J-Plane Structure of the Veneziano Model

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The Veneziano model for $\pi^+\pi^-$ elastic scattering is subjected to J-plane anaylsis, and it is shown in that in the finite J plane it consists only of Regge poles and fixed poles at nonsense wrong-signature points. The residues of the leading and first two subsidiary trajectories are computed explicitly, and an algorithm is derived for the computation of all others. In deriving the algorithm, the Mandelstam form of the Regge-pole series is obtained. It is shown that this series is asympotic in a certain domain, but not convergent, thus demonstrating the essential role of the background integration in producing crossing symmetry. The relation between the Veneziano residues and those of the Khuri model is discussed, as well as some other features of the J-plane structure.

I. INTRODUCTION

THERE has been considerable interest recently in a simple and elegant model of the hadron S matrix proposed by Veneziano.[1] This model is explicitly crossing-symmetric, gives rise to asymptotic almost-Regge behavior,[2] and contains an infinite number of zero-width resonances. It is also a solution, albeit by no means unique,[3] of the finite-energy sum rules in the narrow-resonance approximation. A model with so many theoretically desirable features evidently warrants further attention, particularly with a view to relieving the restriction to infinitely narrow resonances and the consequent loss of unitarity.

Perhaps the most interesting heuristic feature of the Veneziano amplitude is that it enables one to learn how the Regge residues may be constructed so as to yield a crossing-symmetric amplitude. As was noted by Khuri,[4] the residues have to behave in a rather complicated fashion in a crossing-symmetric model with infinitely rising trajectories. In fact, as we shall observe later on, the mechanism which builds crossing symmetry into the Veneziano amplitude through the residues is very different from that in Khuri's model.

It is thus tempting to regard the Veneziano amplitude as a guide to the construction of the Regge residues, so as to ensure crossing symmetry within the accuracy of the narrow-resonance approximation. With the residues thus computed, one then departs from the Veneziano form by permitting nonlinear and distinct imaginary parts for the leading and daughter trajectories. One may then assume the approximate validity of elastic unitarity at the resonance points, to compute the imaginary parts of the trajectory functions and thereby the resonance widths. This unambitious program, requires only the residues at the resonance points themselves, which can be obtained directly from the amplitude. If, on the other hand, one wishes to study the implications of the unitarity requirement at arbitrary energies, one must first subject the Veneziano amplitude to a J-plane analysis.

In this paper we shall show that in spite of the presence of asymptotic oscillations in the one-term Veneziano model for $\pi^+\pi^-$ elastic scattering, it is nonetheless built out of Regge poles and fixed poles at nonsense wrong-signature points. The $I=1$ amplitude displays pure Regge behavior, whilst the $I=0$ and $I=2$ amplitudes have fixed poles in addition to the Regge poles. The leading Regge pole and the first two secondary poles are explicitly exhibited and the Regge residues computed. An algorithm is then derived for the computation of all higher residues. In deriving the algorithm for the residues we also obtain the Mandelstam form of the series of Regge-pole contributions. The series is asymptotic in a certain domain but, as we show in the Appendix, it is not convergent. This demonstrates the essential role of the background integral in a model with crossing symmetry. Finally we remark several theoretical features of the Veneziano model which become clear from the J-plane analysis.

II. ANALYTICITY IN FINITE J PLANE

In this section we exhibit the singularities in the complex j plane of a one-term Veneziano amplitude appropriate to elastic $\pi^+\pi^-$ scattering in the s channel.[5] Omitting an inessential over-all constant, we write

$$A(s,t)=\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}. \quad (1)$$

Here $\alpha(s)$ is identified with the ρ trajectory (the f being considered exchange-degenerate with it). For linearly rising trajectories in the zero-width approximation,

$$\alpha(t)=a+a't, \quad a'>0. \quad (2)$$

The amplitudes for the various isospin channels are

$$\begin{aligned} A^{(0)}(s,t)&=\tfrac{3}{2}[A(s,t)+A(s,u)]-\tfrac{1}{2}A(t,u),\\ A^{(1)}(s,t)&=A(s,t)-A(s,u),\\ A^{(2)}(s,t)&=A(t,u), \end{aligned} \quad (3)$$

where the superscripts label the isospin.

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] See Ref. 1 and also R. Roskies, Phys. Rev. Letters **21**, 1851 (1968).
[3] See, e.g., M. Virasoro, Phys. Rev. **177**, 2309 (1969).
[4] N. N. Khuri, Phys. Rev. **176**, 2026 (1968). We shall find the techniques of this paper useful in Sec. III.
[5] C. Lovelace, Phys. Letters **28B**, 264 (1968); J. Yellin and J. Shapiro (to be published).

We shall consider separately the s-channel partial-wave projections of $A(s,t)$ and $A(t,u)$. From the series decomposition of the beta function,[6] we obtain for (1)

$$A(s,t)=-\frac{1}{\Gamma(\alpha(s))}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{\Gamma(n+1)}\frac{1}{1-\alpha(t)+n} \tag{4}$$

and with (2) this becomes

$$A(s,t)=\frac{1}{\Gamma(\alpha(s))}\frac{1}{2a'q_s^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{\Gamma(n+1)}\frac{1}{z_s-z_n(s)}, \tag{5}$$

in which

$$q_s^2=\tfrac{1}{4}(s-4m_\pi^2),\quad z_s=1+t/2q_s^2,\quad z_n(s)=1+(n+1-a)/2a'q_s^2. \tag{6}$$

The series (5) is convergent for $\mathrm{Re}\alpha(s)<0$, and in this region we can stay off the poles of $[z_s-z_n(s)]^{-1}$. The partial-wave projection is then

$$f_J(s)=\frac{1}{2}\int_{-1}^{+1}dz_s\,P_J(z_s)A(s,t)=\frac{1}{2a'q_s^2\Gamma(\alpha(s))}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{\Gamma(n+1)}Q_J(z_n(s)). \tag{7}$$

Writing (7) in the form

$$f_J(s)=\sum_{n=0}^{\infty}f_{Jn}(s), \tag{8}$$

we note that for $n\to\infty$ the functions f_{Jn} have the asymptotic form

$$f_{Jn}(s)\sim\frac{\beta_0(s,J)}{(n+1)^{J+1-\alpha(s)}}+\frac{\beta_1(s,J)}{(n+1)^{J+2-\alpha(s)}}+\cdots, \tag{9}$$

where the first two coefficients are explicitly

$$\beta_0(s,J)=\frac{\sqrt{\pi}}{2}\frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})}(a'q_s^2)^J\frac{1}{\Gamma(\alpha(s))},$$

$$\beta_1(s,J)=\frac{\sqrt{\pi}}{2}\frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})}(a'q_s^2)^J\frac{1}{\Gamma(\alpha(s))}\times\left\{\frac{\alpha(s)[\alpha(s)-1]}{2}-(J+1)(2a'q_s^2-a)\right\}.$$

For future reference, we note that for $J=\alpha(s)$,

$$\beta_0(s)=\frac{\sqrt{\pi}}{2}(a'q_s^2)^{\alpha(s)}\frac{\alpha(s)}{\Gamma(\alpha(s)+\frac{3}{2})},$$

and for $J=\alpha(s)-1$,

$$\beta_1(s)=(\sqrt{\pi})(a'q_s^2)^{\alpha(s)}\left[-z_0(s)+\frac{\alpha(s)+1}{4a'q_s^2}\right]\times\frac{\alpha(s)}{\Gamma(\alpha(s)+\frac{1}{2})}. \tag{10}$$

[6] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. 1, p. 8 (2).

For $J\neq-1,\,-2,\,\ldots$, each term of (7) is analytic in J and from (9) the series converges provided

$$\mathrm{Re}\alpha(s)<\mathrm{Re}J. \tag{11}$$

Furthermore, the functions $Q_J(z_n(s))$ have the necessary bounds[7] for $f_J(s)$ to satisfy the requirements of Carlson's theorem.[7] Thus, in the region with $J=\alpha(s)-1$, we get a unique analytic continuation of $f_J(s)$ in the complex J plane. To continue to the region where (11) is violated, write

$$f_J(s)=g_{JN}(s)+\sum_{n=0}^{\infty}\left(f_{Jn}-\sum_{\nu=0}^{N-1}\frac{\beta_\nu(s,J)}{(n+1)^{J+\nu+1-\alpha(s)}}\right), \tag{12}$$

with

$$g_{JN}(s)=\sum_{\nu=0}^{N-1}\beta_\nu(s,J)\sum_{n=0}^{\infty}\frac{1}{(n+1)^{J+\nu+1-\alpha(s)}}=\sum_{\nu=0}^{N-1}\beta_\nu(s,J)\zeta(J+\nu+1-\alpha(s)), \tag{13}$$

where $\zeta(\sigma)$ is the Riemann zeta function.

Since each term in the series on the right side of (12) tends to $n^{-J-N-1+\alpha(s)}$, this series converges for

$$\mathrm{Re}\alpha(s)-N<\mathrm{Re}J$$

and hence is analytic in J, except possibly for $J=-1$, -2, ... The function $g_{JN}(s)$ is a finite sum of ζ-functions. Now $\zeta(\sigma)$ is known[8] to be an entire function of σ, except for a pole with unit residue at $\sigma=1$. Hence $g_{JN}(s)$ is an entire function of J except for poles at

$$J+\nu-\alpha(s)=0,\quad \nu=0,\,1,\,2,\,\ldots,\,N-1$$

with residues $\beta_\nu(s)$. Thus in the J plane the poles occur for

$$J=\alpha(s),\quad \alpha(s)-1,\quad \alpha(s)-2,\quad \ldots,\quad \alpha(s)-(N-1). \tag{14}$$

Since N can be arbitrarily large but finite, it follows that except possibly for the points $J=-1,\,-2,\,\ldots$, the only singularities of $f_J(s)$ in the finite J plane are poles at

$$J=\alpha(s)-\nu,\quad \nu=0,\,1,\,2,\,\cdots \tag{15}$$

whose residues are the $\beta_\nu(s)$ of (10).

Next we note that the apparent fixed poles at $J=-1,\,-2,\,\ldots$ arising from the Q_J functions in (7) are spurious, i.e., they have zero residues: Consider the case $J=-1$. To use the series (7) assume first that $\mathrm{Re}\alpha(s)<-1$ so that (6) converges when J is near -1. The residue of $Q_J(z_n(s))$ at $J=-1$ is $P_0(z_n(s))=1$. Hence the residue of $f_J(s)$ at $J=-1$ is proportional to

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{\Gamma(n+1)}=\Gamma(\alpha(s)+1)(1-z)^{-1-\alpha(s)}\big|_{z=1}=0,$$

for

$$\mathrm{Re}\alpha(s)<-1. \tag{16}$$

[7] See, e.g., E. J. Squires, *Complex Angular Momentum and Particle Physics* (W. A. Benjamin, Inc., New York, 1964).

[8] E. C. Titchmarsh, *Theory of Functions* (Oxford University Press, London, 1939), 2nd ed., 4.43, p. 151.

By analytic continuation, it follows that the residue at $J=-1$ is zero for all $\alpha(s)$. The argument for $J=-2, -3, \ldots$ is a trivial generalization of the above. Thus (16) gives all singularities of $f_J(s)$ in the finite J plane. We note that the same results hold for the amplitude $A(s,u)$, since it merely involves the change $z_s \to -z_s$ in (5).

We next consider the partial-wave projection of the amplitude $A(t,u)$,

$$B(x,y)=\sum_n \binom{-x-y}{n}\left(\frac{1}{n+x}+\frac{1}{n+y}\right), \tag{17}$$

derived from the representation

$$B(x,y)=\int_0^1 (v^{x-1}+v^{y-1})(1+v)^{-x-y}dv. \tag{18}$$

We obtain

$$A(u,t)=-\frac{(4a'q_s^2z_0-1)}{2a'q_s^2}\sum_{n=0}^{\infty}\binom{-4a'q_s^2z_0}{n} \times\left(\frac{1}{z_s-z_n}-\frac{1}{z_s+z_n}\right). \tag{19}$$

Taking the partial-wave projection, we get the signatured amplitudes

$$a_J^-(s)=0,$$

$$a_J^+(s)=\frac{(1-4a'q_s^2z_0)}{2a'q_s^2}\sum_{n=0}^{\infty}\binom{-4a'q_s^2z_0}{n}Q_J(z_n)$$
$$=\frac{(1-4a'q_s^2z_0)}{\Gamma(1+4a'q_s^2z_0)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+4a'q_s^2z_0)}{\Gamma(n+1)} \times Q_J(z_n(s)). \tag{20}$$

This representation leads, as before, to a unique analytic contribution in J. It is easy to see that $a_J^+(s)$ has no Regge poles. Proceeding along the same line of argument as developed from Eq. (7) to Eq. (13), we obtain, by taking n large in (20), the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n+1)^{\sigma(J)}}=(1-2^{1-\sigma})\zeta(\sigma), \tag{21}$$

where

$$\sigma=J+2-4a'q_s^2z_0(s)\,.$$

Since the right-hand side is now an entire function of σ, there can be no Regge poles present. However, fixed poles are present in (20) at the points $J=-1, -3, \ldots$; and when $J\to -N$, $Q_J\to P_{N-1}/(J+N)$. Consider a possible fixed pole at $J=-1$ in (20). Its residue is

$$\left(\frac{1-4a'q_s^2z_0}{2a'q_s^2}\right)2^{-4a'q_s^2z_0},$$

which is nonzero. Similarly, the fixed poles are also present at $J=-3, -5, \ldots$. We now show that the possible fixed poles at $J=-2, -4, \ldots$ are spurious, since their residues vanish. The residues of the fixed poles at $J=-(2r+2), r=0, 1, 2, \ldots$, are proportional to

$$\sum_{n=0}^{\infty}\binom{-4a'q_s^2z_0}{n}P_{2r+1}\left(z_0+\frac{n}{2a'q_s^2}\right), \tag{22}$$

which we will show to vanish.

Since a Legendre polynomial of odd order has only odd powers present, it is sufficient to show that

$$\sum_{n=0}^{\infty}\binom{-x}{n}(x+2n)^m=0 \tag{23}$$

for odd positive integers m, where $x=4a'q_s^2z_0$. Indeed, the left-hand side of (23) is

$$\sum_n\binom{-x}{n}\frac{d^m}{d\alpha^m}e^{\alpha(x+2n)}\Big|_{\alpha=0}=2^{-x}\frac{d^m}{d\alpha^m}(\cosh\alpha)^{-x}\Big|_{\alpha=0}=0,$$

for odd m.

Thus we have shown that in the s channel the term $A(u,t)$ has no Regge poles and only fixed poles at nonsense wrong-signature points.[9] On the other hand, from our earlier discussion, the terms $A(s,t)$ and $A(s,u)$ have only Regge poles present. From (3), it is evident that the $I=1$ amplitude displays pure Regge behavior, while the $I=0$, 2 amplitudes have fixed poles.

III. A SIMPLE ALGORITHM FOR COMPUTING RESIDUES

The computation of the residues by the method described in Sec. II involves the working out of the product of the asymptotic series for Γ functions and the Q_J in (7), and is rather messy. The residues are more readily computed by the following device[4]:

We use the integral representation[10]

$$B(\nu,p)=\int_0^{\infty}dy\, e^{-py}(1-e^{-y})^{\nu-1}, \qquad \mathrm{Re}\,p>0, \quad \mathrm{Re}\,\nu>0. \tag{24}$$

Setting

$$p=\tfrac{1}{2}-\alpha(t), \quad \nu=-\alpha(s),$$

we obtain from (1) and (2)

$$A(s,t)=-\alpha(s)\int_0^{\infty}dy\, e^{-[1-\alpha(t)]y}(1-e^{-y})^{-\alpha(s)-1}, \tag{25a}$$

$$A(s,t)=(\tfrac{1}{2}\pi)^{1/2}\int_0^{\infty}dy\, e^{-z_sy}y^{-1/2}H(s,y) \tag{25b}$$

[9] The presence of fixed poles in the Veneziano model has also been shown by J. Yellin and D. Silvers (to be published) and V. Alessandrini and D. Amati, Phys. Letters **29B**, 193 (1969). The latter authors, however, do not make the Froissart-Gribov continuation. We thank these authors for sending us reports of their work prior to publication.

[10] Reference 6, p. 11 (24).

for $\mathrm{Re}\alpha(s)<0$, $\mathrm{Re}\alpha(t)<1$, where

$$H(s,y)=y^{-\alpha(s)-1/2}f(s,y)\,,$$

$$f(s,y)=-\left(\frac{2}{\pi}\right)^{1/2}\alpha(s)(-2a'q_s^2)^{\alpha(s)}e^{z_0(s)y}\times\left(\frac{1-e^{y/2a'q_s^2}}{-y/2a'q_s^2}\right)^{-\alpha(s)-1}, \quad (26)$$

$$z_0(s)=1+(1-a)/2a'q_s^2.$$

Note that $f(s,y)$ is analytic in the finite y plane except for branch points at

$$y/2a'q_s^2=\pm 2\pi i\,,\quad \pm 4\pi i\,,\ \ldots. \quad (27)$$

For s fixed and $\mathrm{Re}\alpha(t)\to-\infty$, we note, from the representation (25a), that all of the contribution to $A(s,t)$ will come from the lower part of the integration. Since with $\mathrm{Re}\alpha(s)<0$, $\mathrm{Re}\alpha(t)\to-\infty$ we get $\mathrm{Re}z_s\to+\infty$, we shall generate an *asymptotic* series for $A(s,t)$ by formally replacing $f(s,y)$ in (25b) by its power series about $y=0$, even though the radius of convergence of this power series is only $4\pi a'|q_s^2|$.

To make this more precise, write

$$f(s,y)=\sum_{m=0}^{N-1}\frac{a_m(s)}{m!}y^m+f_N(s,y)\,, \quad (28)$$

so that

$$A(s,t)=I_1+I_2\,,$$

where

$$I_1=\sum_{m=0}^{N-1}(\tfrac{1}{2}\pi)^{1/2}\frac{a_m(s)}{m!}\int_0^\infty dy\,e^{-z_sy}y^{m+\alpha(s)-1}, \quad (29)$$

$$I_2=(\tfrac{1}{2}\pi)^{1/2}\int_0^\infty dy\,e^{-z_sy}y^{-\alpha(s)-1}f_N(s,y)\,. \quad (30)$$

Since the damping in y at $y=\infty$ in the integrand of (25a) or (25b) is exponential for $\mathrm{Re}\alpha(t)<1$, it follows that both I_1 and I_2 are convergent at the upper limit of integration, so long as $\mathrm{Re}\alpha(t)<1$. However, the singularity at the lower endpoint in (25a) or (25b), which required $\mathrm{Re}\alpha(s)<0$ for convergence, now imposes this restriction only on I_1. For I_2, since $f_N\sim y^N$ at $y=0$, the restriction takes the weaker form

$$\mathrm{Re}\alpha(s)<N\,.$$

It is clear, therefore, that all poles of the amplitude in the s-plane region with $\mathrm{Re}\alpha(s)<N$ arise from I_1. Therefore, since N is arbitrary, the formal replacement of $f(s,y)$ by its power series must reproduce accurately all s-plane resonance poles and their residues.

We next insert the Sonine-Gegenbauer expansion[11] for $y^{m-\alpha-1/2}$ in (29), i.e.,

$$y^\nu=2^\nu\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)C_n^\nu(0)I_{n+\nu}(y)\,, \quad (31)$$

valid for

$$\nu\neq 0,\ -1,\ -2,\ \ldots,$$

where $C_n^\nu(x)$ is a Gegenbauer function. Then, since

$$Q_\nu(z)=(\tfrac{1}{2}\pi)^{1/2}\int_0^\infty dy\,e^{-yz}I_{\nu+1/2}(y)y^{-1/2} \quad (32)$$

for $\mathrm{Re}\nu>-1$, $\mathrm{Re}z>1$, we obtain

$$I_1=\sum_{p=0}^{\infty}\gamma_p(s)Q_{p-\alpha(s)-1}(z_s)\,, \quad (33)$$

in which for $p<N$,

$$\gamma_p(s)=[p-\alpha(s)-\tfrac{1}{2}]\sum_{m=0}^{p}\frac{a_m(s)}{m!}2^{m-\alpha(s)-1/2}\times\Gamma(m-\alpha(s)-\tfrac{1}{2})C_{p-m}^{m-\alpha(s)-1/2}(0)\,. \quad (34)$$

Equation (33) now constitutes an asymptotic expansion for $A(s,t)$ of the Mandelstam-Regge form,[12] which has the property that it contains correctly all resonance poles in the region $\mathrm{Re}\alpha(s)<N$, and N can be taken arbitrarily large. We may therefore compute the residues $\beta_p(s)$ by writing (33) in the form

$$A(s,t)\sim\sum_{p=0}^{\infty}\frac{\beta_p(s)[2\alpha_p(s)+1]}{\cos\pi\alpha_p(s)}Q_{p-\alpha(s)-1}(-z_s)\,, \quad (35)$$

from which one readily obtains the formula

$$\beta_p(s)=\tfrac{1}{2}\cos\pi\alpha(s)\,e^{i\pi\alpha(s)}\sum_{m=0}^{p}\frac{a_m(s)}{m!}2^{m-\alpha(s)-1/2}\times\Gamma(m-\alpha(s)-\tfrac{1}{2})C_{p-m}^{m-\alpha(s)-1/2}(0)\,, \quad (36)$$

$$p=0,\ 1,\ 2,\ \cdots,$$

where

$$a_m(s)=(2/\pi)^{1/2}\alpha(s)(2a'q_s^2)^{\alpha(s)}\times\frac{d^m}{dy^m}\left[e^{z_0(s)y}\left(\frac{1-e^{y/2a'q_s^2}}{y/2a'q_s^2}\right)^{-\alpha(z)-1}\right]_{y=0} \quad (37)$$

and

$$C_\nu^n(0)=0\,,\quad \nu\text{ odd}$$
$$=(-1)^{\nu/2}\frac{\Gamma(n+\frac{1}{2}\nu)}{\Gamma(1+\frac{1}{2}\nu)(n)}\,,\quad \text{even }\nu\,, \quad (38)$$

and

$$z_0(s)=1+(1-a)/2a'q_s^2. \quad (39)$$

Formulas (36)-(39) provide a simple algorithm for computing the residues. It may be readily verified that $\beta_0(s)$ and $\beta_1(s)$ are as given in Eq. (10). $\beta_2(s)$ is given by

$$\beta_2(s)=\frac{\sqrt{\pi}}{2}\frac{\alpha(s)(a'q_s^2)^{\alpha(s)}}{\Gamma(\alpha(s)+\frac{1}{2})}\times\left\{1+[2a(s)-1]\left[z_0(s)\left(z_0(s)-\frac{\alpha(s)+1}{2a'q_s^2}\right)+\frac{\alpha(s)+1}{(2a'q_s^2)^2}\left(\frac{\alpha(s)+2}{4}-\frac{1}{3}\right)\right]\right\}. \quad (40)$$

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[11] Volume 2 of Ref. 6.

[12] S. Mandelstam, Ann. Phys. (N. Y.) 19, 254 (1962).

We have noted that (35) is an asymptotic series for $A(s,t)$ in a certain domain. That it is in fact *only* asymptotic, and not a convergent series in any domain, is shown in the Appendix.

IV. SOME REMARKS ON THE VENEZIANO AMPLITUDE SUGGESTED BY THE J-PLANE ANALYSIS

It has been noted[1,2] that the asymptotic behavior of Eq. (1) in the narrow-resonance approximation is not of pure Regge type, owing to an oscillating factor $\cot\pi\alpha(t)$ which enters when the limit is taken parallel to the positive real axis of $\alpha(t)$. Now our J-plane analysis shows that there are only a finite number of Regge poles to the right of any line in the J plane parallel to the line $\mathrm{Re}J=0$ and so the contribution of these terms is of *pure* Regge type for $t\to\infty$. Hence the oscillation can only arise from the background integral and therefore the condition required for the elimination of the background at large $|t|$ cannot be satisfied by the Veneziano amplitude for all directions.

It is known[13] that a unitary amplitude with the usual analyticity properties must exhibit an accumulation point of Regge poles at threshold when J has the value $-\frac{1}{2}$. As we have seen, the Veneziano amplitude is meromorphic in J for all s and so this phenomenon is absent.[14] It is also known that at the point $J=-\frac{1}{2}$, the residues $\beta_n(s)$ of trajectories which do *not* accumulate at threshold should be regular in q_s as $q_s^2\to 0$. However, the opposite is the case with the Veneziano residues associated with all but the leading trajectory. Since the Veneziano model is nonunitary, this state of affairs is not surprising. However, in constructing a *unitary* theory, it is clear that at least the nonleading Veneziano residues would have to be substantially modified.

Another interesting feature of the Veneziano model which is brought into focus by the J-plane analysis is the asymptotic behavior of the residues $\beta_p(s)$ for $s\to\infty$. Recently, Khuri[4] attempted to account for the behavior of the residues in a model with rising trajectories by first assuming that the amplitude could be written as a *convergent* series in the Mandelstam-Regge form. As we show in the Appendix, this requires that $f(s,y)$ in Eq. (26) be an entire function of y. In the Veneziano model, this is true for (almost) *no* values of s. Khuri takes $f(s,y)$ to be of the form

$$f(s,y)=f(s)e^{\eta(s)y}, \tag{41}$$

with arbitrary functions $f(s)$ and $\eta(s)$. He then finds that crossing symmetry, which requires Regge behavior in the direct channel, implies that $\beta_p(s)$ contains a logarithmic dependence on s for $s\to\infty$ but a simple dependence on p. Now it is easy to see from our formulas that $\beta_p(s)$ has no such logarithmic s dependence in the Veneziano model. For example

$$\beta_0(s)\underset{s\to\infty}{\longrightarrow}\mathrm{const}\times e^{-\nu s},\quad \nu>0 \tag{42}$$

which is a behavior recently suggested for rising trajectories.[15]

On the other hand, the dependence of $\beta_p(s)$ on p in the Veneziano model is far more complicated, as is evident from contrasting our equation (36) with Eq. (38) in Khuri's paper.[4] Khuri was unable to obtain a choice of $f(s)$ and $\eta(s)$ which would make his model exhibit exact crossing symmetry at finite s. The lesson to be learned from this is that crossing symmetry and a convergent Mandelstam-Regge series may perhaps be incompatible requirements. Thus, the inclusion of some piece of the background integral, which is well known to be necessary for correct analyticity and threshold behavior, may also be essential for the inclusion of crossing symmetry. We also learn that it is perfectly possible to have Regge behavior in both channels and crossing symmetry with residue functions free of logarithmic singularities.

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APPENDIX

We here show that for almost all values of s the series Eq. (35) is at most asymptotic, i.e., there is no finite region of the z_s plane in which it is convergent.

Consider a series of the form

$$F(z)=\sum_{n=0}^{\infty}\zeta_p Q_{p-\alpha-1}(z). \tag{A1}$$

From the behavior of the $Q_{p+\nu}(z)$ for fixed z and $p\to\infty$ it follows that (A1) converges for those (and only those) z for which

$$G(y)\equiv\sum_{p=0}^{\infty}\zeta_p y^p,\quad y=z-(z^2-1)^{1/2} \tag{A2}$$

is convergent, i.e., in the region of the z plane exterior to an ellipse with foci ±1 and semimajor axis $\cosh\xi_0$ with $e^{-\xi_0}\le r_0$, where r_0 is the radius of convergence of (A2). Suppose that $r_0\neq0$. Then the function

$$\bar{G}(y)\equiv y^{\alpha+1/2}\sum_{p=0}^{\infty}\zeta_p I_{p-\alpha-1/2}(y) \tag{A3}$$

is an entire function of y as may be seen from the be-

[13] V. Gribov and I. Pomeranchuk, Phys. Rev. Letters 9, 238 (1962).

[14] We are grateful to A. Mueller for calling this point to our attention.

[15] C. E. Jones and V. L. Teplitz, Phys. Rev. Letters 19, 135 (1967); H. Goldberg, *ibid.* 19, 1391 (1967).

havior of the Bessel function $I_\nu(y)$ for large ν. From the representation (32), we have

$$F(z)=(\tfrac{1}{2}\pi)^{1/2}\int_0^\infty dy\, e^{-yz}y^{-1/2}\sum_{p=0}^{\infty}\zeta_p I_{p-\alpha-1/2}(y). \quad \text{(A4)}$$

Comparing this with (25b) and (26), (A3) shows that $f(s,y)$ must be an entire function of y. Since $f(s,y)$ in (26) has branch points where

$$y/2a'q_s^2=\pm 2\pi i,\quad \pm 4\pi i,\ \ldots,$$

unless $\alpha(s)$ is an integer, it follows that $A(s,t)$ cannot have a convergent series of the form (A1) except for integer α. The points for which $\alpha(s)=1, 2, \cdots$ are singular points of $A(s,t)$. Hence $A(s,t)$ may only have a convergent series of the form (A1) when

$$\alpha(s)=-1, -2, \ldots. \quad \text{(A5)}$$

(It is interesting to note that these would be the so-called indeterminacy points of a potential theory.) It is trivial to show that $A(s,t)$ does indeed have a convergent series of the form (A1) for the points (A5).

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Chiral $SU(3)\otimes SU(3)$ as a Symmetry of the Strong Interactions

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Starting with the modern developments of current algebra and the hypothesis of partially conserved axial-vector current, it has gradually become apparent that the strong interactions are almost invariant under the group $SU(3)\otimes SU(3)$. In the limit that symmetry breaking is neglected, $SU(3)\otimes SU(3)$ does not appear as a symmetry of the particle states as $SU(3)$ does, but rather as a symmetry realized by eight Goldstone bosons, i.e., the pseudoscalar octet. Most papers on $SU(3)\otimes SU(3)$ symmetry have been concerned with soft-meson theorems and their connection with effective Lagrangians. This paper is devoted to other aspects of the symmetry. Part of the paper is frankly pedagogical. The physics behind a symmetry realized by way of Goldstone bosons is brought out through a study of the σ model. Then the general principles are stated abstractly and applied to the hadrons. One of the new results presented here is that there are two distinct ways in which $SU(3)\otimes SU(3)$ can be realized. In both cases there is an octet of massless pseudoscalar mesons. The two possibilities differ in the residual symmetry of the hadron spectrum: In one case, it is only $SU(3)$; in the other, it is $SU(3)$ times a discrete symmetry, which leads to parity doublets. It is conjectured that some of the observed parity doubling in nucleon resonances is a consequence of this new discrete symmetry. Symmetry breaking is discussed in detail and is found to be very complex. In particular, it is shown that, at least for the pseudoscalar-meson masses, octet enhancement can never occur for first-order perturbations around an $SU(3)\otimes SU(3)$-symmetrical limit. Since octet enhancement is an empirical fact, one is forced to conclude that lowest-order perturbation theory is not a good approximation. In connection with octet enhancement, we show how one can use a principle of pole dominance in the angular momentum plane to replace scalar "tadpole" mesons with Regge trajectories.

I. INTRODUCTION

FOR some time it has been apparent that the strong interactions are approximately $SU(3)$-symmetric. More recently, the joint successes of current algebra and partially conserved axial-vector current (PCAC)[1] have indicated that the strong interactions are nearly symmetrical under the bigger group $SU(3)\otimes SU(3)$. The larger symmetry does not, however, manifest itself in multiplets of particles as does $SU(3)$, but through the appearance of eight nearly zero-mass pseudoscalar mesons, i.e., Goldstone bosons.[2]

Historically, Nambu and his collaborators[3] were the first to suggest that both the small mass of the pion and PCAC might be consequences of an approximate symmetry of the strong interactions. The next major steps came out of Gell-Mann's suggestion[4] that the vector and axial-vector currents of the hadrons generate the algebra of $SU(3)\otimes SU(3)$. The combination of current algebra and PCAC lead to a large number of low-energy theorems[1] for processes involving soft pions and, occasionally, kaons. These low-energy theorems which are only approximate in the real world would become exact in a limit where the pseudoscalar-meson masses vanish and the axial-vector currents are conserved. Thus, the soft-meson theorems may be thought of as consequences of approximate symmetry. This

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[1] See, e.g., S. Adler and R. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).

[2] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[3] Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962), and references therein.

[4] M. Gell-Mann, Physics **1**, 74 (1964).