PHYSICAL REVIEW VOLUME 183, NUMBER 5 25 JULY 1969

Quantum Electrodynamics at Small Distances*

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(Received 6 February 1969)

In an earlier paper, it was shown that if the vacuum polarization in quantum electrodynamics is calculated neglecting corrections to internal photon lines, then the charge renormalization diverges as a single power of the logarithm of a cutoff in every order of perturbation theory. It was further shown that if the coefficient of the logarithm $f(\alpha_0)$ has a zero, and a further positivity condition of f is satisfied, then the complete unrenormalized-photon Green's function is finite. These results we now relate to the Gell-Mann–Low investigation of renormalized quantum electrodynamics at small distances. In particular, the vanishing of the Gell-Mann–Low function ψ is shown to follow from the vanishing of f. This demonstrates why it is sufficient, from the point of view of the renormalized theory, to study quantum electrodynamics without photon self-energy corrections in order to examine the predictions of the theory at small distances.

I. INTRODUCTION

IN a previous paper,[1] we showed that quantum electrodynamics, as a closed theory, is consistent with finite charge renormalization if the bare charge e_0 is determined as a zero of a certain function. Let us briefly summarize the argument. If we assume that e_0 is finite, we can justify the computation of the "most divergent part" of vacuum polarization omitting terms with vacuum-polarization corrections and using e_0^2 as the coupling constant. This is because one can imagine writing all the vacuum-polarization diagrams, and first "summing up" all the ones with vacuum-polarization insertions. If e_0 is finite, these vanish at small distances. Since the "most divergent part" of the vacuum polarization is dominated by the contributions at small distances, we may therefore consider only the diagrams without such corrections. One can then show that the resulting expression, to all orders in e_0, diverges like a single power of the logarithm of a cutoff. The coefficient is the function $f(e_0^2)$. The original hypothesis of finite e_0, that is, finite vacuum polarization, is consistent only if $f(e_0^2)=0$.

Some years ago, Gell-Mann and Low[2] also investigated quantum electrodynamics at small distances. They found that with a certain set of plausible assumptions, e_0 could be finite only if it satisfied an eigenvalue equation $\psi(e_0^2)=0$. In their work, ψ was also related to the coefficient of a logarithmic term in the vacuum polarization. However, Gell-Mann and Low did not make the argument given above, but rather considered all terms of perturbation theory, without a prior summing up of vacuum-polarization insertions. They used a renormalized version of the theory in which they introduced a cutoff λ. If the vacuum polarization is expressed as a power series in the bare charge e_λ appropriate to that cutoff, they postulated that each term in this power series would remain finite if the electron mass were put equal to zero.

The intuitive reason for this is that the momentum integrals which yield the vacuum polarization need a cutoff in order to converge in the ultraviolet region, because there are only a limited number of denominators. As a consequence, even when the electron mass m is zero, so that all denominators vanish when the internal momenta are zero, the resulting integrals converge in the infrared region. Gell-Mann and Low then showed that if the coefficient $\psi(e_\lambda^2)$ of the single logarithmic term in the resulting expression for the vacuum polarization were known, one could construct the complete form of the photon propagator at small distances. The resulting expression is consistent with finite e_0 only if $\psi(e_0^2)=0$.

In this paper, we shall first review the work of Gell-Mann and Low, putting it in a form which will be most suitable for making a comparison with our earlier work. We shall then show that although f and ψ are distinct functions, a relationship between them can be established from which it follows that if $f(e_0^2)=0$, then $\psi(e_0^2)=0$. The eigenvalue equations, therefore, are consistent. We shall also show that the hypothesis of Gell-Mann and Low concerning the $m\to 0$ behavior of vacuum polarization follows as a consequence of the theorem established in our earlier paper; namely, that the vacuum polarization computed without vacuum-polarization insertions, diverges like the single power of the logarithm of the cutoff. We shall also show that the form of the photon Green's function at small distances, as given by the formula of Gell-Mann and Low, is the same as the form given by us in our previous paper. In Appendix A, we shall give an expression for ψ

* This work is supported in part through funds provided by the Atomic Energy Commission under Contract Nos. AT(30-1)2098, AT(45-1)1388B, and AT(30-1)3829.

[1] K. Johnson, R. Willey, and M. Baker, Phys. Rev. 163, 1699 (1967).

[2] M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

valid to sixth order. In Appendix B, we shall discuss a criticism of our work given by Jackiw.[3]

In concluding this Introduction, we may mention that although the eigenvalue equations $f=0$ and $\psi=0$ are equivalent, the tasks of calculating $f(x)$ and $\psi(x)$ are by no means equivalent. f is given in terms of a model theory with no corrections to the photon Green's function. ψ involves the complete theory.

II. ψ OF GELL-MANN AND LOW

We shall begin by introducing the function

$$d(q^2)\equiv e_0{}^2q^2D(q^2). \tag{1}$$

Here, e_0 is the bare charge,[4] and D is the unrenormalized photon Green's function including all of its radiative corrections. The quantity d has the following formal properties:

$$d(0)=e^2\,, \tag{2}$$

where e is the renormalized charge, and

$$d(\infty)=e_0{}^2. \tag{3}$$

Following Gell-Mann and Low,[2] we introduce an "intermediate charge" e_λ, defined by the equation

$$d(\lambda^2)=e_\lambda{}^2. \tag{4}$$

Here $q^2=\lambda^2$ is a spacelike momentum. In this domain, d is a real positive function and, consequently, e_λ is a real number. Gell-Mann and Low showed that it was plausible that $d(q^2)$ have the following property: If the radiative corrections are computed as a power series in e_λ, then when $q^2,\lambda^2\gg m^2$ (where m is the physical mass of the electron) the coefficient of every term in the power series will approach a finite limit. Thus, $d(q^2) = d(q^2/\lambda^2;\, m^2/\lambda^2;\, e_\lambda{}^2)$ has the property that it approaches a finite limit as $m\to 0$ for fixed $e_\lambda{}^2$ and q^2/λ^2.

In this limit, we put

$$\frac{1}{d(q^2)}=\frac{1}{e_\lambda{}^2}-\psi(e_\lambda{}^2)\ln\frac{q^2}{\lambda^2}+\sum_{n=2}^{\infty}\psi_n(e_\lambda{}^2)\left(\ln\frac{q^2}{\lambda^2}\right)^n, \tag{5}$$

thus defining the function $\psi(e_\lambda{}^2)$,[5] which may be computed as a power series. If we differentiate (5) with respect to q^2 and then put $\lambda^2=q^2$, we find that

$$q^2\frac{\partial}{\partial q^2}\frac{1}{d(q^2)}=-\psi(e_q{}^2). \tag{6}$$

However, when $\lambda^2=q^2$ we have from (5)

$$\frac{1}{d(q^2)}=\frac{1}{e_q{}^2}.$$

We consequently obtain the differential equation

$$q^2\frac{\partial}{\partial q^2}\frac{1}{d(q^2)}=-\psi(d(q^2))\,, \tag{7}$$

with the solution

$$\ln\frac{q^2}{\lambda^2}=\int_{e_\lambda{}^2}^{d(q^2)}\frac{dx}{x^2\psi(x)}=\int_{d(\lambda^2)}^{d(q^2)}\frac{dx}{x^2\psi(x)}. \tag{8}$$

Consequently, we find that all of the functions $\psi_n(e_\lambda{}^2)$, which are in (5) are determined by $\psi(e_\lambda{}^2)$, the coefficient of the single logarithm in d. This equation was first obtained by Gell-Mann and Low.[2] Since (8) is valid when $q^2,\lambda^2\gg m^2$, we see that if $d(q^2)$ is computed as a power series in e, the conventionally renormalized charge, then $d(q^2,e^2)$ must be such that the expression

$$\ln\frac{q^2}{m^2}-\int_{\text{const}}^{d(q^2,e^2)}\frac{dx}{x^2\psi(x)}$$

is independent of q^2 when $q^2\gg m^2$. Let us call the asymptotic form of d (that is, all of the logarithms and *constant* terms when $q^2\gg m^2$), $d_A(q^2/m^2)$. Then,

$$\ln\frac{q^2}{m^2}=\int_{q(e^2)}^{d_A(q^2/m^2,e^2)}\frac{dx}{x^2\psi(x)}, \tag{9}$$

where $q(e^2)\equiv d_A(1,e^2)$. Here, $q(e^2)$ is a finite power series in the renormalized charge, which may be obtained by computing d as a power series in e^2, finding its asymptotic form valid when $q^2\gg m^2$ (i.e., dropping all terms in d which vanish when $m\to 0$), and then evaluating this asymptotic function when $q^2=m^2$. Thus, $q(e^2)$ is given by the constants added to the logarithms in d. The Gell-Mann–Low eigenvalue equation for the bare charge then follows from the requirement that $d(\infty) = e_0{}^2$. Thus, if as $q^2\to\infty$,

$$d(q^2)\to d(\infty)=e_0{}^2\,,$$

where e_0 is *finite*, then the integration in (8) takes place over a bounded interval. Since the left side grows without bound, a singularity of the integrand at $x=e_0{}^2$ is required. Thus, the bare charge must be a zero of $\psi(x)$, that is

$$\psi(e_0{}^2)=0. \tag{10}$$

Since renormalized perturbation theory is defined for arbitrarily small e^2, and the function $q(e^2)$ vanishes as $e^2\to 0$, $e_0{}^2$ must be given by the first positive zero of the function ψ is there if more than one. Since the form of ψ is independent of the value of the renormalized charge e, the bare charge must be a number *independent* of e, except that $e^2<e_0{}^2$. Even if $x\,\psi(x)$ does not have a

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[3] R. Jackiw, Nucl. Phys. **B5**, 158 (1968).

[4] In this paper to minimize the proliferation of π's, we denote by e_0 a "very irrational" charge, namely, $e_0=e_0'/(8\pi^2)^{1/2}$, where e_0' is the conventional, rationalized charge. The normal irrational charge is $e_0'/(4\pi)^{1/2}$.

[5] This $\psi(e_\lambda)^2$ differs from that of Gell-Mann and Low (Ref. 2) by a factor of $e_\lambda{}^2$.

zero for a finite value of e_0^2, it still must remain logarithmically bounded, if the renormalized theory taken to all orders is not to have a ghost pole in the photon Green's function. Thus, if the value of the integral is finite as $d\to\infty$, when $\ln(q^2/m^2)$ attains this finite value as q^2 increases, d will be infinite. This finite value of q^2 will accordingly produce a singularity in d; that is, a ghost. If this does not happen, then $d\to\infty$ as $q^2\to\infty$. A sufficient and (for practical purposes) necessary condition for the integral to diverge is that $x\psi(x)<(\ln x)$. Consequently, $\psi(x)$ must vanish as $x\to\infty$. Thus, the bare charge still must be a zero of ψ, even if in this case the zero appears only at infinity.

III. $f=0$ IMPLIES $\psi=0$

In this section, we shall show that the assertion of Gell-Mann and Low which leads to all of these remarkable properties is indeed true; that d, when computed as a power series in $e_\lambda{}^2$, is such that the coefficient of $(e_\lambda{}^2)^n$ in every order approaches a finite limit as $m\to 0$. We shall show that this follows as an immediate corollary of one of our earlier results.[1] We proved that the photon self-energy part, computed in an arbitrary order, omitting all the Feynman graphs which have insertions in single-photon lines, diverges like the single power of the logarithm of a cutoff.

We begin by defining a functional $\rho^*[q^2;d(k)]$. Let

$$\pi_{\mu\nu}{}^*=(g_{\mu\nu}q^2-q_\mu q_\nu)\rho^*[q^2;d(k)]$$

be given by all the photon self-energy part graphs of perturbation theory, omitting graphs with insertions in photon lines (Fig. 1). The lines in the graphs are interpreted as follows: Each vertex contains a factor of 1 instead of e. Each internal photon line stands for $d(k^2)/k^2$, so as $k^2\to 0$, $d(k^2)/k^2\to e^2/k^2$; as $k^2\to\lambda^2$, $d(k^2)/k^2\to e_\lambda{}^2/k^2$; and (formally) as $k^2\to\infty$, $d(k^2)/k^2\to e_0{}^2/k^2$. Each electron line stands for $(\gamma\cdot p+m-mZ_m)^{-1}$, where $Z_m=\delta m/m$ is a mass counterterm defined so that when the electron self-energy is computed, using the photon Green's function $d(k)/k^2$, it is equal to δm on the mass shell. In ρ^* we imagine making an expansion of Z_m, and then of the Z_m dependence of S, as a power series in d. We then group these terms together with the terms of the same order in d which appear in Fig. (1).

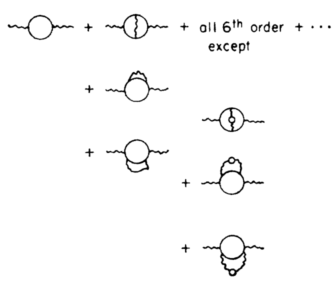

FIG. 1. Graphs of the functional $\rho^*(q^2;d(k^2))$. The internal wavy lines refer to the photon Green's function $d(k^2)/k^2$.

With these definitions, one can make the following statements: *All* the (divergent) graphs of the unrenormalized theory are generated formally by solving the functional equation

$$1/d(q^2)=1/e_0{}^2+\rho^*[q^2;d(k^2)] \tag{11}$$

as a power series in $e_0{}^2$; all of the conventionally renormalized (finite) graphs are generated by solving

$$1/d(q^2)=1/e^2+\rho^*[q^2;d(k^2)]-\rho^*[0;d(k^2)] \tag{12}$$

as a power series e^2. The Gell-Mann–Low series is generated by solving

$$1/d(q^2)=1/e_\lambda{}^2+\rho^*[q^2;d(k^2)]-\rho^*[\lambda^2;d(k^2)] \tag{13}$$

as a power series in $e_\lambda{}^2$ (also finite order by order). We shall consider this alternative. However, instead of iterating immediately in $e_\lambda{}^2$, let us first expand (13) as a functional series in $d(k^2)-e_\lambda{}^2$. Thus, we put

$$d(k^2)=e_\lambda{}^2+[d(k^2)-e_\lambda{}^2] \tag{14}$$

in ρ^*, and expand in $[d(k^2)-e_\lambda{}^2]$. We find[6]

$$\frac{1}{d(q^2)}=\frac{1}{e_\lambda{}^2}+\rho^*(q^2;e_\lambda{}^2)-\rho^*(\lambda^2;e_\lambda{}^2)$$
$$+\int\left[\frac{\delta\rho^*(q^2)}{\delta d(k^2)}-\frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right]_{d=e_\lambda{}^2}[d(k^2)-e_\lambda{}^2]d^4k$$
$$+\frac{1}{2}\int\left[\frac{\delta^2\rho^*(q^2)}{\delta d(k_1{}^2)\delta d(k_2{}^2)}-\frac{\delta^2\rho^*(\lambda^2)}{\delta d(k_1{}^2)\delta d(k_2{}^2)}\right]_{d=e_\lambda{}^2}$$
$$\times[d(k_1{}^2)-e_\lambda{}^2][d(k_2{}^2)-e_\lambda{}^2]d^4k_1d^4k_2+\cdots. \tag{15}$$

We may regard (15) as a nonlinear integral equation for $d(q^2)$ which yields the effects of all insertions into single-photon lines, in terms of kernels

$$[\delta^{(n)}\rho^*/\delta d(k_1{}^2)\cdots\delta d(k_n{}^2)]_{d=e_\lambda{}^2},$$

which are defined by graphs with *no* insertions in single-photon lines.

We may now apply the theorem about single logarithms[1] to Eq. (15). The first term in (15), namely, $\rho^*(q^2;e_\lambda{}^2)-\rho^*(\lambda^2;e_\lambda{}^2)$, contains no graphs with insertions in single-photon lines. Consequently, $\rho^*(q^2;e_\lambda{}^2)$ computed with a cutoff diverges like the single power of the logarithm of a cutoff. This theorem was established by showing that $\rho^*(0;e_\lambda{}^2)$ computed with a cut-

[6] The dependence of ρ^* upon d arises partially through the mass counter-term δmZ_m. This gives a contribution to (15) of the form

$$\int\frac{\delta\rho^*(q^2)}{\delta S(p)}\frac{\delta S(p)}{\delta d(k^2)}[d(k^2)-e_\lambda{}^2]d^4pd^4k$$
$$=\int\frac{\delta\rho^*(q^2)}{\delta S(p)}[S(p)]^2m\frac{\delta Z_m}{\delta d(k)^2}[d(k^2)-e_\lambda{}^2]d^4pd^4k+\cdots,$$

since S contains the self-energy corrections which are also functionals of d. However, it is easy to see that these terms are all finite, and vanish as $m\to 0$. (They are at most of order $(m^2/q^2)\times[\ln(q^2/m^2)]^n$ in nth order for $q^2\gg m^2$.) Therefore, we need not consider such terms.

off Λ^2 diverged like a single power of $\ln(\Lambda^2/m^2)$ as $m\to 0$; indeed,

$$\rho^*(0;e_\lambda^2)\to f(e_\lambda^2)\ln(\Lambda^2/m^2) \tag{16}$$

as $m\to 0$. Some formulas for f were given in our paper. When $q\neq 0$, the ultraviolet convergence properties of ρ^* are unaltered, so

$$\rho^*(q^2;e_\lambda^2)=f(e_\lambda^2)\ln(\Lambda^2/m^2)+g(e_\lambda^2)F(q^2/m^2). \tag{17}$$

Since, when $q\neq 0$, ρ^* is finite as $m\to 0$ (the possible infrared divergences being cut off by q, the external momentum), we see that as $m\to 0$,

$$g(e_\lambda^2)F(q^2/m^2)\to -f(e_\lambda^2)\ln(q^2/m^2)+\text{const},$$

so

$$\rho^*(q^2)\to f(e_\lambda^2)\ln(\Lambda^2/q^2)+\text{const}, \tag{18}$$

and consequently,

$$\rho^*(q^2;e_\lambda^2)-\rho^*(\lambda^2;e_\lambda^2)\to +f(e_\lambda^2)\ln(\lambda^2/q^2) \tag{19}$$

as $m\to 0$. We see that the effect of making the subtraction at λ^2 is to introduce a *precisely defined* ultraviolet cutoff. This is of no great importance in the first term of (15), since there is only a single logarithm, but is of crucial importance in the remaining terms, since there multiple powers of logarithms will appear. In the limit $m\to 0$, using the same theorem as was used in proving that $\rho^*(0;e_\lambda^2)$ diverges at most like a single logarithm, one may also show that $\delta^{(n)}\rho^*(0)/\delta d(k_1^2)\cdots\delta d(k_n^2)$ remains finite. The reason is that these quantities are related to ρ^* by removing internal photon lines and integrations, all of which can only make the infrared convergence properties of the graphs worse. Thus, we see that it is a consequence of the theorem proven about $\rho^*(0)$ that as $m\to 0$ all of the "kernels" appearing in (15) go to finite limits. Thus as $m\to 0$, Eq. (15) becomes

$$\frac{1}{d(q^2)}=\frac{1}{e_\lambda^2}-f(e_\lambda^2)\ln(q^2/\lambda^2)+\int\left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)}-\frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right)_{d=e_\lambda^2,m=0}[d(k^2)-e_\lambda^2](d^4k)$$
$$+\frac{1}{2}\int\left(\frac{\delta^2\rho^*(q^2)}{\delta d(k_1)\delta d(k_2)}\right)_{d=e_\lambda^2,m=0}[d(k_1^2)-e_\lambda^2]\times[d(k_2^2)-e_\lambda^2]d^4k_1d^4k_2+\cdots. \tag{20}$$

We may now solve (20) by iteration in the inhomogeneous term to obtain a solution which has the form (5). This establishes the validity of the Gell-Mann–Low hypothesis about d. In addition, we obtain a relation between $\psi(e_\lambda^2)$, defined as the coefficient of the single logarithm in the *solution* of (20), and f, the coefficient of the single logarithm which appears in the inhomogeneous term of the rather complicated equation. This relation, obtained by iteration of (20), may be written as

$$\psi(e_\lambda^2)=\sum_{n=1}^{\infty}[f(e_\lambda^2)]^nC_n(e_\lambda^2)=f(e_\lambda^2)C_1(e_\lambda^2)+\cdots. \tag{21}$$

Equation (21) shows that our eigenvalue condition $f=0$ implies the Gell-Mann–Low condition $\psi=0$. That is, in order to establish that ψ has a zero, it is *sufficient* to show that f vanishes. The necessity of this, of course, is not given by these arguments. However, because f is a vastly simpler function to compute, since it is defined in terms of a model theory without corrections to single-photon lines, it may be at least "thinkable" to try to calculate f to all orders. However, if the bare charge is infinite, that is if $\psi\to 0$ only as $x\to\infty$, we cannot assert that $f(x)\to 0$ is sufficient to guarantee that $\psi\to 0$. One would also have to establish that the coefficients C_n in (21) remain sufficiently bounded as $x\to\infty$. It should be remarked that in the neighborhood of $f=0$, a solution of (20) based on an iteration in f may be expected to converge much more rapidly than a solution based on an expansion in e_λ^2 on which renormalized perturbation theory is based.

In Sec. IV, we shall need a formula for $C_1(e_\lambda^2)$. To obtain it we need the terms in $1/d-1/e_\lambda^2$ of order f. Let us put

$$1/d-1/e_\lambda^2=-f(e_\lambda^2)\gamma(q^2/\lambda^2)+O(f^2),$$

so

$$d-e_\lambda^2=e_\lambda^4=e_\lambda^4f(e_\lambda^2)\gamma(q^2/\lambda^2)+O(f^2). \tag{22}$$

Then, to order f, (20) becomes linear,

$$-f(e_\lambda^2)\gamma(q^2/\lambda^2)=-f(e_\lambda^2)\ln(q^2/\lambda^2)+e_\lambda^4f(e_\lambda^2)\times\int d^4k\left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)}-\frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right)_{d=e_\lambda^2,m=0}\gamma(k^2/\lambda^2).$$

We then obtain for γ, the linear integral equation

$$\gamma(q^2/\lambda^2)=\ln(q^2/\lambda^2)-e_\lambda^2\int d^4k\left[\frac{\delta\rho^*(q^2)}{\delta d(k^2)}-\frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right]_{d=e_\lambda^2,m=0}\gamma(k^2/\lambda^2). \tag{23}$$

From (22), (5), and (21), we see that $C_1(e_\lambda^2)$ is the coefficient of the single logarithm in the solution of (23). Thus,

$$C_1(e_\lambda^2)=q^2(\partial/\partial q^2)\gamma(q^2/\lambda^2)\,|_{q^2=\lambda^2}. \tag{24}$$

To solve (23), we first introduce the function $G(\delta,e_\lambda^2)$ for $\delta>0$;

$$-\int d^4k\left[\frac{\delta\rho^*(q^2)}{\delta d(k^2)}\right]_{d=e_\lambda^2,m=0}(\lambda^2/k^2)^\delta\equiv(\lambda^2/q^2)^\delta G(\delta,e_\lambda^2). \tag{25}$$

When $\delta>0$, the integral converges in perturbation theory, so G is a finite quantity. We define

$$-\left.\frac{\delta\rho^*(q^2)}{\delta d(k^2)}\right|_{d=e_\lambda^2, m=0} \equiv \frac{1}{\pi^2 k^4} k\left(\frac{q^2}{k^2}\right), \quad k>q$$

$$\equiv \frac{1}{\pi^2 q^4} k\left(\frac{k^2}{q^2}\right), \quad q>k \tag{26}$$

which has the indicated symmetries, since $-\delta\rho^*/\delta d$ is essentially the Bethe-Salpeter kernel for the scattering of light by light. Then,

$$G(\delta, e_\lambda^2) = \int_0^1 dx (x^{\delta-1} + x^{1-\delta}) k(x; e_\lambda^2). \tag{27}$$

We see that if we put

$$\gamma(q^2/\lambda^2) = A[(q^2/\lambda^2)^{-\epsilon} - 1], \tag{28}$$

our equation becomes

$$A[(\lambda^2/q^2)^\epsilon - 1] = \ln(q^2/\lambda^2)$$
$$-e_\lambda^4 \int d^4k \left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)} - \frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right) A[(\lambda^2/k^2)^\epsilon]$$
$$+ A e_\lambda^4 \int d^4k \left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)} - \frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right).$$

Since

$$\rho^*(q^2; e_\lambda^2) - \rho^*(\lambda^2; e_\lambda^2) = -f(e_\lambda^2) \ln(q^2/\lambda^2),$$

we find, on differentiation with respect to e_λ^2,

$$-f'(e_\lambda^2) \ln(q^2/\lambda^2) = \frac{\partial}{\partial e_\lambda^2}[\rho^*(q^2; e_\lambda^2) - \rho^*(\lambda^2; e_\lambda^2)]$$
$$= \int d^4k \left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)} - \frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right)_{d=e_\lambda^2}. \tag{29}$$

Therefore, the right side of (23) becomes

$$= (1 - e_\lambda^4 A f') \ln(q^2/\lambda^2) + e_\lambda^4 A[(\lambda^2/q^2)^\epsilon - 1] G(\epsilon, e^2).$$

Consequently, we have a solution if

$$A = +1/e_\lambda^4 f'(e_\lambda^2)$$

and if

$$e_\lambda^4 G(\epsilon, e_\lambda^2) = 1 = e_\lambda^4 \int_0^1 dx\, (x^{1-\epsilon} + x^{\epsilon-1}) k(x; e_\lambda^2). \tag{30}$$

Thus, using (24) and (28), we find

$$C_1(e_\lambda^2) = [q^2(\partial/\partial q^2)\gamma(q^2/\lambda^2)]_{q^2=\lambda^2} = -\epsilon A.$$

So,

$$C_1(e_\lambda^2) = -\epsilon(e_\lambda^2)/e_\lambda^4 f'(e_\lambda^2), \tag{31}$$

where

$$e_\lambda^4 \int_0^1 dx\, (x^{1-\epsilon} + x^{\epsilon-1}) k(x) = 1 \tag{32}$$

is the equation for $\epsilon(e_\lambda^2)$.

Of course, the considerations above apply in the form given only if the solution to (30) is such that ϵ comes out positive, since otherwise the integral in (30) diverges. However, one can verify that (28) is also the solution of (23) when $\epsilon<0$. However, (30), which determines ϵ, must be replaced by its analytic continuation, valid also when $\epsilon<0$, i.e., by the equation

$$1 = \frac{e_\lambda^4 k(0; e_\lambda^2)}{\epsilon} + e_\lambda^4 \int_0^1 dx\, \{x^{\epsilon-1}[k(x) - k(0)] + x^{1-\epsilon} k(x)\}.$$

If we substitute (26) into (29), we may easily find that

$$f'(e_\lambda^2) = -k(0; e_\lambda^2).$$

Consequently, we may also write (31) and (32) in the equivalent form, valid for both positive and negative ϵ;

$$C_1(e_\lambda^2) = 1 + e_\lambda^4 C_1(e_\lambda^2) \int_0^1 dx$$
$$\times \{x^{\epsilon-1}[k(x) - k(0)] + x^{1-\epsilon} k(x)\}, \tag{33}$$

where

$$\epsilon = -e_\lambda^4 f'(e_\lambda^2) C_1(e_\lambda^2).$$

These equations indicate more clearly the behavior of C_1 for small e_λ.

IV. ASYMPTOTIC FORM OF d

Finally, for completeness we would also like to establish the equivalence between our expression for the asymptotic form of d valid in the region where $d \to e_0^2$, and the expression (8) for d given by Gell-Mann and Low.[2,7] In our earlier paper,[1] we solved a linearized version of (20) valid when $e_\lambda^2 = e_0^2$ and $\lambda^2 = \infty$. This was supposed to give a correct computation of the first-order deviation between d and e_0^2. We found that

$$d(q^2) - e_0^2 = C(m^2/q^2)^{\epsilon(e_0^2)}, \tag{34}$$

where C was undetermined [since Eq. (20) becomes homogeneous when $f=0$]. The exponent ϵ was determined in terms of the kernel (Fig. 2)

$$-\left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)}\right)_{d=e_\lambda^2, m=0} = \frac{1}{\pi^2} k(q^2, k^2) \frac{1}{k^4}$$

by the equation

$$1 = e_0^4 \int_0^1 dx\, k(x)(x^{1-\epsilon} + x^{-1+\epsilon}). \tag{35}$$

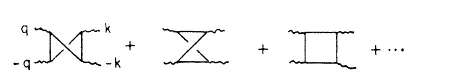

FIG. 2. Graphs of $\delta\rho^*(q)/\delta d(k)$. Aside from normalization these are just the graphs for scattering of light by light.

[7] A slightly different method for making this comparison is outlined in the article by K. Johnson in *Solid State Physics, Particle Physics, Nuclear Physics* (W. A. Benjamin, Inc., New York, 1968).

We shall now show that with equivalent assumptions exactly the same result may be obtained from (8). This should not be too surprising since (8) is the *exact* solution of the complete integral equation (20). When q^2 and λ^2 are such that both $d(q^2)$ and $e_\lambda{}^2$ are near $e_0{}^2$ (the zero of ψ), we may approximate the function $x^2\psi$ by

$$x^2\psi(x)\sim e_0{}^4\psi'(e_0{}^2)(x-e_0{}^2),$$

since $\psi(e_0{}^2)=0$. (We assume that $\psi'\neq 0$.) In this case, the integral in (8) is easily carried out, and we find

$$d(q^2)-e_0{}^2=(\lambda^2/q^2)^{-e_0{}^4\psi'(e_0{}^2)}(e_\lambda{}^2-e_0{}^2). \qquad (36)$$

Now d, when computed in terms of e^2, the renormalized charge, depends only on q^2/m^2, and therefore this equation is equivalent to

$$d(q^2)-e_0{}^2=C(e^2)(m^2/q^2)^{-e_0{}^4\psi'(e_0{}^2)}, \qquad (37)$$

where $C(e^2)$ depends only on the conventionally renormalized charge, but is left undetermined by these considerations. If $\epsilon=-e_0{}^4\psi'(e_0{}^2)$, (37) is equivalent to our result (34). However, since $f=0$ when $x=e_0{}^2$, we find from (21) that

$$\psi'(e_0{}^2)=f'(e_0{}^2)C_1(e_0{}^2),$$

and if we then use (33), we get

$$-e_0{}^4\psi'(e_0{}^2)=\epsilon(e_0{}^2),$$

where $\epsilon(e_0{}^2)$ obeys (32) when $\epsilon>0$. ψ' is naturally negative where ψ vanishes, since ψ is positive when $x<e_0{}^2$. Since this is identical to (35), we have established that our earlier result is the same as that of Gell-Mann and Low.

V. CONCLUSIONS

It has been shown that the famous statement of Gell-Mann and Low, namely, that D calculated as a power series in a suitably defined charge e_λ, approaches a finite limit as $m\to 0$, follows from the result established in our earlier paper; the photon self-energy diverges like a single power of the logarithm of an ultraviolet cutoff, if all graphs with insertions in single-photon lines are omitted. We have also shown that the Gell-Mann–Low eigenvalue equation for the bare charge, $\psi(e_0{}^2)=0$, is satisfied if $f(e_0{}^2)=0$, where $f(e_0{}^2)$ is the coefficient of the single logarithm in the photon self-energy part calculated without photon self-energy insertions. This was the eigenvalue equation obtained by us in our earlier publication. Since $f(x)$ is a vastly simpler function than $\psi(x)$, the task one is presented with is at least slightly simpler: Calculate $f(x)$ to all orders. We shall begin to address ourselves to this question in a subsequent publication.

APPENDIX A

For completeness we shall work out the form of ψ which is correct to sixth order. Although we may evaluate ψ to sixth order by expanding the exact formula (21) which relates ψ and f, a more elementary, and perhaps more instructive, approach is to work directly with (20).

We see that to order $e_\lambda{}^4$ ("sixth" order),

$$1/d(q^2)\equiv 1/e_\lambda{}^2-f(e_\lambda{}^2)\ln(q^2/\lambda^2)+e_\lambda{}^4 f_0\int\left[\left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)}-\frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right)_{d=0,m=0}\right]\times\ln(k^2/\lambda^2)d^4k. \qquad (A1)$$

To obtain ψ, we need the coefficient of the single ln under the $(\ln)^2$ in the integral. Thus, with

$$-\int\left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)}-\frac{\delta\rho^*(\lambda^2)}{\delta d(k^2)}\right)_{d=0,m=0}\ln(k^2/\lambda^2)d^4k=-\tfrac{1}{2}a[\ln(q^2/\lambda^2)]^2+b\ln(q^2/\lambda^2), \qquad (A2)$$

we find that, to the needed accuracy,

$$\begin{aligned}\psi(e_\lambda{}^2)&=f(e_\lambda{}^2)+e_\lambda{}^4f_0b\\&=f_0+f_1e_\lambda{}^2+(f_2+f_0b)e_\lambda{}^4+\cdots,\end{aligned} \qquad (A3)$$

where f_0, f_1, and f_2 are the coefficients appearing in the second-, fourth-, and sixth-order contributions to f. These constants are known:

$$f_0=\tfrac{2}{3},$$
$$f_1=1 \text{ (Ref. 8)},$$
$$f_2=-\tfrac{1}{4} \text{ (Ref. 9)}.$$

(Recall the definition of e.)

We may easily obtain a more explicit formula for b, by using, in (A2), the expression (25) for the kernel. We then find that the left side of (A2) becomes

$$\int_{q^2}^\infty\frac{dk^2}{k^2}[k(q^2/k^2)-k(0)]\ln\frac{k^2}{\lambda^2}+\int_{\lambda^2}^\infty\frac{dk^2}{k^2}[-k(\lambda^2/k^2)+k(0)]\ln\frac{k^2}{\lambda^2}-k(0)\int_{\lambda^2}^{q^2}\frac{dk^2}{k^2}\ln\frac{k^2}{\lambda^2}+\int_0^{q^2}dk^2\frac{k^2}{q^4}k\left(\frac{k^2}{q^2}\right)\ln\frac{k^2}{\lambda^2}-\int_0^{\lambda^2}dk^2\frac{k^2}{\lambda^4}k\left(\frac{k^2}{\lambda^2}\right)\ln\frac{k^2}{\lambda^2},$$

which simplifies to

$$=-\tfrac{1}{2}k(0)\left[\ln\frac{q^2}{\lambda^2}\right]^2+\left(\ln\frac{q^2}{\lambda^2}\right)\int_0^1dx\left[\frac{k(x)-k(0)}{x}+xk(x)\right].$$

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[8] R. Jost and J. M. Luttinger, Helv. Phys. Acta 23, 201 (1950).

[9] J. Rosner, Phys. Rev. Letters 17, 1190 (1966); Ann. Phys. (N. Y.) 44, 11 (1967).

Therefore, comparing with the right side of (A2), we obtain

$$a=k(0)\,,\qquad b=\int_0^1 dx\left(\frac{k(x)-k(0)}{x}+xk(x)\right). \tag{A4}$$

An explicit evaluation of the graphs in Fig. 2 yields the expression

$$k(x)=-1-\tfrac{4}{3}\int_0^x dy\left(1-\frac{y}{x}\right)^2\frac{\ln y}{1+y}. \tag{A5}$$

Accordingly, we find that on evaluation of b and substitution in (A3),

$$\psi=\tfrac{2}{3}+x+x^2[(8/3)\zeta(3)-101/36]+\cdots. \tag{A6}$$

Note also that according to (A5), $k(0)=-1$, and since in general $k(0;e_\lambda^2)=-f'(e_\lambda^2)$, we see in particular that $k(0;0)=-f'(0)=-f_2$. Thus, $f_2=+1$, which agrees with Jost and Luttinger and is a check on the calculation of $k(x;0)$. This expression for ψ does not agree with that of Hagen and Samuel.[10] We shall discuss the reason for the difference in Appendix B. We do not regard the fact that the coefficients of all three powers are positive as any more definitive concerning the existence of solutions to $\psi=0$ than the fact that f_2 is negative. However, since a zero of f insures a zero of ψ, we feel that the fact that no theorem such as "all f_n are positive" can be true is at least weakly encouraging for the optimists who would like to believe in the existence of a finite canonical field theory.

APPENDIX B

In a recent publication,[3] it was stated that our eigenvalue equation $f(e_0^2)=0$ was necessary, but not sufficient to ensure a zero in ψ, which from the point of view of the renormalized theory is the function of fundamental importance. However, in that work f was defined as the coefficient of the single logarithm in $Z_3(e_\Lambda^2)=e^2/e_\Lambda^2$, where Λ is some cutoff and Z_3 was computed as a power series in e_Λ^2, the bare charge, in the cutoff theory. Our function f was not defined in this way, and it is by no means obvious that it will appear in Z_3 in this way. To make the distinction clear, we shall denote the coefficient of the single logarithm in Z_3 by f_J. We shall find below that f_J is not the same as f.

In the publication of Jackiw, a formula appears which relates f_J, ψ, and q which appears here in Eq. (9). Since $q(e^2)$ clearly depends upon the definition of the renormalized charge as being the value of $d(q^2)$ at $q^2=0$, this formula might be suspect, since $q(e^2)$ is left undetermined by considerations of the asymptotic form of d expressed in terms of e_λ^2. In contrast, f and ψ are expressed simply in terms of e_λ^2.

Unfortunately, since f and f_J are not the same, the result of Rosner,[9] who calculated f in sixth order, cannot be used to compute ψ with these formulas. Therefore, the expression for ψ to sixth order, computed by Hagen and Samuels[10] is incorrect. If we compare (A3) with Hagen and Samuel's Eq. (3), we see that only if b and the *constant* added to the fourth-order logarithmic term in the conventionally renormalized vacuum polarization are equal, will f_J and f coincide in sixth order. Thus, in Hagen and Samuel, the constant b_0 which appears instead of b is given by

$$-\int\left[\frac{\delta\rho^*(q)}{\delta d(k)}-\frac{\delta\rho^*(0)}{\delta d(k)}\right]_{d=0,m\neq0}d^4k =-a_0\ln(q^2/m^2)+b_0+O(m^2/q^2)\,, \tag{B1}$$

where $q^2\gg m^2$. As in Appendix A, we put $d=0$ to pick out the fourth-order contribution. We shall see in a moment that although $a=a_0$, $b\neq b_0$. Note that in (B1), m is finite, and the subtraction is made at $q^2=0$, the point at which the conventionally renormalized charge e is defined.

Let us now turn to an explicit computation of a_0 and b_0. The left-hand side of (B1) becomes

$$-\pi^2\int_{q^2}^\infty dk^2\,k^2\left(\frac{\delta\rho^*(q^2)}{\delta d(k^2)}-\frac{\delta\rho^*(0)}{\delta d(k^2)}\right)_{d=0,m\neq0} -\pi^2\int_0^{q^2}dk^2\,k^2\left(\frac{\delta\rho^*(q^2)}{\delta d(k)}\right)_{d=0,m\neq0} +\pi^2\int_0^{q^2}dk^2\,k^2\left(\frac{\delta\rho^*(0)}{\delta d(k)}\right)_{m\neq0}. \tag{B2}$$

In the first and second terms we may put $m=0$ and get finite results, i.e., the first and second terms in (B2) are equal to

$$\int_0^1 dx\frac{k(x)-k(0)}{x}+\int_0^1 dx\,xk(x)\,,$$

where $k(x)=k(x,0)$ is independent of e_λ^2. We must handle the last term more carefully. Let us define

$$-\left[\frac{\delta\rho^*(0)}{\delta d(k)}\right]_{d=0,m\neq0}=(1/\pi^2)(1/k^4)K(m^2/k^2)\,.$$

We see that $K(0)=k(0)$. Furthermore, at $k=0$, the integral in (B1) converges at $k\to0$ at least like k^4, as $k\to0$. Therefore, the negative of the last term in

[10] C. R. Hagen and M. A. Samuel, Phys. Rev. Letters **20**, 1405 (1968).

(B2) is

$$\int_0^{q^2} \frac{dk^2}{k^2} K(m^2/k^2) = \int_0^{q^2} dk^2 \frac{d}{dk^2}\left[\ln\frac{k^2}{m^2} K(m^2/k^2)\right] - \int_0^{q^2} dk^2 \ln\frac{k^2}{m^2}\frac{d}{dk^2}K(m^2/k^2)$$

$$= \ln\frac{q^2}{m^2} K(m^2/q^2) - \int_0^{q^2/m^2} dx(\ln x)\frac{d}{dx}K(1/x),$$

and when $q^2 \gg m^2$, since $K(1/x) = k(0) + O[(1/x)\ln x]$, the integral converges, so this becomes

$$\ln\frac{q^2}{m^2} k(0) - \int_0^{\infty} dx \ln x \frac{d}{dx}[K(1/x)].$$

Consequently, we obtain the result

$$\int\left[\frac{\delta\rho^*(q^2)}{\delta d(k^2)} - \frac{\delta\rho^*(0)}{\delta d(k^2)}\right]_{d=0, m=0} d^4k$$

$$= k(0)\ln(q^2/m^2) - \int_0^1 dx\left(\frac{k(x)-k(0)}{x} + xk(x)\right) - \int_0^{\infty} dx \ln x \frac{d}{dx}K(1/x),$$

so

$$a_0 = k(0), \quad b_0 = b + \int_0^{\infty} dx \ln x \frac{d}{dx}K(1/x). \quad \text{(B3)}$$

Thus, the value of the constant b_0, associated with the conventionally renormalized fourth-order vacuum polarization, does not yield the needed constant b, which should appear in (A3). With (A4) and (A5), we find for b the value,

$$b = 4\zeta(3) - 23/6. \quad \text{(B4)}$$

In contrast, Hagen and Samuel, who determined b_0 from the expression of Källén and Sabry[11] for fourth-order vacuum polarization, find that

$$b_0 = -5/6 + 4\zeta(3). \quad \text{(B5)}$$

It is not difficult to understand why the transcendental number which appears in b_0 is the same as that in b, for according to (B3), b and b_0 differ only by the integral

$$\int_0^{\infty} dx \ln x \frac{d}{dx}K(1/x),$$

and the function K is defined in terms of the fourth-order Bethe-Salpeter kernel for scattering of light by light, with only one nonvanishing external momentum entering the graphs. The resulting Feynman integrals can then give rise only to elementary functions. In contrast, $k(x)$ is related to the lowest-order scattering amplitude for light by light with two nonvanishing external momenta, and these integrals give rise to Spence functions, which are the source of the transcendental number in both b and b_0.

However, since b and b_0 are different, and since

$$f_{J2} + bf_{J0} = f_2 + bf_0$$

(because both are valid expressions for the sixth-order term in ψ), we see that f_{J2} and f_2 are different. Hence, f and f_J are not the same functions. Since Jackiw, and then Hagen and Samuel, assumed that $f_J = f$ to compute ψ, we find the reason for the discrepancy between their result and ours for ψ in sixth order. Finally, since Jackiw's criticism applies to f_J and not to f, we do not need to make further comment on it.

[11] G. Källén and A. Sabry, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **29**, No. 17 (1955).