known to be important.[9] Also, since the $\pi\pi$ channel is coupled to *all* particle-antiparticle states, it would seem very likely that pion-pion scattering is more dependent on inelastic contributions than most other processes. Certainly, when some effect of inelasticity is included in dispersion calculations, the ρ-meson parameters are somewhat more closely reproduced[10,11]; but for $\pi\pi$ s waves the proximity of the elastic threshold to the left-hand cut and the lack of a centrifugal barrier probably make the effect of short-range forces also important in an accurate calculation. These short-range forces are closely related to the inelastic contributions in the crossed channels which, although most difficult to calculate in a partial-wave dispersion relation, can be estimated in the set of equations developed in the next section.

One of the more important of these low-energy inelastic contributions to $\pi\pi$ scattering is likely to be the $\pi\pi\rightarrow K\bar{K}$ reaction, both in the direct and crossed channels. On the other hand, recent calculations[12] have also shown that the two-pion exchange terms arising from the unitarity condition in the crossed channel are probably also important in a determination of the low-energy $\pi K\rightarrow\pi K$ amplitudes. For these reasons we present in Sec. III numerical solutions for the coupled $\pi\pi$ and πK system. These solutions are only approximate, since, for ease of computation, the effects of essentially all other coupled reactions are taken to be constant. This in turn necessitates some modification of the resulting amplitudes to ensure their known cut-plane analyticity. Even with these approximations, we find the resulting ρ and K^* resonance parameters as well as the low-energy πK phase shifts to be in fair agreement with the experimental data. The $\pi\pi$ s waves,

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[4] D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N.Y.) **37**, 77 (1966).
[5] W. Zimmermann, Nuovo Cimento **21**, 249 (1961).
[6] W. D. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967); L. W. Jones *et al.*, Phys. Letters **21**, 590 (1966); L. J. Gutay *et al.*, Phys. Rev. Letters **18**, 142 (1967).
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[8] C. F. Kyle, A. W. Martin, and H. R. Pagels, Stanford University report (unpublished).
[9] P. W. Coulter and G. L. Shaw, Phys. Rev. **141**, 1419 (1966).
[10] P. W. Coulter and G. L. Shaw, Phys. Rev. **138**, B1273 (1965).
[11] J. Fulco, G. Shaw, and D. Wong, Phys. Rev. **137**, B1242 (1965).
[12] J. L. Gervais, Phys. Rev. **138**, B1457 (1965).

however, are still found to be strongly bound, which is perhaps not really surprising if, as we believe, $\pi\pi$ scattering is very intimately dependent on a number of coupled reactions both in the direct and crossed channels. This is verified in Sec. IV, where the explicit consideration of a third channel, $\pi\pi\rightarrow\pi\omega$, in the $\pi\pi$ and πK system can, at least to some extent, apparently account for the structure of $\pi\pi$ s-wave scattering.

II. DYNAMICAL EQUATIONS

In this section we shall develop a system of equations for any number of coupled two-body reactions which incorporate the S-matrix properties of Lorentz invariance, unitarity, analyticity, and crossing. To do so, it is advantageous to define the S matrix as

$$S=1+2iT=(1+iK)(1-iK)^{-1}, \tag{1}$$

where Hermiticity of the K matrix ensures that S is unitary. In terms of T, Eq. (1) can be written as

$$T=K+iTK, \tag{2}$$

which, for an arbitrary process $a+b\rightarrow c+d$ with isospin I, becomes

$$T_I^{ab,cd}(s,t,u)=K_I^{abcd}(s,t,u)+i[T(s,t,u)\underset{s}{\times}K(s,t,u)]_I^{abcd}, \tag{3}$$

$$\begin{aligned}[T\underset{s}{\times}K]_I^{abcd}&=\sum_{\alpha\beta}\int\frac{d^3p_\alpha}{2\omega_\alpha}\frac{d^3p_\beta}{2\omega_\beta}T^{ab,\alpha\beta}(p_\alpha,p_\beta,p_a,p_b)\\&\quad\times K^{\alpha\beta,cd}(p_c,p_d,p_\alpha,p_\beta)\delta^4(p_\alpha+p_\beta-p_a-p_b)\\&=\sum_{\alpha\beta}\frac{k^{\alpha\beta}}{4\sqrt{s}}\int d\Omega_s T_I^{ab,\alpha\beta}(s,t',u')\\&\quad\times K_I^{\alpha\beta,cd}(s,t'',u'').\end{aligned} \tag{4}$$

The Mandelstam variables s, t, u are defined in the usual way, $k^{\alpha\beta}$ is the relative momentum of α or β in the $a+b$ c.m. system, and $d\Omega_s$ is the element of solid angle subtended by $\mathbf{p}_\alpha$ also in that system.

Equation (3) replaces the more usual unitarity condition, with the sum in Eq. (4) theoretically including all intermediate states coupled to ab and cd.[13] In practice, however, this sum will be truncated to a small number of low-mass two-particle states, so that the resulting functions $K(s,t,u)$, while being regular at the corresponding normal thresholds, will still possess those singularities of $T(s,t,u)$ associated with the remaining normal thresholds, as well as possible anomalous thresholds, and all physical-region singularities in the t and u channels.[14] The extension of Eq. (3) to incorporate the crossed-channel thresholds was first suggested by Zimmermann.[5] The basis for this is to define a "symmetry function" $\mathfrak{T}(s,t,u)$ by the relation

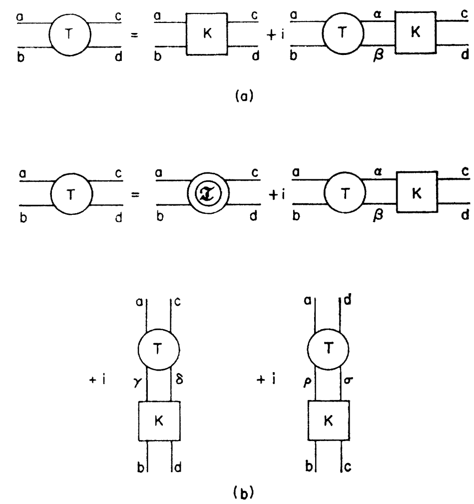

FIG. 1. Bubble-notational form of (a) Eq. (3), defining the K-matrix elements, and (b) Eq. (5), defining the symmetry functions $\mathfrak{T}_I(s,t,u)$.

shown diagramatically in Fig. 1(b). Writing this out fully for our arbitrary process $a+b\rightarrow c+d$, we obtain

$$\begin{aligned}T_I^1(s,t,u)=\mathfrak{T}_I^1(s,t,u)&+i[T(s,t,u)\underset{s}{\times}K(s,t,u)]_I^1\\&+i\theta_{IJ}[T(t,u,s)\underset{t}{\times}K(t,u,s)]_J^2\\&+i\phi_{IJ}[T(u,s,t)\underset{u}{\times}K(u,s,t)]_J^3,\end{aligned} \tag{5}$$

where θ and ϕ are the crossing matrices such that

$$T_I^1(s,t,u)=\theta_{IJ}T_J^2(t,u,s)=\phi_{IJ}T_J^3(u,s,t), \tag{6}$$

and the superscripts 1, 2, 3 refer to the three processes

$$\begin{aligned}1:&\quad a+b\rightarrow c+d,\\2:&\quad a+\bar{c}\rightarrow d+\bar{b},\\3:&\quad a+\bar{d}\rightarrow\bar{b}+c.\end{aligned} \tag{7}$$

The third and fourth terms on the right-hand side of Eq. (5) are given by relations such as Eq. (4) defined in the t and u channels, respectively. The isospin combination for these crossed terms was determined to ensure that $\mathfrak{T}(s,t,u)$ was free from the two-body singularities in all three channels. To see this, let us insert Eq. (3) and its crossed-channel versions into Eq. (5). Using Eq. (6), this becomes

$$\begin{aligned}2T_I^1(s,t,u)=K_I^1(s,t,u)&+\theta_{IJ}K_J^2(t,u,s)\\&+\phi_{IJ}K_J^3(u,s,t)-\mathfrak{T}_I^1(s,t,u),\end{aligned} \tag{8}$$

from which we deduce that $\mathfrak{T}_I^1(s,t,u)$ is regular for s, t, or $u>0$, wherever $K^1(s,t,u)$, $K^2(t,u,s)$, or $K^3(u,s,t)$ is regular, respectively.[15] Also, we find from Eq. (8)

[13] Equation (3) is similar to the Bethe-Salpeter equation, except that here the intermediate particles are on the mass shell.

[14] R. H. Dalitz, *Strange Particles and Strong Interactions* (Oxford University Press, New York, 1962).

[15] Note that $K(s,t,u)$ has the same t and u singularities as $T(s,t,u)$. Similarly, $K(u,s,t)$ has the same s and t singularities as $T(u,s,t)$, etc.

that $\mathfrak{T}_I$ satisfies the same crossing relations, [Eq. (6)], as $T(s,t,u)$.

Now substituting $T(s,t,u)$, as defined above, into Eq. (3) and its crossed versions, and defining a three-vector

$$\mathbf{K}_I=\begin{bmatrix} K_I^1(s,t,u) \\ K_I^2(t,u,s) \\ K_I^3(u,s,t) \end{bmatrix}, \tag{9}$$

we find that $\mathbf{K}$ satisfies the equation

$$\begin{bmatrix} -1 & \theta & \phi \\ \theta^{-1} & -1 & \theta^{-1}\phi \\ \phi^{-1} & \phi^{-1}\theta & -1 \end{bmatrix}_{IJ} \mathbf{K}_J=\mathfrak{T}_I+i\mathbf{X}_I, \tag{10}$$

where the three-vector $\mathfrak{T}_I$ is defined in the same manner as $\mathbf{K}_I$, and

$$\mathbf{X}_I=\begin{bmatrix} [Y(s,t,u)\underset{s}{\times} K(s,t,u)]_I^1 \\ [Y(t,u,s)\underset{t}{\times} K(t,u,s)]_I^2 \\ [Y(u,s,t)\underset{u}{\times} K(u,s,t)]_I^3 \end{bmatrix}, \tag{11}$$

with

$$\mathbf{Y}_I=-\mathfrak{T}_I+\begin{bmatrix} 1 & \tilde{\theta} & \tilde{\phi} \\ \tilde{\theta}^{-1} & 1 & \tilde{\theta}^{-1}\tilde{\phi} \\ \tilde{\phi}^{-1} & \tilde{\phi}^{-1}\tilde{\theta} & 1 \end{bmatrix}_{IJ} \mathbf{K}_J. \tag{12}$$

Inverting Eq. (10) and using the crossing properties of $\mathfrak{T}_I$ given by Eq. (6), we have

$$\mathbf{K}_I=\mathfrak{T}_I+\tfrac{1}{2}i\begin{bmatrix} 0 & \theta & \phi \\ \theta^{-1} & 0 & \theta^{-1}\phi \\ \phi^{-1} & \phi^{-1}\theta & 0 \end{bmatrix}_{IJ} \mathbf{X}_J, \tag{13}$$

which is our defining equation for K_I and hence, by Eqs. (3) or (8), for $T(s,t,u)$.

It should be noted that the effect of the coupled processes enters via the definition of $\mathbf{X}_I$, where for instance,

$$\begin{aligned} X_I^{abcd}(s,t,u)&=\sum_{\alpha\beta}\frac{k^{\alpha\beta}}{4\sqrt{s}}\int d\Omega_s\, Y_I^{abcd}(s,t,'u') \\ &\qquad\times K_I^{\alpha\beta cd}(s,t'',u''), \\ X_I^{a\bar{c}d\bar{b}}(t,u,s)&=\sum_{\gamma\delta}\frac{k^{\gamma\delta}}{4\sqrt{t}}\int d\Omega_t\, Y_I^{a\bar{c}\gamma\delta}(t,u',s') \\ &\qquad\times K_I^{\gamma\sigma d\bar{b}}(t,u'',s''), \\ X_I^{a\bar{d}\bar{b}c}(u,s,t)&=\sum_{\rho\sigma}\frac{k^{\rho\sigma}}{4\sqrt{u}}\int d\Omega_u\, Y_I^{a\bar{d}\rho\sigma}(u,s',t') \\ &\qquad\times K_I^{\rho\sigma\bar{b}c}(u,s'',t''), \end{aligned} \tag{14}$$

with $d\Omega_t$ and $d\Omega_u$ the elements of solid scattering angle defined in the t and u c.m. systems, respectively; $\tilde{\theta}$ and $\tilde{\phi}$ in Eq. (12) are the appropriate crossing matrices for the various processes $a+b\to\alpha+\beta$, $a+\bar{\delta}\to c+\gamma$, $a+\bar{\rho}\to\sigma+d$. Furthermore, although not explicitly described, Eq. (13), besides taking account of isospin, can also be extended to particles of any spin if we simply define the superscripts a, b, c, d, etc., as the corresponding helicities and take the last term of Eq. (13) to be a sum over helicities as well as the isospin index J. In this case, θ and ϕ, etc., are the combinations of isospin-crossing matrices and the helicity-crossing matrices of Trueman and Wick.[16]

Equation (13) and similar equations for the processes contributing to $\mathbf{X}_I$ form a coupled system for the various K-matrix elements given in terms of the functions $\mathfrak{T}_I$ corresponding to each of the coupled processes. These functions incorporate the effect of the contributions arising from the higher-order thresholds than from the two-particle ones considered and, at least theoretically, can be deduced in the following manner. We know that $\mathfrak{T}_I(s,t,u)$ must have those singularities not explicitly "extracted" by Eq. (3) and its crossed versions. Also, it is well known[17] that the integral in Eq. (3) gives rise to an unphysical singularity along the negative s axis which, since it is not present in the amplitude $T(s,t,u)$, must be annihilated by a similar singularity in K. Such a singularity arises in two ways: first, it is due to the $k^{\alpha\beta}/\sqrt{s}$ term in the phase-space factor; but also, it is due to a pinch singularity arising from the normal thresholds in t and u. The kinematic singularity can be removed by defining a new K matrix $\bar{K}(s,t,u)$ so that the integral in Eq. (3) becomes[18]

$$\int F^{\alpha\beta}(s)d\Omega_s\, T_I^{ab\alpha\beta}(s,t',u')K_I^{\alpha\beta cd}(s,t'',u''), \tag{15}$$

where

$$F^{\alpha\beta}(s)=-\frac{i}{8\pi}(s-x)\int_x^\infty \frac{[(s'-x)(s'-y)]^{1/2}}{s'(s'-x)(s'-s)}ds' \tag{16}$$

and

$$x=(m_\alpha+m_\beta)^2,\quad y=(m_\alpha-m_\beta)^2.$$

With this description of the unitarity-type products, Eqs. (3)–(14) follow through without modification, and we shall henceforth implicitly be working with $\bar{K}$. The dynamical pinch singularity cannot be removed so easily (if at all),[17] so from Eq. (8) we see that in order to remove this singularity and the corresponding ones along the negative t and u axes from the scattering amplitudes, similar singularities must also be present in $\mathfrak{T}_I(s,t,u)$.

In a complete solution of Eq. (13), therefore, which satisfies analyticity, crossing, and unitarity at least in the low-energy region, these symmetry functions would be defined by the crossing relations together with the discontinuities arising both from the pinch singularities and the higher-mass thresholds. If the effect of these higher-mass processes could be inserted phenomenologically, there would be no need to determine the reciprocal effects of a system of coupled channels. However, in the example we shall consider in the remainder of this paper of pion-pion and kaon-pion

[16] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).
[17] P. G. Freund and R. Karplus, Nuovo Cimento **21**, 519 (1961).
[18] J. G. Cordes, Phys. Rev. **156**, 1707 (1967).

scattering, we have no knowledge of these higher-order processes. Thus, we shall be forced to assume some simple form for the symmetry functions, i.e., constants, and inquire if such an approximation is adequate to describe the probably complicated nature of low-energy scattering. There is some reason to believe this might be a very plausible assumption provided the inclusion of one or two coupled processes introduces most of the required energy-dependent effects. Unfortunately, to preserve analyticity with such an approximation we shall find it necessary to use only partial solutions to the dynamical equation. Hence, to have any confidence in these solutions we should hope that, like the complete solutions, they would satisfy crossing, at least in the neighborhood of the symmetry point.

The idea that the Zimmermann equation could be used in a calculatory scheme for strong interactions was first suggested by Landshoff and Olive,[19] and a pion-pion scattering amplitude uncoupled from other processes was previously determined by Cordes[18] using this equation with approximations similar to those discussed here. This solution is characterized by the production of a "ρ" resonance about 800 MeV and width 250 MeV, together with large negative s-wave phase shifts, particularly δ_0^0, which drops rapidly below $-\pi$. However, as mentioned in the introduction, the dipion interaction is probably very dependent on the coupled channels, even in the elastic region. Therefore, as an initial investigation using the extension to the formalism described above, we shall study in the next section the approximate solution of the coupled $\pi\pi$ and πK system.

III. APPROXIMATE PERTURBATIVE SOLUTIONS TO THE COUPLED $\pi\pi$ AND πK SYSTEM

It was noted in the last section that the functions $\mathfrak{T}(s,t,u)$ could be determined by a knowledge of their singularities. In practice, however, such a knowledge is denied us and, although it might be possible to find arbitrary functions to approximate $\mathfrak{T}(s,t,u)$ and examine which singularities are necessary to produce the physical cross sections, such a program would incorporate a large number of parameters. Therefore, to keep the number of parameters as small as possible, we shall consider here only the explicit coupling of the $\pi\pi$ and πK channels and approximate the corresponding symmetry functions by constants.

With this approximation, the energy-momentum dependence of the K-matrix elements arises entirely from the modified phase-space factors of Eq. (16). Thus, solving Eq. (13) by successive iterations starting with $K_I(s,t,u)=\mathfrak{T}_I(s,t,u)=a_I$(constant), etc., we are led to the result

$$K_I(s,t,u)=\sum c_{m,n,I}{}^{x,y,z}\int [F^{\alpha\beta}(x)]^m \times[F^{\alpha'\beta'}(y)]^n d\Omega_z+\cdots\,, \quad (17)$$

where $c_{mn}{}^{xyz}$ are constants, $\alpha\beta$, $\alpha'\beta'$ represent the various intermediate states, and x, y, z stand for any of the three variables s, t, or u. By expanding $F(x)$ and $F(y)$ as partial waves in the z channel and by considering the resulting singularity structure, we find that, unless $x=y=z$, the remaining terms of Eq. (17) give rise to branch cuts along the negative s, t, and u axes. As discussed in the last section, this is to be expected because of the remaining pinch singularities. However, by taking the various symmetry functions to be constant, these singularities will normally also be present in the scattering amplitudes given by Eqs. (3) or (8). In order to maintain the known cut-plane analyticity of $T(s,t,u)$, we can do one of two things: either perform a finite number of iterations with, say, $n+m=N$ and adjust the parameters to minimize these unwanted singularities; or neglect those terms in Eq. (17) which give rise to them, i.e., assume that their effect is absorbed into the a_I. In the first case, it is unlikely that the singularities in s, t, and u can be simultaneously minimized, and it would be difficult to estimate the effect of the missing higher perturbations. In the second case, those terms with $x=y=z$ correspond to chain diagrams which can be summed completely, while it is known that in an exact solution of Eqs. (8) and (13) the singularities in the remaining terms must cancel, and hopefully, therefore, might contribute a rather smaller effect. For these reasons we shall adopt the second approach and concentrate on the so-called "chain approximation."

In the case of the coupled $\pi\pi$ and πK system, this means we can write the K-matrix elements for the process $\pi\pi\to\pi\pi$ in the form[20]

$$\mathbf{K}_I{}^{\pi\pi,\pi\pi}=a_I+\begin{bmatrix}0&\theta&\phi\\ \theta^{-1}&0&\theta^{-1}\phi\\ \phi^{-1}&\phi^{-1}\theta&0\end{bmatrix}_{IJ}\begin{bmatrix}A(s)\\A(t)\\A(u)\end{bmatrix}_J, \quad (18)$$

where

$$\theta_{IJ}=(-1)^J\Lambda_{IJ}\,,\quad \phi_{IJ}=(-1)^I\Lambda_{IJ}\,,$$

and Λ is the $\pi\pi$ crossing matrix

$$\Lambda=\begin{bmatrix}\frac{1}{3}&1&5/3\\ \frac{1}{3}&\frac{1}{2}&-\frac{5}{6}\\ \frac{1}{3}&-\frac{1}{2}&\frac{1}{6}\end{bmatrix}.$$

Similarly, for the πK system we have

$$\mathbf{K}_I{}^{\pi K,\pi K}=\begin{bmatrix}b\\c\\d\end{bmatrix}_I+\begin{bmatrix}0&\rho&\pi\\ \rho^{-1}&0&\rho^{-1}\pi\\ \pi^{-1}&\pi^{-1}\rho&0\end{bmatrix}_{IJ}\begin{bmatrix}B(s)\\C(t)\\D(u)\end{bmatrix}_J,$$

$$\mathbf{K}_I{}^{\pi\pi,K\bar{K}}=\begin{bmatrix}c\\d\\b\end{bmatrix}_I+\begin{bmatrix}0&\rho^{-1}\pi&\rho^{-1}\\ \pi^{-1}\rho&0&\pi^{-1}\\ \rho&0&0\end{bmatrix}_{IJ}\begin{bmatrix}C(s)\\D(t)\\B(u)\end{bmatrix}_J, \quad (19)$$

$$\mathbf{K}_I{}^{\pi K,K\pi}=\begin{bmatrix}d\\b\\c\end{bmatrix}_I+\begin{bmatrix}0&\pi^{-1}&\pi^{-1}\rho\\ \pi&0&\rho\\ \rho^{-1}\pi&\rho^{-1}&0\end{bmatrix}_{IJ}\begin{bmatrix}D(s)\\B(t)\\C(u)\end{bmatrix}_J,$$

[19] P. V. Landshoff and D. I. Olive, J. Math. Phys. 7, 1464 (1966).

[20] This can be deduced by writing out the first few iterations of Eq. (13) in this approximation.

where

$$\rho=\alpha^{-1}\beta\,,\quad \pi=\alpha^{-1}\gamma\,,$$

and[21]

$$\alpha=\tfrac{1}{3}\begin{pmatrix}-1 & 4\\ 2 & 1\end{pmatrix},\quad \beta=\gamma=\begin{pmatrix}1/\sqrt{6} & -1\\ 1/\sqrt{6} & -\tfrac{1}{2}\end{pmatrix}.$$

To determine the functions A, B, C, and D, we substitute the above expressions into relations similar to Eq. (13) for the various processes involving pions and kaons, and neglect those integrals involving different Mandelstam variables. The remaining integrals give rise to factors of 4π, and since Eq. (13) in this approximation is found to be linear in the functions A, B, C, and D, this procedure leads to a set of simultaneous linear equations for these functions in terms of modified phase-space factors and the constants a_I, b_I, c_I, and d_I. The solution of these equations for the coupled $\pi\pi$ and πK channels which is described in the Appendix is simplified if we note that, to satisfy the crossing relations for the symmetry functions, we must have

$$\begin{aligned}a_I&=\theta_{IJ}a_J=\phi_{IJ}a_J\,,\\ b_I&=g_{IJ}b_J=\beta_{IJ}c_J=\gamma_{IJ}d_J\,,\end{aligned}\tag{20}$$

which reduces the number of free parameters to two, one for each process involving the four particles $(\pi\pi,\pi\pi)$ and $(\pi K,\pi K)$.

This chain approximation to Eq. (13), unlike the full solution, cannot be expected to satisfy simultaneously both Eqs. (3) and (8) for the scattering amplitude $T(s,t,u)$; i.e., it cannot satisfy both unitarity and crossing. However, since the resulting K-matrix elements *already contain the effects of coupled processes in the crossed channels as well as similar effects for the coupled processes themselves*, we shall concentrate on obtaining solutions which satisfy the unitarity condition Eq. (3), using Eq. (8) only to test the validity of these solutions at least in the region of the symmetry point $s=t=u=\frac{4}{3}\mu^2$.

Numerical solutions were computed by performing a rather extensive search in the two-parameter space. In a complete solution of Eq. (13), the parameters would be arbitrary and could be adjusted so that the partial waves $f_{l,I}(s)$ calculated from Eq. (3) in the form

$$f_{l,I}{}^{ab,cd}(s)=k_{l,I}{}^{ab,cd}(s)+\sum_{\alpha\beta}2\pi iF^{\alpha\beta}(s)\times f_{l,I}{}^{ab,\alpha\beta}(s)k_{l,I}{}^{\alpha\beta,cd}(s)\,,$$

with

$$k_{l,I}(s)=\frac{1}{2}\int K_I(s,t',u')P_l(\cos\theta_s)d\cos\theta_s\tag{21}$$

and

$$f_{l,I}{}^{ab,ab}(s)=\frac{\sqrt{s}}{2\pi ik^{\alpha\beta}}(\eta_l{}^Ie^{2i\delta_l{}^I}-1)\,,$$

give the gross features of the low-energy data. However,

[21] A. O. Barut, *Theory of the Scattering Matrix* (The Macmillan Co., New York, 1967).

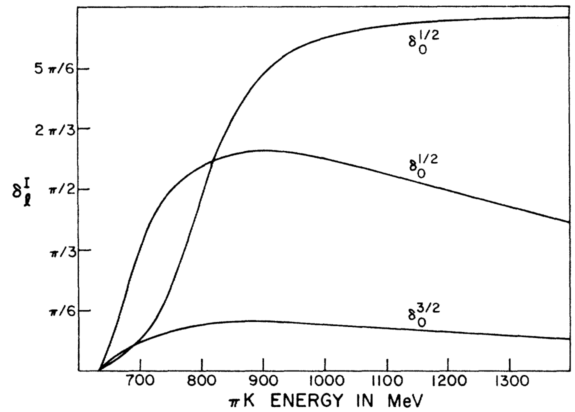

FIG. 2. Low-energy πK phase shifts computed from the chain approximation with $a_0=-0.72$, $c_0=0.96$.

with the above chain approximation having determined the region of the two-parameter space which produces both a p-wave $\pi\pi$ resonance and a p-wave $I=\frac{1}{2}$ πK resonance, the optimum approximate solution was found by varying the parameters within this subspace so as to minimize the differences in the pion-pion amplitudes T_0 $(=\frac{5}{2}T_2)$ given by Eq. (18)[22] and the similar quantities given by Eq. (3) at the point $s=t=u=\frac{4}{3}\mu^2$, with

$$T_0{}^{\pi\pi}(\tfrac{4}{3}\mu^2,\tfrac{4}{3}\mu^2,\tfrac{4}{3}\mu^2)=f_{0,0}(\tfrac{4}{3}\mu^2)-2.5f_{2,0}(\tfrac{4}{3}\mu^2)+\cdots.$$

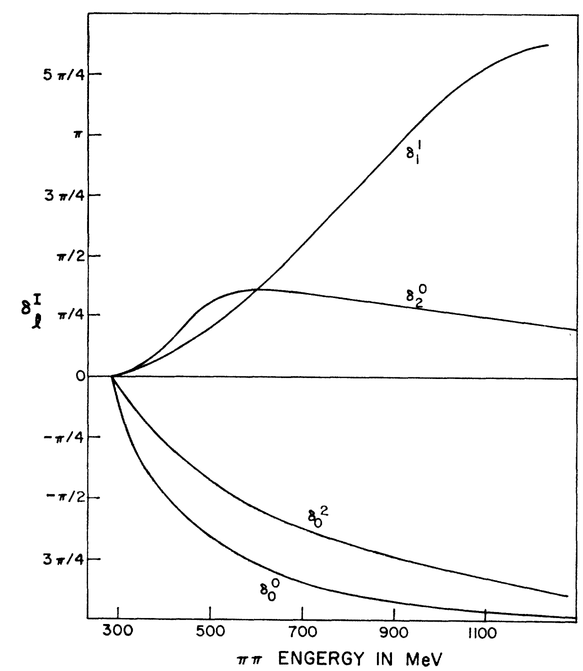

FIG. 3. Low-energy $\pi\pi$ phase shifts computed from the chain approximation with $a_0=-0.72$, $c_0=0.96$.

[22] The crossing-symmetric form of Eq. (8) already ensures that $T_0=\frac{5}{2}T_2$ at the symmetry point, but this is not the case for the amplitudes calculated from Eq. (3).

TABLE I. Resonance parameters obtained from the coupled $\pi\pi$ and πK calculation.

Resonance	Mass in MeV	Width in MeV
ρ	687	150
K^*	806	56
κ	750	230

In this manner, if the differences between these amplitudes are small, we may have some confidence that this partial solution to the scattering Eq. (13) is a good approximation to the full crossing-symmetric solution, at least in the region of the symmetry point. The optimum values for the parameters were found to be

$$a_0=-0.72\,,\quad c_0=0.96\,,$$

with the corresponding $\pi\pi$ amplitudes at the symmetry point given by

$$\text{Eq. (3)}:\quad T_0=-2.21\mu^{-1},\quad \tfrac{5}{2}T_2=-2.39\mu^{-1};$$
$$\text{Eq. (8)}:\quad T_0=-2.35\mu^{-1},$$

which represents a fairly sharp local minimum in the two-parameter space. The πK and $\pi\pi$ phase shifts are sketched in Figs. 2 and 3, respectively, and the resulting resonance parameters are given in Table I.

It will be seen that the πK phase shifts are in fair agreement with a recent dispersion-relation calculation[23] except for the κ resonance in the $T=\frac{1}{2}$ s wave, which is perhaps an unexpected bonus from this simple calculation. In the case of $\pi\pi$ scattering, the ρ-meson width is found to be much smaller than in the usual ρ bootstrap or matrix N/D equations.[11] In part, this is because the resonance has a mass which is 80 MeV below the usually accepted value. But if we relax the consistency condition on the solution at the symmetry point, and instead require that the ρ has its physical mass, the resulting width, although now of the order 300 MeV, still represents some improvement over the corresponding matrix N/D result. This point will be discussed further in Sec. V. However, it will also be seen that the s waves are still strongly bound and give rise to very negative phase shifts which are similar to the corresponding results obtained in the uncoupled solution of the Zimmermann equation.[18] This inability to produce realistic s-wave $\pi\pi$ phase shifts indicates once again that these partial waves are probably strongly dependent on a large number of processes coupled through the unitarity condition both in the direct and crossed channels. However, before discussing the effects of including another channel which is coupled directly to the $\pi\pi$ system, we should first consider the reaction $K\bar{K}\to K\bar{K}$ which, with its crossed versions, completes the pion+kaon system.[24]

[23] K. C. Gupta, R. P. Saxena, and V. S. Mathur, Phys. Rev. 141, 1479 (1966).

[24] The inclusion of $K\bar{K}\to K\bar{K}$ and its crossed versions in the $\pi\pi$ and πK system forms a closed set of amplitudes related to any two-body process involving pions and kaons.

The K-matrix elements for the reaction $K\bar{K}\to K\bar{K}$ may be written, again in the chain approximation, as

$$\mathbf{K}_{\mathrm{I}}{}^{K\bar{K},K\bar{K}}=\begin{bmatrix}e\\f\\g\end{bmatrix}_{\mathrm{I}}+\begin{bmatrix}0 & \psi & \chi\\ \psi^{-1} & 0 & \psi^{-1}\chi\\ \chi^{-1} & \chi^{-1}\psi & 0\end{bmatrix}_{IJ}\begin{bmatrix}E(s)\\F(t)\\G(u)\end{bmatrix},\qquad(22)$$

with relations similar to Eq. (19) for the crossed processes, where

$$\psi=\mu^{-1}\nu\,,\quad \chi=\mu^{-1}\lambda\,,$$

and[21]

$$\mu=\frac{1}{2}\begin{pmatrix}-1 & 3\\ 1 & 1\end{pmatrix},\quad \nu=\frac{1}{2}\begin{pmatrix}1 & 3\\ 1 & -1\end{pmatrix},\quad \lambda=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Substituting Eqs. (18), (19), (22) and its crossed versions into Eq. (13) and using the crossing condition

$$g_I=\mu_{IJ}e_J=\nu_{IJ}f_J=\lambda_{IJ}g_J\,,\qquad(23)$$

the chain approximation is obtained as before by solving the simultaneous equations for the functions A, B, C, D, E, F, and G. These functions are given in the Appendix as an example of the simultaneous solution of a closed coupled system of amplitudes involving two kinds of particles, in this case kaons and pions. It should be noted that using Eq. (23) and putting $e_0=0$ annihilates the contribution to the $\pi\pi$ and πK system arising from $K\bar{K}\to K\bar{K}$ and its crossed reactions, thus reducing the solution for A, B, C, and D to that obtained previously.

It was found that varying the value of e_0 produced only small changes in the values of the πK and $\pi\pi$ amplitudes shown in Figs. 2 and 3, indicating that the $K\bar{K}\to K\bar{K}$ reaction has little dynamical effect on these reactions. This was perhaps to be expected, since in the πK system, the longer-range two-pion exchanged terms are apparently dominant over the $K\bar{K}$ exchanges in determining the low-energy features,[12] and for $\pi\pi$ scattering, the $K\bar{K}\to K\bar{K}$ amplitude is not coupled directly to the dipion system. Thus if we are to obtain physical $\pi\pi$ s-wave solutions, we must look elsewhere, i.e., to processes which are coupled directly to the $\pi\pi$ amplitude. In the next section, therefore, we shall calculate the effect of coupling the $\pi\pi\to\pi\omega$ reaction to the $\pi\pi$ and πK system.

IV. EFFECTS OF THE $\pi\pi\to\pi\omega$ REACTION ON THE $\pi\pi$ AND πK SYSTEM

Since the ω meson has unit spin, the crossing matrices in Eq. (13) for processes involving ω now refer to the helicity-crossing matrices of Trueman and Wick.[16] These processes would normally include $\pi\pi\to\pi\omega$, $\pi\omega\to\pi\omega$, $K\bar{K}\to\pi\omega$, and their crossed versions, but to keep the complications due to spin as small as possible, we shall ignore the $\pi\omega\to\pi\omega$ and $K\bar{K}\to\pi\omega$ processes. In comparison with the analogous reaction $K\bar{K}\to K\bar{K}$ of the last section, we see that this may only have a slight effect on the resulting $\pi\pi$ and πK amplitudes.

In this case the K-matrix elements for the process $\pi\pi\rightarrow\pi\omega$ may be written in the chain approximation as

$$\mathbf{K}_\lambda{}^{\pi\pi,\pi\omega}=h_\lambda+\begin{bmatrix}0 & \theta & \phi\\ \theta^{-1} & 0 & \theta^{-1}\phi\\ \phi^{-1} & \phi^{-1}\theta & 0\end{bmatrix}_{\lambda\mu}\begin{bmatrix}H(s)\\ H(t)\\ H(u)\end{bmatrix}_\mu, \quad (24)$$

where λ, μ refer to the helicities $\pm1,0$ of the particle ω, and[16]

$$\theta_{\lambda\mu}=\theta_{\lambda\mu}(\xi)$$
$$\equiv\begin{bmatrix}\frac{1}{2}(1+\cos\xi) & -(\sqrt{\tfrac{1}{2}})\sin\xi & \frac{1}{2}(1-\cos\xi)\\ (\sqrt{\tfrac{1}{2}})\sin\xi & \cos\xi & -(\sqrt{\tfrac{1}{2}})\sin\xi\\ \frac{1}{2}(1-\cos\xi) & (\sqrt{\tfrac{1}{2}})\sin\xi & \frac{1}{2}(1+\cos\xi)\end{bmatrix}, \quad (25)$$
$$\phi_{\lambda\mu}=\theta_{\lambda\mu}(\zeta),$$

with

$$\cos\xi=[t/(t-4m_\omega^2)]^{1/2}(s+m_\omega^2-\mu^2)/S_{cd},$$
$$\cos\zeta=[u/(u-4\mu^2)]^{1/2}(s+\mu^2-m_\omega^2)/S_{cd},$$

and

$$S_{cd}^2=[s+(m_\omega+\mu)^2][s+(m_\omega-\mu)^2].$$

The functions A_J, B_J, C_J, D_J, and H_λ forming the coupled $\pi\pi$, πK, and $\pi\omega$ system in this case can be determined by substituting Eqs. (18), (19), and (25) into Eq. (13) and solving the resulting simultaneous equations in exactly the same manner as previously. In fact, since the $\pi\pi\rightarrow\pi\omega$ reaction is coupled only to the $I=1$ $\pi\pi$ channel, in which $a_{I=0}=0$, these simultaneous equations take on a much simpler form than in the previous case. Also, using the condition of parity conservation for the helicity amplitudes, we find that we may put

$$h_1=h_{-1},\quad h_0=0,$$

which reduces the number of arbitrary parameters in this example to three.

Numerical solutions were found from Eq. (3) by relating the $\pi\pi\rightarrow\pi\omega$ helicity amplitudes to partial waves, using the well-known formalism of Jacob and Wick.[25] The parameters were again adjusted to produce both a $\pi\pi$ p-wave resonance and a πK $I=\frac{1}{2}$ p-wave resonance, and the best "crossing-symmetric" solution was determined by minimizing the differences in the resulting amplitudes given by Eqs. (3) and (8) at the symmetry point. In scanning this three-parameter space, it was found that probably the smallest local minimum was given by $h_1=0$, and a_0 and c_0 having the values -0.72 and 0.96 obtained previously; i.e., the solution found in Sec. III is again the best chain-approximation solution to Eq. (3) in the neighborhood of the symmetry point. Nevertheless, on carefully inspecting the parametric space, we found that there is another rather shallow local minimum which, while not satisfying the crossing condition so closely, does give rise to more physical-looking $\pi\pi$ s-wave phase shifts. This solution is given by

$$a_0=0.15,\quad c_0=1.42,\quad h_1=0.96, \quad (26)$$

and the corresponding amplitudes at the symmetry point are found to be

$$\text{Eq. (3)}:\quad T_0=1.51\mu^{-1},\quad \tfrac{5}{2}T_2=2.32\mu^{-1};$$
$$\text{Eq. (8)}:\quad T_0=0.85\mu^{-1}.$$

While this is not a very close fit to the crossing relation at this point, it does represent a marked improvement over a random choice for the three parameters which typically can lead either to opposite signs for T_0 and T_2 given by Eq. (3), or to T_2, an order of magnitude larger than T_0, or to similar discrepancies between the values of T_0 calculated from Eqs. (3) and (8). In this context, in view of the already approximate nature of the calculation, we believe the results given by Eq. (26) can be thought to be quite satisfactory solutions to the dynamical equations. The corresponding πK and $\pi\pi$ phase shifts are sketched in Figs. 4 and 5, and the resulting resonance parameters[26] are given in Table II. The inelasticity factors $\eta_l{}^I$ for πK scattering are unity, since no direct coupled channel has been considered, and for $\pi\pi$ scattering, $\eta_0{}^0$ is found to be ≥ 0.8, while the dominant contribution in the $I=1$ partial wave comes from the $\pi\omega$ channel, and $\eta_1{}^1$ is similar to that used in Ref. 10.

It will be seen that the $\pi\pi$ $I=0$ s wave in this case is similar to the phenomenological values obtained by Walker *et al.*[6] up to 600 MeV, although the possible resonance at 900 MeV is not reproduced. Up to 900 MeV, it is also very similar to the nonresonating oslution determined by Fulco and Wong[27] to be consistent with forward dispersion relations, forward-back-

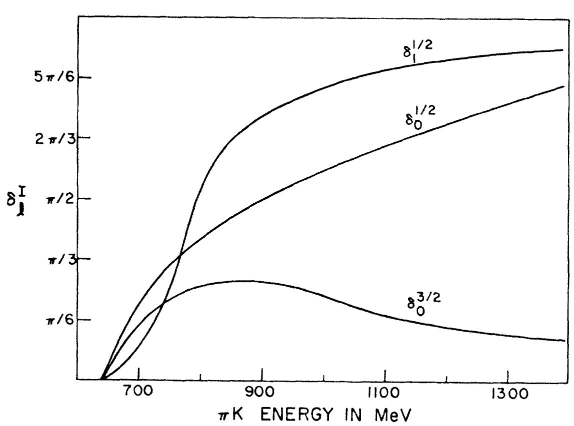

FIG. 4. πK phase shifts obtained by coupling the $\pi\pi\rightarrow\pi\omega$ reaction to the $\pi\pi$ and πK system.

[25] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[26] As in the two-channel case, if we demand that the ρ has a mass of 765 MeV, the width is increased by 100 to 150 MeV, depending on the chosen values of the parameters.

[27] J. R. Fulco and D. Y. Wong, Phys. Rev. Letters 19, 1399 (1968).

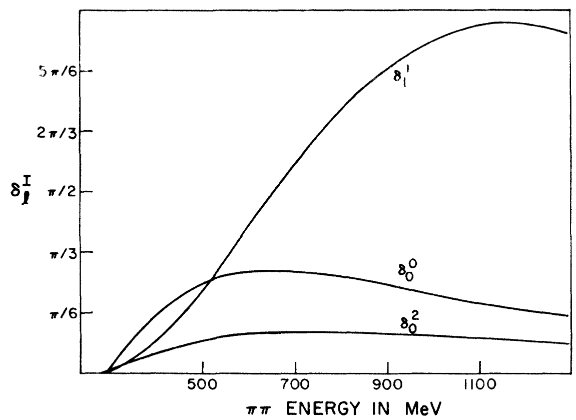

FIG. 5. $\pi\pi$ phase-shifts obtained by coupling the $\pi\pi \to \pi\omega$ reaction to the $\pi\pi$ and πK system.

ward asymmetry in $\pi^-\pi^0$ and $\pi^+\pi^-$ production, the K_1-K_2 mass difference, and $\delta_0^2-\delta_0^0$ in $K_L \to 2\pi$, as well as with the unitarity corrections to current-algebra results and with the Adler sum rule. However, the $I=2$ s-wave phase shift in our calculation is now also positive, which is not consistent with recent phenomenological analyses,[6] and the well-established f^0 resonance in the $I=0$ d wave is not reproduced. Also, it could be argued that the existence of a κ resonance in the $I=\frac{1}{2}$ πK s wave should imply the existence of a similar resonance in the $I=0$ $\pi\pi$ s wave, both being members of an SU_3 octet. Thus, we would not like to make any quantitative statements from this calculation about the detailed structure of low-energy $\pi\pi$ scattering.[28] However, we believe we can safely conclude that before any S-matrix estimate of the $\pi\pi$ interaction can be made, the detailed energy-dependent effects of at least three coupled processes, both in the direct and crossed channels, must be included as well as some averaged effect of higher-order contributions.

V. DISCUSSION

As we have stressed throughout this paper, the symmetry functions have those singularities, among others, of the scattering amplitudes not extracted by the unitarity-type equation [Eq. (3)], and therefore reflect the contributions to the amplitudes arising from the higher-order processes. In our numerical calculation we have taken these functions to be constant,[29] but at the same time have explicitly considered one or two low-mass coupled reactions which may produce strong energy-dependent effects on the low-energy $\pi\pi$ and πK phase shifts. In this manner, not only have we been able to produce the main features of πK scattering, but in $\pi\pi$ scattering, the ρ-meson parameters are more closely produced than in an analogous calculation with the matrix N/D method.[11] This holds even when we require that the ρ has its physical mass. In fact, the calculated width in this case, although up to 150 MeV greater than before, is still fairly similar to the result obtained[10] from the single-channel N/D equations in which some phenomenological effect of an infinite number of inelastic channels was considered. Since in our calculation the physical ρ mass can be reached only by relaxing the consistency condition at the symmetry point, we should not, perhaps, compare the two methods until we can achieve a $\pi\pi$ p-wave resonance at 765 MeV with a solution which is at least approximately crossing-symmetric. However, since this would probably require the parametric freedom of at least one further coupled channel, it would be surprising if the resulting ρ width was found to be any larger, and we might even expect it to be somewhat reduced.

Another interesting feature of the present approach is that, since the dynamical equation [Eq. (13)] is not plagued with the CDD ambiguity of this N/D approach and takes account of coupled processes in all three channels, we have at least been able to attempt a realistic calculation of $\pi\pi$ s waves. With the explicit coupling of the $\pi\pi \to K\bar{K}$ and $\pi\pi \to \pi\omega$ channels, we have seen that this attempt is quite successful, leading to an $I=0$ s-wave phase shift which has some of the features suggested by recent phenomenological studies. We must emphasize, however, that before we can firmly establish the results presented in this paper, we must know the size of the error incurred in considering only chain diagrams in the solution of Eq. (13). To do this it would be advantageous to find some suitable functional form for the symmetry functions for which the dynamical equations can be solved either exactly or to a better approximation than that considered here. We believe that this may be possible, at least in the case of spinless equal-mass particle scattering, and work in this direction is in progress. Nevertheless, the condition that the chain approximation solution is approximately crossing-symmetric in the neighborhood of the symmetry point does give us some confidence in these preliminary results.

Finally, let us briefly discuss two interesting aspects of the Zimmermann equation. It should be noted that

TABLE II. Resonance parameters obtained by coupling the $\pi\pi \to \pi\omega$ reaction to the $\pi\pi$ and πK system.

Resonance	Mass in MeV	Width in MeV
ρ	654	185
K^*	789	49
κ	902	290

[28] There is some evidence [see G. Ellison and S. Humble' Phys. Rev. **173**, 1563 (1968); S. Humble, Nucl. Phys. **B8**, 695 (1968); J. Pisut, *ibid.* **B8**, 159 (1968)] that the $I=0$ scattering length is negative, in which case the inclusion of the $\pi\pi \to \pi\omega$ channel overestimates the effect of the higher-mass coupled channels.

[29] It should be noted that these constants are in some ways analogous to subtraction constants in dispersion relations, which also reflect the contribution of the high-energy effects in the low-energy region.

Eq. (13) is most easily solved by an iteration procedure starting with $K(s,t,u)=\mathfrak{T}(s,t,u)$. In other words, the resulting solution will be a perturbation expansion in what is essentially the contribution of the higher-mass effects to the low-energy amplitude, and no *a priori* knowledge of the total low-energy amplitude is required. This is to be compared with dispersion relations, where the iterative procedure is usually performed in terms of the generalized Born contributions, which does assume some detailed knowledge of the bound-state, resonant or nonresonant features of the low-energy amplitudes.

The other aspect of the present approach to S-matrix calculations which we should like to mention is the use of the analyticity requirement. In order to write down the crossing relations, of course, it has been assumed that the two-body amplitudes have the required cut-plane analytic structures, but explicit use of the analyticity conditions has been made only in determining the symmetry functions and as a consistency condition on the resulting solution of the dynamical equation. The advantage of this approach may be most apparent in the consideration of low-energy production processes, where the usual dispersion-relation techniques even for single-particle production become embroiled in the discussion of the detailed analyticity of the amplitudes defined on a five-dimensional space of variables. If an equation similar to that proposed by Zimmermann can be written down for these production processes, then it may prove possible to formulate models in which the necessary analytic properties may be assumed in order to use the crossing relations of, for instance, Barut and Leung.[30] The detailed analytic structure of the amplitude would still be required to determine the symmetry functions, but in practical application, it may be sufficient to again use some simple approximations for these. Some work in this direction has already been done,[31] but this has consisted solely of considering final-state effects, and the consideration of possible crossed-channel contributions has so far been ignored.

In conclusion, we believe the dynamical system of equations presented here represents a method which is complementary to, and in some respects more appealing than, the usual dispersion-relation techniques of calculating low-energy scattering amplitudes, although for quantitative accuracy, some work must be done in developing more exact solutions to the basic equations. The idea that extensions to these equations may provide a bridge between two-body reactions and production processes is an interesting possibility which we hope to pursue in a subsequent paper.

[30] A. O. Barut and Y. C. Leung, Phys. Rev. **138**, B1119 (1965).

[31] M. O. Taha, Nuovo Cimento **42B**, 201 (1966); F. Riordan, *ibid.* **58B**, 649 (1968).

APPENDIX

The solution to the set of linear simultaneous equations described in Sec. III for the coupled $\pi\pi$, πK, and $K\bar{K}$ system is given by

$$A_I(x)=X(h,a,c;\,f(x),2g(x),0)/Z(a,c,h;\,f(x),2g(x)),$$
$$B_I(x)=Y(a,c,h;\,f(x);\,2g(x),0)/Z(a,c,h;\,f(x),2g(x)),$$

with

$$h_I=\tfrac{1}{2}(e_I+f_I),$$

and

$$\begin{aligned}
C_I(x)&=X(b,b,d;\,h(x),h(x),0)/Z(b,d,b;\,h(x),h(x)),\\
D_I(x)&=Y(b,d,b;\,h(x),h(x),0)/Z(b,d,b;\,h(x),h(x)),\\
E_I(x)&=X(e,e,f;\,g(x),cf(x)C(x))/Z(e,f,e;\,g(x),g(x)),\\
F_I(x)&=Y(e,f,e;\,g(x);\,g(x)cf(x)C(x))/\\
&\qquad Z(e,f,e;\,g(x),g(x)),\\
G_I(x)&=-g^2Xg(x)/[i+gXg(x)],
\end{aligned}\tag{A1}$$

where the isospin indices on the constants have been suppressed. The functions $f(x)$, $g(x)$, $h(x)$ represent the modified phase-space factors

$$\begin{aligned}
f(x)&=2\pi F^{\pi\pi}(x),\\
g(x)&=2\pi F^{K\bar{K}}(x)=2\pi F^{KK}(x),\\
h(x)&=2\pi F^{\pi K}(x),
\end{aligned}\tag{A2}$$

and the functions X, Y, Z are given by

$$\begin{aligned}
X(x,y,z;\,p,q,r)&=(xq+i)(y^2p+z^2q+r)-zq(yzp+zxq+r),\\
Y(x,y,z;\,p,q,r)&=(xp+i)(xyp+yzq+r)\\
&\qquad -yp(x^2p+y^2q+r),\\
Z(x,y,z;\,p,q)&=y^2pq-(xp+i)(zq+i).
\end{aligned}\tag{A3}$$

Putting the constants e, f, and g equal to zero in (A1) reduces the values of A_I, B_I, C_I, D_I to the solution obtained for the coupled $\pi\pi$ and πK system, and equating all constants to zero except a_I reduces the functional form of $A_I(x)$ to

$$A_I(x)=-a_I^2f(x)/[i+a_If(x)],$$

which is equivalent to the solution for the uncoupled $\pi\pi$ system found in Ref. 18.