PHYSICAL REVIEW VOLUME 183, NUMBER 5 25 JULY 1969

Soft Pions, Chiral Symmetry, and Phenomenological Lagrangians

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(Received 9 January 1969)

A method is devised for extracting the entire content of the joint assumptions of partially conserved axial-vector current (PCAC) and current algebra. This technique is used to derive a set of identities which generate all possible soft-pion theorems; moreover, the direct relationship between the PCAC–current-algebra approach and the so-called phenomenological Lagrangian approach to deriving these theorems is explicitly established. The technique used throughout clearly reveals all of the assumptions inherent in the phenomenological Lagrangian approach, and provides formal expressions for the correction terms to the soft-pion theorems following from the Lagrangian formalism. Particular attention is paid to clarifying the relationship between so-called "PCAC correction terms" and "Σ terms," with special emphasis placed upon their role in the calculation of formulas for pion-pion scattering. It is argued that our results strongly suggest that the only appealing explanation for the success of the PCAC hypothesis is that the real world satisfies an approximate $SU(2)\otimes SU(2)$ symmetry; and, moreover, in a world in which such a symmetry is exact, the pion corresponds to a Goldstone boson.

1. INTRODUCTION

In recent years there has been a considerable amount of interest in the derivation and application of soft-pion theorems. While we do not wish to dwell upon the various successes of the soft-pion techniques, we think it is fair to say that, taken as a whole, the low-energy theorems constitute a significant advance in strong-interaction physics.

Historically the first low-energy theorems for pions were obtained by Nambu and Lurié.[1] In order to derive these theorems, they took a model for the strong interactions in which the axial-vector current was conserved. By assuming the existence of this (chiral) symmetry, they were able to relate the amplitude for the process $\pi+N\to\pi+N$ to the amplitude for the process $\pi+N\to N+2\pi$, when one of the final pions was taken to be soft. The more recent developments of the hypothesis of partially conserved axial-vector current (PCAC) and current algebra led to further results of this type, which however, were derived in a rather different way.

The standard PCAC technique, which originates with Goldberger and Treiman[2] and owes much of its recent popularity to Adler,[3] is simply to suppose that the divergence of the axial-vector current (whenever it appears) is dominated by the pion pole. This PCAC technique, which incidentally gives the same answer as Nambu and Lurié's symmetry scheme for the process previously mentioned, is perfectly straightforward. However, as anyone who has tried to do a calculation involving more than two soft pions knows, the PCAC method of reducing in pions and bringing time derivatives through a time-ordered product soon becomes prohibitively complicated.

For this and other reasons, Weinberg suggested that effective Lagrangians could be used to do soft-pion calculations.[4] He constructed a Lagrangian which, because it satisfied PCAC and current algebra, could be assumed to automatically give the same soft-pion theorems as the more cumbersome machinery of PCAC.

Part of the tradition is the statement that all three of the above-mentioned methods—chiral symmetry, PCAC–current-algebra, and effective Lagrangians—give equivalent results for soft pions. One of the principal objectives of this paper is to *show explicitly* that these methods are, in fact, equivalent.

More generally, we give here a complete and unified treatment of all low-energy theorems involving any number of soft pions. As the reader shall see, the general results obtained by a consistent use of PCAC and current algebra suggest very strongly that a very convenient language which can be used to describe the content of all of these results, is that of approximate symmetry. To be more explicit, we show that the usual calculations based upon PCAC and current algebra are mathematically equivalent to assuming that the strong interactions almost possess a chiral symmetry where, in the symmetry limit, the pion is massless and the axial-vector current is conserved. At this point we should stress the words "mathematically equivalent"; the reader who finds it difficult to believe that this symmetry really exists in a physically meaningful sense can take comfort in the fact that ordinary PCAC and current algebra lead to exactly the same formulas for soft-meson theorems. The language of approximate symmetry, nevertheless, is quite useful and allows us to give a precise meaning to PCAC in a natural way. What is perhaps more important, it also provides a systematic scheme for keeping track of corrections to the PCAC approximation.

Our specific results are based upon a certain identity which gives the S-matrix element $\langle\alpha+n\pi|S|\beta+m\pi\rangle$, for arbitrary states $\langle\alpha|$ and $|\beta\rangle$, in terms of matrix elements of time-ordered products of vector and axial-

* Research supported in part by the National Science Foundation.
[1] Y. Nambu and D. Lurié, Phys. Rev. 125, 1429 (1962).
[2] M. L. Goldberger and S. B. Treiman, Phys. Rev. 110, 1178 (1958).
[3] S. L. Adler, Phys. Rev. 137, B1022 (1965); 139, B1638 (1965).
[4] S. Weinberg, Phys. Rev. Letters 18, 188 (1967).

vector currents. In the "symmetry limit" (i.e., when corrections to PCAC are ignored) this identity is so constructed that the soft-pion theorem for $\langle\alpha+n\pi|S\times|\beta+m\pi\rangle$ can be read off directly. The "symmetry-breaking" terms, which contain the corrections to the PCAC approximation, can also be retained in the identity.

A useful feature of the form of this identity with PCAC correction terms retained is that it enables one to see explicitly what is being neglected when one makes the assumption that PCAC is exact. Hopefully, this formally exact equation, which contains expressions for all correction terms, may be used to develop methods with which one could actually calculate or estimate some of these terms.

In addition to isolating the symmetry-breaking terms, we are also able to define exactly what a soft pion is. That is, the term "soft pion" is to be understood in the sense that the low-energy theorems are accurate to some order in the momenta of the pions as all of the momenta of the soft pions approach zero. From our fundamental identity, which is exact for pions of any momenta, we are able to deduce the explicit order to which these soft-pion theorems are correct.

Recently, most of the work on soft pions has been in connection with effective Lagrangians.[5] These necessarily nonlinear Lagrangians have now become a subject in themselves. Our contribution to this subject is to show explicitly how such effective Lagrangians arise in a natural way as soon as one has PCAC and current algebra. Since we do not start from an effective Lagrangian, but in a sense derive one, we are in a position to bring out the exact physical content of the approach.

This paper is organized as follows: In Sec. 2, we discuss the derivation of the Goldberger-Treiman relation and soft-pion theorems with emphasis upon the broken-symmetry point of view. The aim of this section is to review the major points involved in the derivation of these results and to make some observations which we hope will clarify some misconceptions prevalent in the literature. Section 3 is devoted to a discussion of our general results for the "symmetry limit" without any proofs. Section 4 is a discussion of corrections to the symmetry limit with particular attention paid to the treatment of π-π scattering and calculation of the entire first-order symmetry-breaking correction to the results of Sec. 3. The remaining sections are devoted to proving the results discussed in Secs. 2–4, and are best presented as a series of theorems and corollaries. In Sec. 5, we prove the most general form of our basic identity, and in Sec. 6 we show how to use this identity in order to derive an expression for the amplitude $\langle\alpha+n\pi|S|\beta+m\pi\rangle$ in the symmetry limit. Section 7 shows how to derive a phenomenological Lagrangian from these results and Secs. 8 and 9 discuss the question of adding electromagnetic and weak-interaction corrections and the generalization to $SU(3)\otimes SU(3)$, respectively. Finally, Sec. 10 is a discussion of possible ways one could use these results to understand other aspects of the strong interactions.

The proof of the general, broken-symmetry identity for $\langle\alpha+n\pi|S|\beta+m\pi\rangle$ is left to Appendix B.

2. OBSERVATIONS AND ONE- AND TWO-PION THEOREMS

As we mentioned in the Introduction, our general results suggest very strongly that if the usual current algebra holds and PCAC is a consistently good approximation, then it is very convenient to think of the strong interactions as being almost symmetrical under the group $SU(2)\otimes SU(2)$. This is to be understood as a symmetry where, if it were exact, the usual axial-vector currents would be conserved and the pion would be massless. [In fact, one can discuss everything we do here extended to the framework of an approximate $SU(3)\otimes SU(3)$ symmetry which contains an octet of massless pseudoscalar mesons in the symmetry limit,[6] but for reasons of mathematical simplicity we restrict ourselves to $SU(2)\otimes SU(2)$].

A full treatment of this point of view is complex and requires all of the detailed arguments which appear in the later sections of this paper. Fortunately, the general physical features of these arguments can be made clear by discussing a few simple cases which we shall do in this section. Before doing so, however, we would like to make the following point: *PCAC is not simply a consequence of the fact that the pion has a small mass.* What we mean by this statement is that the smallness of m_π is not, by itself, enough to imply that at zero momentum transfer the matrix elements of the axial-vector currents are dominated by the (nearby) pion pole. This, as we shall soon see, has to do with the fact that the operator under consideration is the divergence of the axial-vector current and not just a random pseudoscalar operator. We wish to stress this point because there seems to be a rather widespread belief that PCAC is automatically implied by the small pion mass and the "nearby singularities" approach of dispersion theory. If we take the Goldberger-Treiman relation as our example, we can show that this is not the case.

In order to establish our notation let us first list some standard kinematic relations. The matrix element of the axial-vector current between nucleons can be

[5] A partial list of references on this subject is L. Brown, Phys. Rev. **163**, 1802 (1967); W. Bardeen and B. Lee, in *Nuclear and Particle Physics*, edited by B. Margolis and C. Lam (W. A. Benjamin, Inc., New York, 1968); S. Weinberg, Phys. Rev. **166**, 1568 (1968); J. Schwinger, Phys. Letters **24B**, 47 (1967); J. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967); B. W. Lee and H. T. Nieh, *ibid.* **166**, 1507 (1968); P. Chang and F. Gürsey, *ibid.* **164**, 1752 (1967).

[6] R. F. Dashen, preceding paper Phys. Rev. **182**, 1245 (1969); M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

written as

$$\langle N(p')|\mathbf{A}^\mu(0)|N(p)\rangle = \bar{u}(p')\{[\gamma^\mu\gamma_5 g_A(q^2)+q^\mu\gamma_5 h_A(q^2)]\tfrac{1}{2}\boldsymbol{\tau}\}u(p), \quad (2.1)$$

where $q=p'-p$ and in our normalization $g_A\equiv g_A(0) \approx 1.2$. If we take the divergence of both sides of Eq. (2.1), we get

$$\langle N(p')|\partial_\mu \mathbf{A}^\mu(0)|N(p)\rangle = -i\bar{u}(p')\{\gamma_5 d(q^2)\boldsymbol{\tau}/2\}u(p), \quad (2.2)$$

where

$$d(q^2)=2m_N g_A(q^2)+q^2 h_A(q^2). \quad (2.3)$$

We also need to define

$$\langle \pi_\alpha(q)|A_\beta{}^\mu(0)|0\rangle = -(iq^\mu/2f_\pi)\delta_{\alpha\beta}, \quad (2.4)$$

where f_π is measured in the decay $\pi\to\mu+\nu$ and $(2\sqrt{2}f_\pi)^{-1}=0.96m_\pi$.

With this definition, taking a divergence on both sides of (2.4) leads to the kinematic relation

$$\langle \pi_\alpha(q)|\partial_\mu A_\beta{}^\mu(0)|0\rangle = (m_\pi^2/2f_\pi)\delta_{\alpha\beta}. \quad (2.5)$$

Having established this notation, let us now review the standard derivation of the Goldberger-Treiman relation. Setting $q^2=0$ in Eq. (2.3) gives

$$2m_N g_A(0)\equiv 2m_N g_A \equiv d(0). \quad (2.6)$$

Next, let us write $d(q^2)$ as a pion-pole term plus a remainder,

$$d(q^2)=G_{\pi NN}m_\pi^2/f_\pi(m_\pi^2-q^2)-\bar{d}(q^2), \quad (2.7)$$

where $G_{\pi NN}$ is the pion-nucleon coupling constant ($G_{\pi NN}^2/4\pi\approx 14$) and we have used Eq. (2.5). In Eq. (2.7), $\bar{d}$ is defined to be d minus the pion-pole term and may be written explicitly as

$$\bar{d}(q^2)=\int_{9m_\pi^2}^\infty \frac{\rho(\mu^2)d\mu^2}{\mu^2-q^2} +(\text{possible subtraction terms}), \quad (2.8)$$

where the spectral integral runs over all singularities in $d(q^2)$ except the pion pole. If we now assume that because of the small pole denominator $[(q^2-m_\pi^2)^{-1} = -m_\pi^{-2}$ for $q^2=0]$ the pion pole dominates $\bar{d}(0)$ in Eq. (2.7), then we obtain the Goldberger-Treiman relation. The problem is to justify the pole-dominance hypothesis. Since the residue of the pion pole is itself proportional to m_π^2, we clearly cannot simply rely upon the small pole denominator m_π^{-2} as an argument for pole dominance. The reader should note in this connection that the factor of m_π^2 in the residue is a kinematic constraint which arises from the fact that the operator under consideration is a divergence; if we were discussing the form factor of some general pseudoscalar operator this factor would not be present.

We may further convince ourselves that PCAC is not simply a consequence of small m_π by going through an alternative derivation of the Goldberger-Treiman relation. Extracting the pion poles on both sides of Eq. (2.3), we can write

$$\frac{G_{\pi NN}}{f_\pi}\frac{m_\pi^2}{(m_\pi^2-q^2)}-\bar{d}(q^2)=2m_N g_A(q^2) +\frac{G_{\pi NN}}{f_\pi}\frac{q^2}{m_\pi^2-q^2}+q^2\bar{h}_A \quad (2.9)$$

or

$$G_{\pi NN}/f_\pi-\bar{d}(q^2)=2m_N g_A(q^2)+q^2\bar{h}_A(q^2), \quad (2.10)$$

where $\bar{h}_A$ is h_A less its pion pole. [N.B.: $g_A(q^2)$ does not have a pion pole.] It is worth noting at this point that in going to Eq. (2.10) all explicit dependence upon m_π has disappeared. Setting $q^2=0$ in Eq. (2.10) gives

$$G_{\pi NN}/f_\pi-\bar{d}(0)=2m_N g_A, \quad (2.11)$$

which we recognize as the Goldberger-Treiman relation provided that $\bar{d}(0)$ is negligible. The lack of any explicit m_π dependence in Eq. (2.11) should emphasize the fact that the smallness of m_π does not by itself imply that $\bar{d}(0)$ is small in comparison to all the other terms appearing in Eq. (2.11).

Since we have argued that the small pion mass does not by itself imply PCAC, let us try to see what sort of principle is needed. Since both g_A and $G_{\pi NN}(2m_N f_\pi)^{-1}$ are experimentally known to be of order 1, it is clear from Eq. (2.11) that the Goldberger-Treiman relation requires that

$$\bar{d}(0)/2m_N \ll 1. \quad (2.12)$$

Now $\bar{d}(0)$ is a typical matrix element of the divergence of the axial-vector current $\mathbf{A}^\mu$ (except that the pion pole has been removed), and such matrix elements should have a size characteristic of the part of the strong Hamiltonian which violates the conservation of $\mathbf{A}^\mu$. (This statement will be made more precise in what follows.) On the other hand, m_N is a typical strong-interaction mass, and is therefore of the order of the total strong-interaction Hamiltonian. Evidently, if we let ϵ be the strength of the part of the strong-interaction Hamiltonian which violates the conservation of $\mathbf{A}^\mu$ relative to the total Hamiltonian, then we have

$$\epsilon\sim\bar{d}/2m_N\ll 1. \quad (2.13)$$

This then, is a suggestion that the part of the strong Hamiltonian which violates the conservation of the axial-vector currents is rather small. Let us explore this point more fully.

Assuming the usual algebra of currents,[7] the charges Q_α and $Q_\alpha{}^5$ defined by

$$Q_\alpha(t)=\int d^3x\, V_\alpha{}^0(\mathbf{x},t),\quad Q_\alpha{}^5(t)=\int d^3x\, A_\alpha{}^0(\mathbf{x},t) \quad (2.14)$$

generate, under equal-time commutation, the algebra of

[7] M. Gell-Mann, Physics 1, 63 (1964).

$SU(2)\otimes SU(2)$. Although the total Hamiltonian in the Heisenberg picture is a constant in time, the fact that at any single time the charges form a closed algebra implies that the Hamiltonian can always be uniquely decomposed into the form

$$H=H_0(t)+\epsilon H_1(t)\,, \tag{2.15}$$

where $H_0(t)$ is the largest piece of H which transforms as a scalar under the $SU(2)\otimes SU(2)$ group generated by the charges at time t, and $\epsilon H_1(t)$ transforms as some sum of ireducible tensors under the same group. The factor ϵ in front of H_1 is assumed to give the over-all scale of $H_1(t)$ relative to H.

Isospin invariance gives

$$[Q_\alpha(t),H]=0 \tag{2.16}$$

or $dQ_\alpha/dt=0$. On the other hand,

$$\begin{aligned}[Q_\alpha{}^5(t),H]&=\epsilon[Q_\alpha{}^5(t),H_1(t)]\\ &=-i\frac{d}{dt}Q_\alpha{}^5(t)=-i\int\partial_\mu A_\alpha{}^\mu(\mathbf{x},t)d^3x\,,\end{aligned} \tag{2.17}$$

so that ϵ also sets the scale of nonconservation of the axial-vector currents. Of course, this decomposition of H into H_0 and ϵH_1 is content free as far as physics is concerned unless we have additional information, for example, that ϵ is small. In fact, we have already seen that PCAC suggests that this is so; let us therefore make the hypothesis that ϵ is an adjustable small parameter and ask the question: What would the world look like for $\epsilon=0$?

Mathematically, there are three possibilities which we can list; as we shall see, only one of these possibilities is clearly consistent with small ϵ in the real world. From Eq. (2.5) one easily sees that when $\epsilon=0$ and therefore $\partial_\mu\mathbf{A}^\mu=0$, either $m_\pi=0$ or $f_\pi{}^{-1}=0$. If $m_\pi=0$ and $f_\pi{}^{-1}\neq0$, then Eq. (2.11) gives an exact Goldberger-Treiman relation but no further restrictions upon the theory. On the other hand, if $m_\pi\neq0$ so that $f_\pi{}^{-1}=0$, then Eq. (2.11), with $\bar{d}$ set equal to zero, can be satisfied only if $g_A=0$ or $m_N=0$. In a world corresponding to the case $f_\pi{}^{-1}=0$ and $g_A=0$, it can be shown that there must be a negative-parity baryon degenerate in mass with the nucleon, and this has not been seen. There are then three mathematical possibilities for the world $\epsilon=0$; they are the following:

(i) $m_\pi=0$, $f_\pi{}^{-1}\neq0$, $g_A\neq0$, and $m_N\neq0$;
(ii) $f_\pi{}^{-1}=0$, $m_N=0$, $m_\pi\neq0$, and $g_A\neq0$;
(iii) $f_\pi{}^{-1}=0$, $g_A=0$, $m_\pi\neq0$, and $m_N\neq0$, with an opposite-parity partner for the nucleon.

Clearly, only case (i) is consistent with our present experimental knowledge, small ϵ, and an assumed smooth transition to the limit $\epsilon=0$. Moreover, as we already pointed out, case (i) has the additional appealing feature that in the limit $\epsilon=0$ the Goldberger-Treiman relation holds exactly and its approximate validity in the real world can be understood as a consequence of ϵ's being small. In the other two cases, there is no *a priori* reason why the Goldberger-Treiman relation should hold at all for any value of ϵ.

Let us pause for a moment in order to briefly summarize our position. We have argued at length that PCAC cannot be understood simply on the grounds that m_π is small. Barring some extremely complex dynamical justification of PCAC, which is well beyond our present understanding of strong interactions, the only rational explanation of PCAC seems to be that the parameter ϵ in Eq. (2.15) is small. Furthermore, it seems that if the limit $\epsilon\to0$ makes sense, the pion must be massless in this limit: The other two possibilities we reject on the grounds that they are not experimentally consistent with small ϵ and do not lead to an explanation of PCAC.

We are therefore led to a picture of the strong interactions in which $H=H_0+\epsilon H_1$, with H_0 invariant under $SU(2)\otimes SU(2)$ and where ϵ is small; and where in the symmetry limit $\epsilon=0$, the symmetry is realized by having a massless pion (i.e., a Goldstone boson). Later in this section we shall give additional arguments in support of this picture of the strong interactions, but for now let us temporarily accept it.

At this point we would like to continue the discussion of this section with an analysis of some other applications of PCAC. Besides eventually using this analysis to establish additional connections between the success of these applications and the idea of approximate $SU(2)\otimes SU(2)$ symmetry, we shall introduce various concepts and techniques which will be useful in later sections.

First let us consider a direct generalization of the Goldberger-Treiman relation. We begin by writing the trivial identity

$$\langle\alpha|\partial_\gamma(q)|\beta\rangle=-iq_\mu\langle\alpha|A_\gamma{}^\mu(q)|\beta\rangle\,, \tag{2.18}$$

where $A_\gamma{}^\mu(q)$ and $\partial_\gamma(q)$ are short-hand notations for $\int d^4x\, e^{+iq\cdot x}A_\gamma{}^\mu(x)$ and $\int d^4x\, e^{+iq\cdot x}\partial_\mu A_\gamma{}^u(x)$, respectively, and where $\langle\alpha|$ and $|\beta\rangle$ are arbitrary hadron states. Both sides of Eq. (2.18) have poles at $q^2=m_\pi{}^2$ with residues given by

$$\begin{aligned}\langle\alpha|A_\gamma{}^\mu(q)|\beta\rangle_{\text{pole}}&=\frac{-q_\mu}{2f_\pi(q^2-m_\pi{}^2)}\langle\alpha+\pi_\gamma(q)|S|\beta\rangle\,,\\ \langle\alpha|\partial_\gamma(q)|\beta\rangle_{\text{pole}}&=\frac{+im_\pi{}^2}{2f_\pi(q^2-m_\pi{}^2)}\langle\alpha+\pi_\gamma(q)|S|\beta\rangle\,,\end{aligned} \tag{2.19}$$

where S is the strong interaction S matrix. We may then write Eq. (2.18) as

$$\begin{aligned}(1/2f_\pi)&\langle\alpha+\pi_\gamma(q)|S|\beta\rangle\\ &=q_\mu\langle\alpha|\bar{A}_\gamma{}^\mu(q)|\beta\rangle+\langle\alpha|\bar{\partial}_\gamma(q)|\beta\rangle\,,\end{aligned} \tag{2.20}$$

where, as before, the barred quantities are defined as being equal to the corresponding unbarred quantity the pion pole removed. The hypothesis that ϵ is small means that the second term on the right-hand side of Eq. (2.20) can be neglected. Since the quantity $q_\mu\langle\alpha|\hat{A}_\gamma{}^\mu(q)|\beta\rangle$ is not known for arbitrary states α and β, this generalized Goldberger-Treiman relation cannot be tested except in the limit $q_\mu \to 0$. In this event, due to the presence of the explicit factor of q_μ, only those terms in $\langle\alpha|\hat{A}_\gamma{}^\mu(q)|\beta\rangle$ which are singular as $q_\mu \to 0$ need be kept. In the case in which α is a one nucleon state and β is a pion plus nucleon state only the "Born terms" or more correctly the pole terms, in the sense of dispersion theory, are singular in this limit. For this particular case taking the limit $q_\mu \to 0$ and neglecting the term $\langle N|\bar{\partial}_\gamma(0)|N\rangle$ leads, as is well known, to Adler's self-consistency condition for π-N scattering.[3] The observation that this relation is satisfied to roughly the same accuracy as the Goldberger-Treiman relation, increases our confidence in the assumption that ϵ is small.

At this juncture, we should discuss one point which could lead to confusion. The reader may be wondering why we keep emphasizing the statement that $\bar{\partial}_\gamma(q)$ is small, rather than $\partial_\gamma(q)$, which is also formally of order ϵ. This can be easily understood if one pursues the following argument.

For small ϵ we may (by our previous assumption that $m_\pi{}^2=0$ when $\epsilon=0$) set $m_\pi{}^2=\epsilon\mu_0{}^2$, where μ_0 is some scale mass, in which case we can write $\partial(q)$ as

$$\langle\alpha|\partial_\gamma(q)|\beta\rangle=\frac{\epsilon\mu_0{}^2}{2f_\pi(q^2-\epsilon\mu_0{}^2)}\langle\alpha+\pi_\gamma(q)|S|\beta\rangle+\langle\alpha|\bar{\partial}_\gamma(q)|\beta\rangle. \quad (2.21)$$

Evidently, the pion-pole term in $\partial_\gamma(q)$ is of order ϵ, as is $\bar{\partial}_\gamma(q)$, if q^2 is not too small. However, if q^2 is on the order of $\epsilon\mu_0{}^2$, or smaller, the pion-pole term is effectively of order unity. For this reason, so long as ϵ is finite, no matter how small, one must be careful about neglecting pion-pole terms in $\partial_\gamma(q)$: The quantity $\bar{\partial}_\gamma(q)$ can, of course, always be neglected for sufficiently small ϵ. The reader will note that this fact never caused any trouble in our derivations since the various pion-pole terms combine so as to cancel out the pole denominator, as they do in Eq. (2.18) to give Eq. (2.20).

Just to dispel any doubts which might remain in the reader's mind about the equivalence of the PCAC hypothesis and calculating in the limit $\epsilon=0$, we will now show how to derive Eq. (2.20) when we set $\epsilon=0$.

For $\epsilon=0$, we have $\partial_\gamma(q)=0$ identically and thus Eq. (2.18) reads

$$-iq^\mu\langle\alpha|A_\gamma{}^\mu(q)|\beta\rangle=0. \quad (2.22)$$

Then, using Eq. (2.19) with $m_\pi{}^2=0$, one obtains

$$-iq_\mu[(-q^\mu/2f_\pi q^2)\langle\alpha+\pi_\gamma(q)|S|\beta\rangle+\langle\alpha|\hat{A}_\gamma{}^\mu(q)|\beta\rangle]=0 \quad (2.23)$$

or

$$(1/2f_\pi)\langle\alpha+\pi_\gamma(q)|S|\beta\rangle=q_\mu\langle\alpha|\hat{A}_\gamma{}^\mu(q)|\beta\rangle, \quad (2.24)$$

which is just Eq. (2.20) with $\bar{\partial}(q)=0$ corresponding to setting ϵ equal to zero. It is a general feature of $SU(2)\otimes SU(2)$ symmetry, realized by having a massless pion, that one always gets the same answer when calculating pion scattering amplitudes by starting directly with $\epsilon=0$ or by starting with $\epsilon\neq 0$ and then taking the limit $\epsilon\to 0$, *provided that one is careful to take all pion poles into account.* Another way to say this is to note that setting $\bar{\partial}\approx 0$ for small but nonzero ϵ is the same thing as the usual statement of PCAC that $\partial(q)$ is dominated by the pion pole. In the symmetric case ($\epsilon=0$), since $\partial(q)$ vanishes identically, the assumption of pole dominance clearly is meaningless; however, the interplay of the pion poles in $A_\gamma{}^\mu(q)$ and $\partial_\gamma(q)$ illustrated above always works out so that the predictions of pole dominance for nonzero ϵ are the same as the predictions of the symmetric theory.

The second item we would like to discuss is the derivation of the familiar two soft-pion theorem. Our interest in this problem is to show explicitly how the various pion poles cancel as they did in going from Eq. (2.18) to Eq. (2.20). We will also analyze the content of the so-called "Σ terms" which first appear in this case.

We begin with the time-ordered product

$$\langle\alpha|T(\partial_\mu A_\delta{}^\mu(q)\partial_\nu A_\gamma{}^\nu(k))|\beta\rangle.$$

Separating off terms corresponding to the diagrams

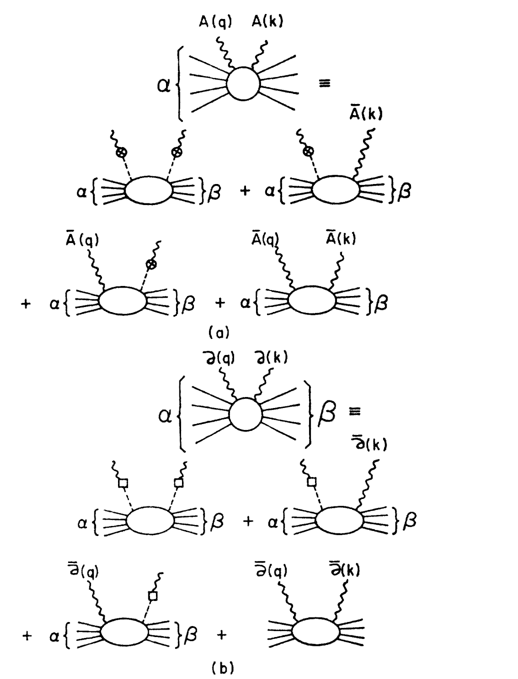

FIG. 1. (a) The wavy lines in all of these diagrams stand for axial currents. The circles with crosses in them stand for specific axial-vector current matrix elements, and the first three diagrams on the right stand for terms in which the axial current creates a single pion from vacuum. These diagrams give the pion-pole terms in Eq. (2.25). (b) The boxes stand for specific matrix elements of the divergence of the axial-vector current.

shown in Fig. 1(b), we get

$$\langle\alpha|T(\partial_\mu A_\delta{}^\mu(q)\partial_\nu A_\gamma{}^\nu(k))|\beta\rangle$$
$$=(+i)^2\Bigg[\frac{m_\pi{}^4\langle\alpha+\pi_\delta(q)+\pi_\gamma(k)|S|\beta\rangle}{(2f_\pi)^2(q^2-m_\pi{}^2)(k^2-m_\pi{}^2)}+\frac{m_\pi{}^2\langle\alpha+\pi_\delta(q)|\bar\partial_\gamma(k)|\beta\rangle}{2f_\pi(q^2-m_\pi{}^2)}+\frac{m_\pi{}^2\langle\alpha+\pi_\gamma(k)|\bar\partial_\delta(q)|\beta\rangle}{2f_\pi(k^2-m_\pi{}^2)}+\langle\alpha|T(\bar\partial_\delta(q)\bar\partial_\gamma(k))|\beta\rangle\Bigg]. \quad (2.25)$$

Starting with the same time-ordered product and pulling all derivatives through the time-ordering instruction, using the usual current algebra in order to evaluate the resulting equal-time commutators, we get

$$\langle\alpha|T(\partial_\mu A_\delta{}^\mu(q)\partial_\nu A_\gamma{}^\nu(k))|\beta\rangle=(+i)^2[q_\mu k_\nu\langle\alpha|T(A_\delta{}^\mu(q)A_\gamma{}^\nu(k))|\beta\rangle+\varepsilon_{\gamma\delta\rho}q_\mu\langle\alpha|V_\rho{}^\mu(q+k)|\beta\rangle-i\langle\alpha|\Sigma_{\delta\gamma}(q+k)|\beta\rangle], \quad (2.26)$$

where

$$+i\Sigma_{\gamma\delta}(q+k)=\int d^4x d^4y\, e^{+iq\cdot x}e^{+ik\cdot y}\delta(x_0-y_0)[\partial_\mu A_\gamma{}^\mu(y),A_\delta{}^0(x)]\equiv i\int d^4x\, e^{+i(q+k)\cdot x}\Sigma_{\gamma\delta}(x). \quad (2.27)$$

Now, as we did in deriving the generalized Goldberger-Treiman relation, we can expand

$$\langle\alpha|T(A_\gamma{}^\mu(q)A_\delta{}^\nu(k))|\beta\rangle$$

into double-pion-pole, single-pion-pole, and no-pole terms, corresponding to the diagrams of Fig. 1(a). The resulting expression can be combined with Eq. (2.25) to give

$$\frac{m_\pi{}^4\langle\alpha+\pi_\delta(q)+\pi_\gamma(k)|S|\beta\rangle}{(2f_\pi)^2(q^2-m_\pi{}^2)(k^2-m_\pi{}^2)}=\frac{q^2k^2\langle\alpha+\pi_\delta(q)+\pi_\gamma(k)|S|\beta\rangle}{(2f_\pi)^2(q^2-m_\pi{}^2)(k^2-m_\pi{}^2)}-\frac{k^2q_\mu\langle\alpha+\pi_\gamma(k)|\hat A_\delta{}^\mu(q)|\beta\rangle}{2f_\pi(k^2-m_\pi{}^2)}-\frac{q^2k_\mu\langle\alpha+\pi_\gamma(q)|\hat A_\delta{}^\mu(k)|\beta\rangle}{2f_\pi(q^2-m_\pi{}^2)}$$
$$+q_\mu k_\nu\langle\alpha|T(\hat A_\delta{}^\mu(q)\hat A_\gamma{}^\nu(k))|\beta\rangle-\frac{m_\pi{}^2\langle\alpha+\pi_\delta(q)|\bar\partial_\gamma(k)|\beta\rangle}{2f_\pi(q^2-m_\pi{}^2)}-\frac{m_\pi{}^2\langle\alpha+\pi_\gamma(k)|\bar\partial_\delta(q)|\beta\rangle}{2f_\pi(k^2-m_\pi{}^2)}-\langle\alpha|T(\bar\partial_\delta(q)\bar\partial_\gamma(k))|\beta\rangle$$
$$+q_\mu\varepsilon_{\gamma\delta\rho}\langle\alpha|V_\rho{}^\mu(q+k)|\beta\rangle-i\langle\alpha|\Sigma_{\gamma\delta}(q+k)|\beta\rangle. \quad (2.28)$$

Using Eq. (2.20) with the state $\langle\alpha|$ replaced by $\langle\alpha+\pi_\delta(q)|$ or $\langle\alpha+\pi_\gamma(k)|$, we get

$$(1/2f_\pi)\langle\alpha+\pi_\delta(q)+\pi_\gamma(k)|S|\beta\rangle-\langle\alpha+\pi_\delta(q)|\bar\partial_\gamma(k)|\beta\rangle=k_\nu\langle\alpha+\pi_\delta(q)|\hat A_\gamma{}^\nu(k)|\beta\rangle, \quad (2.29)$$

$$(1/2f_\pi)\langle\alpha+\pi_\delta(q)+\pi_\gamma(k)|S|\beta\rangle-\langle\alpha+\pi_\gamma(k)|\bar\partial_\delta(q)|\beta\rangle=q_\mu\langle\alpha+\pi_\gamma(k)|\hat A_\delta{}^\mu(q)|\beta\rangle. \quad (2.30)$$

Combining this with Eq. (2.28) and collecting terms, we get the identity

$$[1/(2f_\pi)^2]\langle\alpha+\pi_\delta(q)+\pi_\gamma(k)|S|\beta\rangle=q_\mu k_\nu\langle\alpha|T(\hat A_\delta{}^\mu(q)\hat A_\gamma{}^\nu(k))|\beta\rangle+q_\mu\varepsilon_{\gamma\delta\rho}\langle\alpha|V_\rho{}^\mu(q+k)|\beta\rangle$$
$$-i\langle\alpha|\Sigma_{\gamma\delta}(q+k)|\beta\rangle+(1/2f_\pi)(\langle\alpha+\pi_\gamma(k)|\bar\partial_\delta(q)|\beta\rangle+\langle\alpha+\pi_\delta(q)|\partial_\gamma(k)|\beta\rangle)-\langle\alpha|T(\bar\partial_\delta(q)\bar\partial_\gamma(k))|\beta\rangle. \quad (2.31)$$

Having cancelled out all pion poles in the variables k^2 and q^2, we can now apply our hypothesis that ϵ is small. As before, all terms in Eq. (2.31) containing $\bar\partial$ are of order ϵ (the term $\langle\alpha|T(\bar\partial(k)\bar\partial(q))|\beta\rangle$ is actually of order ϵ^2). Also, the operator $\Sigma_{\gamma\delta}$ is of order ϵ, as may be seen from the definition in Eq. (2.27). If we neglect terms of order ϵ, or equivalently set $\epsilon=0$, we obtain

$$[1/(2f_\pi)^2]\langle\alpha+\pi_\delta(q)+\pi_\gamma(k)|S|\beta\rangle=q_\mu k_\nu\langle\alpha|T(\hat A_\delta{}^\mu(q)\hat A_\gamma{}^\nu(k))|\beta\rangle+q_\mu\varepsilon_{\gamma\delta\rho}\langle\alpha|V_\rho{}^\mu(k+q)|\beta\rangle. \quad (2.32)$$

We leave it as an (instructive) exercise for the reader to show that Eq. (2.32) be obtained directly, when one starts out by setting $\epsilon=0$ and $m_\pi{}^2=0$.

If we let the α and β be one nucleon states, then crossing a pion in Eq. (2.32) gives a formula for pion-nucleon scattering. In the limit of small k and q, when only the singular term in $\langle\alpha|T(\hat A_\delta{}^\mu\hat A_\gamma{}^\nu)|\beta\rangle$ need be kept, Eq. (2.32) reproduces Weinberg's calculation of the π-N scattering lengths.[8] Since the predicted scattering lengths agree well with experiment, we can take this as further support for our contention that the approximation $\epsilon\approx0$ is a good one.

There is a possible source of confusion in Eqs. (2.31) and (2.32) which we should discuss here. We have not defined the object $\hat A$ and $\bar\partial$ in a way that makes them local operators. For this reason, quantities like $\langle\alpha|T(\hat A_\delta{}^\mu(q)\hat A_\gamma{}^\nu(k))|\beta\rangle$ are not to be thought of as literally taking the time-ordered product of two nonlocal objects. Rather, $\langle\alpha|T(\hat A_\delta{}^\mu(q)\hat A_\gamma{}^\nu(k))|\beta\rangle$ is defined to be $\langle\alpha|T(A_\delta{}^\mu(q)A_\gamma{}^\nu(k))|\beta\rangle$ minus the relevant pion-pole terms: That is, one forms the time-ordered product and then removes the poles, not the other way around.

Now that we have discussed these low-energy theorems for one and two pions, we would like to keep our promise and present additional arguments in support of our statement that the only rational way to understand

[8] S. Weinberg, Phys. Rev. Letters 17, 168 (1966).

PCAC is that ϵ is small and $m_\pi \to 0$ as $\epsilon \to 0$. This is done in the next two paragraphs.

It is important to appreciate the fact that Eqs. (2.20) and (2.31) are exact, content-free identities which hold in any theory and, in particular, are independent of any assumption such as PCAC or $\epsilon \approx 0$. These equations acquire physical significance only when the terms involving $\bar{\partial}$ and Σ are known to be small in comparison to other terms, PCAC in its usual form in fact just tells us to drop these terms. (The reason why PCAC says one must neglect the Σ term will be discussed in a moment.) As we have seen, this is identical to the way one handles these terms if one hypothesizes that the term ϵH_1, occurring in the formal decomposition $H=H_0+\epsilon H_1$, is negligible. Thus, these soft-pion theorems provide another strong indication that the physical content of PCAC is simply the statement that ϵ is small. Another interesting point to take note of is that the validity of these soft-pion theorems gives the same sort of information about the behavior of m_π in the limit $\epsilon \to 0$ as we obtained from a consideration of the Goldberger-Treiman relation. The point is that, as we have seen, when $\epsilon \to 0$ either $m_\pi \to 0$ or $f_\pi^{-1} \to 0$. In the former case, when f_π^{-1} remains finite one obtains the nontrivial identities given in Eqs. (2.24) and (2.32) which give information about pion scattering which does, in fact, agree with experiment. If, on the other hand, $f_\pi^{-1} \to 0$ as $\epsilon \to 0$, Eqs. (2.20) and (2.31) remain content-free identities for all values of ϵ, since in this case there is no obvious reason why the terms $\bar{\partial}$ and Σ should be small in comparison to the other terms in these equations even when $\epsilon=0$. The reason for this should be obvious; namely, when $f_\pi^{-1} \to 0$ as $\epsilon \to 0$, the terms in Eqs. (2.20) and (2.31) involving the S-matrix element, which are explicitly multiplied by f_π^{-1}, vanish along with $\bar{\partial}$ and Σ. The remaining terms vanish trivially; in fact, the term $q_\mu\langle\alpha|\hat{A}_\delta{}^\mu(q)|\beta\rangle$ in Eq. (2.20) becomes equal to $q_\mu\langle\alpha|A_\delta{}^\mu(q)|\beta\rangle$ since $\hat{A}_\delta{}^\mu = A_\delta{}^\mu$ when $f_\pi^{-1}=0$, and therefore this term vanishes by current conservation. Similarly in Eq. (2.31) the term $q_\mu k_\nu\langle\alpha|T(\hat{A}_\delta{}^\mu(q)\hat{A}_\gamma{}^\nu(k))|\beta\rangle + q_\mu \varepsilon_{\gamma\delta\rho}\langle\alpha|V_\gamma{}^\mu(q+k)|\beta\rangle$ also vanishes when $\partial_\mu A_\delta{}^\mu=0$ and $f_\pi^{-1}=0$, again because in this case $\hat{A}_\delta{}^\mu = A_\delta{}^\mu$ since the pion decouples from the axial-vector current. Thus, only the alternative in which $m_\pi \to 0$ as $\epsilon \to 0$ provides an explanation for the success of the Adler consistency relation and Weinberg's calculation of the pion-nucleon scattering lengths.

With these points out of the way, let us now show that PCAC implies that the operator Σ is small; as we shall see, this will provide still another argument that it is reasonable to assume that ϵ is a small number.

Let us start with Eq. (2.20), replacing $|\alpha\rangle$ by the state $|\alpha+\pi_\gamma(k)\rangle$, and obtain

$$\begin{aligned}(1/2f_\pi)&\langle\alpha+\pi_\gamma(k)+\pi_\delta(q)|S|\beta\rangle\\ &=q_\mu\langle\alpha+\pi_\gamma(k)|\hat{A}_\delta{}^\mu(q)|\beta\rangle+\langle\alpha+\pi_\gamma(k)|\bar{\partial}_\delta(q)|\beta\rangle\\ &\approx q_\mu\langle\alpha+\pi_\gamma(k)|\hat{A}_\delta{}^\mu(q)|\beta\rangle,\end{aligned}\tag{2.33}$$

according to the PCAC hypothesis that $\bar{\partial}(q)$ is negligible. On the other hand, we can use Eq. (2.31), dropping the small $\bar{\partial}$ terms, to obtain another approximate expression for $\langle\alpha+\pi_\gamma(k)+\pi_\delta(q)|S|\beta\rangle$. We then have the consistency condition

$$\begin{aligned}q_\mu&\langle\alpha+\pi_\gamma(k)|\hat{A}_\delta{}^\mu(q)|\beta\rangle\\ &\approx 2f_\pi[q_\mu k_\nu\langle\alpha|T(\hat{A}_\delta{}^\mu(q)\hat{A}_\gamma{}^\nu(k))|\beta\rangle\\ &\quad+\varepsilon_{\gamma\delta\rho}q_\mu\langle\alpha|V_\rho{}^\mu(k+q)|\beta\rangle-i\langle\alpha|\Sigma_{\gamma\delta}(k+q)|\beta\rangle],\end{aligned}\tag{2.34}$$

which must hold in any theory in which PCAC is a good approximation. Any disagreement between the two sides of Eq. (2.34) must be of order $\bar{\partial}$; that is, of the order of the corrections to PCAC. To learn something more about the term $\Sigma_{\gamma\delta}$, one need only consider Eq. (2.34) in the limit $q^\mu \to 0$. In this limit two important things happen. First, the term involving $V_\rho{}^\mu$ drops out. Second, one need keep only those terms in

$$\langle\alpha|T(\hat{A}_\delta{}^\mu(q)\hat{A}_\gamma{}^\nu(k))|\beta\rangle$$

which are singular as $q^\mu \to 0$. Thus, using a self-explanatory notation, we have

$$\begin{aligned}i\langle\alpha|\Sigma_{\gamma\delta}(k)|\beta\rangle\approx\lim_{q\to0}&[q_\mu k_\nu\langle\alpha|T(\hat{A}_\delta{}^\mu(q)\hat{A}_\gamma{}^\nu(k))|\beta\rangle_{\text{poles}}\\ &-(q_\mu/2f_\pi)\langle\alpha+\pi_\gamma(k)|\hat{A}_\delta{}^\mu(q)|\beta\rangle_{\text{poles}}].\end{aligned}\tag{2.35}$$

It will be shown below that the right-hand side of Eq. (2.35) is of order $\bar{\partial}(k)$, so that we get

$$\Sigma_{\gamma\delta}(k)\sim\bar{\partial}(k)\tag{2.36}$$

in any consistent theory in which $\bar{\partial}$ is small. Equation (2.36) is, of course, not to be taken to mean that $\Sigma_{\gamma\delta}$ and $\bar{\partial}$ are in any way identical as operators but merely that their matrix elements are of the same order of magnitude.

The fact that the right-hand side of Eq. (2.35) is of order $\bar{\partial}$, is most easily seen by considering a specific example, although the argument is readily generalized. Suppose that α and β are one-nucleon states; then the pole diagrams for Eq. (2.35) are shown in Fig. 2. It should be clear that the difference between the two terms in Fig. 2 is of order

$$k_\nu\langle N|\hat{A}_\gamma{}^\nu(k)|N\rangle-(1/2f_\pi)\langle N+\pi_\gamma(k)|S|N\rangle$$

which, according to Eq. (2.20), is just $\langle N|\bar{\partial}_\gamma(k)|N\rangle$.

The argument just presented thus establishes the fact that PCAC correction terms and $\Sigma_{\gamma\delta}$ are of the same order of magnitude, and that one is consistent in their application of the PCAC hypothesis only if one drops both kinds of terms simultaneously. Using this result, one can now construct another argument for the fact that ϵ must be small.

Suppose, for the sake of argument, that ϵH_1 belongs to a single irreducible representation (n,n) of the group $SU(2)\otimes SU(2)$ generated by the charges at time t. This

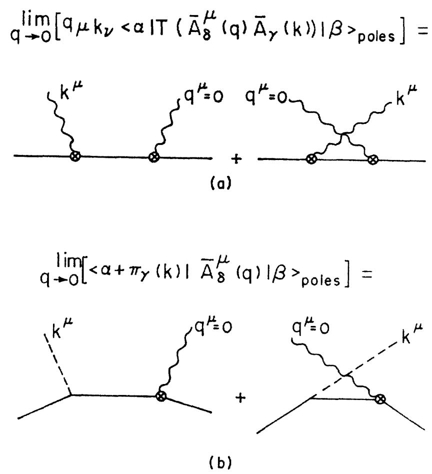

FIG. 2. The circles with crosses stand for the matrix elements of the axial-vector current between nucleons.

assumption, coupled with the fact that H_1 is an isospin singlet, can be shown to imply that

$$\sum_{\gamma=1}^{3} \Sigma_{\gamma\gamma}(k_0, \mathbf{k}=0) = C_n \epsilon \int e^{ik_0 t} H_1(t) dt , \qquad (2.37)$$

where C_n is a c number which depends only on the representation (n,n) to which H_1 belongs; for example, if $(n,n)=(\frac{1}{2},\frac{1}{2})$, then $C_n=3$. Equations (2.36) and (2.37) obviously demonstrate directly that to say corrections to PCAC are small is the same thing as saying ϵ is small. Our simplifying assumption that H_1 belongs to a single irreducible representation of $SU(2)\otimes SU(2)$ is clearly not crucial to establishing the basic result that PCAC tells us to ignore terms of order ϵ^1 and higher.

By now we have discussed, in some detail, our reasons for saying that the success of the PCAC approximation can be taken as a strong indication that ϵH_1 can be thought of as a small $SU(2)\otimes SU(2)$-symmetry breaking term. If we have failed to convince the reader that this is reasonable, we can only add at this point that aside from the concrete arguments which we have already presented there is what we consider a rather compelling aesthetic reason for adopting this point of view, namely, that, in its usual form, PCAC is at best an amorphous principle. The introduction of the idea of $SU(2)\otimes SU(2)$ symmetry gives an exact meaning to PCAC; the PCAC approximation is then equivalent to the statement that, in calculating pion scattering amplitudes, one need only calculate to order ϵ^0. (Actually, as we shall see at a later point, the case of π-π scattering is rather special in that the first order in ϵ correction term at threshold can be expected to be as large as the amplitude calculated in the symmetric limit; however, π-π scattering possesses the additional important property that, in this case, we can completely calculate the most important correction term to the low-energy theorem.) When we view the PCAC hypothesis as a statement about a broken symmetry, there is some hope that corrections to PCAC might someday be calculable, whereas, without the symmetry concept, one does not really know what PCAC means and the problem of calculating corrections is less well defined.

The only point which we have not yet discussed is the broken-symmetry interpretation of the ambiguities encountered in the familiar applications of PCAC. In order to be better able to compare the two points of view, let us first review the main features of the conventional statement of the ambiguities encountered.

As we have already noted, the statement of PCAC involves extracting the pion-pole term from matrix elements of the form $\langle\alpha|\partial_\mu A_\alpha{}^\mu(q)|\beta\rangle$. In the event that the states $|\alpha\rangle$ and $|\beta\rangle$ are both one-particle states, this matrix element is a function of q^2 alone, and in this case we can unambiguously define the pole term as the value of the residue of the pole at $q^2=m_\pi{}^2$ divided by $q^2-m_\pi{}^2$. However, when $|\alpha\rangle$ and $|\beta\rangle$ are states containing more than one particle no such clean-cut statement can be made. The reason for this is that in the many-particle case the matrix element $\langle\alpha|\partial_\mu A_\alpha{}^\mu(q)|\beta\rangle$ does not depend upon q^2 alone, but also depends upon a set of energy- and momentum-transfer variables. In such a case, when one makes an assumption of pole dominance, it is necessary to tell which variables (or combination of variables) should be held fixed as we allow q^2 to vary. The point is that because of conservation of momentum the different variables satisfy various constraints and different definitions give different meanings to PCAC. For example, suppose the state contains two particles of masses m_1 and m_2 and momenta p_1 and p_2. Moreover, let $|\alpha\rangle$ be a one-particle state of mass m_3 and momentum p_3. Then, the matrix element

$$\langle p_3|\partial_\mu A_\alpha{}^\mu(q)|p_1,p_2\rangle$$

is a function of the variables s, t, and u defined as

$$s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad \text{and} \quad u=(p_2-p_3)^2. \qquad (2.38)$$

Besides these definitions there is the constraint of conservation of momentum which gives

$$p_3+q=p_1+p_2 , \qquad (2.39)$$

which implies

$$s+t+u=m_1{}^2+m_2{}^2+m_3{}^3+q^2 . \qquad (2.40)$$

The important point is that since this constraint depends upon q^2, different choices of the definition of the residue of the pole term give different off-mass-shell definitions of PCAC. To see this, suppose that we define the pole term in the scattering amplitude choosing s and t as independent variables; that is,

$$\text{pole term} = \frac{1}{q^2-m_\pi{}^2}\langle\alpha,\pi_\alpha(q)|S|\beta\rangle|_{s,t} , \qquad (2.41)$$

where by $\langle\alpha+\pi|S|\beta\rangle|_{s,t}$ we mean the on-mass-shell amplitude for physical pions with given s and t. Equation (2.41) gives us a definite prescription for going off mass shell in q^2, holding s and t fixed. We see that since $u=m_1^2+m_2^2+m_3^2+q^2-s-t$, going off mass shell in q^2 corresponds to having the value of u change. Clearly, this is not the same as defining

$$\text{pole term}=\frac{1}{q^2-m_\pi^2}\langle\alpha+\pi_\alpha(\bar{q})|S|\beta\rangle|_{u,t}, \quad (2.42)$$

since in definition (2.41) by letting $q^2=0$ we are defining the amplitude to have the value $-(1/m_\pi^2)\times\langle\alpha+\pi_\alpha(\bar{q})|S|\beta\rangle$ when $u=m_1^2+m_2^2+m_3^2-s-t$, whereas in Eq. (2.42) we define it to have the same value at $u=m_1^2+m_2^2+m_3^2+m_\pi^2-s-t$. Thus, we see that any definition of PCAC must be accompanied by a prescription telling us which independent dynamical variables must be held fixed. As we observed before, this was not an important point in deriving our results since they only depended upon the assumption that some prescription existed for which the pole term dominated the divergence when $q^2\approx m_\pi^2$.

At this point we would like to point out that this ambiguity is not peculiar to PCAC but is, in fact, the same difficulty encountered in comparing the results of any broken symmetry with experiment. For example, consider the case of pion-nucleon scattering; $SU(2)$ symmetry gives relations between the processes $\pi+N\to\rho+N$ and $\pi+N\to K^*+\Lambda$. However, since $SU(3)$ is actually a broken symmetry, comparison of these predictions with experiment is difficult. The reason for this is that the relationships between these amplitudes is strictly true only in the limit that $SU(3)$ is not broken, which means that all particles in the same multiplet have the same mass. Thus, the results of a symmetric calculation tell us something about the scattering amplitude when $\bar{m}_1=m_\Lambda=m_N$ and $\bar{m}_2=m_{K^*}=m_\rho$, which corresponds to $s+t+u=m_\pi^2+2\bar{m}_1^2+\bar{m}_2^2$. In order to compare these symmetry results with experiment, we have to compare them at values of $s+t+u=m_\pi^2+m_N^2+m_{K^*}^2+m_\Lambda^2$ for one case and $s+t+u=m_\pi^2+2m_N^2+m_\rho^2$ for the other. Thus, the question arises exactly as in the PCAC case, whether the results of the symmetry calculation should be expected to be correct for the amplitudes in which s and t are those of the symmetry calculation, or should we expect the symmetry predictions to be true when t and u are fixed, etc. The actual comparison of the predictions of a broken symmetry with experimental scattering amplitudes is a delicate question involving a detailed study of a particular process and, as we have said, is really more of an art than a science. Fubini and Furlan[9] have recently studied this question in some detail and their work suggests a possible systematic prescription for discussing this question for any given process.

This last section finishes our recapitulation of essentially familiar results so as to point out their content with respect to the idea that the strong interactions are essentially $SU(2)\otimes SU(2)$-symmetric. What we hope we have made clear in this section is that all of the conventional results following from the joint application of PCAC and current algebra can be very naturally stated using the familiar language of broken symmetry. In particular, we should like to emphasize that the following series of statements provides a natural and complete definition of the PCAC hypothesis, completely equivalent to the familiar one.

(i) The strong-interaction Hamiltonian can be broken into an $SU(2)\otimes SU(2)$-symmetric part H_0 plus a symmetry-breaking term ϵH_1.

(ii) ϵ is small in the sense that a perturbation expansion about the symmetric theory defined by H_0 makes sense.

(iii) The symmetric theory is one in which isospin shows up as a conventional symmetry giving rise to degenerate multiplets of particles, but the symmetry generated by the axial-vector charges is realized by the appearance of massless pseudoscalar pions (i.e., Goldstone bosons).

(iv) All pion scattering amplitudes may, for small pion energies, be correctly calculated by working to order ϵ^0; moreover, in general, calculating ϵ^1 correction terms is very difficult.

(v) Scattering amplitudes involving only pions can be calculated correctly to order ϵ^1 if one knows something about the $SU(2)\otimes SU(2)$ transformation properties of H_1. (More will be said about this in Sec. 4.)

3. DISCUSSION OF GENERAL RESULTS

In Sec. 2, we showed that the usual one- and two-soft-pion theorems, arising from the joint application of PCAC and current algebra, correspond to calculating these amplitudes in the limit $\epsilon=0$. Appendix B is devoted to showing that, in fact, this is true for processes involving any number of soft pions. Assuming, for the present, that the symmetric limit of the real world is the correct one in which to calculate any low-energy pion scattering amplitude, the problem of calculating these amplitudes still remains. This section is devoted to a discussion of the general results, proven in later sections, which tell us how to calculate all possible low-energy theorems for the scattering of pions off anything when $\epsilon=0$ and $m_\pi^2=0$.

Before stating our general results, there is one point which must be discussed, namely, we must explain a formal device which we use throughout this paper. The observation which leads directly to the introduction of this device is the following: *The starting point of every soft-pion theorem is an identity of the form*

$$0=\int d^4x_1\cdots d^4x_n e^{+iq_1\cdot x_1}\cdots e^{+iq_n\cdot x_n}\times\langle\alpha|T(\partial_\mu A_{\alpha_1}{}^\mu(x_1)\cdots\partial_\mu A_{\alpha_n}{}^\mu(x_n))|\beta\rangle. \quad (3.1)$$

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[9] S. Fubini and G. Furlan (unpublished).

As we have seen in Sec. 2, we make use of this identity by pulling all derivatives through the time-ordering instruction and using current algebra to evaluate the resulting equal-time commutators. The trick which makes these manipulations simpler is to notice that the following proposition is generally true.

Proposition. If we know how to compute expressions of the form

$$\int d^4x_1\cdots d^4x_n \times\langle\alpha|T(\boldsymbol{\varphi}(x_1)\cdot\mathbf{D}(x_1)\cdots\boldsymbol{\varphi}(x_n)\cdot\mathbf{D}(x_n))|\beta\rangle \quad (3.2)$$

for an arbitrary c-number isospinor function $\boldsymbol{\varphi}(x)$ and an arbitrary boson field $\mathbf{D}(x)$, then we can compute all expressions of the form

$$\int d^4x_1\cdots d^4x_n e^{+iq_1\cdot x_1}\cdots e^{+iq_n\cdot x_n} \times\langle\alpha|T(D_{\alpha_1}(x_1)\cdots D_{\alpha_n}(x_n))|\beta\rangle. \quad (3.3)$$

[We should explain that we state this proposition for an arbitrary boson field $\mathbf{D}(x)$ rather than for $\partial_\mu\mathbf{A}^\mu(x)$ so that there will be no confusion over the fact that when $\epsilon=0$, $\partial_\mu\mathbf{A}^\mu(x)$ vanishes identically.]

The general proof of this proposition is not difficult and is left as an instructive exercise to the reader; we shall content ourselves with showing how things work in the simplest cases $n=1$ and $n=2$.

Case 1: $n=1$. In this case if we know how to evaluate $\int d^4x\,\boldsymbol{\varphi}(x)\cdot\mathbf{D}(x)$ for arbitrary $\boldsymbol{\varphi}(x)$, we need only take $\boldsymbol{\varphi}(x)=e^{+iq\cdot x}\boldsymbol{\varepsilon}$ (where $\boldsymbol{\varepsilon}$ is an arbitrary isovector) to see that the proposition is trivially true.

Case 2: $n=2$. In this case we assume that we know how to calculate

$$\int d^4xd^4y\langle\alpha|T(\boldsymbol{\varphi}(x)\cdot\mathbf{D}(x)\boldsymbol{\varphi}(y)\cdot\mathbf{D}(y))|\beta\rangle, \quad (3.4)$$

and wish to know how to calculate

$$\int d^4xd^4y\; e^{+iq_1\cdot x}e^{+iq_2\cdot y}\langle\alpha|T(D_{\alpha_1}(x_1)D_{\alpha_2}(x_2))|\beta\rangle. \quad (3.5)$$

To do this let $\boldsymbol{\varphi}(x)$ be given by

$$\boldsymbol{\varphi}(x)=\boldsymbol{\varepsilon}_1e^{+iq_1\cdot x}+\boldsymbol{\varepsilon}_2e^{+iq_2\cdot x}.$$

Substituting this into Eq. (3.4), we get

$$\begin{aligned}\int d^4xd^4y[&e^{+iq_1\cdot x}e^{+iq_1\cdot y}\langle\alpha|T(\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(x)\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(y))|\beta\rangle\\ &+e^{+iq_1\cdot x}e^{+iq_2\cdot y}\langle\alpha|T(\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(x)\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(y))|\beta\rangle\\ &+e^{+iq_2\cdot x}e^{+iq_1\cdot y}\langle\alpha|T(\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(x)\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(y))|\beta\rangle\\ &+e^{+iq_2\cdot x}e^{+iq_2\cdot y}\langle\alpha|T(\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(x)\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(y))|\beta\rangle]. \quad (3.6)\end{aligned}$$

Letting $x=u+\frac{1}{2}z$ and $y=u-\frac{1}{2}z$, we get

$$\begin{aligned}(2\pi)^4\Big[&\delta^4(p_\alpha+q_1+q_1-p_\beta)\int d^4z\langle\alpha|T(\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(\tfrac{1}{2}z)\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(-\tfrac{1}{2}z))|\beta\rangle\\ &+\delta^4(p_\alpha+q_2+q_1-p_\beta)\int d^4z\; e^{+i(q_1-q_2)z/2}\langle\alpha|T(\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(\tfrac{1}{2}z)\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(-\tfrac{1}{2}z))|\beta\rangle\\ &+\delta^4(p_\alpha+q_2+q_1-p_\beta)\int d^4z\; e^{+i(q_2-q_1)z/2}\langle\alpha|T(\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(\tfrac{1}{2}z)\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(-\tfrac{1}{2}z))|\beta\rangle\\ &+\delta^4(p_\alpha+q_2+q_2-p_\beta)\int d^4z\langle\alpha|T(\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(\tfrac{1}{2}z)\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(-\tfrac{1}{2}z))|\beta\rangle\Big]. \quad (3.7)\end{aligned}$$

Clearly, as long as we choose q_1 and q_2 such that $q_1\neq q_2$ only the middle two terms in Eq. (3.7) contribute, because of the momentum-conservation δ functions. Letting $z'=-z$ in the third term and noting that $\mathbf{D}(x)$ is a boson field, we see that these terms are equal, and with this choice of $\boldsymbol{\varphi}(x)$, we get

$$\int d^4xd^4y\langle\alpha|T(\boldsymbol{\varphi}(x)\cdot\mathbf{D}(x)\boldsymbol{\varphi}(y)\mathbf{D}(y))|\beta\rangle=2\int d^4xd^4y \times e^{+iq_1\cdot x}e^{+iq_2\cdot y}\langle\alpha|T(\boldsymbol{\varepsilon}_1\cdot\mathbf{D}(x)\boldsymbol{\varepsilon}_2\cdot\mathbf{D}(y))|\beta\rangle, \quad (3.8)$$

which gives the result we wish by appropriately choosing $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$. The points $q_1=q_2$ can clearly be reached by taking the appropriate limit and using the continuity in the variables q_1 and q_2. As we said, the more general proof for arbitrary n is exactly the same except one finds constraints that the momenta $q_1,\cdots,q_n$ must obey in order that only the δ functions multiplying the desired terms contribute. In general, the reader will find that this amounts to having to exclude certain $4(n-m)$-dimensional planes in the space of n-tuples of the form $(q_1,\cdots,q_n)$, where m is always greater than or equal to 1. The values on these planes, as we have seen for the case $n=2$, are determined by continuity.

Using the notation just established we can write an identity, which we shall prove in Secs. 5 and 6, for the general n-pion scattering amplitudes when $\epsilon=0$. This can be stated as follows.

A. Basic Identity

If we assume

(i) $\partial_\mu A_\alpha{}^\mu(x)=0,\quad m_\pi{}^2=0.$

(ii)
$$\begin{aligned}
&[V_\alpha{}^0(x),V_\beta{}^\mu(y)]_{x_0=y_0}\\
&\qquad=i\delta^3(\mathbf{x}-\mathbf{y})\varepsilon_{\alpha\beta\gamma}V_\gamma{}^\mu(x)+\text{S.T.}\,,\\
&[V_\alpha{}^0(x),A_\beta{}^\mu(y)]_{x_0=y_0}\\
&\qquad=i\delta^3(\mathbf{x}-\mathbf{y})\varepsilon_{\alpha\beta\gamma}A_\gamma{}^\mu(x)+\text{S.T.}\,,\\
&[A_\alpha{}^0(x),A_\beta{}^\mu(y)]_{x_0=y_0}\\
&\qquad=i\delta^3(\mathbf{x}-\mathbf{y})\varepsilon_{\alpha\beta\gamma}V_\gamma{}^\mu(x)+\text{S.T.}\,,
\end{aligned}\tag{3.9}$$

(where S.T. stands for possible Schwinger terms).

(iii) All time-ordered products have been appropriately redefined to be covariant so that all Schwinger terms cancel and can be ignored.[10]

Then we can prove, for S_0 equal to the S matrix in the symmetrical theory:

$$\begin{aligned}
&\langle\alpha+\pi(\varepsilon_1,q_1)\cdots+\pi(\varepsilon_n,q_n)|S_0|\beta\rangle\\
&\qquad\qquad=f_\pi{}^n\langle\alpha|U^n(q_1,\cdots,q_n)|\beta\rangle\,,
\end{aligned}\tag{3.10}$$

where $U^n(q_1,\cdots,q_n)$ is defined by taking the coefficient of $f_\pi{}^n$ in the expansion of the exponential

$$\begin{aligned}
T\bigg\{\exp\bigg[&+i\int d^4x(-2f_\pi)(\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu\\
&+\frac{f_\pi}{1+f_\pi{}^2\varphi^2}[(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu-f_\pi\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu)\bigg]\bigg\}
\end{aligned}\tag{3.11}$$

and letting

$$\boldsymbol{\varphi}(x)=\sum_{j=1}^n\boldsymbol{\varepsilon}_je^{+iq_j\cdot x}\,,$$

keeping only those terms in the resulting expression for which all of the q_j's are distinct. (As before, the bar above the axial-vector current tells us to leave out all terms giving rise to a pole in any one of the momenta $q_j{}^2$.)

Clearly, the cases $n=1$ and $n=2$ give us the two formulas already calculated explicitly,

$$\langle\alpha+\pi(\varepsilon,k)|S_0|\beta\rangle=f_\pi\bigg[-2i\langle\alpha|\int d^4x(\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu)|\beta\rangle\bigg]$$

and

$$\begin{aligned}
&\langle\alpha+\pi(\varepsilon,q)+\pi(\varepsilon,k)|S_0|\beta\rangle\\
&\quad=f_\pi{}^2[2(+i)^2\langle\alpha|T(\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu\partial_\nu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\nu)|\beta\rangle\\
&\qquad\qquad-2i\langle\alpha|(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu|\beta\rangle].
\end{aligned}$$

We should point out that the form of this general identity can be modified if we redefine the function $\boldsymbol{\varphi}(x)$ by multiplying it by an arbitrary function $G(\varphi^2(x))$; what we have done in deriving Eqs. (3.10) and (3.11) is to choose $G(\varphi^2)$ so as to get the simplest result. It is also worth noting that the identity expressed in Eqs. (3.10) and (3.11) is an exact formula for zero-mass pion scattering amplitudes, and if we knew how to evaluate the resulting time-ordered products of currents, then in principle we could calculate the amplitudes for pions of arbitrary energies.

[10] L. Brown, Phys. Rev. **150**, 1338 (1966); S. Y. Lee (unpublished).

Since we have no reliable way of evaluating such expressions, however, the best we can do is use this identity to derive formulas which tell us something about the behavior of these general scattering amplitudes as we allow all of the pion momenta to vanish. In the next few paragraphs we describe the general results which can be proven for this case, which we call the tree approximation.

B. Tree Approximation

In order to discuss this approximation, it becomes convenient to introduce a scale into the problem by letting each $q_i=\xi Q_i$; the soft-pion limit then corresponds to the limit $\xi\to0$. We show in detail in Sec. 7 how to derive a phenomenological Lagrangian which we can use to reproduce the resulting low-energy theorem. In this section we present and discuss the general features of the results obtained in Sec. 7.

The starting point for deriving all low-energy theorems is the basic identity given in Eqs. (3.10) and (3.11).

The most important result in Sec. 7 is the fact that the basic identity [Eqs. (3.10) and (3.11)] allows us to calculate the coefficient of the lowest power of ξ, appearing in an expansion of the amplitude about $\xi=0$, in terms of tree diagrams, whose vertices are given by measurable vector and axial-vector coupling constants.

In order to better understand a statement of our general results concerning the tree approximation, there is one point that must be discussed; that is the fact that the full π-π scattering amplitude is involved in Eqs. (3.10) and (3.11) in a rather complicated way, and before we can construct, starting from our identity, a phenomenological Lagrangian, we must learn to separate out explicitly an expression which generates only the π-π scattering amplitudes to the leading order in ξ.

The process $N\to N+3\pi$, which is related by crossing to the physical process $\pi+N\to N+2\pi$, illustrates completely the various points to be made about the tree approximation. As we shall see, this process vanishes as ξ in the limit $\xi\to0$ as does any process of the form $\langle N+m\pi|S|N+p\pi\rangle$ when all the pion momenta vanish; the basic identity allows us to compute the coefficient of the term going as ξ in the scattering amplitude without knowing anything about the currents except certain measurable physical coupling constants.

In order to get an idea of how this works, let us use the basic identity to get the following:

$$\begin{aligned}
&\langle N+3\pi|S_0|N\rangle\\
&\quad=f_\pi{}^3[+\tfrac{4}{3}i\langle N|T(\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu\partial_\nu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\nu\partial_\sigma\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\sigma)|N\rangle\\
&\qquad\quad-2\langle N|T(\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu(\boldsymbol{\varphi}\times\partial_\sigma\boldsymbol{\varphi})\cdot\mathbf{V}^\sigma)|N\rangle\\
&\qquad\qquad+2i\langle N|\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu|N\rangle].
\end{aligned}\tag{3.12}$$

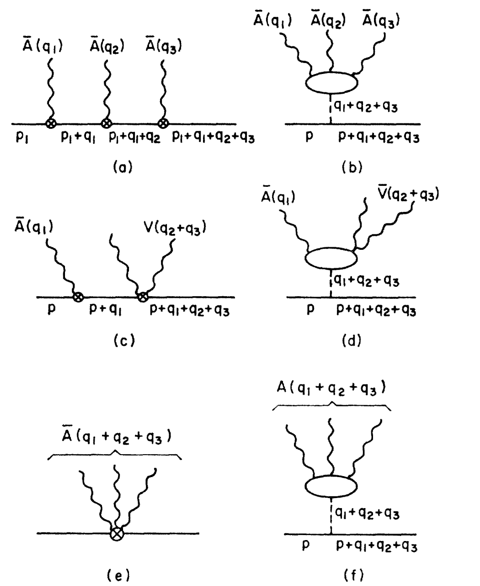

FIG. 3. Tree diagrams for the process $N+\pi \rightarrow N+2\pi$.

Calculating the term of order ξ is not difficult at this point, provided we observe that every factor of $\partial_\mu \varphi$ gives rise to an explicit factor of ξ, multiplying any term in which it appears. Thus, the first term in Eq. (3.12) is explicitly of order ξ^3; the second is of order ξ^2, and the third is of order ξ. If the explicit ξ dependence were everything, then only the last term could contribute to the leading order of ξ; however, this is not the case. For example, the first term in Eq. (3.12) has a piece which goes as ξ^{-2}, coming from diagrams such as the one shown in Fig. 3(a), corresponding to insertions of the axial-vector current on the external nucleon lines, since the propagators which appear in Fig. 3(a) clearly go as $(p\cdot q)^{-1} \sim \xi^{-1}$. Note, that there is another diagram which has an explicit factor of ξ^{-2} associated with it, namely, one in which the three axial-vector currents create a one-pion intermediate state from vacuum. It is shown in Fig. 3(b); we shall discuss this kind of diagram in a moment. Clearly, in the limit $\xi \rightarrow 0$, the vertices in Fig. 3(a) are given once one knows the axial-vector coupling constant g_A.

Similarly, the second term in Eq. (3.12) gives rise to diagrams which have singular ξ dependence, namely, the diagrams shown in Figs. 3(c) and 3(d); Figs. 3(e) and 3(f) correspond to the singular terms coming from the third term in Eq. (3.12). Whereas Figs. 3(a), 3(c), and 3(e) are easily evaluated in terms of known axial-vector and vector coupling constants, something remains to be said about the diagrams appearing in Figs. 3(b), 3(d), and 3(f). Although we cannot say much about these terms individually, we can say what the coefficient of the leading power of ξ in their sum must be in the limit $\xi \rightarrow 0$. The reason for this is that unitarity tells us the residue of the one-pion pole term which appears in the scattering amplitude in question must factor into the product of the on-mass-shell π-π scattering amplitude times the π-N vertex function. As we shall see later, the leading part of the π-π amplitude vanishes as ξ^2 (when $\epsilon=0$), and the π-N vertex goes as ξ, which tells us that the sum of all of these diagrams goes as ξ and therefore contributes to the coefficient of the ξ terms in the π-N vertex function.

To summarize, we find that in order to calculate the ξ term in the pion-nucleon scattering amplitude, we need to calculate two types of terms. First, we must calculate nucleon-pole terms like those shown in Figs. 3(a), 3(c), and 3(e); and second, we must calculate graphs which involve a π-π scattering subgraph, as shown in Figs. 3(b), 3(d), and 3(f).

Clearly, the vertices in Fig. 3(a) are given by an expression of the form

$$\bar{u}_N(g_A\gamma^\mu\gamma_5\boldsymbol{\tau})u_N\,, \tag{3.13}$$

and from Eq. (3.12) we see that whenever such a vertex appears it gets multiplied by a factor of $\partial_\mu\boldsymbol{\varphi}$, so that the first term in Eq. (3.12), which gives us Fig. 3(a), is the same as the term of order f_π^3 calculated using a pion-nucleon effective interaction Lagrangian of the form

$$\mathcal{L}_1{}^{(x)} = -f_\pi\partial_\mu\boldsymbol{\varphi}(x)\bar{\psi}_N(x)(g_A\gamma^\mu\gamma_5\boldsymbol{\tau})\psi_N(x)\,, \tag{3.14}$$

where $\boldsymbol{\varphi}(x)$ now stands for the canonical pion field and $\psi_N(x)$ for the canonical nucleon field. Similarly, the vector vertex to be evaluated in Fig. 3(c) is given by an expression of the form

$$\bar{u}_N(\gamma^\mu\boldsymbol{\tau})u_N\,, \tag{3.15}$$

and a glance at Eq. (3.12) shows us that this must be multiplied by a pion wave function obtained by multiplying this vertex factor by $\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}$. Again, it is clear that such a vertex comes from adding a term to $\mathcal{L}_1(x)$ [Eq. (3.14)] of the form

$$\mathcal{L}_2(x) = -f_\pi{}^2[\boldsymbol{\varphi}(x)\times\partial_\mu\boldsymbol{\varphi}(x)]\bar{\psi}_N(x)(\gamma^\mu\boldsymbol{\tau})\psi_N(x)\,. \tag{3.16}$$

Using the effective Lagrangian $\mathcal{L}_1+\mathcal{L}_2$ and calculating to order $f_\pi{}^3$, we obviously generate Figs. 3(a) and 3(c). The diagram in Fig. 3(e) is generated by a term of the form

$$\mathcal{L}_3(x) = +f_\pi{}^3\varphi^2(x)\partial_\mu\boldsymbol{\varphi}(x)\cdot\bar{\psi}_N(x)(g_A\gamma^\mu\gamma_5\boldsymbol{\tau})\psi_N(x)\,, \tag{3.17}$$

and therefore the Lagrangian defined by $\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3$ generates all of the diagrams of interest except the ones involving π-π scattering amplitudes. A study of the basic identity, Eqs. (3.10) and (3.11), allows us to write the four-pion vertex as

$$+f_\pi{}^2\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}\,, \tag{3.18}$$

so that the contribution of the sum of Figs. 3(b), 3(d), and 3(f) is obtained by multiplying Eq. (3.18) by the order f_π vertex coming from $\mathcal{L}_1(x)$ [Eq. (3.14)] and multiplying by a pion propagator. From these considerations we find that the low-energy theorem for $\pi+N\rightarrow 2\pi+N$ can be reproduced by calculating to order $f_\pi{}^3$

with the effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}}=-\bar{\psi}_N\{i\gamma^\mu\partial_\mu+m_N+\gamma^\mu[f_\pi g_A\gamma_5\boldsymbol{\tau}\cdot(\partial_\mu\boldsymbol{\varphi}-f_\pi^2\varphi^2\partial_\mu\boldsymbol{\varphi})+f_\pi^2\boldsymbol{\tau}\cdot(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})]\}\psi_N+f_\pi^2\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}, \quad (3.19)$$

where $\boldsymbol{\varphi}(x)$ and $\psi_N(x)$ stand for the canonical pion and nucleon fields, respectively.

As we shall see in later sections, a study of the basic identity [Eqs. (3.10) and (3.11)] shows that the low-energy theorem for the process $N+m\pi\rightarrow N+m\pi$ can be most straightforwardly calculated using the phenomenological Lagrangian

$$\mathcal{L}_{\mathrm{eff}}=-\bar{\psi}_N\left[i\gamma^\mu\partial_\mu+m_N\frac{\gamma^\mu}{(1+f_\pi^2\varphi^2)}\times[f_\pi g_A\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\varphi}+f_\pi^2\boldsymbol{\tau}\cdot(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})]\right]\psi_N-\tfrac{1}{2}\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}/(1+f_\pi^2\varphi^2)^2, \quad (3.20)$$

keeping only tree diagrams and calculating to order $f_\pi^{(n+m)}$. Moreover, processes of the form $N+N\rightarrow N+N+m\pi$ can be calculated using the effective Lagrangian of Eq. (3.20) in order to calculate the vertices when pions attach to the external lines of the process $N+N\rightarrow N+N$ as shown in Fig. 4(c), where the shaded blob signifies the strong-interaction amplitude $N+N\rightarrow N+N$ in the absence of soft pions. Obviously Eq. (3.20) says nothing about such amplitudes and they must be known in advance.

One can proceed to study the implications of Eq. (3.10) and (3.11) for arbitrary processes of the form $\alpha+m\pi\rightarrow\beta+n\pi$ in a manner patterned exactly after the preceding discussion. The general result is that one can always calculate the leading term in ξ for any amplitude, by calculating the relevant tree diagrams generated by an effective Lagrangian of the general form

$$\mathcal{L}=\mathcal{L}_{\mathrm{free}}+\mathcal{L}_{\mathrm{int}}, \quad (3.21)$$

where

$$\mathcal{L}_{\mathrm{int}}=(-1)\int d^4x\left[\frac{2f_\pi}{[1+f_\pi^2\varphi(x)]^2}\times\{\partial_\mu\boldsymbol{\varphi}(x)\cdot\boldsymbol{\mathcal{A}}^\mu(x)+f_\pi[\boldsymbol{\varphi}(x)\times\partial_\mu\boldsymbol{\varphi}(x)]\cdot\boldsymbol{\mathcal{V}}^\mu(x)\}+\frac{1}{2}\frac{\partial_\mu\boldsymbol{\varphi}(x)\cdot\partial^\mu\boldsymbol{\varphi}(x)}{1+f_\pi^2\varphi^2(x)}\right], \quad (3.22)$$

where $\boldsymbol{\varphi}(x)$ stands for the pion field, $\boldsymbol{\mathcal{A}}^\mu(x)$ stands for a phenomenologically defined axial-vector current which cannot create a single pion from vacuum, and $\boldsymbol{\mathcal{V}}^\mu(x)$ stands for a phenomenologically defined vector current. By the term "phenomenologically defined" we mean that both currents are assumed to be built out of phenomenological fields for the external particles involved, exactly as we did in the case of π-N scattering.

In addition to these results, we also find that the following statements are true:

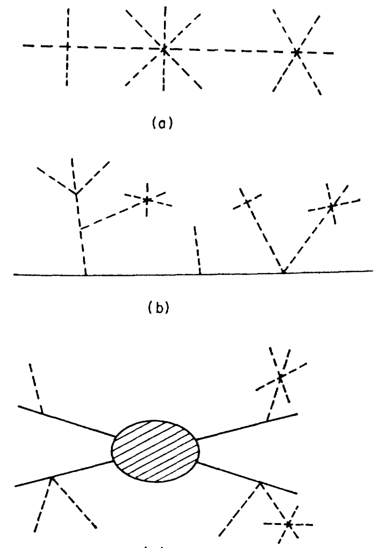

FIG. 4. Examples of tree diagrams which contribute to processes involving many pions.

(i) The rate at which various scattering amplitudes vanish in the limit $\xi\rightarrow 0$ is (a) ξ^2 for processes involving pions alone; (b) ξ^1 for process of the form $\alpha\rightarrow\alpha+\pi$, where α stands for any one-particle state; (c) ξ^0 for processes of the form $\alpha\rightarrow\beta+n\pi$, where α and β are two different hadron states which can scatter into each other through the strong process $\langle\alpha|S|\beta\rangle$.

(ii) Using the general effective Lagrangian and the prescription that one only calculates with tree diagrams (which comes from the way in which the Lagrangian is derived), one can correctly calculate the coefficient of the term of order (a) ξ^2 for processes involving only pions, where the trees correspond to diagrams of the form shown in Fig. 4(a); (b) ξ^1 for processes of the form $\alpha\rightarrow\alpha+m\pi$, where the trees correspond to diagrams of the form shown Fig. 4(b) and α is an arbitrary one-particle state; (c) ξ^0 for processes of the form $\alpha\rightarrow\beta+m\pi$, where the trees are of the form shown in Fig. 4(c). (Note that the cross-hatched blob in this diagram stands for the strong-interaction scattering amplitude $\langle\alpha|S|\beta\rangle$, where no soft pions are present, and that it must already be known, as it cannot be calculated from the phenomenological Lagrangian.)

(iii) Various phenomenological Lagrangians can be derived starting from the different possible forms for the basic identity, but they can be made the same by appropriately redefining the pion field.

(iv) If one wishes to calculate more than the coefficient of the leading term in ξ in the various soft-pion scattering amplitudes, one must go back to the basic identity and make a model for the time-ordered products. [Note that this says that even in the symmetric world ($\epsilon=0$) the effective Lagrangians *do not* tell us everything about pion scattering amplitudes.]

C. Comments on Nonlinear Lagrangians

An interesting corollary of the fact that one can derive these nonlinear phenomenological Lagrangians starting only from the joint assumptions $\partial_\mu \mathbf{A}^\mu = 0$ and current algebra, is the following observation: *If any one of the various nonlinear Lagrangian theories actually define solvable theories, then the only effect higher-order diagrams can have on the calculation of soft-pion processes is to renormalize the trees.*

To see that this is the case, one need only observe that these Lagrangians define conserved currents which satisfy current algebra. One can therefore directly apply our results and derive a phenomenological Lagrangian which must give the same results as the full theory when calculating the leading order in ξ of any amplitudes. Moreover, in constructing this phenomenological Lagrangian, we derive the fact that, when using it, one should only calculate using tree diagrams whose vertices are given by measured physical coupling constants. Thus, if the original Lagrangian theory is solvable, our results tell us that our phenomenological Lagrangian can be obtained from the original one by using the original one to lowest order (i.e., calculating only tree diagrams) and changing the bare coupling constants to the physical ones.

4. CORRECTIONS TO SYMMETRIC LIMIT

Although the principal objective of this paper is to discuss the extraction of information from an $SU(2)\otimes SU(2)$-symmetric theory with Goldstone bosons, we also have something to say about a theory in which $\partial_\mu \mathbf{A}^\mu \neq 0$. Our results in this area serve to establish what promises to be a comprehensive framework, within which one can begin to methodically study the problems associated with symmetry breaking. Foremost among these results is an extension of the basic identity to apply to the case in which $\partial_\mu \mathbf{A}^\mu \neq 0$. This extension is best stated as two theorems, which are proven in Sec. 5 and Appendix B, respectively. The first theorem tells us how to evaluate arbitrary expressions of the form[11]

$$\langle\alpha|\int d^4x_1\cdots d^4x_n \times T(\boldsymbol{\varphi}\cdot\partial_\mu \mathbf{A}^\mu(x_1)\cdots\boldsymbol{\varphi}\cdot\partial_\mu \mathbf{A}^\mu(x_n))|\beta\rangle, \quad (4.1)$$

and it can be written as

$$\langle\alpha|T\left(\exp\left[+i2f_\pi\int d^4x\boldsymbol{\varphi}\cdot\partial_\mu \mathbf{A}^\mu(x)\right]\right)|\beta\rangle = \langle\alpha|T\left(\exp\left[+i\int d^4x L(x)\right]\right)|\beta\rangle, \quad (4.2)$$

where

$$L(\mathbf{x},x_0) = \frac{-2f_\pi}{1+f_\pi^2\varphi^2}[\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(\mathbf{x},x_0) + f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu(\mathbf{x},x_0)] - \int_0^1 du\, S^{-1}(u,x_0)[f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(\mathbf{x},x_0)]S(u,x_0) + S^{-1}(1,x_0)[2f_\pi\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(\mathbf{x},x_0)]S(1,x_0), \quad (4.3)$$

and

$$S(u,t) = \exp\left(+iuf_\pi\int d^3x\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{A}^0(\mathbf{x},x_0)\right)$$

and

$$G(\varphi^2) = 2\tan^{-1}(f_\pi\sqrt{\varphi^2})/f_\pi\sqrt{\varphi^2}.$$

As we shall see, the first set of bracketed terms in (4.3) gives rise to precisely the same type of terms encountered in the case $\epsilon=0$; the rest of the terms can be identified as correction terms to higher order in ϵ (e.g., Σ terms).

The second identity proven in Appendix B tells us how to rearrange the one-pion pole terms occurring on both sides of equality (4.2) when one expands the term

$$\frac{(+i2f_\pi)^n}{n!}\langle\alpha|\int d^4x_1\cdots d^4x_n \times T(\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x_1)\cdots\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x_n))|\beta\rangle \quad (4.4)$$

(as we must do in order to derive the identities satisfied by the massive pion scattering amplitudes), and then does the same for the term of order f_π^n in the expansion of the exponential involving L. Briefly, the result is that

$$\langle\alpha+n\pi|S|\beta\rangle = f_\pi^n\langle\alpha|U^n|\beta\rangle + (\text{PCAC correction terms}), \quad (4.5)$$

where U^n is the coefficient of f_π^n in the expansion of the exponential $\exp[+i\int d^4x\,\bar{L}(x)]$ and where $\bar{L}(x)$ is defined exactly as $L(\mathbf{x},x_0)$ in Eq. (4.3) except that the first bracketed term is replaced by

$$= -2f_\pi^n\left(\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu + \frac{f_\pi}{1+f_\pi^2\varphi^2} \times[(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu - f_\pi\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu]\right). \quad (4.6)$$

By PCAC correction terms we mean all of the terms involving $\bar{\partial}$'s coming from the expansion of Eq. (4.4) into pieces having n pion poles, $n-1$ pion poles, etc. The explicit form of the expression which we have denoted as (PCAC correction terms) is calculated in Appendix B.

At this point we would like to keep our promise and show why it is possible to calculate the $\epsilon\xi^0$ correction term to the low-energy theorem for π-π scattering. This will be the subject of the next part of this section; before we start, however, it is worth pointing out that the method we discuss is merely a restatement of what is

[11] In the following sections all formulas of the form $\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)$, $G(\varphi^2)(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu(x)$, etc., are to be understood to mean $\partial_\mu\boldsymbol{\varphi}(x)\,\mathbf{A}^\mu(x)$, $(G(\varphi^2)(x))[\boldsymbol{\varphi}(x)\times\partial_\mu\boldsymbol{\varphi}(x)]\cdot\mathbf{V}^\mu(x)$, etc.

essentially Weinberg's derivation of the π-π scattering lengths.[8] Our contribution to this subject is to show how one can derive his results quite straightforwardly from the point of view of broken symmetry. More explicitly, what we shall show is that for a certain range of pion energies, with assumptions about the transformation properties of ϵH_1, we can calculate a formula for π-π scattering which is correct to order $\epsilon\xi^0$ and $\epsilon^0\xi$.

Discussion of this result breaks naturally into two sections. First, we need to discuss the nature of the approximations used to get it; second, we shall discuss the way one calculates the first-order effects of symmetry breaking.

In order to derive this result, we must first define an off-mass-shell continuation $\mathcal{T}(q_1,q_2,q_3,q_4)$ of the pion-pion scattering amplitude as follows:

$$(2\pi)^4\delta^4(q_1+q_2+q_3+q_4)i\mathcal{T}(q_1q_2q_3q_4)=(q_1^2-m_\pi^2)(q_2^2-m_\pi^2)(q_3^2-m_\pi^2)(q_4^2-m_\pi^2)$$
$$\times\int d^4x_1\cdots d^4x_4(2f_\pi)^4e^{+iq_1\cdot x_1}e^{+iq_2\cdot x_2}e^{+iq_3\cdot x_3}e^{+iq_4\cdot x_4}\langle 0|T\left(\frac{\partial_\mu A_\alpha{}^\mu(x_1)}{m_\pi^2}\frac{\partial_\lambda A_\beta{}^\lambda(x_2)}{m_\pi^2}\frac{\partial_\rho A_\gamma{}^\rho(x_3)}{m_\pi^2}\frac{\partial_\sigma A_\delta{}^\sigma(x_4)}{m_\pi^2}\right)|0\rangle, \quad (4.7)$$

where we have temporarily suppressed the isospin indices $\alpha\beta\gamma\delta$ on $\mathcal{T}$. Clearly, this expression is just the π-π scattering amplitude when all of the q_i are on the mass shell $q_i^2=m_\pi^2$. Moreover, as we have seen for the case of one- and two-soft-pion theorems, this expression has the property that as we let ϵ go to zero (so that m_π^2 also goes to zero) this goes smoothly to an expression for the same scattering amplitude in the symmetric theory. In other words, if we take the limit $\epsilon\to 0$, $m_\pi^2\to 0$ and then let all of the $q_i^2\to 0$, the function $\mathcal{T}(q_1,q_2,q_3,q_4)$ is just equivalent to the soft-pion amplitude for the scattering of zero-mass pions calculated using the symmetric Hamiltonian H_0. Since we shall be discussing the behavior of the function $\mathcal{T}(q_1,q_2,q_3,q_4)$ as we let all of the four-momenta go to zero, let us again introduce a scaling parameter ξ such that $q_i=\xi Q_i$ $(i=1, 2, 3, 4)$ for fixed four-momenta Q_i.

It is clear that, in general, the expression $\mathcal{T}(q_1,q_2,q_3,q_4)$ can go to a constant as ξ goes to zero, plus a term which vanishes as ξ^2, plus terms which vanish as higher powers of ξ^2. Thus, for small ξ, we can approximate $\mathcal{T}(q_1,q_2,q_3,q_4)$ as

$$\mathcal{T}\cong A+B\xi^2+O(\xi^4). \quad (4.8)$$

In order to avoid confusion, let us point out that this is not meant to be a power-series expansion of $\mathcal{T}(q_1,q_2,q_3,q_4)$, especially around the physical threshold, since the function $\mathcal{T}$ has a cut at this point. However, we shall now present an argument showing that for our purposes the effects of the cut can be ignored. (It is worth pointing out at this stage that it is the assumption that this approximation can be used up to the physical threshold which is the heart of Weinberg's calculation of π-π scattering lengths.)

Let us first argue that the cut is unimportant in the case $\epsilon=0$. In this case, as we shall see in a later section, the π-π scattering amplitude, and therefore the function $\mathcal{T}(q_1,q_2,q_3,q_4)$, vanishes as ξ^2. Furthermore, we shall show that the coefficient, which we shall call B_0, of the ξ^2 term can be calculated completely in the case $\epsilon=0$. Using unitarity, we see that the imaginary part of the scattering amplitude along the cut is obtained by evaluating the term $\langle 2\pi|T^\dagger T|2\pi\rangle$; this can be done approximately by inserting a complete set of states and keeping the first term as shown in Fig. 5. Clearly, the lowest state which can be inserted is the two-pion intermediate state, and since the two-pion scattering amplitude vanishes as ξ^2, and the two-particle phase space as ξ, the lowest power of ξ appearing in the imaginary part of the scattering amplitude is ξ^5. Thus, the presence of the cut makes no contribution to the scattering amplitude, which affects our calculation[12] of the term $B_0\xi^2$.

In the case $\epsilon\neq 0$, we can use the same approximation, except that the coefficients must be expanded in powers of ϵ. For example,

$$\mathcal{T}(q_i)\cong(A_0+\epsilon A_1+\epsilon^2A_2\cdots)+(B_0+\epsilon B_1+\cdots)\xi^2+O(\xi^4). \quad (4.9)$$

Since in the limit $\epsilon=0$ the amplitude must vanish as ξ^2, it follows that $A_0=0$. Clearly, if $\epsilon\ll 1$, we can ignore terms of order ϵ^2 with respect to those of order ϵ, so that for our purposes we need only consider

$$\mathcal{T}(q_i)\cong\epsilon A_1+(B_0+\epsilon B_1)\xi^2+\cdots. \quad (4.10)$$

As before, we can estimate the contributions coming from the cut in $\mathcal{T}(q_1,\cdots,q_4)$ and, in this case, they go as $\xi[A_1^2\epsilon^2+B_0^2\xi^4+A(B_0+\epsilon B_1)\epsilon\xi^2+\cdots]$. What this tells us, in effect, is that we can ignore the contributions from the cut whenever all of the following conditions are satisfied: (i) The value of ξ is small enough so that terms of order ξ^4 can be ignored with respect to those of order ξ^2. (ii) ϵ is small enough so that we can ignore terms of order ϵ^2, $\epsilon^2\xi$, $\epsilon\xi^2$, ϵ^4, etc., with respect to terms of order $\epsilon^0\xi^2$ and $\epsilon\xi^0$.

The important question at this point is how small does

$$\langle 2\pi|T^+T|2\pi\rangle =$$

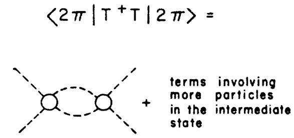

FIG. 5. Unitarity for π-π scattering.

[12] Note that this would not be true if the amplitude went as $A+B\xi^2+\cdots$ since then there would be a term which went as $A^2\xi$ in the imaginary part of ξ.

ξ have to be so that condition (i) is satisfied. Since the pion mass is proportional to ϵ, we see that in keeping terms of order ϵ we are keeping terms of order m_π^2. At threshold the terms of order ξ^2 are also of order m_π^2; that is, they are of order ϵ. Thus, it is reasonable to assume that for ϵ small enough, conditions (i) and (ii) will be satisfied and the approximate expression

$$\mathcal{T}(q_1,q_2,q_3,q_4)\cong\epsilon A_1+B_0\xi^2 \tag{4.11}$$

can be expected to be a good formula for the physical scattering up to and slightly above the physical threshold. This, however, is exactly the statement we wished to justify.

We now come to the second part of the problem. Let us point out at this stage that we know how to calculate B_0, as we shall see, by calculating it in the symmetric theory; the outstanding problem is to calculate the term ϵA_1. We shall now show how this term can be determined.

The first step in calculating ϵA_1 is to note that when $q_j=0$, Bose symmetry and crossing imply (where we explicitly exhibit all isospin dependence) that

$$i\mathcal{T}_{\alpha\beta\gamma\delta}(q_1=q_2=q_3=q_4=0) =i(\epsilon A_1)(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{4.12}$$

If we set $\alpha=\beta$, sum over α, and then do the same for γ and δ, we get an identity which we can write as

$$i\sum_{\alpha=\beta,\gamma=\delta}^{3}\mathcal{T}_{\alpha\beta\gamma\delta}(0,0,0,0)=15i(\epsilon A_1). \tag{4.13}$$

Going back to Eq. (4.7) and setting all of the $q_j=0$ and performing the necessary integrations by parts, we find that there are only two terms which do not vanish identically, since when $\epsilon\neq0$, $m_\pi^2\neq0$. One of these terms is a time-ordered product of Σ-terms and is of order ϵ^2; since we are only keeping terms of order ϵ, this can be ignored. The remaining term reduces to the form

$$(2\pi)^4\Big[+i\int dt\{\langle0|[Q_\delta^5(x_0),[Q_\gamma^5(x_0),[Q_\alpha^5(x_0),[Q_\beta^5(x_0),\epsilon\mathcal{H}_1(x_0)]]]]|0\rangle$$
$$+\langle0|[Q_\delta^5(x_0),[Q_\beta^5(x_0)[Q_\alpha^5,(x_0)[Q_\gamma^5(x_0),\epsilon\mathcal{H}_1(x_0)]]]]|0\rangle$$
$$+\langle0|[Q_\gamma^5(x_0),[Q_\beta^5(x_0),[Q_\alpha^5(x_0),[Q_\delta^5(x_0),\epsilon\mathcal{H}_1(x_0)]]]]|0\rangle\}\Big]. \tag{4.14}$$

If we now assume that

$$\epsilon\mathcal{H}_1(x_0)=\int d^3x(\epsilon\mathcal{H}_1(\mathbf{x},x_0)), \tag{4.15}$$

then translation invariance of the vacuum in both space and time allows us to rewrite Eq. (4.14) as

$$i(2\pi)^4\delta^4(0)\{\langle0|[Q_\delta^5(0),[Q_\gamma^5(0),[Q_\alpha^5(0),[Q_\beta^5(0),\epsilon\mathcal{H}_1(0)]]]]|0\rangle$$
$$+\langle0|[Q_\delta^5(0),[Q_\beta^5(0),[Q_\alpha^5(0),[Q_\gamma^5(0),\epsilon\mathcal{H}_1(0)]]]]|0\rangle$$
$$+\langle0|[Q_\gamma^5(0),[Q_\beta^5(0),[Q_\alpha^5(0),[Q_\delta^5(0),\epsilon\mathcal{H}_1(0)]]]]|0\rangle\}. \tag{4.16}$$

It is at this stage that we must, in order to proceed any further, make an explicit assumption about the transformation properties of the Hamiltonian density $\epsilon\mathcal{H}_1(0)$. Such an assumption allows us to get the expression into a form which can be evaluated to lowest order in ϵ.[13] For example, let us suppose that $\epsilon\mathcal{H}_1(0)$ belongs to a single irreducible representation of the group $SU(2)\otimes SU(2)$. As we noted in Sec. 2, the fact that $\epsilon\mathcal{H}_1$ is an isosinglet implies that this representation must be of the form (n,n). Moreover, the formula for one of the Casimir operators of $SU(2)\otimes SU(2)$ acting on an arbitrary operator O in the representation (n,n) is given by

$$c_1O=\sum_{\alpha=1}^{3}([Q_\alpha,[Q_\alpha,O]]+[Q_\alpha^5[Q_\alpha^5,O]]), \tag{4.17}$$

where

$$c_1=4n(n+1). \tag{4.18}$$

[13] Clearly this is equivalent to making an assumption about the transformation properties of the Σ term.

Since $\epsilon\mathcal{H}_1(0)$ is an isosinglet, the Casimir operator acting on $\epsilon\mathcal{H}_1$ is given by

$$c_1(\epsilon\mathcal{H}_1(0))=\sum_{\alpha=1}^{3}[Q_\alpha^5,[Q_\alpha^5,\epsilon\mathcal{H}_1(0)]]. \tag{4.19}$$

However, although $[Q_\gamma^5,\epsilon\mathcal{H}_1(0)]$ is also a member of the representation (n,n) by assumption, it clearly is not an isosinglet; therefore, the term

$$\sum_{\alpha=1}^{3}[Q_\alpha^5[Q_\alpha^5,[Q_\gamma^5,\epsilon\mathcal{H}_1(0)]]$$

is not equal to $c_1[Q_\gamma^5,\epsilon\mathcal{H}_1(0)]$ but rather

$$\sum_{\alpha=1}^{3}[Q_\alpha^5,[Q_\alpha^5,[Q_\gamma^5,\epsilon\mathcal{H}_1(0)]]] =(c_1-2)[Q_\gamma^5,\epsilon\mathcal{H}_1(0)], \tag{4.20}$$

which follows directly from Eq. (4.17) and a straightforward application of the Jacobi identity.

Returning to Eq. (4.16), setting $\alpha=\beta$ and summing over $\alpha=1, 2, 3$, and subsequently setting $\gamma=\delta$ and summing once more, we get

$$i(2\pi)^4\delta^4(0)[c_1^2+2(c_1-2)c_1]\langle 0|\epsilon\mathcal{H}_1|0\rangle. \quad (4.21)$$

Equating this result to that in Eq. (4.13), we get

$$(\epsilon A_1)=\left(\frac{c_1^2+2(c_1-2)c_1}{15}\right)\langle 0|\epsilon\mathcal{H}_1(0)|0\rangle. \quad (4.22)$$

Thus, once we learn how to evaluate $\langle 0|\epsilon\mathcal{H}_1(0)|0\rangle$, we have completely calculated the term ϵA_1. This can be done in the following way. Consider the time-ordered product

$$\int d^4x d^4y \langle 0|T(\partial_\mu A_\alpha{}^\mu(x)\partial_\nu A_\beta{}^\nu(y))|0\rangle. \quad (4.23)$$

Isolating the one-pion contribution and dropping the resulting term of the form $\langle 0|T(\bar\partial\bar\partial)|0\rangle$, which is of order ϵ^2, we get

$$\int d^4x d^4y \langle 0|T(\partial_\mu A_\alpha{}^\mu(x)\partial_\nu A_\beta{}^\nu(y))|0\rangle \cong (2\pi)^4\delta^4(0)\frac{(-i)m_\pi^2}{4f_\pi^2}\delta_{\alpha\beta}, \quad (4.24)$$

correct to order ϵ. If we now evaluate Eq. (4.23) by pulling all derivatives through the time-ordering instruction, we get

$$-(2\pi)^4 i\delta^4(0)\langle 0|[Q_\alpha{}^5[Q_\beta{}^5,\epsilon\mathcal{H}_1(0)]]|0\rangle, \quad (4.25)$$

and if we set $\alpha=\beta$, sum over α, and equate the results of Eqs. (4.24) and (4.25), we get

$$3m_\pi^2/4f_\pi^2=c_1\langle 0|\epsilon\mathcal{H}_1|0\rangle. \quad (4.26)$$

Thus, the general formula for (ϵA_1) becomes

$$(\epsilon A_1)=\frac{c_1^2+2(c_1-2)c_1}{15c_1}\left(\frac{3m_\pi^2}{4f_\pi^2}\right). \quad (4.27)$$

If, for example, we take $(n,n)=(\frac{1}{2},\frac{1}{2})$, then $c_1=3$ and we get

$$(\epsilon A_1)=m_\pi^2/4f_\pi^2, \quad (4.28)$$

which agrees with Weinberg's calculation of the π-π scattering lengths in which he assumes that the Σ term belongs to a $(\frac{1}{2},\frac{1}{2})$ representation of $SU(2)\otimes SU(2)$.

By viewing the results which we have obtained in Sec. 3, we see that the amplitude for pion-pion scattering has the rather special property that we can calculate an expression for this amplitude, which is correct up to terms of the order $\epsilon\xi^2$, once we assume something about the transformation properties of the symmetry-breaking term. This is not true for amplitudes involving particles other than pions; however, it is true in a generalized form for all amplitudes involving pions alone, regardless of the number of pions. To be more specific, we find that the corrections of order $\epsilon\xi^0$ can be calculated correctly by adding a term to the symmetric Lagrangian given in Eq. (3.13) of the form $m_\pi^2F(f_\pi^2\varphi^2)$. The specific form of the function $m_\pi^2F(f_\pi^2\varphi^2)$ can be calculated exactly once one decides upon the transformation properties of the symmetry breaking term $H_1(0)$ by evaluating the expression

$$-m_\pi^2F(f_\pi^2\varphi^2) =\langle 0|\left[\int_0^1 du\, S^{-1}(u,x_0)[f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\boldsymbol{\partial}]S(u,x_0)\right]|0\rangle, \quad (4.29)$$

where the terms in this equation are defined in Eq. (4.3). It should be clear that only those terms in the expansion of the vacuum expectation value which are even in f_π can contribute due to G parity.

This comment concludes our summary of the general results following from the theorems to be proven in the remaining sections of this paper.

5. BASIC IDENTITY

In this section we prove the most general form of the basic identity, discussed in Sec. 4, namely, Theorem 1.

Theorem 1. Let $\partial_\mu A_\alpha{}^\mu(x)$ be the divergence of the strangeness-nonchanging axial-vector currents, and let $\boldsymbol{\varphi}(x)$ be an arbitrary C-number isospinor function; then the following identity is true:

$$\langle\alpha|T\left(\exp\left[+i2f_\pi\int d^4x\, \boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x)\right]\right)|\beta\rangle =\langle\alpha|T\left(\exp\left[+i\int d^4x\, L(x)\right]\right)|\beta\rangle, \quad (5.1)$$

where $L(x)$ is defined by

$$L(x)=\frac{-2f_\pi}{1+f_\pi^2\varphi^2}[+\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)+f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu(x)] -\int_0^1 du\, S^{-1}(u,x_0)[f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x)]S(u,x_0) +S^{-1}(1,x_0)[2f_\pi\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x)]S(1,x_0) \quad (5.2)$$

and

$$S(u,x_0)=\exp\left[+iuf_\pi\int d^3x\, G(\varphi^2)(\boldsymbol{\varphi}\cdot\mathbf{A}^0)(x)\right] \quad (5.3)$$

and

$$G(\varphi^2)=+2\tan^{-1}(f_\pi\sqrt{\varphi^2})/f_\pi\sqrt{\varphi^2}. \quad (5.4)$$

Proof. In order to simplify the notation used, let us define

$$h(x_0)=2f_\pi\int d^3x\ \boldsymbol{\varphi}(x)\cdot\partial_\mu\mathbf{A}^\mu(x)\,,$$
$$U(x_0)=T\left(\exp\left[+i\int_{-\infty}^{x_0}dx_0'h(x_0')\right]\right),\tag{5.5}$$
$$g(x_0)=f_\pi\int d^3x\ G(\varphi^2)\boldsymbol{\varphi}(x)\cdot\mathbf{A}^0(x)\,,$$
$$S(x_0)=\exp[+ig(x_0)]\,,$$

where for the moment $G(\varphi^2)$ stands for an arbitrary function of φ^2. Let us further define

$$V(x_0)=S^{-1}(x_0)U(x_0)\,.\tag{5.6}$$

Clearly, if we choose $\boldsymbol{\varphi}(x)$ to vanish (or oscillate) sufficiently rapidly as $x_0\to+\infty$, we have

$$\lim_{x_0\to\infty}S(x_0)=1\,.\tag{5.7}$$

Thus, using these definitions, we have

$$\langle\alpha|T\left(\exp\left[+i2f_\pi\int d^4x\ \boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x)\right]\right)|\beta\rangle$$
$$\equiv\lim_{x_0\to\infty}\langle\alpha|U(x_0)|\beta\rangle\equiv\langle\alpha|V(x_0)|\beta\rangle\,.\tag{5.8}$$

From the definition of $V(x_0)$, it follows that

$$\frac{d}{dx_0}V(x_0)=+i\left\{-\left[-iS^{-1}(x_0)\frac{dS}{dx_0}(x_0)\right]\right.$$
$$\left.+S^{-1}(x_0)h(x_0)S(x_0)\right\}V(x_0)\equiv+iL(x_0)\,.\tag{5.9}$$

This equation has the familiar solution

$$V(x_0)=T\left(\exp\left[+i\int_{-\infty}^{x_0}dx_0'L(x_0')\right]\right)\tag{5.10}$$

and thus

$$\lim_{x_0\to\infty}\langle\alpha|V(x_0)|\beta\rangle$$
$$=\langle\alpha|T\left(\exp\left[+i\int dx_0L(x_0)\right]\right)|\beta\rangle\,.\tag{5.11}$$

Our proof reduces to finding the form of $L(x_0)$, which will involve the following lemma:

Lemma 1. If $S(x_0)=e^{+ig(x_0)}$, then

$$-iS^{-1}(x_0)\frac{dS(t)}{dx_0}=\int_0^1du\ e^{-iug(x_0)}\frac{dg}{dx_0}(x_0)e^{+iug(x_0)}\,.\tag{5.12}$$

(For the proof of this lemma see Appendix A.)

Using Lemma 1 and substituting the expression for $G(x)$, we get

$$-iS^{-1}\frac{dS}{dx_0}$$
$$=+f_\pi\int_0^1du\left\{\exp\left[-if_\pi u\int d^3x\ G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{A}^0(x)\right]\right.$$
$$\times\int d^3y\ \partial_\mu[G(\varphi^2)\boldsymbol{\varphi}\cdot A^\mu(y)]$$
$$\left.\times\exp\left[+if_\pi u\int d^3z\ G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{A}^0(z)\right]\right\},\tag{5.13}$$

where we have assumed that $G(\varphi^2)\boldsymbol{\varphi}$ vanishes sufficiently rapidly as $|\bar{y}|\to\infty$. This expression breaks into the sum of two terms:

$$f_\pi\int_0^1du\left\{\exp\left[-if_\pi u\int d^3x\ G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{A}^0\right]\right.$$
$$\times\left(\int d^3y\ \partial_\mu(G(\varphi^2)\boldsymbol{\varphi})\cdot\mathbf{A}^\mu\right)$$
$$\left.\times\exp\left[+if_\pi u\int d^3z\ G(\boldsymbol{\varphi}^2)\cdot\mathbf{A}^0\right]\right\}\tag{5.14}$$

and

$$f_\pi\int_0^1du\left\{\exp\left[+if_\pi u\int d^3x\ G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{A}^0\right]\right.$$
$$\times\left(\int d^3y\ G(\varphi^2)\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu\right)$$
$$\left.\times\exp\left[+if_\pi\int d^3z\ G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{A}^0\right]\right\}.\tag{5.15}$$

Fixing attention on the term (5.14), we see that it can be evaluated explicitly using the equal-time current commutation relations. This is done most easily by introducing the two commuting currents

$$J_{+\alpha}{}^\mu(x)=\tfrac{1}{2}[V_\alpha{}^\mu(x)+A_\alpha{}^\mu(x)]\tag{5.16}$$

and

$$J_{-\alpha}{}^\mu(x)=\tfrac{1}{2}[V_\alpha{}^\mu(x)-A_\alpha{}^\mu(x)]\,,$$

where both $J_{+\alpha}{}^\mu(x)$ and $J_{-\alpha}{}^\mu(x)$ separately satisfy an $SU(2)$ algebra. In terms of these, we can rewrite (5.14) as

$$=f_\pi\int du\Big\{\exp\Big[-if_\pi u\int d^3x\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{J}_+{}^0\Big]\Big(\int d^3y\,\partial_\mu(G(\varphi^2)\boldsymbol{\varphi})\cdot\mathbf{J}_+{}^\mu\Big)\exp\Big[+if_\pi u\int d^3z\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{J}_+{}^0\Big]\Big\}$$

$$-f_\pi\int_0^1 du\Big\{\exp\Big[+iuf_\pi\int d^3x\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{J}_-{}^0\Big]\Big(\int d^3y\,\partial_\mu(G(\varphi^2)\boldsymbol{\varphi})\cdot\mathbf{J}_-{}^\mu\Big)\exp\Big[-if_\pi u\int d^3z\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{J}_-{}^0\Big]\Big\}. \quad (5.17)$$

It is now easy to show that the exponentials bracketing the term $\int d^3y\,\partial_\mu(G(\varphi^2)\boldsymbol{\varphi})\cdot\mathbf{J}^\mu(y)$ only generate an $SU(2)$ rotation of the operator $\mathbf{J}_+{}^\mu(y)$ about the axis determined by $\boldsymbol{\varphi}(y)$ through an angle θ, whose magnitude is given by

$$\theta=(-1)\,|uf_\pi G(\varphi^2)\sqrt{\varphi^2}|\,.$$

The same is true for the $\mathbf{J}_-{}^\mu$ term, except that the rotation is through the angle $-\theta$. This is clear if one examines the first few terms of the expansion of the $\mathbf{J}_+{}^\mu$ term in multiple commutators, that is,

$$\exp\Big[-iuf_\pi\int d^3x\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{J}_+{}^0(x)\Big]\Big(\int d^3y\,\partial_\mu(G(\varphi^2)\boldsymbol{\varphi})\cdot\mathbf{J}_+{}^\mu(y)\Big)\exp\Big[+iuf_\pi\int d^3z\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{J}_+{}^0(z)\Big]$$

$$=\int d^3y\,\partial_\mu(G(\varphi^2)\boldsymbol{\varphi})\cdot\mathbf{J}_+{}^\mu(y)+iuf_\pi\int d^3y\partial_\mu(G(\varphi^2)\varphi_\alpha)\cdot\int d^3x\, G(\varphi^2)[\boldsymbol{\varphi}\cdot\mathbf{J}_+{}^0(x)J_{+\alpha}{}^\mu(y)]+\cdots$$

$$=\int d^3y\,\partial_\mu(G(\varphi^2)\varphi_\alpha)(y)\{J_{+\beta}{}^\alpha(y)-iuf_\pi G(\varphi^2)[\varphi^\beta i\varepsilon_{\beta\alpha\gamma}J_{+\gamma}{}^\mu(y)]$$
$$+(-i)^2[uf_\pi G(\varphi^2)]^2(i\varphi^\beta\varepsilon_{\beta\alpha\gamma})(i\varphi^\sigma\varepsilon_{\sigma\gamma\rho})J_{+\rho}{}^\mu(y)+\cdots\}, \quad (5.18)$$

where the $\delta^3(\mathbf{x}-\mathbf{y})$ appearing in the equal-time commutators has been used to do the extra integrations. Since, in three dimensions, it is trivial to do a rotation about the $\boldsymbol{\varphi}(y)$ direction through the angle $\theta=+|uf_\pi G^2(\varphi^2)\sqrt{\varphi^2}|$, we can explicitly work out the rotations and then do the integral over u which appears as the argument of the various sine and cosine functions which result. Depending upon the choice of $G(\varphi^2)$, we can get many different results; however, if we choose

$$G(\varphi^2)=2\tan^{-1}(f_\pi\sqrt{\varphi^2})/f_\pi\sqrt{\varphi^2}, \quad (5.19)$$

we get

$$\int d^3x\,\frac{2f_\pi}{1+f_\pi{}^2\varphi^2}\{\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)+f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu\}. \quad (5.20)$$

Noting that the definition of $S(u,x_0)$ is

$$S(u,x_0)=\exp\Big[-if_\pi\int d^3x\, G(\varphi^2)\boldsymbol{\varphi}\cdot\mathbf{A}^0(x)\Big],$$

we see that Eq. (5.15) gives the term

$$\int_0^1 du\, S^{-1}(u,x_0)[f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x)]S(u,x_0),$$

and the term $S^{-1}(x_0)h(x_0)S(x_0)$ gives

$$S^{-1}(1,x_0)[2f_\pi\boldsymbol{\varphi}\cdot\partial_\mu\mathbf{A}^\mu(x)]S(1,x_0),$$

which proves our theorem. Our theorem has an immediate corollary.

Corollary 1.1. In the case $\epsilon=0$ the basic identity becomes

$$1=T\Big(\exp\Big[+i\int d^4x\,\frac{(-2f_\pi)}{1+f_\pi{}^2\varphi^2}\times[\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu+(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu]\Big]\Big).$$

Proof. Note that when $\epsilon=0$, we are in the symmetry limit and $\partial_\mu\mathbf{A}^\mu$ is identically zero, and thus so is $h(x_0)$. The remainder of the proof follows directly by substituting this fact into Theorem 1.

An examination of the results of Theorem 1 and comparison with the same identity for the symmetry limit given in Corollary 1 reveals some interesting features. First of all, the first bracketed set of terms in Eq. (5.2) is formally the same as the entire expression obtained in the symmetry limit. Second, by expanding the rest of the terms in Eq. (5.2), and proceeding to evaluate the resulting multiple commutators, we see that these generate the expected Σ terms. Note, however, that this expression cannot be put into a simple closed form unless one knows how the symmetry-breaking piece of the Hamiltonian transforms.

In Appendix B, we make use of a slight modification of Theorem 1 to show how the general formulas for multipion scattering can be obtained in the case of broken symmetry; however, we shall now fix attention upon the basic identity stated in Corollary 1 and show how it naturally leads one to phenomenological Lagrangians as a device for calculating soft-pion theorems in the symmetry limit.

6. GENERAL METHOD FOR CANCELLING POLE TERMS

This section is devoted to the statement and proof of a theorem which allows us to write a rigorous identity giving the scattering amplitude for any process of the form $\alpha+m\pi\rightarrow\beta+n\pi$. As in Sec. 2, we shall use the $(n+m)$th-order term in f_π coming from an expansion of the basic identity of Corollary 1 and extract all pole terms in the momentum squared of a single pion; recombining these will give the desired result.

Theorem 2. The scattering amplitude for the process $\beta\rightarrow\alpha+n\pi$ is given by

$$\langle\alpha+\pi(\varepsilon_1q_1)\cdots\pi(\varepsilon_nq_n)|S_0|\beta\rangle = f_\pi^n\langle\alpha|U^n(q_1,\cdots,q_n)|\beta\rangle\,, \quad (6.1)$$

where $U^n(q_1,\cdots,q_n)$ is defined by taking the coefficient of f_π^n in the expansion of the exponential

$$T\biggl(\exp\biggl\{-i\int d^4x\, 2f_\pi\biggl[\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu(x)+\frac{f_\pi}{1+f_\pi^2\varphi^2}\times[(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu(x)-\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)]\biggr]\biggr\}\biggr)\,, \quad (6.2)$$

where we take

$$\boldsymbol{\varphi}(x)=\sum_{j=1}^n \boldsymbol{\varepsilon}_j e^{+iq_j\cdot x}$$

and keep only those terms in the resulting expression in which all of the q_j's appear. (The caret above the axial-vector current, as before, means that in evaluating any time-ordered products in which it may appear, we leave out all graphs in which it creates a single-pion state from the vacuum.)

Proof. Using the basic identity of corollary 1, we have

$$1=T\biggl(\exp\biggl[-i\int d^4x\,\frac{2f_\pi}{1+f_\pi^2\varphi^2}\times[\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu+f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu]\biggr]\biggr)\,. \quad (6.3)$$

From this we take out the coefficient of f_π^n by expanding both the exponential and the denominator $(1+f_\pi^2\varphi^2)^{-1}$. We should note, however, that terms which have more than one power of $\boldsymbol{\varphi}(x)$ at the same point can never give rise to poles when only one of the q_j^2 is zero. For example, the third-order identity is

$$0=(-i)^3\frac{8f_\pi^3}{3!}\int d^4x_1d^4x_2d^4x_3 \times T((\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu)(x_1)(\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu)(x_2)(\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu)(x_3)) +(-i)(2f_\pi^3)\int d^4x_1\varphi^2(x_1)(\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu)(x_1)\,. \quad (6.4)$$

Letting $\varphi(x)=e^{+iq_1\cdot x}\boldsymbol{\varepsilon}_1+e^{+iq_2\cdot x}\boldsymbol{\varepsilon}_2+e^{+iq_3\cdot x}\boldsymbol{\varepsilon}_3$, we see that the first term can give rise to a set of diagrams in which an axial-vector current creates a pion (from vacuum) of momentum q_1 or q_2 or q_3, thus giving rise to terms which have poles if any one of the $q_i^2=0$. This is not true for the second term, however, since in this case the pion which the axial-vector current creates will have momentum $q_1+q_2+q_3$ and clearly its square is not zero as we let one of the q_i^2 go to zero. Thus, terms involving products of $\boldsymbol{\varphi}$'s at the same point do not give rise to the type of pole terms which we shall wish to separate off. With this in mind let us rewrite Eq. (6.3) as

$$1=T\biggl(\exp\biggl\{-i\int d^4x\, 2f_\pi\biggl[\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)+\frac{f_\pi}{1+f_\pi^2\varphi^2}\times[(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu-f_\pi\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu]\biggr]\biggr\}\biggr)\,. \quad (6.5)$$

Using this form, it is clear that only terms coming from powers of $\int d^4x[\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)]$ can contribute pole diagrams. Generally speaking, a term of the form $[\int d^4x\,\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)]^n$ when it appears in a time-ordered product will contribute terms which have poles when any of the $q_j^2=0$, and terms in which there is no pole if one of the momenta squared is zero, etc. In order to facilitate keeping track of such pole diagrams, let us use the purely formal device of writing

$$\mathbf{A}^\mu(x)=\mathbf{\Pi}^\mu(x)+\hat{\mathbf{A}}^\mu(x)\,. \quad (6.6)$$

This notation is well defined in the diagram sense, the rule being that whenever a $\mathbf{\Pi}^\mu(x)$ appears in a time-ordered product, keep only those diagrams in which it creates a one-pion state from vacuum, and whenever $\hat{\mathbf{A}}^\mu(x)$ appears, keep all diagrams except the ones in which it creates a one-pion intermediate state from vacuum. (Thus the definition of $\mathbf{A}_\alpha{}^\mu$ agrees with our previous definition.) Adopting this notation, we rewrite Eq. (6.5) as

$$1=T\biggl(\exp\biggl\{-i\int d^4x\biggl[2f_\pi\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{\Pi}^\mu(x) +2f_\pi\biggl(\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu+\frac{f_\pi^2}{1+f_\pi^2\varphi^2}[(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu -f_\pi\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)]\biggr)\biggr]\biggr\}\biggr)\,. \quad (6.7)$$

Because of the presence of the time-ordered product instruction, this can be rewritten as

$$1=T\biggl(\exp\biggl[-i\int d^4x\, 2f_\pi\partial_\mu\boldsymbol{\varphi}_\mu\cdot\mathbf{\Pi}^\mu\biggr] \times\exp\biggl[-i\int d^4y\, 2f_\pi\{\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu+f_\pi[(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu -f_\pi\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu]\}\biggr]\biggr)\,, \quad (6.8)$$

and thus the term of order f_π^p can be written as

$$0=T\left[f_\pi^p \sum_{n=0}^{p} \frac{(-i)^n 2^n}{n!}\left(\int d^4x\, \partial_\mu\varphi\cdot\mathbf{\Pi}^\mu\right)^n \times U^{p-n}\right], \quad (6.9)$$

where U^{p-n} is the coefficient of the term of order f_π^{p-n} in the expansion of the second exponential.

Taking this expression between states $\langle\alpha|$ and $|\beta\rangle$ and letting

$$\varphi(x)=\sum_{j=1}^{p} e^{+iq_j\cdot x}\varepsilon_j\,,$$

we get an expression for case $p=1$.

$p=1$, $\varphi(x)=\varepsilon_1 e^{+iq\cdot x}$.

$$0=f_\pi\left[\langle\alpha|U^1|\beta\rangle+\frac{(-i)2}{1!}(+iq_\mu)\times\int d^4x\, e^{+iq\cdot x}\langle\alpha|\varepsilon\cdot\mathbf{\Pi}^\mu(x)|\beta\rangle\right], \quad (6.10)$$

but by the definition of $\mathbf{\Pi}^\mu(x)$,

$$\int d^4x\langle\alpha|\varepsilon\cdot\mathbf{\Pi}^\mu|\beta\rangle=(-q^\mu/2f_\pi q^2)\langle\alpha+\pi(\varepsilon,q)|S_0|\beta\rangle\,,$$

which gives

$$\langle\alpha+\pi(\varepsilon\cdot q)|S_0|\beta\rangle=f_\pi\langle\alpha|U^1|0\rangle. \quad (6.11)$$

$p=2$, $\varphi(x)=\varepsilon_1 e^{+iq_1\cdot x}+\varepsilon_2 e^{+iq_2\cdot x}$. We have

$$\begin{aligned}0=f_\pi^2\Bigg[&\langle\alpha|U^2(q_1,q_2)|\beta\rangle+\frac{(-i)2(+iq_{1\mu})(-q_1^\mu)}{2f_\pi q_1^2}\langle\alpha+\pi(\varepsilon_1,q_1)|U^1(q_2)|\beta\rangle\Bigg]\\ &+\frac{(-i)2(+iq_{2\mu})(-q_2^\mu)}{2f_\pi q_2^2}\langle\alpha+\pi(\varepsilon_2,q_2)|U^1(q_1)|\beta\rangle\\ &+\frac{(-i)^2(+iq_{1\mu})(+iq_{2\mu})(-q_1^\mu)(-q_2^\mu)}{(2f_\pi)^2q_1^2q_2^2}\cdot\frac{(2!)}{(2!)}\langle\alpha+\pi(\varepsilon_1q_1)+\pi(\varepsilon_1q_2)|S_0|\beta\rangle\\ =f_\pi^2\langle\alpha|U^2(q_1,q_2)|\beta\rangle&-f_\pi\langle\alpha+\pi(\varepsilon_1,q_1)|U^1(q_2)|\beta\rangle-f_\pi\langle\alpha+\pi(\varepsilon_2,q_2)|U^1(q_2)|\beta\rangle+\langle\alpha+\pi(\varepsilon_1q_1)+\pi(\varepsilon_2q_2)|S_0|\beta\rangle. \quad (6.12)\end{aligned}$$

Using the result of $p=1$ for the state $\langle\alpha|\to\langle\alpha+\pi|$, we get

$$\begin{aligned}f_\pi\langle\alpha+\pi(\varepsilon_1,q_1)|U^1(q_2)|\beta\rangle&=\langle\alpha+\pi(\varepsilon_1q_1)+\pi(\varepsilon_2q_2)|S_0|\beta\rangle\\ &=f_\pi\langle\alpha+\pi(\varepsilon_2q_2)|U^1(q_1)|\beta\rangle, \quad (6.13)\end{aligned}$$

so that our final result following from Eq. (6.12) is

$$\langle\alpha+\pi(\varepsilon_1q_1)+\pi(\varepsilon_2q_2)|S_0|\beta\rangle=f_\pi^2\langle\alpha|U^2(q_1,q_2)|\beta\rangle. \quad (6.14)$$

Thus, we have shown that our theorem is true for $p=1$, 2. The remainder of the proof proceeds by induction.

Make the inductive hypothesis that the theorem is true for all $p<p_0$. Then consider the identity.

$$0=\langle\alpha|T\left(f_\pi^{p_0}\sum_{n=0}^{p_0}\frac{(-2i)^n}{n!}\times\left[\int d^4x\,\partial_\mu\varphi\cdot\mathbf{\Pi}^\mu\right]^n U^{p_0-n}\right)|\beta\rangle, \quad (6.15)$$

where

$$\varphi(x)=\sum_{j=1}^{p_0} e^{+iq_j\cdot x}\varepsilon_j.$$

For each n such that $1\leq n<p_0$, partition the momenta $q_1,\cdots,q_{p_0}$ into two sets $q_{\sigma(1)}\cdots q_{\sigma(n)}$ and $q_{\sigma(n+1)}\cdots q_{\sigma(p)}$ [where $\sigma(i)$ denotes a permutation of the numbers 1, $\cdots$, n and $n+1$, $\cdots$, p_0], such that the first n momenta are associated with the term $[\int d^4x[\partial_\mu\varphi\cdot\mathbf{\Pi})]^n$. The set of terms coming from such a partition gives a term of the form

$$f_\pi^p\frac{(-2i)^n}{n!}\left[n!\frac{(+iq_1^\mu)\cdots(+iq_n^\mu)(-q_{1\mu})\cdots(-q_{n\mu})}{(2f_\pi)^n q_1^2\cdots q_n^2}\times\langle\alpha+\pi(\varepsilon_1q_1)+\cdots\pi(\varepsilon_nq_n)|U^{p-n}(q_{n+1}\cdots q_{p_0})|\beta\rangle\right]. \quad (6.16)$$

{The factor of $n!$ comes from the fact that for each partitioning of the momenta the term $[\int d^4x(\partial_\mu\varphi\cdot\mathbf{\Pi}^\mu)]^n$ gives one term for each possible ordering of the n momenta associated with it.}

Using the inductive hypothesis, we get that this term is equal to

$$(-1)^n\langle\alpha+\pi(\varepsilon_1,q_n)+\cdots\pi(\varepsilon_{p_0},q_{p_0})|S_0|\beta\rangle. \quad (6.17)$$

The number of ways of partitioning the set $(q_1,\cdots,q_{p_0})$ into a set of n momenta and (p_0-n) momenta is given by the binomial coefficient

$$\binom{p_0}{n}=\frac{p_0!}{(p_0-n)!n!}.$$

Therefore, using the inductive hypothesis for all terms in Eq. (6.15) for which $1\leq n<p_0$ gives

$$0=f_\pi^{p_0}\langle\alpha|U^{p_0}(q_1\cdots q_{p_0})|\beta\rangle+\sum_{n=1}^{p_0}(-1)^n\binom{p_0}{n}\times\langle\alpha+\pi(\varepsilon_1 q_1)\cdots\pi(\varepsilon_{p_0}q_{p_0})|S_0|\beta\rangle. \quad (6.18)$$

Noting that

$$(1+X)^{p_0}=\sum_{n=0}^{p_0}\binom{p_0}{n}X^n,$$

we get for $X=-1$

$$0=1+\sum_{n=1}^{p_0}(-1)^n\binom{p_0}{n} \quad (6.19)$$

or

$$-1=\sum_{n=1}^{p_0}(-1)^n\binom{p_0}{n}.$$

Substituting this in Eq. (6.18), we get

$$\langle\alpha+\pi(\varepsilon_1 q_1)\cdots\pi(\varepsilon_{p_0}q_{p_0})|S|\beta\rangle=f_\pi^{p_0}\langle\alpha|U^p(q_1\cdots q_{p_0})|\beta\rangle. \quad (6.20)$$

This completes the proof of Theorem 2 and gives the basic identity which we spoke about in Sec. 3. We shall now show how to use this basic identity to derive low-energy theorems.

7. PHENOMENOLOGICAL LAGRANGIANS AND TREE DIAGRAMS

This section is devoted to proving the following theorem:

Theorem 3. Let $\beta\to\alpha+n$ pions be any process involving the production of n pions whose momenta are q_j ($j=1,\cdots,n$). Further, let $q_j=\xi Q_j$ for a set of n fixed four-momenta Q_j. Then the coefficient of the lowest power of ξ appearing in the expansion of the scattering amplitude about $\xi=0$ is correctly calculated (in the case $\epsilon=0$) by using the phenomenological Lagrangian

$$\mathcal{L}(x)=\mathcal{L}_{\text{free}}(x)+\mathcal{L}_{\text{int}}(x), \quad (7.1)$$

where

$$\mathcal{L}_{\text{int}}(x)=-\left[\frac{2f_\pi}{1+f_\pi^2\varphi^2}(\partial_\mu\boldsymbol{\varphi}\cdot\mathcal{A}^\mu(x)+f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathcal{V}^\mu(x))\right]-\frac{1}{2}\frac{\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}}{(1+f_\pi^2\varphi^2)^2}, \quad (7.2)$$

and calculating with it to nth order in f_π according to the usual Feynman rules, but subject to the restriction that one keeps only tree diagrams.

As far as the notation in Eq. (7.2) goes, $\boldsymbol{\varphi}(x)$ stands for the pion field, and $\mathcal{A}^\mu(x)$ and $\mathcal{V}^\mu(x)$ stand for phenomenologically defined vector and axial-vector fields built out of the fields of the external particles. The only restrictions upon the form of $\mathcal{A}^\mu(x)$ and $\mathcal{V}^\mu(x)$ is that they be constructed to give the appropriate vector and axial-vector coupling constants.[14]

Proof. The proof of this theorem uses several important lemmas which we must first prove.

Lemma 1. Scattering amplitudes for processes involving only pions vanish as ξ^2, and the coefficient of the ξ^2 term is correctly calculated by using the phenomenological Lagrangian,

$$\mathcal{L}_{\pi-\pi}(x)=-\frac{1}{2}\frac{\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}}{(1+f_\pi^2\varphi^2)^2}. \quad (7.3)$$

Proof of lemma 1. The proof of this lemma is reasonably involved and will divide naturally into a study of the four- and six-pion scattering amplitudes followed by a generalization of the observations made for these cases.

Step 1. Our first step in this proof is to show that the four-pion scattering amplitude is given (to order ξ^2) by

$$\langle4\pi|S_0|0\rangle=-if_\pi^2\int d^4x\ \varphi^2(x)\partial_\mu\boldsymbol{\varphi}(x)\cdot\partial^\mu\boldsymbol{\varphi}(x), \quad (7.4)$$

where

$$\boldsymbol{\varphi}(x)=\sum_{j=1}^4\boldsymbol{\varepsilon}_j e^{+iq_j\cdot x}.$$

The point is that this is the same result which would be obtained by calculating to order f_π^2 with the phenomenological Lagrangian

$$\mathcal{L}_{\pi-\pi}=-\frac{1}{2}\frac{1}{(1+f_\pi^2\varphi^2)^2}(\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}).$$

Deriving this result is quite straightforward. We start with the basic identity

$$\langle4\pi|S_0|0\rangle=f_\pi^4\langle0|U^4(q_1\cdots q_4)|0\rangle, \quad (7.5)$$

which, using the definition of U^4, becomes

$$\begin{aligned}\langle4\pi|S_0|0\rangle=f_\pi^4\Bigg\{&\frac{(-2i)^4}{4!}\langle0|T\left(\int d^4x_1\cdots d^4x_4\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}(x_1)\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu(x_2)\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu(x_3)\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu(x_4)\right)|0\rangle\\&+\frac{(-2i)^3}{3!}\left[\binom{3}{1}\langle0|T\left(\int d^4x_1d^4x_2d^4x_3\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}(x_1)\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}(x_2)(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu(x_3)\right)|0\rangle\right]\\&+\frac{(-2i)^2}{2!}\left[\binom{2}{1}\langle0|T\left(\int d^4x_1d^4x_2\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}(x_1)(\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu)(x_2)\right)|0\rangle\right]+(-2i)\left[\langle0|\int d^4x(\varphi^2(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu(x))|0\rangle\right]\Bigg\}. \quad (7.6)\end{aligned}$$

[14] This point is discussed in greater detail in the body of the proof.

Since each factor of $\partial_\mu \varphi$ introduces an explicit factor of ξ, the first term is explicitly of order ξ^4, the second ξ^3, the third ξ^2, and the fourth ξ. However, by isospin invariance of the vacuum the fourth term vanishes identically, and therefore the term with the lowest explicit ξ dependence is the third term. As we shall show, this term, in fact, goes as ξ^2, but before doing this let us examine the ξ^2 dependence of the remaining terms. Clearly, if the explicit ξ dependence was all one had to know about a term, the first and second terms could not contribute a part which is of order ξ^2; however, there can be terms which go as ξ^{-2}, due to the presence of one-pion states, when one expands the time-ordered products by inserting a complete set of intermediate states between the various operators. We must therefore examine the possibility of such singular behavior in ξ for the first two terms.

The first term is of the general form $\langle 0|\hat{A}\hat{A}\hat{A}\hat{A}|0\rangle$. Since G parity forbids terms like $\langle 0|\hat{A}\hat{A}|\pi\rangle\langle\pi|\hat{A}\hat{A}|0\rangle$ and the definition forbids terms like $\langle 0|\hat{A}|\pi\rangle\langle\pi|\hat{A}\hat{A}|\pi\rangle \times\langle\pi|\hat{A}|0\rangle$, we see that the first term can have no intermediate one-pion states inserted anywhere in an expansion of the T product and therefore vanishes faster than ξ^2. Similarly, G parity and the definition of $\hat{A}$ prevent the occurrence of pole terms in the second term which is of the form $\langle 0|\hat{A}\hat{A}V|0\rangle$. Thus, it is in fact true that for the four-pion amplitude only the term with explicit ξ^2 dependence can contribute to the coefficient of ξ^2 term in the scattering amplitude. In order to see that this does happen, we shall now show that the $\langle 0|T(\hat{A}^\mu\hat{A}^\nu)|0\rangle$ term has a piece which is equal to $-ig^{\mu\nu}/(2f_\pi)^2$.

Evaluating the leading power of ξ coming from the third term is easy. Using the definition of $\varphi(x)$, we see that we must evaluate terms of the form

$$\langle 0|T\bigg(\int d^4x_1 d^4x_2 e^{+iq_1\cdot x_1}(\boldsymbol{\varepsilon}_1\cdot\mathbf{A}^\mu)(x_1) \times e^{+i(q_2+q_3+q_4)\cdot x_2}(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_3)(\boldsymbol{\varepsilon}_4\cdot\mathbf{A}^\nu)(x_2)\bigg)|0\rangle. \quad (7.7)$$

Since $q_1=-(q_2+q_3+q_4)$ and $q_1^2=0$, our assumption that all time-ordered products have been appropriately redefined to be covariant tells us that the most general form the sum terms of this form can take is

$$(cg^{\mu\nu}+dq_1^\mu q_1^\nu)[(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_4)(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_3)+(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2)(\boldsymbol{\varepsilon}_3\cdot\boldsymbol{\varepsilon}_4)+(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_3)(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_4)]. \quad (7.8)$$

The constant c in this expression can be determined as follows. We know, using the fact $\partial_\mu\mathbf{A}^\mu=0$ and extracting the one-pion pole, that

$$\begin{aligned}0&=q_{1\mu}\langle 0|T\bigg(\int d^4x_1 d^4x_2 \times e^{+iq_1\cdot x_1}e^{-iq_1\cdot x_2}(\boldsymbol{\varepsilon}_1\cdot\hat{\mathbf{A}}^\mu)(x_1)(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_3)(\boldsymbol{\varepsilon}_4\cdot\mathbf{A}^\nu)\bigg)|0\rangle\\ &=(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_4)(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_3)\frac{(iq_{1\mu})(-iq_1^\mu)(+iq_1^\nu)}{4f_\pi^2 q_1^2}+q_{1\mu}\langle 0|T(\hat{\mathbf{A}}^\mu\hat{\mathbf{A}}^\nu|0\rangle. \quad (7.9)\end{aligned}$$

Evaluating the $\langle 0|T(\hat{\mathbf{A}}\hat{\mathbf{A}})|0\rangle$ term by means of Eq. (7.8), noting that $q_1^2=0$, we have

$$0=+i(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_4)(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_3)q_1^\nu+cq_1^\nu(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_4)(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_3), \quad (7.10)$$

and therefore

$$c=-i(4f_\pi^2)^{-1}. \quad (7.11)$$

Combining these results and rewriting everything in terms of the functions $\varphi(x)$, we get

$$\langle 4\pi|S_0|0\rangle=-if_\pi^2\int d^4x\ \varphi^2(x)\partial_\mu\varphi(x)\partial^\mu\varphi(x), \quad (7.12)$$

which proves the first part of the lemma.

Step 2. The next step in the proof of this lemma is to show that the six-pion scattering is given by using the phenomenological Lagrangian given in the statement of the theorem to order f_π^4. As before, the proof of this result involves determining the coefficient of the term going as ξ^2 in the basic identity. For our purposes it will not be necessary to write out the entire expression arising from the basic identity; instead, we shall adopt an abbreviated notation. What we shall do is to drop all φ's and their derivatives and symbolically denote a term by an expression of the form $\langle 0|T(\hat{A}_1A_3V_2)|0\rangle$, where the subscripts tell how many of the external pion momenta are associated with a given current. Adopting these conventions, we have

$$\begin{aligned}\langle 6\pi|S_0|0\rangle=f_\pi^6\bigg\{&\frac{(-2i)^2}{2!}[\langle 0|T(A_3A_3)|0\rangle-2\langle 0|T(V_2V_4)|0\rangle+2\langle 0|T(\hat{A}_1A_5)|0\rangle]\\ &+\frac{(-2i)^3}{3!}\bigg[-\binom{3}{1}\langle 0|T(\hat{A}_1\hat{A}_1V_4)|0\rangle-\binom{3}{2}\langle 0|T(\hat{A}_1V_2A_3)|0\rangle+\langle 0|T(V_2V_2V_2)|0\rangle\bigg]\\ &+\frac{(-2i)^4}{4!}\bigg[-\binom{4}{1}\langle 0|T(\hat{A}_1\hat{A}_1\hat{A}_1\hat{A}_3)|0\rangle+\binom{4}{2}\langle 0|T(\hat{A}_1\hat{A}_1V_2V_2)|0\rangle\bigg]\\ &+\frac{(-2i)^5}{5!}\bigg[\binom{5}{1}\langle 0|T(\hat{A}_1\hat{A}_1\hat{A}_1\hat{A}_1V_2)|0\rangle\bigg]+\frac{(-2i)^6}{6!}\langle 0|T(\hat{A}_1\hat{A}_1\hat{A}_1\hat{A}_1\hat{A}_1\hat{A}_1)|0\rangle\bigg\}. \quad (7.13)\end{aligned}$$

As before, only a limited number of these terms contribute to the ξ^2 part of the amplitude. For example the last term containing six $\hat{A}$'s can only have one pole, as is clear from G parity and the definition of $\hat{A}$; thus it vanishes as ξ^4. Similarly, the term with five currents vanishes as ξ^3. The term $\langle 0|T(\hat{A}\hat{A}\hat{A}A)|0\rangle$ also has a pole but it comes from an insertion of the form $\langle 0|T(\hat{A}\hat{A}\hat{A})|\pi\rangle\langle\pi|A|0\rangle$, and since the term $\langle\pi|A|0\rangle$ introduces another factor of ξ, this term also vanishes as ξ^3. As the terms $\langle 0|T(\hat{A}\hat{A}V)|0\rangle$ and $\langle 0|T(VVV)|0\rangle$ admit no one-pion intermediate states, they vanish as ξ^3, and thus only the terms which vanish explicitly as ξ^2 and the pole diagrams occurring in the fifth and eighth terms contribute to the ξ^2 part of the amplitude.

Rewriting Eq. (7.13) in terms of the ξ^2 terms, we have

$$\langle 6\pi|S|0\rangle$$
$$=f_\pi{}^6\left\{\frac{(-2i)^2}{2!}[\langle 0|T(A_3A_3)|0\rangle+2\langle 0|T(A_1A_5)|0\rangle]\right.$$
$$+\frac{(-2i)^3}{3!}\left[-\binom{3}{2}\langle 0|T(\hat{A}_1V_2A_3)|0\rangle_{\text{pole term}}\right]$$
$$\left.+\frac{(-2i)^4}{4!}\binom{4}{2}\langle 0|T(\hat{A}_1V_2\hat{A}_1V_2)|0\rangle_{\text{pole term}}\right\}. \quad (7.14)$$

Note that we have dropped the term $\langle 0|T(V_3V_3)|0\rangle$; the reason for this is that although it is only multiplied by two explicit factors of ξ, the indicated vacuum expectation value starts off as ξ^2. To see this, first observe that since the vector current cannot create a one-pion state from vacuum, there cannot be a $1/\xi^2$ term in the expansion of the vacuum expectation value; but then by Lorentz invariance the most general form it can take is

$$cg^{\mu\nu}+O(\xi^2). \quad (7.15)$$

If we explicitly let

$$\boldsymbol{\varphi}(x)=\sum_{j=1}^{n}\boldsymbol{\varepsilon}_j e^{+iq_j\cdot x},$$

a typical term of the type $\langle 0|T(VV)|0\rangle$ is of the form

$$\int d^4xd^4y\ e^{+iq\cdot x}e^{+ik\cdot y}\langle 0|T(\mathbf{V}^\mu(x)\mathbf{V}^\nu(y))|0\rangle, \quad (7.16)$$

where

$$q^\mu+k^\mu=0.$$

Multiplying Eq. (7.16) by q_μ and integrating by parts, we get two terms; one involves $\partial_\mu V^\mu$ which is zero, and the second is proportional to $\langle 0|V^\mu|0\rangle$ which vanishes because of the isospin invariance of the vacuum. Using the general form dictated by Lorentz invariance to evaluate this expression, we get $cq^\mu=0$ or $c=0$, which proves that this term vanishes as ξ^4. Aside from these considerations, we also have the fact that terms like $\langle 0|T(AV)|0\rangle$ and $\langle 0|T(\hat{A}V)|0\rangle$ vanish identically by G parity.

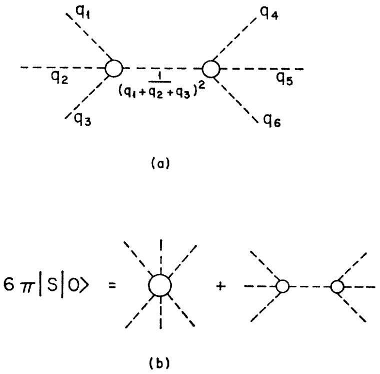

FIG. 6. Pole terms occurring in pion amplitudes.

Proving the necessary result can now be accomplished by explicitly separating the pole term in $\langle 0|T(A_3A_3)|0\rangle$, which gives us

$$\langle 6\pi|S|0\rangle=f_\pi{}^6[(\text{explicit } \xi^2 \text{ terms})$$
$$+(\text{terms with pion poles which go as } \xi^2)$$
$$+(\text{terms which vanish as } \xi^4 \text{ or faster})]. \quad (7.17)$$

Unitarity implies that the only pole terms in the six-pion amplitude come from diagrams such as the one in Fig. 6(a) where each vertex is given by an off-mass-shell continuation of the four-pion scattering amplitude, and the intermediate one-pion states gives the factor $+iq^{-2}$. At this point we make the important observation that even though there is an ambiguity in the definition of these vertices due to the fact that they are being continued off the mass shell, this ambiguity does not affect the order ξ^2 part of the six-pion amplitude. The reason for this is that the residue of the pole is unambiguously defined to be the product of two on-mass-shell four-pion amplitudes, each of which is proportional to ξ^2. In taking these amplitudes off the mass shell, we can add an arbitrary term which vanishes faster than ξ^2 and no such a term can contribute to the ξ^2 part of the scattering amplitude. This fact tells us that the six-pion amplitude is given by the diagram in Fig. 6(a) with the vertex factors given by the ξ^2 part of the four-pion amplitude $\langle 4\pi|S|0\rangle$, which we already determined, plus the terms which explicitly go as ξ^2.

Calculating the contribution of the terms which are explicitly of order ξ^2 is easy, as they take the form

$$f_\pi{}^6\frac{(-2i)^2}{2!}\langle 0|T(\hat{A}_3\hat{A}_3)|0\rangle$$
$$=f_\pi{}^6\frac{(-2i)^2}{4f_\pi{}^2}(-i)\int d^4x\ \varphi^4\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}$$
$$=+\tfrac{1}{2}if_\pi{}^4\int d^4x\ \varphi^4\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}(x) \quad (7.18)$$

and

$$f_\pi^6 \frac{(-2i)^2}{2!}(+2)\langle 0|T(\hat{A}_1\hat{A}_5)|0\rangle = \frac{f_\pi^6(-2i)^2(+2)(-i)}{2\times 4f_\pi^2}\int d^4x\ \varphi^4\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}$$

$$=+if_\pi^4\int d^4x\ \varphi^4\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}. \quad (7.19)$$

Combining these, we get

$$+i\tfrac{3}{2}f_\pi^4\int d^4x\ \varphi^4\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}, \quad (7.20)$$

which is just the f_π^4 term in the expansion of the effective Lagrangian given in the statement of the theorem.

These considerations tell us that sixth-order pion scattering is given by calculating the sum of the two diagrams drawn in Fig. 6(b). Clearly, these are the same terms obtained by calculating to order f_π^4 with the effective Lagrangian and the usual Feynman rules.

This completes the discussion of Step 2.

Step 3. The final step is to extend the results already obtained to processes containing any even number of pions. It is not more difficult to see that the same technique used in Steps 1 and 2 works in general. As before, one starts with the basic identity; as before, there are explicit ξ^2 terms arising from expressions of the form $\langle 0|T(\hat{A}A)|0\rangle$ and $\langle 0|T(AA)|0\rangle$. The exact form of these terms, which we shall hereafter call direct terms, can be easily computed from the basic identity given in Theorem 2. All of the remaining ξ^2 terms correspond to pole diagrams which must sum up to be the set of all pole terms given by unitarity. For example, see the types of diagrams one must get for the eight- and ten-pion case as shown in Fig. 7. The vertices in these diagrams must be given by the direct terms arising in the lower-order amplitudes. Thus, the nondirect terms

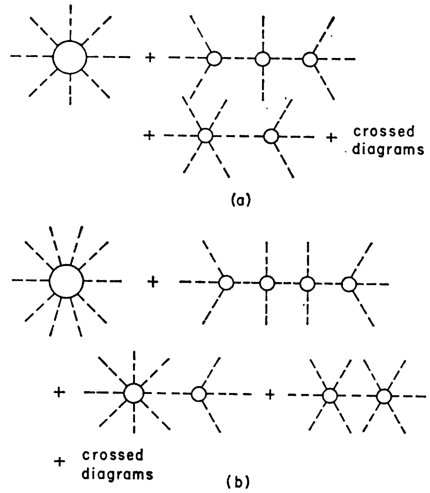

FIG. 7. (a) The diagrams that give the ξ^2 part of the eight-pion amplitude. (b) The diagrams that give the ξ^2 part of the ten-pion amplitude.

are all generated by calculating all possible direct terms and iterating them to give all relevant tree diagrams. As in step 2, we remark that to order ξ^2 this is an unambiguous process.

Obtaining the correct direct terms is not difficult. We need only note that the terms which contribute explicit ξ^2 dependence must be of the type $\langle 0|T(\hat{A}A)|0\rangle$ and $\langle 0|T(AA)|0\rangle$. Since the prescription is to remove the pole from the term $\langle 0|T(AA)|0\rangle$, we need only note that we can get the nonpole piece by replacing the term

$$-f_\pi\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu \quad (7.21)$$

in the basic identity of theorem 2 by the term

$$-f_\pi\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu. \quad (7.22)$$

Combining the $\hat{A}$ terms, we see that the direct term of the two-pion amplitude is given by

$$\tfrac{1}{2}i(-2i)^2f_\pi^2\left(\text{expansion to order } f_\pi^{2(m-1)} \text{ of } \frac{1}{(1+f_\pi^2\varphi^2)^2}\right)\frac{\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}}{4f_\pi^2}$$
$$=\tfrac{1}{2}i\left(\text{expansion to order } f_\pi^{2(m-1)} \text{ of } \frac{1}{(1+f_\pi^2\varphi^2)^2}\right)\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}. \quad (7.23)$$

This is clearly equivalent to the statement that the direct terms are generated by expanding the effective Lagrangian

$$\mathcal{L}_{\text{eff}}=-\frac{1}{2}\frac{1}{(1+f_\pi^2\varphi^2)^2}(\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}) \quad (7.24)$$

to the appropriate order, and the pole diagrams are obtained by iterating this same Lagrangian, keeping only tree diagrams. This then finishes the proof of Lemma 1.

We now go on to prove two more lemmas which tell us how to deal with amplitudes involving particles other than pions.

Lemma 2. The amplitude for the production of n soft pions from a one-particle state $|\alpha\rangle$ vanishes as ξ. Moreover, the coefficient of the term of order ξ can be correctly calculated by using the phenomenological Lagrangian given in Eq. (7.2) keeping only tree diagrams.

Proof. In order to prove Lemma 2, we must once again use the basic identity

$$\langle\alpha+n\pi|S_0|\alpha\rangle=f_\pi^n\langle\alpha|U^n|\alpha\rangle.$$

Since the state $|\alpha\rangle$ contains hadrons, the terms with the lowest explicit power of ξ are of the form

$$\langle\alpha|\hat{\mathbf{A}}^{\mu}|\alpha\rangle,\quad \langle\alpha|\mathbf{V}^{\mu}|\alpha\rangle,\quad \text{or}\quad \langle\alpha|\mathbf{A}^{\mu}|\alpha\rangle.$$

We claim that each of these terms, in fact, vanishes as ξ. Before giving a general argument proving this assertion, let us see how it works in a special case. Let us consider, for example, the case in which $\langle\alpha| = \langle N|$, a one-nucleon state, and $n=1$. For this case the only relevant matrix element is $\langle N|\mathbf{A}^{\mu}|N\rangle$ and this is quite generally given by

$$\langle N|\mathbf{A}^{\mu}(a)|N\rangle = \bar{u}_N\{[g_A(q)\gamma_5\gamma^{\mu}+q^{\mu}h_A(q^2)]\tfrac{1}{2}\boldsymbol{\tau}\}u_N. \quad (7.25)$$

If we include the q_μ factor coming from $\partial_\mu\boldsymbol{\varphi}$, we see that the vertex goes as

$$2m_N g_A \bar{u}_N(\gamma_5 \tfrac{1}{2}\boldsymbol{\tau})u_N \quad (7.26)$$

(where we used the fact that $q^2=0$), and this vertex vanishes as ξ due to the γ_5. So much for the axial-vector matrix element.

The matrix element of the vector current clearly is proportional to the nucleon momentum times the isospin operator in the limit $q\to 0$, and thus the explicit factor of ξ in front of the vector current makes this term vanish as ξ.

As before, all terms multiplied by more explicit powers of ξ can contribute to the order ξ part of the amplitude only through diagrams in which massive one-particle intermediate states, whose propagators go as ξ^{-1}, occur.

We can now generalize this argument to the case of states involving particles with spin greater than $\frac{1}{2}$ as follows. For spin $\frac{1}{2}$ or greater, matrix elements of the form $\langle\alpha|\mathbf{A}^{\mu}(q)|\alpha\rangle$ must, in the limit $q\to 0$, be proportional to the spin vector of the state $|\alpha\rangle$ since this is the only possible axial vector. As we are calculating matrix elements of $\mathbf{A}^{\mu}$ multiplied by explicit factors of the momentum transfer, these terms must go to zero as ξ^1. (This argument fails only when the external states have spin zero, and then $\langle\alpha|\mathbf{A}^{\mu}|\alpha\rangle$ is identically zero.) The coefficient of proportionality between this matrix element and the expectation value of the spin operator is what we define to be the axial-vector coupling constant for such a vertex. Similarly, the matrix element of the vector current goes to a constant times the total four-momentum of the external state times the expectation value of the isospin operator and, as before, the constant of proportionality is defined to be the relevant vector coupling constant. Thus, owing to the explicit factor of ξ the vector terms also vanish as ξ^1. Evidently the amplitudes for processes involving particles other than pions vanish as ξ^1, as opposed to the pion-pion amplitudes, which vanish as ξ^2.

Once again we note that the tree diagrams, which are the only other terms in the basic identity which can contribute to order ξ^1, must sum up to give the ξ^{-1} pole terms coming from unitarity. For example, Figs. 8(a)

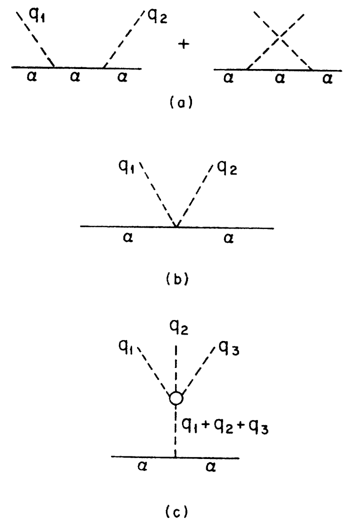

FIG. 8. Tree diagrams for the process $\pi+\alpha\to\pi+\alpha$ and an example of a pion-pole tree for the process $\pi+\alpha\to 2\pi+\alpha$.

and 8(b) gives the tree diagrams for the process $\alpha\to\alpha+2\pi$. We must take careful note, however, of the fact that when there are three or more pions involved, as in the process $\alpha\to\alpha+3\pi$, the presence of $\mathbf{A}^{\mu}$ terms as well as $\hat{\mathbf{A}}$ terms implies that diagrams such as Fig. 8(c) must be kept. Fortunately, Lemma 1 tells us how to compute all the π-π scattering vertices to order ξ^2, which is all we need to know.

Clearly, the diagrams for the processes which do not involve possible π-π scatterings are generated by the expression

$$[-2f_\pi/(1+f_\pi^2\varphi^2)][\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^{\mu}+f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^{\mu}] \quad (7.27)$$

expanded to the appropriate order in f_π. As we have already argued, all such diagrams vanish as order ξ.

Diagrams involving π-π scatterings are of the form given in Figs. 9(a) and 9(b) and involve knowing the π-π scattering amplitude and the scattering amplitudes for processes in which single pions attach to massive particles. Since the pion propagator goes as ξ^{-2}, we generate terms of order ξ only by keeping nothing but the leading behavior in ξ for each vertex. Obviously, all diagrams involving π-π scatterings take the form of clusters of pions hanging from massive particle lines and thus there is always a pion propagator for each π-π vertex. Thus, it is generally true that the ξ^2 part of the π-π vertex cancels against the ξ^{-2} part of the propagator and the ξ dependence comes completely from the rest of the tree diagram.

These arguments tell us that we can get the order-ξ part of the scattering amplitude by iterating the phenomenological Lagrangian

$$\mathcal{L}=\mathcal{L}_{\text{free}}+\mathcal{L}_{\text{int}}, \quad (7.28)$$

where

$$\mathcal{L}_{\text{int}}=\frac{-2f_\pi}{1+f_\pi^2\varphi^2}[\partial_\mu\boldsymbol{\varphi}\cdot\mathcal{A}^\mu+f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathcal{V}^\mu]-\frac{1}{2}\frac{(\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi})}{(1+f_\pi^2\varphi^2)^2}.$$

Clearly, whenever the state $|\alpha\rangle$ contains more than one massive particle, the relevant tree diagrams correspond to the places in the graph shown in Fig. 10 where the phenomenological axial-vector and vector currents act on each external line.

This finishes the discussion of the proof of Lemma 2. All we need to discuss is the case of inelastic processes involving particles other than pions in order to prove the theorem. We do this in the proof of the following lemma.

Lemma 3. The inelastic production of n pions in processes like $\beta\rightarrow\alpha+n\pi$ goes as ξ^0 as ξ goes to zero. Moreover, the constant term can be correctly calculated using the phenomenological Lagrangian given in Eq. (7.2) keeping only tree diagrams, assuming we already know the scattering amplitude for the process $\beta\rightarrow\alpha$.

Proof. The proof of this lemma follows easily from the results of Lemma 2. In effect, the only thing to be proved is the statement about the rate at which the amplitude vanishes. One can see this by noting that the arguments presented in the proofs of Lemma 2 apply to the inelastic processes as well, in so far as the calculation of effects to lowest order in ξ involves keeping only tree diagrams such as those in Fig. 10 (the cross-hatched blob is again used to represent the supposedly known matrix element $\langle\alpha|S|\beta\rangle$). These tree diagrams correspond to the insertion of the vector and axial-vector vertices on the external legs of the diagrams for the strong-interaction process $\beta\rightarrow\alpha$. As before, we can use the phenomenological Lagrangian given in Eq. (7.2) to generate all of the tree diagrams; however, now for each vector and axial-vector current which acts, there is a propagator entering the vertex for the process, given by the cross-hatched blob in Fig. 10, and this cancels the ξ factor coming from the vector or axial-vector vertex. This then finishes the proof of Lemma 3.

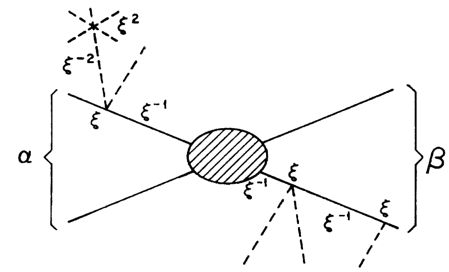

FIG. 10. Illustration of the various powers in ξ coming from different parts of a tree diagram.

Combining the results of Lemmas 1–3, we see that we have proved Theorem 3; however, there are some comments worth making at this point. They are most easily stated as corollaries to Theorem 3.

Corollary 3.1. In general, the correction terms to the scattering amplitude calculated by using the phenomenological Lagrangian of Eq. (7.2) and keeping only tree diagrams occur to the next order in ξ.

Proof. First let us consider π-π scattering. As we have seen in the proof of Lemma 1, there is no reason why terms of the form $\langle 0|T(\hat{A}\hat{A}V)|0\rangle$ should vanish, and therefore it can contribute a ξ^3 piece to the pion-pion scattering amplitudes.

With regard to processes involving particles other than pions, there is no reason why diagrams in which the axial-vector current attaches to a blob rather than a single external line [Figs. 11(a) and 11(b)] should not contribute (except perhaps for special spin combinations). Clearly, whenever these diagrams do not vanish, they furnish examples of corrections to the amplitude of next order in ξ.

This corollary should not be taken as a proof that the corrections to all processes must be of the next order in ξ, but rather as a statement that if this is not the case, the correction term must vanish for some dynamical reason, and such a prediction awaits the development of a theory which allows us to calculate time-ordered products of currents when $\xi\neq 0$.

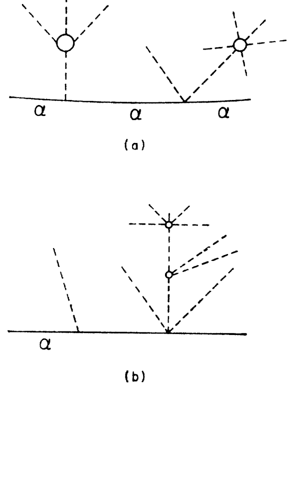

FIG. 9. Example of tree diagrams which must be kept when computing process involving many pions.

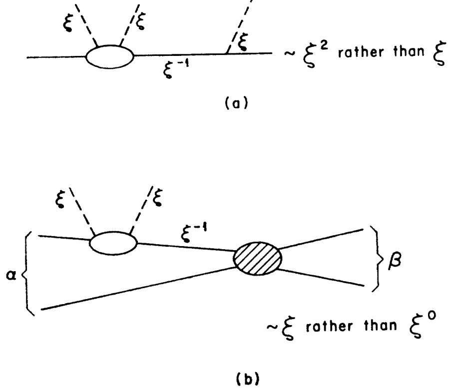

FIG. 11. Examples of terms contributing to the next order in ξ for any amplitude.

Corollary 3.2. In processes for which all tree diagrams vanish for reason of some symmetry, the phenomenological Lagrangians cannot be used to calculate anything about the process. We can, however, say that the amplitude vanishes at least one power of ξ faster than the trees would have vanished.

Proof. The proof of this corollary is obvious from the way in which we derived the phenomenological Lagrangians and will not be discussed further. But, it is worth giving one example of a process to which this corollary applies. Let us consider η-K scattering. In this case the matrix elements $\langle\eta|\mathbf{A}^\mu|\eta\rangle$ and $\langle K|\mathbf{A}^\mu|K\rangle$ vanish by parity and therefore there can be no axial-vector currents inserted on external lines. Thus, in the process $\eta+K\to\eta+K+\pi$ the only tree diagrams which could occur vanish identically.

8. ADDING ELECTROMAGNETIC AND WEAK INTERACTIONS

It is not our purpose, in this section, to develop a general formalism for the addition of weak and electromagnetic effects to these phenomenological Lagrangians as this is a formidable problem. Rather, we wish to show how the calculations of pion photoproduction and similar effects fit naturally into the framework which we have established.

Throughout this section we shall assume that with the addition of electromagnetism the identity $\partial_\mu\mathbf{A}^\mu=0$ becomes

$$\partial_\mu A_\alpha{}^\mu - ie\mathcal{A}_\mu[Q_3,A_\alpha{}^\mu]=0\,, \tag{8.1}$$

where $\mathcal{A}_\mu$ stands for the electromagnetic field. There are many ways to use this assumption; one way is to use this expression for $\partial_\mu\mathbf{A}^\mu$ in the basic identity of Theorem 1 [Eqs. (5.1) and (5.2)]. This, however, has the disadvantage of separating the PCAC correction terms from the Σ terms, as they appear on different sides of the equation. As we show in Appendix B, it is more convenient to rederive Theorem 1 with the following modification.

Let

$$h(x_0)=2f_\pi\int d^3x\,[\boldsymbol{\varphi}\cdot(x)\cdot\partial_\mu\mathbf{A}^\mu - ie\varepsilon_{3\alpha\beta}\varphi_\alpha A_\beta{}^\mu\mathcal{A}_\mu].$$

We then have $h(x_0)$ as identically zero, and get an identity of the form given in Corollary 1.1,

$$1=\exp\left[i\int d^4x\,L(x)\right], \tag{8.2}$$

where

$$\begin{aligned}L(x)=[2f_\pi/(1+f_\pi{}^2\varphi^2)]\{&\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)\\ &-ie\mathcal{A}_\mu[\boldsymbol{\varepsilon}_3\times\boldsymbol{\varphi}]\cdot\mathbf{A}^\mu(x)+f_\pi\mathcal{A}_\mu[\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}]\cdot\mathbf{V}^\mu(x)\\ &-if_\pi\mathcal{A}_\mu[\boldsymbol{\varphi}\times(\boldsymbol{\varepsilon}_3\times\partial_\mu\boldsymbol{\varphi})]\cdot\mathbf{V}^\mu(x)\}\,,\end{aligned} \tag{8.3}$$

where $\boldsymbol{\varepsilon}_3$ stands for the unit vector in the 3 direction in the isospin space determined by the function φ_α.

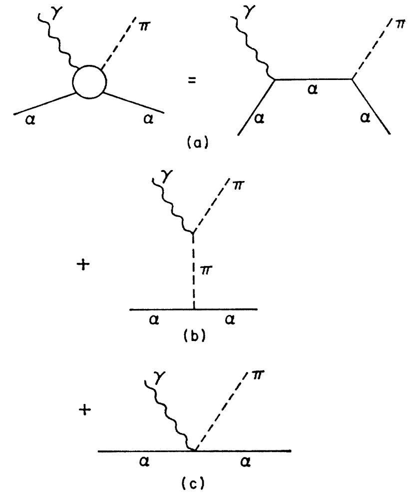

FIG. 12. Tree diagrams for $\gamma+\alpha\to\alpha+\pi$.

Clearly, this corresponds to making a minimal substitution in the identity without electromagnetism.

Before discussing the extraction of the terms corresponding to poles in the momentum of a single pion, we must now introduce the fundamental restriction to be observed in all applications of the results of this section. Briefly, the restriction is that when calculating any process having n photons in the external state, we should keep only terms of order e^n. This means that one never keeps terms in which photon bubbles, or vertex corrections, etc., can occur. Clearly, this is a very strong restriction, since it means we are still working in the limit in which the pion mass is zero and in fact, we are only keeping diagrams of the form shown in Fig. 12(a). These diagrams are easily seen to be the only ones which contribute if we notice that the usual Feynman rules tell us that, with the restriction we have stated, we may as well treat $\mathcal{A}^\mu(x)$ as an external potential; that is, $\mathcal{A}^\mu(x)$ is taken, for processes involving m photons, to have the form

$$\mathcal{A}^\mu(x)=\Sigma_e{}^{+ik_j\cdot x}\varepsilon_j{}^\mu\,, \tag{8.4}$$

and we only keep terms when expanding the basic identity in which all of the k_j's appear.

With this prescription in mind, it is evident that terms involving $\mathcal{A}_\mu\mathbf{V}^\mu$ and $\mathcal{A}_\mu\hat{\mathbf{A}}^\mu$ do not generate poles when one of the pion four-momenta squared is set equal to zero, since they correspond to terms of the form $\mathbf{A}^\mu(q+k_j)$. Thus the pole-cancellation arguments of Sec. 7 go through as before and result in the identity

$$\langle\alpha+n\pi+m\gamma|S_0|\beta\rangle=f_\pi{}^n\langle\alpha+m\gamma|U^n|\beta\rangle\,, \tag{8.5}$$

where U^n is defined as before, but now using the expression for $L(x)$ given in Eq. (8.3), except that $\mathbf{A}^\mu$ is replaced by $\hat{\mathbf{A}}^\mu$.

Getting the tree diagrams is very simple. The terms containing explicit factors of $\mathcal{A}^\mu$ are explicitly of order e and therefore generate contact terms of the form shown in Fig. 12(c); the other two diagrams in Fig. 12 arise as the result of evaluating matrix elements of the form $\langle\alpha+\gamma|\mathbf{V}^\mu|\beta\rangle$. If we follow our previous observation that a photon in the external state corresponds to calculating the matrix element $\langle\alpha|\mathbf{V}^\mu|\beta\rangle$ in the presence of the external potential, allowing it to act once for each photon, it is clear that diagrams in Figs. 12(a) and 12(b) occur.

These brief considerations make it clear that, in the presence of electromagnetism, processes involving photons can be correctly calculated by adding to the phenomenological Lagrangians already described the term $[J_{\mathrm{em}}{}^\mu\cdot\mathcal{A}_\mu(x)]$. Note that as before when adding a phenomenological expression for $(J_{\mathrm{em}}{}^\mu)$, we must use the correct physical value of the matrix element including the anomalous moments for the various particles which appear. Thus, the effective Lagrangian for pion-nucleon scattering becomes

$$\mathcal{L}_{\mathrm{eff}}=\bar\psi_N(x)\bigg[i\gamma^\mu\bigg(\partial_\mu-ie\frac{1+\tau_3}{2}\mathcal{A}_\mu\bigg)+m_N+\frac{\gamma_\mu}{1+f_\pi{}^2\varphi^2}\{f_\pi g_A\gamma_5\boldsymbol{\tau}\cdot[\partial_\mu\boldsymbol{\varphi}-ie\mathcal{A}_\mu(\boldsymbol{\varepsilon}_3\times\boldsymbol{\varphi})]\}+f_\pi{}^2\{\boldsymbol{\varphi}\times[\partial_\mu\boldsymbol{\varphi}-ie\mathcal{A}_\mu(\boldsymbol{\varepsilon}_3\times\boldsymbol{\varphi})]\}\bigg]\psi_N(x)$$
$$+\frac{[\partial_\mu\boldsymbol{\varphi}-ie\mathcal{A}_\mu(\boldsymbol{\varepsilon}_3\times\boldsymbol{\varphi})]\cdot[\partial^\mu\boldsymbol{\varphi}-ie\mathcal{A}^\mu(\boldsymbol{\varepsilon}_3\times\boldsymbol{\varphi})]}{(1+f_\pi{}^2\varphi^2)^2}+\bar\psi_N(x)\bigg[\sigma_{\mu\nu}F^{\mu\nu}\bigg(\mu_p\frac{1+\tau_3}{2}+\mu_n\frac{1-\tau_3}{2}\bigg)\bigg]\psi_N(x).\quad(8.6)$$

Evidently the same methods can be used to consider the addition of the strangeness-changing and semileptonic weak interactions, under the same restriction that we keep only terms which do not change the value of $m_\pi{}^2$.

9. GENERALIZATION TO $SU(3)\otimes SU(3)$

Except for the fact that we used the $SU(2)\otimes SU(2)$ commutation relations to do explicitly the rotation encountered in the derivation of Theorem 1, there is nothing in what we have done which restricts us to the consideration of $SU(2)\otimes SU(2)$ symmetry. In particular, we could just as easily have discussed $SU(3)\otimes SU(3)$ symmetry, except that it is not very easy to do the necessary $SU(3)$ rotation. The only necessary modification encountered in this case is that now the entire octet of pseudoscalar mesons correspond to Goldstone bosons in the symmetry limit.

If we let $\varphi_\alpha(x)$, $\alpha=1,\cdots,8$, now stand for an arbitrary c-number octet vector, and change $\varepsilon_{\alpha\beta\gamma}$ to $f_{\alpha\beta\gamma}$ wherever they appear in the commutation relations, the same arguments as in the $SU(2)\otimes SU(2)$ case lead us directly to a set of equivalent phenomenological Lagrangians which differ only up to a redefinition of the fields corresponding to the pseudoscalar mesons.

As we have said, getting a *simple* closed form for the resulting Lagrangians involves finding some clever choice of a function $G_\alpha(\boldsymbol{\varphi})$, where $G_\alpha(\boldsymbol{\varphi})$ is now an arbitrary octet-vector function of φ_α which replaces the function $G(\varphi^2)\boldsymbol{\varphi}$ in our $SU(2)\otimes SU(2)$ derivation; we have not been able to find such a simple expression. However, if one merely wishes to have a power-series expansion of the resulting Lagrangian in order of f_π, one need only choose for the analog of $S(x_0)$ in Theorem 1,

$$S(x_0)=\exp\bigg[+i2f_\pi\int d^3x\ \boldsymbol{\varphi}\cdot\mathbf{A}^0(x)\bigg],\qquad(9.1)$$

where all vectors are now octet vectors and $\mathbf{A}(x)$ includes the entire octet of strangeness-changing and strangeness-nonchanging currents.

With this choice we are led naturally to a phenomenological Lagrangian of the form

$$\begin{aligned}\mathcal{L}_{\mathrm{int}}=&-[2f_\pi\partial_\mu\boldsymbol{\varphi}\cdot\mathcal{A}^\mu+4f_\pi{}^2(f_{\alpha\beta\gamma}\varphi_\alpha\partial_\mu\varphi_\beta)\mathcal{V}_\gamma{}^\mu\\&+(8/2)f_\pi{}^3(\varphi_\beta\partial_\mu\varphi_\alpha f_{\beta\alpha\gamma})(\varphi_\sigma f_{\sigma\gamma\rho})\mathcal{A}_\rho{}^\mu+\cdots]\\&-[\tfrac{4}{3}f_\pi{}^2(\varphi_\beta\partial_\mu\varphi_\alpha f_{\beta\alpha\gamma})(\varphi_\sigma f_{\sigma\gamma\rho}\partial_\mu\varphi_\rho)+\cdots],\end{aligned}\qquad(9.2)$$

where script $\mathcal{A}$'s and $\mathcal{V}$'s have the same meaning as before.

10. SUMMARY

In the preceding sections we have developed a rather extensive formalism which enabled us to extract the full content of the joint assumptions of PCAC and current algebra. We then proceeded to use this formalism to show how one is led to the phenomenological Lagrangian techniques for deriving soft-pion theorems. It should be emphasized at this point that we have not derived results which allow one to calculate soft-pion theorems more readily than by the phenomenological Lagrangian techniques; in fact, from a calculational point of view there is nothing new in our results. However, we believe that the formalism which we have developed does bring to light all of the assumptions which go into the phenomenological Lagrangians and exhibits explicitly all of their limitations. In addition, in our section on symmetry breaking, we have shown why the lowest order in symmetry-breaking corrections to the π-π scattering amplitude assume a special position in deriving low-energy theorems. This is an interesting point since it clarifies the reason why one conventionally adds a symmetry-breaking term to the usual phenomenological Lagrangians used, which involves the pion field alone.

One final point is that perhaps this formalism will point the way towards calculation of the correction terms to the low-energy theorems. The point here is that one might be able to make models for evaluating the necessary time-ordered products which appear in our most general formula as correction terms to the limit $\epsilon=0$.

ACKNOWLEDGMENT

The authors would like to thank the Aspen Center for Physics where most of this work was completed.

APPENDIX A

Theorem. Let $S(t)=e^{+ig(t)}$. Then

$$-iS^{-1}\frac{dS}{dt}=\int_0^1 d\varphi e^{-i\varphi g(t)}\frac{dg}{dt}e^{+i\varphi g(t)}.$$

Proof. Define $S(\varphi,t)=e^{+i\varphi g(t)}$. Then

$$\frac{d}{d\varphi}S(\varphi,t)=ig(t)S(\varphi,t),$$

$$\frac{d}{d\varphi}\frac{d}{dt}S(\varphi,t)=i\frac{dg}{dt}S(\varphi,t)+ig(t)\frac{dS}{dt}(\varphi,t).$$

Let $dS/dt=S(\varphi,t)V(\varphi,t)$. Then

$$\frac{d}{d\varphi}(SV)=\frac{dS}{d\varphi}V+S\frac{dV}{d\varphi}=+igSV+S\frac{dV}{d\varphi}$$

$$=\frac{idg}{dt}S+igSV,$$

which implies that

$$\frac{dV}{d\varphi}=+i\left(S^{-1}\frac{dg}{dt}S\right)(\varphi,t),$$

$$V(1,t)=+i\int_0^1 e^{-i\varphi g(t)}\frac{dg}{dt}e^{+i\varphi g(t)}d\varphi$$

or

$$-iS^{-1}(1,t)\frac{dS}{dt}(1,t)=+i\int_0^1 e^{-i\varphi g(t)}\frac{dg}{dt}e^{+i\varphi g(t)}d\varphi,$$

and since $S(1,t)=S(t)$, we have completed the proof.

APPENDIX B

The aim of this Appendix is to derive a general treatment of formula for the n-pion scattering amplitudes in the case $\epsilon\neq 0$. This may be done starting directly with the formula obtained in Theorem 1; however, the separation of the PCAC correction terms ($\bar{\partial}$) from the rest of the terms introduces needless complications. What we shall do in this Appendix is derive an alternative form of Theorem 1, which puts it more into the form of Corollary 1.1. This form will allow us to do the necessary pole cancellations trivially.

Motivating the steps that we shall now take is not difficult if we realize that the fundamental trick used throughout all previous treatments is the fact that we use two different expressions for $\partial_\mu \mathbf{A}^\mu$ and rotate these expressions in different ways. To be more explicit, let us define

$$\partial_\mu \mathbf{A}^\mu(x)=\boldsymbol{\partial}(x).$$

If we let

$$H(x_0)=2f_\pi\int d^3x[\boldsymbol{\varphi}(x)\cdot\partial_\mu\mathbf{A}^\mu(x)-\boldsymbol{\varphi}\cdot\boldsymbol{\partial}(x)]$$

in Theorem 1, letting all other terms be defined as before, we get the identity

$$1=T\left(\exp\left[i\int d^4x\, L(x)\right]\right),$$

where $L(x)$ is given by

$$L(x)=-\frac{2f_\pi}{1+f_\pi^2\varphi^2}[\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu(x)+f_\pi(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu(x)]$$

$$-\int_0^1 du\, S^{-1}(u,x_0)[f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\boldsymbol{\partial}(x)]S(u,x_0).$$

If we recall that we have a power-series expansion for the term

$$-\int_0^1 du\, S^{-1}(u,x_0)[f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\boldsymbol{\partial}(x)]S(u,x_0)$$

$$=-f_\pi G(\varphi^2)\bigg\{\boldsymbol{\varphi}\cdot\boldsymbol{\partial}(x)-\tfrac{1}{2}if_\pi G(\varphi^2)\varphi_\alpha\varphi_\beta$$

$$\times\int d^4y\,\delta(x^0-y^0)[F_\alpha{}^0(x),\partial_\beta(y)]+\cdots\bigg\}.$$

Expanding $G(\varphi^2)$ in f_π^2, we see that we get one term equal to

$$-2f_\pi\boldsymbol{\varphi}\cdot\boldsymbol{\partial}(x),$$

and a series of terms multiplied by more factors of $\boldsymbol{\varphi}$. Clearly, only the term multiplied by one factor of $\boldsymbol{\varphi}$ gives a pole in the momentum of a single pion; the rest of these terms give contributions correction of order $\boldsymbol{\partial}^2$ and greater to π-π scattering.

Following the same procedure as in the proof of Theorem 2 (when $\epsilon=0$), we separate off the one-pion pole terms. However, in this case we see that the term $-2f_\pi\boldsymbol{\varphi}\cdot\boldsymbol{\partial}(x)$ must be grouped with the term $-2f_\pi\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu$ because they both generate pole terms. If we do this methodically when separating off pole terms in the time-ordered products, we get from each pole term a factor

$$\left(\frac{-q^2+m_\pi^2}{q^2-m_\pi^2}\right)=-1$$

taking the place of every factor of

$$(-q^2)/q^2=-1$$

in Eq. (6.16). Otherwise the derivation goes through unchanged and we find that we get a basic identity of the form

$$\langle\alpha+n\pi|S|\beta\rangle=f_\pi{}^n\langle\alpha|U^n(q_1\cdots q_n)|\beta\rangle,$$

where $U^n(q_1\cdots q_n)$ is defined as before, but for the function $L(x)$ given by

$$\begin{aligned}L(x)=-2f_\pi\bigg[&\partial_\mu\boldsymbol{\varphi}\cdot\hat{\mathbf{A}}^\mu(x)\\&+\frac{f_\pi}{1+f_\pi{}^2\varphi^2}(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})\cdot\mathbf{V}^\mu-\varphi^2\partial_\mu\boldsymbol{\varphi}\cdot\mathbf{A}^\mu\bigg]\\&-[f_\pi(G(\varphi^2)-2)\boldsymbol{\varphi}\cdot\boldsymbol{\partial}(x)+2f_\pi\boldsymbol{\varphi}\cdot\bar{\boldsymbol{\partial}}(x)]\\&-\bigg\{\int du\,S^{-1}(u,x_0)[f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\boldsymbol{\partial}]S(u,x_0)\\&\qquad-f_\pi G(\varphi^2)\boldsymbol{\varphi}\cdot\boldsymbol{\partial}\bigg\}.\end{aligned}$$

The last two sets of terms in $L(x)$ are clearly the ones responsible for generating PCAC correction terms and Σ terms, whereas the first set of terms is identical to the expression derived in the case $\epsilon=0$, $\partial_\mu\mathbf{A}^\mu=0$, and $m_\pi{}^2=0$.

The usefulness of this general identity lies in the fact that it allows us to read off for any process an expression for the on-mass-shell scattering of pions off anything. Perhaps, working from this identity and making models for the time-ordered products involved, one can hope to be able someday to calculate pion scattering amplitudes in the presence of symmetry breaking.