$$\langle N^*|S_0|p\rangle = -4\times\frac{2}{3}\frac{\xi}{(1-\xi^2)^{1/2}}\sin\phi, \qquad \text{(B14)}$$

and $\langle p|S_3|p\rangle = 1$.

Using again the approximation of singlet exchange to calculate the ratios of N^* production to ealstic scattering, we find

$$\left|\frac{T_{N^*p}}{T_{pp}{}^{\mathrm{el}}}\right|^2 \simeq \frac{8}{9}\frac{\xi^2}{\Gamma}\frac{\lambda-1}{[1+\frac{2}{3}(\lambda-1)/\Gamma]^2}. \qquad \text{(B15)}$$

For small Γ, Eq. (B15) is approximately $2\Gamma\xi^2/(\lambda-1)$, consistent with a correspondingly small rate for N^* production.[26] A consistent solution is found, in fact, for $\sin^2\phi = 0.1$ [a "physical" proton of 90% $(56,1)_+$], a $\hat{\chi}$ which is 99.8% χ and an N^* which is 99% $b\hat{\chi}^2$.

With these parameters ($\Gamma = 0.0734$, $\xi^2 = 0.99$) and the parameters of the CHKN fit a, calculation including all of the exchanges considered in CHN provides essentially the same results obtained in Sec. 7.

[26] Note that our conclusion concerning the suitably of admixture with a single level would be completely reversed if λ were close to unity. This might occur if, e.g., axial-vector exchange could be as important as the exchange coupling to S_8.

PHYSICAL REVIEW VOLUME 183, NUMBER 5 25 JULY 1969

Helicity Crossing Matrix for Production Processes*

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(Received 12 March 1969)

The crossing matrix of the helicity amplitude for a production process is worked out with the conventional definition of helicity states. This crossing matrix has the same form in different c.m. frames except for particles moving along the z axis, which have additional phases. It can thus be used for successive crossings. The generalization to N-to-N' processes is discussed, and application to the Regge formalism is considered.

I. INTRODUCTION

THE crossing matrix, which relates the direct-channel and the crossed-channel helicity amplitudes, has been calculated in a two-body–to–two-body process (2-to-2) by Trueman and Wick,[1] Muzinich,[2] and Cohen-Tannoudji, Morel, and Navelet (CMN).[3] Capella,[4] using a technique introduced by Moussa and Stora[5] and utilized by CMN,[3] has derived the crossing matrix through an unconventional definition of helicity states. By his definition, Capella avoids certain phase angles in the crossing matrix. From the point of view of a conventional helicity[6] definition, Capella's helicity states are defined in an "unnatural" frame, though the frame has not yet been worked out explicitly. On the other hand, the boost used in CMN's[3] paper can easily be shown to coincide with the conventional boost in the c.m. frame. The advantage of the conventional approach is shown by considering two successive crossings, as in Fig. 1. The total crossing matrix from the direct channel (d) to the second crossed channel (c_2) is composed of two sub-crossing matrices, one from the channel (d) to (c_1) and a second from (c_1) to (c_2), together with a transformation matrix to connect the frame used to calculate the crossing matrix from (d) to (c_1) and the frame used to calculate the crossing matrix from (c_1) to (c_2). For Capella's method, the transformation matrix will be very complicated and has not been worked out in his paper. If we calculate the two crossing matrices in the same frame, then we do not need to perform the transformation. But for one of the two sub-crossing matrices, the simplicity of Capella's result will disappear, and that matrix must be calculated independently. Of course, since Capella's definition of helicity states is not the conventional one, the crossing matrix from Capella's method and that from the con-

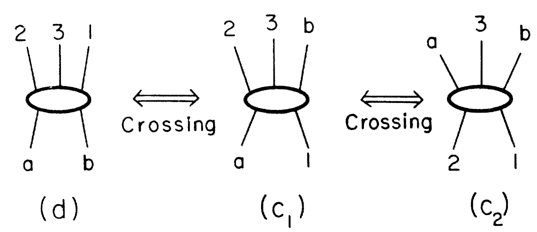

FIG. 1. An example of successive crossings: The direct channel (s_{ab} channel) is crossed into the intermediate channel (t_{a1} channel), and then the final crossed channel (s_{12} channel). The total crossing matrix is $C_I C_r$. The intermediate channel may be t_{a2}, t_{b1}, or t_{b2} channel. However, the total crossing matrix is independent of the intermediate channel.

* Work supported in part by the U. S. Atomic Energy Commission.
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ventional definition of helicity states not only differ in form but also have a different physical interpretation. In this paper, we consider the crossing matrix of the conventional helicity amplitudes for a 2-to-N process, and obtain a result from which we can easily write down two or more successive crossings.

In Sec. II, we start from the analytic properties and the crossing relation of the spinor amplitudes (M function[7,8]) for a 2-to-3 process, and then derive the crossing matrix. The method to calculate the crossing matrix is discussed and the process with two successive crossings is also considered. The results are then generalizable to a 2-to-N process. In Sec. III, the explicit expressions for the crossing angles and the transformation matrix due to change of the choice of x-z plane are worked out for a 2-to-3 process. They are easily generalized to a 2-to-N process. In Sec. IV, some applications of the crossing matrix are discussed. In particular, we consider the case when the direct-channel helicity amplitude, dominated by exchange of crossed-channel Reggeons and/or saturated by the direct-channel resonances, can be expressed in terms of the double Regge terms of the appropriate crossed-channel helicity amplitude through the crossing matrix. In the Appendix, the asymptotic expressions for the crossing angles when the total energy goes to infinity are written down explicitly.

II. KINEMATICS AND CROSSING MATRIX

Consider the process $a+b\rightarrow 1+2+3$ (s_{ab} channel). The particle state i is described by its four-momentum p_i, spin s_i, and the helicity λ_i. We define the c.m. frame of the s_{ab} channel such that the incoming particles a and b move along the z axis and the crossed particle 1 is in the x-z plane with a positive x component of its momentum. Later on, we will show that a different choice of the x-z plane will result in a difference of the phase factor. The invariant variables are defined as

$$s_{ab}=(p_a+p_b)^2\,,\quad s_{ij}=(p_i+p_j)^2\equiv s_k\,,\qquad (i,j,k=1,2,3\text{ cyclic})\qquad(1)$$
$$t_{ia}=(p_i-p_a)^2\,,\quad t_{ib}=(p_i-p_b)^2\equiv t_j(i,j=2,3,i\neq j)\,.$$

We choose s_{ab}, s_{12}, s_{23}, t_{1a}, and t_{3b} as independent variables. The four-momenta p_i in the c.m. frame of the s_{ab} channel are parametrized by

$$\begin{aligned}
p_a&=m_a(\cosh\alpha_a,\,0,\,0,\,\sinh\alpha_a)\,,\\
p_b&=m_b(\cosh\alpha_b,\,\sinh\alpha_b\sin\pi,\,0,\,\sinh\alpha_b\cos\pi)\,,\\
p_1&=m_1(\cosh\alpha_1,\,\sinh\alpha_1\sin\theta_1,\,0,\,\sinh\alpha_1\cos\theta_1)\,,\\
p_i&=m_i(\cosh\alpha_i,\,\sinh\alpha_i\cos\theta_i,\,\sinh\alpha_i\sin\theta_i\cos\phi_i,\\
&\qquad\sinh\alpha_i\sin\theta_i\sin\phi_i)\quad(i=2,3)\,.
\end{aligned}\qquad(2)$$

The invariant variables are related by

$$\begin{aligned}
&t_{ia}+t_{ib}+s_{ab}=m_a^2+m_b^2+m_i^2+s_{jk}\,,\qquad (i,j,k=1,2,3\text{ cyclic})\\
&t_{ia(b)}+t_{ja(b)}+s_{ij}=m_i^2+m_j^2+m_{a(b)}^2+t_{kb(a)}\,,\\
&s_{12}+s_{23}+s_{31}=m_1^2+m_2^2+m_3^2+s_{ab}\,.
\end{aligned}\qquad(3)$$

The crossed channel $a+\bar{1}\rightarrow\bar{b}+2+3$ is called the t_{ia} channel. We shall calculate the crossing matrix between s_{ab} and t_{1a} channels. The crossing matrices between any other two channels can be obtained either by similar methods or by crossing twice. The four-momenta are denoted by $\bar{q}_i$, and parametrized by $(\bar{\alpha}_i,\bar{\theta}_i,\bar{\phi}_i)$ in a similar way as in the s_{ab}-channel c.m. frame.

Before calculating the crossing matrix, we have to make the following two assumptions[3,4]: (a) There exists an analytic domain such that the spinor amplitude can be continued analytically in the invariant variables from the physical region of the s_{ab} channel to that of the t_{ia} channel with all four-momenta fixed on the mass shell. (b) The spinor amplitude $M_{\lambda_1\lambda_2\lambda_3;\lambda_a\lambda_b}(p_1p_2p_3;p_ap_b)$ satisfies the crossing relation[9]

$$M_{\lambda_1\lambda_2\lambda_3;\lambda_a\lambda_b}{}^{(s_{ab})}(p_1p_2p_3;p_ap_b)=(-1)^\sigma M_{\lambda_b\lambda_2\lambda_3;\lambda_a\lambda_1}{}^{(t_{1a})}(-p_bp_2p_3;p_a-p_1)\,.\qquad(4)$$

The phase label σ is unity if the two crossed particles are fermions, and zero otherwise. The spinor amplitude[3,7] is defined to transform under a complex Lorentz transformation Λ as

$$M_{\lambda_b\lambda_2\lambda_3;\lambda_a\lambda_1}{}^{(t_{1a})}(\bar{p})=\sum_{\bar{\lambda}}D_{\bar{\lambda}_b\lambda_b}{}^{s_b}(\Lambda)D_{\bar{\lambda}_2\lambda_2}{}^{s_2}(\Lambda)D_{\bar{\lambda}_3\lambda_3}{}^{s_3}(\Lambda)\times M_{\bar{\lambda}_b\bar{\lambda}_2\bar{\lambda}_3;\bar{\lambda}_a\bar{\lambda}_1}{}^{(t_{1a})}(\Lambda\bar{p})D_{\bar{\lambda}_a\lambda_a}{}^{s_a}(\Lambda)D_{\bar{\lambda}_1\lambda_1}{}^{s_1}(\Lambda)\,,\qquad(5)$$

and similarly for $M^{(s_{ab})}(p)$. The spinor amplitudes $M^{(s_{ab})}(p)$ and $M^{(t_{1a})}(\bar{p})$ are related to the helicity amplitudes $H^{(s_{ab})}$ and $H^{(t_{1a})}$, respectively, by

$$H_{\lambda_1\lambda_2\lambda_3;\lambda_a\lambda_b}{}^{(s_{ab})}(p)=\sum_{\bar{\lambda}}D_{\bar{\lambda}_1\lambda_1}{}^{s_1}(L_{p_1}\epsilon)D_{\bar{\lambda}_2\lambda_2}{}^{s_2}(L_{p_2}\epsilon)\times D_{\bar{\lambda}_3\lambda_3}{}^{s_3}(L_{p_3}\epsilon)M_{\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3;\bar{\lambda}_a\bar{\lambda}_b}{}^{(s_{ab})}(p)\times D_{\bar{\lambda}_a\lambda_a}{}^{s_a}(L_{p_a})D_{\bar{\lambda}_b\lambda_b}{}^{s_b}(L_{p_b})\qquad(6)$$

and

$$H_{\lambda_b\lambda_2\lambda_3;\lambda_a\lambda_1}{}^{(t_{1a})}(\bar{p})=\sum_{\bar{\lambda}}D_{\bar{\lambda}_b\lambda_b}{}^{s_b}(L_{\bar{p}_b}\epsilon)D_{\bar{\lambda}_2\lambda_2}{}^{s_2}(L_{\bar{p}_2}\epsilon)\times D_{\bar{\lambda}_3\lambda_3}{}^{s_3}(L_{\bar{p}_3}\epsilon)M_{\bar{\lambda}_b\bar{\lambda}_2\bar{\lambda}_3;\bar{\lambda}_a\bar{\lambda}_1}{}^{(t_{1a})}(\bar{p})\times D_{\bar{\lambda}_a\lambda_a}{}^{s_a}(L_{\bar{p}_a})D_{\bar{\lambda}_1\lambda_1}{}^{s_1}(L_{\bar{p}_1})\,,\qquad(7)$$

where $\epsilon=i\sigma_2$ (σ_2 is the Pauli matrix), and the boost L_{p_i} is uniquely defined through the equation

$$L_{p_i}=e^{-i\phi_iJ_3}e^{-i\theta_iJ_2}e^{-i\alpha_iK_3}\,.\qquad(8)$$

We now continue Eq. (6) from the physical region of the t_{1a} channel to that of the s_{ab} channel along the path

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stated in the assumptions. The discussion of the analytic properties of the amplitudes under the continuation in CMN's paper[3] is also applicable to a 2-to-3 process. At any point on the generalized Mandelstam diagram[10] of a five-body process in the physical region of s_{ab} channel, there exists[11] one unique complex Lorentz transformation Λ^{-1} which carries the continued set of four-momenta

$$\{\bar{z}\}=\{\bar{p}_a{}^c,\bar{p}_b{}^c,\bar{p}_1{}^c,\bar{p}_2{}^c,\bar{p}_3{}^c\}$$

to the set

$$\{z\}=\{p_a,\,-p_b,\,-p_1,\,p_2,\,p_3\}\,,$$

since $z_i\cdot z_j=\bar{z}_i\cdot\bar{z}_j$ for $\bar{z}_i$ and z_i in the barred and unbarred sets of four-momenta, respectively. From Eqs. (4)–(7), we have

$$\begin{aligned}
&H_{\{\lambda\}}{}^{(s_{ab})}(s_{ab},s_{12},s_{23},t_{1a},t_{3b})\\
&\quad=(-1)^\sigma\sum_{\{\bar{\lambda}\}}(-1)^{\lambda_2-\bar{\lambda}_2+\lambda_3-\bar{\lambda}_3+s_1+\lambda_1-s_b-\bar{\lambda}_b}\\
&\qquad\times D_{\bar{\lambda}_1-\lambda_1}{}^{s_1}(L_{\bar{p}_1{}^c}{}^{-1}\Lambda L_{p_1})D_{-\bar{\lambda}_2-\lambda_2}{}^{s_2}(L_{\bar{p}_2{}^c}{}^{-1}\Lambda L_{p_2})\\
&\qquad\times D_{-\bar{\lambda}_3-\lambda_3}{}^{s_3}(L_{\bar{p}_3{}^c}{}^{-1}\Lambda L_{p_3})H_{\{\bar{\lambda}\}}{}^{(t_{1a})}(s_{ab},s_{12},s_{23},t_{1a},t_{3b})\\
&\qquad\qquad\times D_{\bar{\lambda}_a\lambda_a}{}^{s_a}(L_{\bar{p}_3{}^c}{}^{-1}\Lambda L_{p_a})D_{-\bar{\lambda}_b\lambda_b}{}^{s_b}(L_{\bar{p}_b{}^c}{}^{-1}\Lambda L_{p_b})\,,
\end{aligned}\tag{9}$$

where $H^{(s_{ab})}$ and $H^{(t_{1a})}$ are the c.m. helicity amplitudes in the s_{ab} and t_{1a} channels, respectively. In Eq. (9), there is an over-all phase factor $\eta^{(s)}$ undetermined, as shown in Ref. 3. We have two kinds of techniques to calculate the crossing angles. One is used by Trueman and Wick[1] and by Muzinich,[2] and the other by CMN[3] and by Capella.[4] We shall use the latter method. The boost $L_{ab}(i)$ in the s_{ab} channel in Ref. 3 is defined as

$$L_{ab}(i)\{\dot{t},\dot{\eta}_1,\dot{\eta}_2,\dot{\eta}_3\}=\{q_i/m_i,\eta_1(i),\eta_2(i),\eta_3(i)\}\,,\tag{10}$$

where $\{\dot{t},\dot{\eta}_1,\dot{\eta}_2,\dot{\eta}_3\}$ is a standard tetrad, and

$$\begin{aligned}
\eta_2(i)_\nu&=-\epsilon_{\mu\nu\rho\sigma}q_a{}^\nu q_b{}^\rho q_1{}^\sigma/[G(ab1,ab1)]^{1/2}\\
&\qquad\qquad(\text{for } i=a,b,1)\\
&=-\epsilon_{\mu\nu\rho\sigma}q_a{}^\nu q_b{}^\rho q_i{}^\sigma/[G(abi,abi)]^{1/2}\,,\\
&\qquad\qquad(\text{for } i=2,3)
\end{aligned}$$

$$G(ijk,lmn)=\det\begin{vmatrix} q_i\cdot q_l & q_j\cdot q_l & q_k\cdot q_l\\ q_i\cdot q_m & q_j\cdot q_m & q_k\cdot q_m\\ q_i\cdot q_n & q_j\cdot q_n & q_k\cdot q_n\end{vmatrix}\,,\tag{11}$$

$$\eta_3(i)=-\frac{m_i{}^2(q_a+q_b)-[q_i\cdot(q_a+q_b)]q_i}{m_i\{[q_i\cdot(q_a+q_b)]^2-m_i{}^2(q_a+q_b)^2\}^{\frac{1}{2}}}\,,$$

$$\eta_1(i)_\mu=\epsilon_{\mu\nu\rho\sigma}\eta_2(i)^\nu\eta_3(i)^\rho q_i{}^\sigma/m_i\,,$$

the four-momenta q_i being in an arbitrary frame in the s_{ab} channel. The $\eta_j(i)$'s are orthogonal to each other and normalized to positive or negative unity depending on whether they are spacelike or timelike. Similarly, $L_{ab}(i)$, in an arbitrary frame in the t_{1a} channel, is defined by

$$L_{a1}(i)\{\dot{t},\dot{\eta}_1,\dot{\eta}_2,\dot{\eta}_3\}=\{\bar{q}_i/m_i,\bar{\eta}_1(i),\bar{\eta}_2(i),\bar{\eta}_3(i)\}\,,\tag{12}$$

$$\begin{aligned}
\bar{\eta}_2(i)_\mu&=-\epsilon_{\mu\nu\rho\sigma}\bar{q}_a{}^\nu\bar{q}_1{}^\rho\bar{q}_b{}^\sigma/[G(a1b,a1b)]^{1/2}\quad(\text{for } i=a,1,b)\\
&=-\epsilon_{\mu\nu\rho\sigma}\bar{q}_a{}^\nu\bar{q}_1{}^\rho\bar{q}_i{}^\sigma/[G(a1i,a1i)]^{1/2},\quad(\text{for } i=2,3)
\end{aligned}$$

$$\bar{\eta}_3(i)=-\frac{m_i{}^2(\bar{q}_a+\bar{q}_1)-[\bar{q}_i\cdot(\bar{q}_a+\bar{q}_1)]\bar{q}_i}{m_i\{[\bar{q}_i\cdot(\bar{q}_a+\bar{q}_1)]^2-m_i{}^2(\bar{q}_a+\bar{q}_1)^2\}^{\frac{1}{2}}}\,,\tag{13}$$

$$\bar{\eta}_1(i)_\mu=\epsilon_{\mu\nu\rho\sigma}\bar{\eta}_2(i)^\nu\bar{\eta}_3(i)^\rho\bar{q}_i{}^\sigma/m_i\,.$$

By explicit calculation we can show that $L_{ab}{}^{\text{c.m.}}(i)$ and $L_{a1}{}^{\text{c.m.}}(i)$ in the c.m. frames of s_{ab} and t_{1a} channels with the plane of particles a, b, and 1 as x-z plane are equal to L_{p_i} and $L_{\bar{p}_i}$, respectively; i.e.,

$$L_{ab}{}^{\text{c.m.}}(i)=L_{p_i}\,,\quad L_{a1}{}^{\text{c.m.}}(i)=L_{\bar{p}_i}\,.\tag{14}$$

This is the essential reason why we define $\eta_2(i)$ and $\bar{\eta}_2(i)$ in Eqs. (7) and (9) differently from Capella's definition.[4] Further, the relation between the continued $L_{a1}{}^C(i)$ and $L_{\bar{p}_i{}^c}$ can be written as

$$L_{a1}{}^C(i)=\Lambda^{-1}L_{a1}{}^{\text{c.m.}}(\bar{p}_i{}^c)=\Lambda^{-1}L_{\bar{p}_1{}^c}\,.\tag{15}$$

It means that the crossing angle $L_{\bar{p}_i{}^c}{}^{-1}\Lambda L_{p_i}$ is equal to $L_{a1}{}^C(i)^{-1}L_{ab}(i)$, and we will calculate the crossing angle from $L_{a1}{}^C(i)^{-1}L_{ab}(i)$. Thus, the method introduced by CMN[3] is applicable to a 2-to-3 process. For 2-to-N processes $a+b\to 1+2+\cdots+N$ and $a+\bar{1}\to\bar{b}+2+\cdots+N$, L_{ab} and L_{a1} still have the forms given by Eqs. (10)–(13) except that the index i runs from 2 to N and the invariant variables s_i and t_i are redefined as

$$s_i=(p_a+p_b-p_i)^2\,,\quad t_i=(p_a+p_1-p_i)^2\,.\tag{1'}$$

We note that in the above discussion we choose the x-z plane in s_{ab} and t_{1a} channel as the plane specified by the particles a, b, and 1. If we choose the x-z planes as the plane specified by the particles a, b, and 2 in the s_{ab} channel and the plane of a, 1, and 2 in the t_{1a} channel, then we only need to change the definition of $\eta_2(i)(i=a,b)$ and $\bar{\eta}_2(i)(i=a,1)$ to

$$\begin{aligned}
\eta_2(i)_\mu&=-\epsilon_{\mu\nu\rho\sigma}q_a{}^\nu q_b{}^\rho q_2{}^\sigma/[G(ab2,ab2)]^{1/2}\,,\\
&\qquad\qquad(\text{for } i=a,b)
\end{aligned}\tag{16}$$

$$\begin{aligned}
\bar{\eta}_2(i)_\mu&=-\epsilon_{\mu\nu\rho\sigma}\bar{q}_a{}^\nu\bar{q}_1{}^\rho\bar{q}_2{}^\sigma/[G(a12,a12)]^{1/2}\\
&\qquad\qquad(\text{for } i=a,1)\,.
\end{aligned}$$

The difference of the crossing matrices for two different scattering planes will be calculated in Sec. III explicitly. The result is such that we must insert a phase factor $e^{-i\phi J_3}$ only for the particles moving along the z axis in a certain channel, where ϕ is the angle required to rotate from one x-z plane to another. For the particles not moving along the z axis, nothing is changed. By this observation, it is straightforward now to write down the crossing matrix for a process with two successive

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crossings, as in Fig. 1. The total crossing matrix C is just $C=C_1rC_2$, where C_1 is the crossing matrix from, channel (d) to (C_1) with the plane $ab1$ as the x-z plane, and C_2 is that from channel (C_1) to channel (C_2) with $a12$ as the x-z plane. The Wigner rotation r is to change the plane $a12$ to $a1b$ in channel (C_1).

III. CROSSING ANGLES

Following CMN,[3] the Lorentz transformation $\mathcal{L}(i)=(L_{a1}{}^{C})^{-1}(i)L_{ab}(i)$ can be expressed as

$$\mathcal{L}(i)\{\dot{t},\dot{\eta}_1,\dot{\eta}_2,\dot{\eta}_3\}=\{\epsilon_i\dot{t},\eta_1(i),\eta_2(i),\eta_3(i)\}\,, \tag{17}$$

where $\epsilon_i=-1$ for $i=1,b$ and $\epsilon_i=1$ otherwise. For the particles a, 2, and 3, $\mathcal{L}(i)$ is a pure rotation, while for the crossed particles b and 1 it is a pure rotation plus a complex Lorentz transformation which introduces at most a factor $(-1)^{2s}$. The most general rotation $R(i)$ is

$$R(i)=\exp(-i\xi_iJ_3)\,\exp(-i\psi_iJ_2)\,\exp(-i\zeta_iJ_3)\,. \tag{18}$$

We can calculate the angles (ξ_i,ψ_i,η_i) by computing the values of various $(\dot{\eta}_k,R(i)\dot{\eta}_l)$. The representation function D of a rotation group only depends on $\cos\psi_i$, and the formula for $(\dot{\eta}_k,R(i)\dot{\eta}_l)$'s places no restriction on the sign of $\sin\psi_i$. Thus, we assume ψ_i to range between 0 and π. One may derive the following relations from the various formulas for the $(\dot{\eta}_k,R(i)\dot{\eta}_l)$:

$$\begin{aligned}\cos\psi_i&=-(\dot{\eta}_3,R(i)\dot{\eta}_3)=-\bar{\eta}_3{}^C(i)\cdot\eta_3(i)\,,\\ \sin\psi_i\,\sin\xi_i&=-(\dot{\eta}_2,R(i)\dot{\eta}_3)=-\bar{\eta}_2{}^C(i)\cdot\eta_3(i)\,,\\ \sin\psi_i\,\sin\zeta_i&=-\bar{\eta}_3{}^C(i)\cdot\eta_2(i)\,,\\ \sin\psi_i\,\cos\xi_i&=-\bar{\eta}_1{}^C(i)\cdot\eta_3(i)\,,\\ -\sin\psi_i\,\cos\zeta_i&=-\bar{\eta}_3{}^C(i)\cdot\eta_1(i)\,.\end{aligned} \tag{19}$$

If we use Eq. (11) to replace q's by p's, and Eq. (13) to replace $\bar{q}$'s by $\bar{p}^{C}$'s, we obtain, for particles $a,b,1$,

$$\cos\psi_a=\frac{(s_{ab}+m_a{}^2-m_b{}^2)(t_{1a}+m_a{}^2-m_1{}^2)-2m_a{}^2\Delta}{\lambda(s_{ab},m_a{}^2,m_b{}^2)\lambda(t_{1a},m_a{}^2,m_1{}^2)}\,,$$

$$\cos\psi_b=\frac{(s_{ab}+m_b{}^2-m_a{}^2)(t_{1a}+m_b{}^2-t_b)+2m_b{}^2\Delta}{\lambda(s_{ab},m_b{}^2,m_a{}^2)\lambda(t_{1a},m_b{}^2,t_b)}\,,$$

$$\cos\psi_1=-\frac{(s_{ab}+m_1{}^2-s_1)(t_{1a}+m_1{}^2-m_a{}^2)+2m_1{}^2\Delta}{\lambda(s_{ab},m_1{}^2,s_1)\lambda(t_{1a},m_1{}^2,m_a{}^2)}\,,$$

$$\zeta_a=\pi\,,\quad \zeta_b=\pi\,,\quad \zeta_1=0\,,\quad \xi_a=\pi\,,\quad \xi_b=0\,,\quad \xi_1=\pi\,,$$

and, for particles 2 and 3,

$$\begin{aligned}\cos\psi_i&=\frac{(s_{ab}+m_i{}^2-s_i)(t_{1a}+m_i{}^2-t_i)-2m_i{}^2\Delta}{\lambda(s_{ab},m_i{}^2,s_i)\lambda(t_{1a},m_i{}^2,t_i)}\,,\\ \sin\xi_i&=\frac{2m_i[\Phi_{Ti}]^{\frac{1}{2}}\lambda(t_{1a},m_i{}^2,t_i)}{[\tilde{\Phi}_i\cdot\mathcal{K}_i]^{\frac{1}{2}}}\,,\\ \sin\zeta_i&=\frac{2m_i[\Phi_{Ti}]^{\frac{1}{2}}\lambda(s_{ab},m_i{}^2,s_i)}{[\Phi_i\cdot\mathcal{K}_i]^{\frac{1}{2}}}\,,\\ \cos\xi_i&=-\frac{2m_i}{[\tilde{\Phi}_i\cdot\mathcal{K}_i]^{\frac{1}{2}}}\\ &\quad\times\{G(abi,a1i)-G(a1i,a1i)+G(b1i,a1i)\}\,,\\ \cos\zeta_i&=-\frac{2m_i}{[\Phi_i\cdot\mathcal{K}_i]^{\frac{1}{2}}}\\ &\quad\times\{G(abi,a1i)+G(abi,abi)-G(abi,1bi)\}\,.\end{aligned} \tag{20}$$

Here

$$\begin{aligned}\Delta&=m_a{}^2-m_b{}^2-m_1{}^2+s_1\,,\\ \lambda(x^2,y^2,z^2)&=[(x+y+z)(x-y+z)\\ &\qquad\times(x-y-z)(x+y-z)]^{1/2}\,,\\ \mathcal{K}_i&=\lambda^2(s_{ab},m_i{}^2,s_i)\lambda^2(t_{1a},m_i{}^2,t_i)\\ &\quad-[(s_{ab}+m_i{}^2-s_i)(t_{1a}+m_i{}^2-t_i)-2m_i{}^2\Delta]^2.\end{aligned} \tag{21}$$

Φ_{Ti} is the total boundary function of the process and is defined by

$$\Phi_{Ti}=-G(ab1i,ab1i)=-(\tfrac{1}{2})^4\det\begin{vmatrix}2m_a{}^2 & s_{ab}-m_a{}^2-m_b{}^2 & -t_{1a}+m_a{}^2+m_1{}^2 & -t_{ia}+m_a{}^2+m_i{}^2\\ s_{ab}-m_a{}^2-m_b{}^2 & 2m_b{}^2 & -t_{b1}+m_b{}^2+m_1{}^2 & -t_{ib}+m_b{}^2+m_i{}^2\\ -t_{1a}+m_a{}^2+m_1{}^2 & -t_{b1}+m_b{}^2+m_1{}^2 & 2m_1{}^2 & s_{1i}-m_1{}^2-m_i{}^2\\ -t_{ia}+m_a{}^2+m_i{}^2 & -t_{ib}+m_b{}^2+m_i{}^2 & s_{1i}-m_1{}^2-m_i{}^2 & 2m_i{}^2\end{vmatrix}\,,$$

and Φ_i and $\tilde{\Phi}_i$ are the sub-boundary functions[12] for the processes $a+b\rightarrow i+(1,j)$ and $a+\bar{1}\rightarrow i+(\bar{b},j)$, respectively. They are expressed in terms of invariant variables as

$$\begin{aligned}\Phi_i&=\Phi(s_{ab},t_{ia},t_{ib};s_i)\\ &=s_{ab}t_{ia}t_{ib}-s_{ab}(m_a{}^2m_b{}^2+m_i{}^2s_i)-t_{ia}(m_i{}^2m_a{}^2+m_b{}^2s_i)\\ &\quad+2m_a{}^2m_b{}^2m_i{}^2s_i(1/m_a{}^2+1/m_b{}^2+1/m_i{}^2+1/s_i)\,,\end{aligned} \tag{22}$$

and $\tilde{\Phi}_i=\Phi(t_{1a},t_{ia},s_{1i};t_i)$. The expressions for $\cos\xi_i$ and $\cos\zeta_i$ in Eq. (20) are used to determine the range of ξ_i and ζ_i uniquely.

Substituting Eqs. (14) and (17) into Eq. (9), we have

$$\begin{aligned}H_{\{\lambda\}}{}^{(s_{ab})}&=(-1)^{\sigma}\sum_{\bar{\lambda}}(-1)^{\lambda_2-\bar{\lambda}_2+\bar{\lambda}_3+s_1+\lambda_1-s_b-\bar{\lambda}_b}\\ &\quad\times\exp[i(\lambda_2\zeta_2+\lambda_3\zeta_3+\lambda_2\xi_2+\lambda_3\xi_3)]\\ &\quad\times\exp[-i\pi(-\lambda_a+\lambda_a-\lambda_b+\bar{\lambda}_1)]\\ &\quad\times d_{\bar{\lambda}_1-\lambda_1}{}^{s_1}(\psi_1)d_{-\bar{\lambda}_2-\lambda_2}{}^{s_2}(\psi_2)d_{-\bar{\lambda}_3-\lambda_3}{}^{s_3}(\psi_3)\\ &\qquad\times d_{\bar{\lambda}_a\lambda_a}{}^{s_a}(\psi_a)d_{-\bar{\lambda}_b\lambda_b}{}^{s_b}(\psi_b)H_{\{\lambda\}}{}^{(t_{1a})}\,.\end{aligned} \tag{23}$$

[12] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

Using the identities

$$d_{-\bar{\lambda}-\lambda}{}^{s}(\psi)=(-1)^{-\bar{\lambda}+\lambda}d_{\bar{\lambda}\lambda}{}^{s}(\psi)\,,$$

$$d_{-\bar{\lambda}\lambda}{}^{s}(\bar{\psi})=(1)^{s-\lambda}d_{\bar{\lambda}\lambda}{}^{s}(\pi-\psi)\,,$$

and

$$d_{\bar{\lambda}-\lambda}{}^{s}(\psi)=(-1)^{s+\bar{\lambda}}d_{\bar{\lambda}\lambda}{}^{s}(\pi-\psi)\,, \tag{24}$$

and redefining $\pi-\psi_1$ and $\pi-\psi_b$ as $\bar{\psi}_1$ and $\bar{\psi}_b$, we have

$$\begin{aligned}H_{\lambda_1\lambda_2\lambda_3;\lambda_a\lambda_b}{}^{(s_{ab})}&(s_{ab},s_{12},s_{23},t_{1a},t_{3b})\\ =(-1)^{\sigma}\sum_{\bar{\lambda}}&\exp[-i\pi(-\lambda_a+\lambda_a+\lambda_1-\lambda_b)]\\ &\times\exp[i(\lambda_2\zeta_2+\lambda_3\zeta_3+\bar{\lambda}_2\xi_2+\lambda_3\xi_3)]\\ &\times d_{\bar{\lambda}_1\lambda_1}{}^{s_1}(\bar{\psi}_1)d_{\bar{\lambda}_2\lambda_2}{}^{s_2}(\psi_2)d_{\bar{\lambda}_3\lambda_3}{}^{s_3}(\psi_3)\\ &\times d_{\bar{\lambda}_a\lambda_a}{}^{s_a}(\psi_a)d_{\bar{\lambda}_b\lambda_b}{}^{s_b}(\bar{\psi}_b)\\ &\quad\times H_{\bar{\lambda}_b\bar{\lambda}_2\bar{\lambda}_3;\bar{\lambda}_a\bar{\lambda}_1}{}^{(t_{1a})}(s_{ab},s_{12},s_{23},t_{1a},t_{1b})\,.\end{aligned} \tag{25}$$

The crossing matrix may have an over-all phase factor which may be the product of factors like $(-1)^{2s_i}$. We do not specify it, since it cannot be measured experimentally.

If we choose the x-z planes to be the plane defined by the particles a, b, and 2 in the s_{ab} channel and that specified by the particles a, 1, and 2 in the t_{1a} channel, we just change the definition of $\eta_1(i)$ and $\bar{\eta}_2(i)$ as in Eq. (16). By explicit calculation we can show

$$\cos\zeta_b=\bar{\eta}_2{}^{c}(b)\cdot\eta_2(b)$$

$$=\frac{-G(ab1,ab2)}{[G(a1b,a1b)G(ab2,ab2)]^{1/2}}=\frac{\Phi_3-\Phi_2-\Phi_1}{2(\Phi_1\Phi_3)^{1/2}}\,. \tag{26}$$

If we express the angles $(\alpha_i,\theta_i,\phi_i)$ of Eq. (2) in terms of invariant variables, we can see that

$$\cos\phi_2=(\Phi_3-\Phi_2-\Phi_1)/2(\Phi_1\Phi_3)^{1/2}\,.$$

Therefore, we have $\cos\zeta_b=\cos\phi_2$. Similarly, $\cos\xi_1=\cos\bar{\phi}_b$. We also obtain $\cos\zeta_a=\cos\zeta_b$, $\cos\xi_a=\cos\xi_1$. These results verify the statement about the transformation matrix in Sec. II.

In the 2-to-N case, we have the same formula for crossing matrices, but we have to distinguish s_{ij} and s_k, and s_{ib} and s_j. The invariant variables s_{ij} and t_{ib} are still defined as in Eq. (1), while s_k and t_j are defined as in Eq. (1′). In the 2-to-3 case, s_{ij} and t_{ib} are equal to s_k and t_j, respectively. For the N-to-N' process we have found the Lorentz transformations $L(i)$ and $\bar{L}(i)$ such that $L^{\mathrm{c.m.}}(i)$ and $\bar{L}^{\mathrm{c.m.}}(i)$ are equal to L_{p_i} and $L_{\bar{p}_i}$ in the c.m. frame of the direct channel and of the crossed channel, respectively. The same technique can also be applied to calculate the crossing matrix in the N-to-N' process.

It is worth mentioning that the crossing angles ξ_i, ζ_i, and ψ_i are real in the physical region of the s_{ab} channel, because in this region the three orthogonal $\bar{\eta}_j(i)$ are real vectors in the three-dimensional Euclidean space. The relation that brings the standard tetrad $(\dot{\eta}_1,\dot{\eta}_2,\dot{\eta}_3)$ to $(\bar{\eta}_1(i),\bar{\eta}_2(i),\bar{\eta}_3(i))$ is real also.

IV. APPLICATIONS

Capella[4] has made several applications of the crossing matrix for the 2-to-3 helicity amplitudes. We shall discuss the difference which arises from the crossing angles ξ_i and ζ_i, and mention some further applications.

In general, the crossing angles ξ_i and ζ_i are real in the physical region of the s_{ab} channel. Therefore, for an experiment which fails to measure polarizations, the differential cross section may be expressed as

$$d\sigma/d\Omega\propto\sum_{\bar{\lambda}}|H_{\bar{\lambda}_b\bar{\lambda}_2\bar{\lambda}_3;\bar{\lambda}_a\bar{\lambda}_1}{}^{(t_{a1})}|^2\,.$$

For an experiment measuring the helicity orientation of one final particle, the cross section can be expressed in terms of the t_{1a} channel helicity amplitude

$$\begin{aligned}d\sigma_i/d\Omega\propto&\sum_{\lambda\neq\lambda_i}|H_{\lambda_1\lambda_2\lambda_3;\lambda_a\lambda_b}{}^{(s_{ab})}|^2\\ &=\sum_{\bar{\lambda}_i,\dot{\lambda}_i}d_{\bar{\lambda}_i\lambda_i}{}^{s_i}(\psi_i)d_{\dot{\lambda}_i\lambda_i}{}^{s_i}(\psi_i)\\ &\qquad\times\exp[-i(\lambda_i-\lambda_i)\zeta_i]H_{\bar{\lambda}_i}{}^{(t_{1a})}H_{\dot{\lambda}_i}{}^{(t_{1a})*}\,.\end{aligned} \tag{27}$$

The right-hand side of Eq. (27) looks as if it is not real, but if we take the complex conjugate, we get the original form after changing some dummy indices $\bar{\lambda}_i$ and $\dot{\lambda}_i$. In Eq. (27), indices other than λ_i have been neglected. Equation (27) is particularly useful to relate the crossed-channel Regge terms to the asymptotic expression of the direct channel in high-energy polarization experiments. It is easily seen that the phase angle ζ_i must not be omitted since the change of the phase angle will alter the cross section drastically. Following a similar line of argument, we can write down the formula for the differential cross section with polarization measurements of more than one final-state particle.

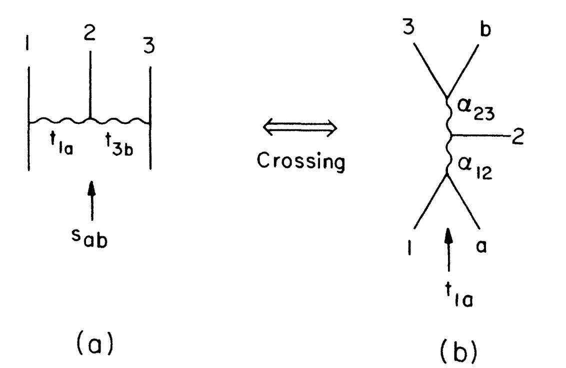

FIG. 2. The s_{ab}-channel c.m. amplitude, dominated by crossed-channel Reggeon exchange can be expressed in terms of the Regge terms of the t_{1a}-channel c.m. amplitude via crossing matrix in a 2-to-3 process.

One of the applications of the crossing matrix is to investigate the kinematical singularities and the kinematical constraints for a 2-to-N process, but we will not discuss this matter further here.

Another application is to express the asymptotic behavior of the s_{ab}-channel c.m. helicity amplitudes in terms of t_{1a}-channel Regge poles for extermal particles with spin. For example, the s_{ab}-channel c.m. helicity amplitude corresponding to the process shown in Fig. 2(a) can be obtained by crossing the t_{1a}-channel amplitude shown in Fig. 2(b), which has the form[13,14]

$$\begin{aligned} H^{(t_{1a})} = \sum_{\mu\bar{\mu}\dot{\mu}\bar{\lambda}_2\bar{\lambda}_3\bar{\lambda}_b} & \Gamma_{\lambda_1\lambda_a\lambda}{}^{s_1 s_a \alpha_{12}}(t_{1a}) D_{\lambda\mu}{}^{\alpha_{12}(t_{1a})}(g^{(12)}) \\ & \times \Gamma_{\mu\bar{\lambda}_2\bar{\mu}}{}^{\alpha_{12} s_2 \alpha_{23}}(t_{1a}, t_{3b}) D_{\bar{\mu}\dot{\mu}}{}^{\alpha_{23}(t_{3b})}(g^{(23)}) \\ & \times \Gamma_{\dot{\mu}\lambda_3\lambda_b}{}^{\alpha_{23} s_3 s_b}(t_{3b}) D_{\bar{\lambda}_2\lambda_2}{}^{s_2}[R_w(\Lambda_2, p_2)] \\ & \times D_{\bar{\lambda}_3\lambda_3}{}^{s_3}[R_w(\Lambda_3, p_3)] D_{\bar{\lambda}_b\lambda_b}{}^{s_b}[R_w(\Lambda_b, p_b)]. \end{aligned} \quad (28)$$

We do not explain the notation here but only mention that $g^{(12)}$ and $g^{(23)}$ are the little groups and Λ_i is the Lorentz transformation which carries the rest frame of the Reggeon to the c.m. frame of the t_{1a} channel. This is referred to the case that both s_{12} and s_{23} become large as s_{ab} becomes large. The s_{ab}-channel c.m. helicity amplitude for a process shown in Fig. 3(a) in which s_{12} becomes large and s_{23} is fixed can be obtained by crossing the t_{1a}-channel c.m. helicity amplitude of the process shown in Fig. 3(b). In this case we need several Regge poles since one of the variables s_{23} or s_{12} is not in the high-energy region. One notes that the amplitudes in Figs. 2(b) and 3(b) have similar formulas except that the labels of the particles are interchanged.

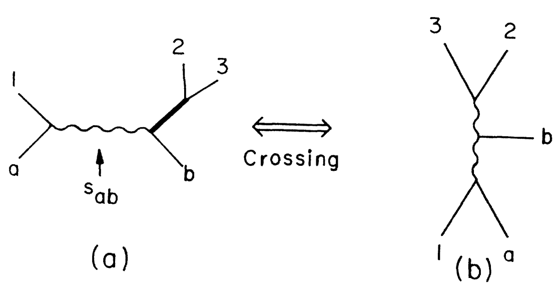

FIG. 3. The s_{ab}-channel c.m. amplitude, dominated by crossed-channel Reggeon exchange and resonances (direct-channel Reggeons), can be expressed in terms of the Regge terms of the t_{1a}-channel c.m. amplitude via crossing matrix in a 2-to-3 process.

[13] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. Letters 19, 614 (1967).

[14] Chan Hong-Mo, J. Loskiewiez, and W. W. M. Allison, Nuovo Cimento 57A, 93 (1968). Further references are cited in this paper.

ACKNOWLEDGMENTS

We are grateful to Professor G. Chew and Professor S. Mandelstam for their constant guidance and encouragement through the development of this work.

APPENDIX

The asymptotic expressions of the crossing angle ξ_i when s_{ab} goes to infinity is explicitly written down. The angle ζ_i will only give an over-all phase factor which does not change the measurable quantities. Therefore, we shall not calculate them. We restrict ourselves to the cases where $s_{12}s_{23}/s_{ab}$ goes to a constant as s_{ab} goes to infinity, since in this region the Regge poles dominate significantly.

(Case 1) $s_{ab} \to \infty$, $s_{23} \to A s_{ab}{}^{1-\epsilon}$, $s_{12} \to B s_{ab}{}^{\epsilon}$ $(0<\epsilon<1)$:

(a) $\epsilon = \frac{1}{2}$,

$$\sin\xi_2 \to \frac{-2m_2[D(A^2+B^2+2AB-4m_2{}^2)]^{1/2}}{[(AB-m_2{}^2)C]^{1/2}},$$

$$\sin\xi_3 \to 0,$$

where

$$\begin{aligned} D = {} & m_2{}^4 - 2m_2{}^2(t_{1a}+t_{3b}) + (t_{1a}+t_{3b})^2 - 4t_{1a}t_{3b} \\ & + AB(2t_{1a}+m_1{}^2-2m_1{}^2+m_a{}^2) + A^2B^2, \\ C = {} & -(m_3{}^2+2m_2{}^2)\lambda^2(t_{1a}, m_2{}^2, t_{3b}) \\ & + 2AB(m_2{}^2-t_{1a}-t_{3b})(m_2{}^2+t_{1a}+t_{3b}) \\ & - 4m_2{}^2 t_{1a} B^2 - 4m_2{}^2 t_{3b} A^2. \end{aligned}$$

(b) $\epsilon > \frac{1}{2}$, $\sin\xi_2 \to 0$, $\sin\xi_2 \to 0$.

(c) $\epsilon < \frac{1}{2}$, $\sin\xi_2 \to \dfrac{D^{1/2}\lambda(t_{1a}, m_2{}^2, t_{3b})}{AB(t_{1a}t_{3b})^{1/2}}$, $\sin\xi_3 \to 0$.

(Case 2) $s_{ab} \to \infty$, $s_{12} \to B s_{ab}$, s_{23} fixed:

$$\mathrm{sun}\,\xi_2 \to 0, \quad \sin\xi_3 \to 0.$$