Current-Algebra Model of $\Delta I=\frac{3}{2}$ Effects in Nonleptonic Kaon Decay

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(Received 20 January 1969; revised manuscript received 24 February 1969)

We construct a two-parameter semiphenomenological model for nonleptonic kaon decay under the following assumptions: (i) All single soft-π limits are satisfied; (ii) aside from pole terms, amplitudes can be expanded as a power series in the pion momenta q; and (iii) in particular, we include in $\langle\pi_q|\mathcal{H}_w|K_k\rangle$ only first- and zeroth-order terms in q. Our $\Delta I=\frac{1}{2}$ results are then identical to the recent work of McNamee. Because of our inclusion of q dependence, nonzero $\Delta I=\frac{3}{2}$ effects are also permitted and are found to be in agreement with the model of Bouchiat and Meyer to order m_π^2/M_K^2. A theoretical estimate of the size of the first-order terms is found to agree with experiment. Uncertainties inherent in our method are discussed.

I. INTRODUCTION

THE techniques of current algebra[1] and partially conserved axial-vector current (PCAC)[2] have been applied to nonleptonic kaon decay by many authors.[3] It is well known that both the $K\to 2\pi$- and $K\to 3\pi$-decay amplitudes satisfy an exact $\Delta I=\frac{1}{2}$ rule in the soft-pion limit.[3] Experimentally, on the other hand, the $\Delta I=\frac{1}{2}$ rule appears to be approximate, with small but definite $\Delta I=\frac{3}{2}$ effects.[4]

Section II generalizes a method of treating the nonleptonic K decays recently discussed by McNamee[5] by expanding to the first order in pion momenta. We show that $\Delta I=\frac{3}{2}$ effects may then be present. In Sec. III, we contrast the results of this model with previous current-algebraic treatments of the $\Delta I=\frac{1}{2}$ amplitude by Nambu and Hara[6] and of the $\Delta I=\frac{3}{2}$ amplitude by Bouchiat and Meyer.[7] Agreement of the model with experimentally observed $\Delta I=\frac{1}{2}$ rule violations is examined in Sec. IV.

Section V employs a modification of the "hard-pion" techniques introduced by Weinberg and Schnitzer[8] in order to justify our method somewhat and also in order to get a specific theoretical prediction for $\Delta I=\frac{3}{2}$ amplitudes, as opposed to the empirical value employed in previous sections. Finally, Sec. VI considers the importance of possible uncertainties on our calculations.

II. GENERALIZATION OF McNAMEE'S MODEL

The theoretical situation regarding the violation of the $\Delta I=\frac{1}{2}$ rule is somewhat unsatisfactory. Strict soft-pion calculations, as remarked in the Introduction, predict that $\Delta I=\frac{3}{2}$ terms vanish.[3] While this is often considered as a great success of current algebra, this result not only disagrees with experimental findings,[4] but also casts doubt upon the validity of a recent current-algebra calculation by Bouchiat and Meyer,[7] who relate $\Delta I=\frac{3}{2}$ effects in $K\to 3\pi$ to those observed in $K\to 2\pi$. One might well wonder whether this is actually a zero-equals-zero result.

In order to gain information concerning possible $\Delta I=\frac{3}{2}$ effects, it is necessary to go a step beyond the soft-pion-limit results. We shall pattern our work after that of McNamee,[5] except that where he approximated all terms other than those contributed by various "pole" diagrams (Figs. 1 and 2) as constants, we shall, in effect, expand such terms in the pion momenta, keeping not only the zeroth-order terms, as he does, but also the first- and some second-order terms. We then evaluate these coefficients by demanding consistency with the various soft-pion limits. In seeing how this program is carried out, we shall recapitulate as well as extend the work of McNamee and put his results into a more convenient form.

We first define our Hamiltonian. Since this work does not treat the problem of CP nonconservation, we shall employ the usual current-current Hamiltonian as proposed by Cabibbo[9]:

$$\mathcal{H}_w(x)=-(G_V/2\sqrt{2})\{\mathcal{J}_\mu(x),\mathcal{J}_\mu^\dagger(x)\}\,,$$

with

$$\mathcal{J}^\mu(x)=\sqrt{2}\cos\theta\,[V_{\pi^-}{}^\mu(x)+A_{\pi^-}{}^\mu(x)]+\sqrt{2}\sin\theta\,[V_{K^-}{}^\mu(x)+A_{K^-}{}^\mu(x)]\,. \quad (1)$$

Our phase convention and normalizations are indicated in Appendix A. We shall need only the $\Delta S=-1$ part of $\mathcal{H}_w$, which we may decompose into $\Delta I=\frac{1}{2}$ and $\Delta I=\frac{3}{2}$

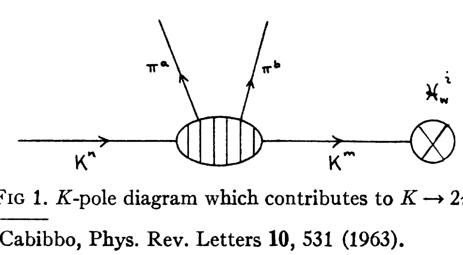

FIG 1. K-pole diagram which contributes to $K\to 2\pi$.

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[6] Y. Nambu and Y. Hara, Phys. Rev. Letters **16**, 87 (1966).
[7] C. Bouchiat and Ph. Meyer, Phys. Letters **25B**, 282 (1967).
[8] S. Weinberg and H. Schnitzer, Phys. Rev. **164**, 1828 (1967).
[9] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

components:

$$\mathcal{H}_w(x)=\mathcal{H}_w^{1/2}(x)+\mathcal{H}_w^{3/2}(x)\,,$$
$$\mathcal{H}_w^{1/2}(x)=(G_V/\sqrt{2})\cos\theta\,\sin\theta\,[\tfrac{2}{3}\mathcal{O}^{(+)}(x)+\tfrac{1}{3}\sqrt{2}\,\mathcal{O}^{(0)}(x)]\,,\quad(2)$$
$$\mathcal{H}_w^{3/2}(x)=(G_V/\sqrt{2})\cos\theta\,\sin\theta\,[\tfrac{1}{3}\mathcal{O}^{(+)}(x)-\tfrac{1}{3}\sqrt{2}\,\mathcal{O}^{(0)}(x)]\,,$$

where

$$\mathcal{O}^{(+)}(x)=\{[V_{\pi^+}{}^{\mu}(x)+A_{\pi^+}{}^{\mu}(x)],\,[V_{K^-}{}^{\mu}(x)+A_{K^-}{}^{\mu}(x)]\}\,,$$
$$\mathcal{O}^{(0)}(x)=\{[V_{\pi^0}{}^{\mu}(x)+A_{\pi^0}{}^{\mu}(x)],\,[V_{\bar{K}^0}{}^{\mu}(x)+A_{\bar{K}^0}{}^{\mu}(x)]\}\,.$$

We now consider the K-vacuum matrix element

$$\langle 0|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\,,$$

where k is the kaon four-momentum, n is an isospin index, and $i=\frac{1}{2}$ or $\frac{3}{2}$, depending upon which component of $\mathcal{H}_w$ we are considering. Note that over-all four-momentum conservation has not been imposed. The weak Hamiltonian is considered as a spurion, which can carry off the appropriate energy and momentum.

Now, if $i=\frac{3}{2}$ such a matrix element vanishes, while if $i=\frac{1}{2}$ we may define[10]

$$\langle 0|\mathcal{H}_w{}^{1/2}(0)|K_k{}^n\rangle=A_{1/2}M_K{}^3\bar{s}_{1/2}K^n\,,\quad(3)$$

where K^n represents a two-component isospinor $\left[K^n=\begin{pmatrix}1\\0\end{pmatrix}\text{ for }K^+\text{ and }K^n=\begin{pmatrix}0\\1\end{pmatrix}\text{ for }K^0\right]$, and $\bar{s}_{1/2}$ is the $\Delta I=\frac{1}{2}$ isospurion $(0\;1)$. $A_{1/2}$ is a constant which could in general be a function of k^2. However, we shall always remain on the kaon mass shell and shall therefore not include such dependence.[11]

When we now consider the K-π matrix, element which McNamee treated as zeroth order in the pion four-momentum, we also include first-order terms:

$$\langle\pi_{q_a}{}^a|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle={}^iA_n{}^a+{}^iB_n{}^ak\cdot q_a\,,\quad(4)$$

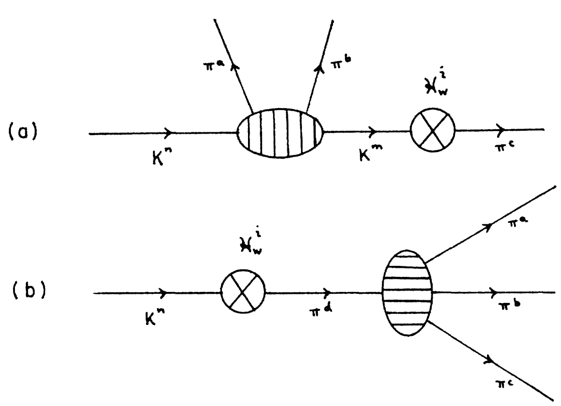

FIG. 2. (a) K-pole diagram which contributes to $K\to 3\pi$. (b) π-pole diagram which contributes to $K\to 3\pi$.

[10] We employ "isospurion" notation, as discussed by J. S. Bell [CERN Report No. 66-29, 1966, Vol. I (unpublished)].

[11] Dependence on k^2 produces nothing new if the parameter λ (see Appendix B) is set equal to zero, as we do in the text. Such dependence *can* become important in the case $\lambda=1$ and will be discussed in a separate paper.

where ${}^iA_n{}^a$, ${}^iB_n{}^a$ are assumed to be constant. In Appendix B, we go one step further and also include in this expansion terms in $q_a{}^2$. However, we shall show in Sec. V that such terms in $q_a{}^2$ vanish in a particular model, and for simplicity we neglect them here.

We require agreement of this form with the soft-pion expression. When we take the limit $q_a\to 0$ and employ PCAC in the form[12]

$$\varphi_a(x)=(1/iF_\pi m_\pi{}^2)\partial_\mu A_a{}^\mu(x)\,,$$

we find

$$\langle\pi_{q_a}{}^a|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle=\frac{m_\pi{}^2-q_a{}^2}{F_\pi m_\pi{}^2}\int d^4x\,e^{iq_a\cdot x}$$
$$\times\langle 0|T(\partial_\mu A_a{}^\mu(x)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle\xrightarrow[q_a\to 0]{}$$
$$-(1/F_\pi)\langle 0|[F_a{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle\,,\quad(5)$$

where

$$F_a{}^5(t)=\int d^3x\,A_a{}^0(\mathbf{x},t)\,.$$

As is well known, for a $V+A$ Hamiltonian as in Eq. (1), we have[3]

$$[F_a{}^5(0),\mathcal{H}_w{}^i(0)]=[F_a(0),\mathcal{H}_w{}^i(0)]\,,$$

where

$$F_a(t)=\int d^3x\,V_a{}^0(\mathbf{x},t)$$

and is just the ath component of the isospin operator. Then, having replaced $F_a{}^5$ by F_a in the matrix element of this commutator between the kaon and the vacuum, instead of carrying out the commutation we allow F_a to operate to the left and right, respectively:

$$\langle 0|[F_a(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle$$
$$=(\langle 0|F_a(0))\mathcal{H}_w{}^i(0)|K_k{}^n\rangle-\langle 0|\mathcal{H}_w{}^i(0)(F_a(0)|K_k{}^n\rangle)$$
$$=0-\tfrac{1}{2}\tau^a{}_{mn}\langle 0|\mathcal{H}_w{}^i(0)|K_k{}^m\rangle$$
$$=-\delta_{i\frac{1}{2}}\tfrac{1}{2}A_{1/2}M_K{}^3\bar{s}_{1/2}\tau^aK^n\,.$$

Thus, the soft-pion limit determines ${}^iA_n{}^a$ but leaves the $k\cdot q_a$ coefficients arbitrary. We find that

$${}^iA_n{}^a=\delta_{i\frac{1}{2}}(A_{1/2}M_K{}^3/2F_\pi)\bar{s}_{1/2}\tau^aK^n\quad(6a)$$

and define two new constants $B_{1/2}$, $B_{3/2}$ by

$${}^{1/2}B_n{}^a=(B_{1/2}M_K/2F_\pi)\bar{s}_{1/2}\tau^aK^n\,,$$
$${}^{3/2}B_n{}^a=(B_{3/2}M_K/2F_\pi)\bar{s}_{3/2}{}^aK^n\,,\quad(6b)$$

where, for the $\Delta I=\frac{3}{2}$ term, we are using a notation introduced by Bouchiat and Meyer,[7] in which $\bar{s}_{3/2}{}^i$ represents a $\Delta I=\frac{3}{2}$ isospurion with vector index i and obeys the subsidiary condition $\mathbf{s}_{3/2}\cdot\boldsymbol{\tau}=0$. We have

$$\bar{s}_{3/2}{}^1=(-\tfrac{1}{2}\;0)\,,\quad\bar{s}_{3/2}{}^2=(-\tfrac{1}{2}i\;0)\,,\quad\bar{s}_{3/2}{}^3=(0\;-1)\,.$$

[12] We use the definition $A_a{}^\mu(x)=\bar{\psi}(x)\frac{1}{2}\lambda_a\gamma^\mu\gamma^5\psi(x)$ so that our Goldberger-Treiman relation is $F_\pi=-iMg_A/g_r$.

We now consider the $K\to 2\pi$ matrix element, for which we define[13]

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle={}^iA_n{}^{ab}+{}^iB_n{}^{ab}k\cdot q_a+{}^iB_n{}^{ba}k\cdot q_b+(K\text{-pole term})^i \quad (7)$$

with, according to Bose statistics, ${}^iA_n{}^{ab}={}^iA_n{}^{ba}$, and where $(K\text{-pole term})^i$ represents the amplitude for K-π scattering, followed by the kaon propagating and disappearing through $\mathcal{H}_w{}^i$ into the vacuum, as in Fig. 1. Following McNamee, we use Weinberg's expression for K-π scattering[14]:

$$\langle\pi_{q_b}{}^bK_{k'}{}^m|T|\pi_{q_a}{}^aK_k{}^n\rangle=(2\pi)^4\delta^4(q_b+k'-q_a-k)\times(-i)(\epsilon^{abc}/4F_\pi{}^2)\tau^c{}_{mn}(q_b+q_a)\cdot(k'+k)\,, \quad (8)$$

where $S=1-iT$. Now since the kaon cannot go into the vacuum via $\mathcal{H}_w{}^{3/2}$, $(K\text{-pole term})^{3/2}=0$, while for $i=\frac{1}{2}$,

$$(K\text{-pole term})^{1/2}=i\epsilon^{abc}\bar{s}_{1/2}\tau^cK^n\frac{A_{1/2}M_K{}^3}{4F_\pi{}^2}\times\frac{2k\cdot(q_b-q_a)-q_b{}^2+q_a{}^2}{2k\cdot(q_b+q_a)-(q_b+q_a)^2}.$$

We demand agreement with the soft-pion limit

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\xrightarrow[q_b\to 0]{}-(1/F_\pi)\langle\pi_{q_a}{}^a|[F_b{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle\,,$$

where π^a is to be kept on its mass shell. Again we replace $F_b{}^5$ by F_b and allow it to operate to the left and right, respectively:

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\xrightarrow[q_b\to 0]{}-i(\epsilon^{abc}/F_\pi)\langle\pi_{q_a}{}^c|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle+(1/2F_\pi)\tau^b{}_{mn}\langle\pi_{q_a}{}^a|\mathcal{H}_w{}^i(0)|K_k{}^m\rangle\,.$$

We may use our previous results for the K-π matrix element, whereupon[15]

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^{1/2}(0)|K_k{}^n\rangle\xrightarrow[q_b\to 0]{}(A_{1/2}M_K{}^3/4F_\pi{}^2)\bar{s}_{1/2}\tau^b\tau^aK^n+(B_{1/2}M_K/4F_\pi{}^2)\times\bar{s}_{1/2}\tau^b\tau^aK^nk\cdot q_a\,,$$

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^{3/2}(0)|K_k{}^n\rangle\xrightarrow[q_b\to 0]{}(B_{3/2}M_K/4F_\pi{}^2)\times[\bar{s}_{3/2}{}^a\tau^bK^n-2i\epsilon^{abc}\bar{s}_{3/2}{}^cK^n]k\cdot q_a\,.$$

Comparison with our expansion (6) yields

$$\begin{aligned}{}^{1/2}A_n{}^{ab}&=\frac{A_{1/2}M_K{}^3}{4F_\pi{}^2}\delta^{ab}\bar{s}_{1/2}K^n\,,\\ {}^{1/2}B_n{}^{ab}&=\frac{B_{1/2}M_K}{4F_\pi{}^2}\bar{s}_{1/2}\tau^b\tau^aK^n\,,\\ {}^{3/2}A_n{}^{ab}&=0\,,\\ {}^{3/2}B_n{}^{ab}&=\frac{B_{3/2}M_K}{4F_\pi{}^2}(\bar{s}_{3/2}{}^a\tau^bK^n-2i\epsilon^{abc}\bar{s}_{3/2}{}^cK^n)\,. \end{aligned} \quad (9)$$

The treatment of the $K\to 3\pi$ matrix element is similar[16]:

$$\begin{aligned}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b\pi_{q_c}{}^c|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle &={}^iA_n{}^{abc}+{}^iB_n{}^{abc}k\cdot q_a+{}^iB_n{}^{bca}k\cdot q_b+{}^iB_n{}^{cab}k\cdot q_c\\ &+{}^iD_n{}^{abc}q_b\cdot q_c+{}^iD_n{}^{bca}q_c\cdot q_a+{}^iD_n{}^{cab}q_a\cdot q_b\\ &+(K\text{-pole term})^i+(\Pi\text{-pole term})^i\,,\end{aligned} \quad (10)$$

where, according to Bose statistics,

$${}^iA_n{}^{abc}={}^iA_n{}^{acb}={}^iA_n{}^{bca}={}^iA_n{}^{bac}={}^iA_n{}^{cab}={}^iA_n{}^{cba}\,,$$
$${}^iB_n{}^{abc}={}^iB_n{}^{acb}\,,\quad\text{and}\quad {}^iD_n{}^{abc}={}^iD_n{}^{acb}\,,$$

and the K- and Π-pole diagrams are the contributions from the diagrams of Figs. 2(a) and 2(b), as emphasized by McNamee.

We employ Weinberg's result for the π-π scattering amplitude[14]:

$$\langle\pi_{q_c}{}^c\pi_{q_d}{}^d|T|\pi_{q_a}{}^a\pi_{q_b}{}^b\rangle=(2\pi)^4\delta^4(q_c+q_d-q_a-q_b)F_\pi{}^{-2}\{\delta^{ab}\delta^{cd}[(q_a+q_b)^2-m_\pi{}^2]+\delta^{ac}\delta^{bd}[(q_a-q_c)^2-m_\pi{}^2]+\delta^{ad}\delta^{bc}[(q_a-q_d)^2-m_\pi{}^2]\}\,. \quad (11)$$

Then

$$(K\text{-pole term})^{1/2}=\tfrac{1}{8}F_\pi{}^{-3}[A_{1/2}M_K{}^3+B_{1/2}M_K(k-q_a-q_b)\cdot q_c]i\epsilon^{abd}\bar{s}_{1/2}\tau^c\tau^dK^n\frac{2k\cdot(q_b-q_a)-q_b{}^2+q_a{}^2}{2k\cdot(q_b+q_a)-(q_b+q_a)^2}+\text{Perm.}\,, \quad (12a)$$

where Perm. indicates terms obtained by the cyclic permutation of the indices (a, b, and c), and

$$(K\text{-pole term})^{3/2}=\tfrac{1}{8}F_\pi{}^{-3}B_{3/2}M_K(k-q_a-q_b)\cdot q_ci\epsilon^{abd}\bar{s}_{3/2}{}^c\tau^dK^n\frac{2k\cdot(q_b-q_a)-q_b{}^2+q_a{}^2}{2k\cdot(q_b+q_a)-(q_b+q_a)^2}+\text{Perm.}\,, \quad (12b)$$

[13] We have omitted possible terms in $q_a{}^2$, $q_b{}^2$, and $q_a\cdot q_b$. In the more general discussion in Appendix B, these terms are included. However, we show in Sec. V that the $q_a{}^2$ and $q_b{}^2$ terms vanish in a particular model, and in Appendix C it is shown that $q_a\cdot q_b$ terms vanish in a certain model.

[14] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[15] We have used the relation $\tau^b\tau^a=\tau^a\tau^b-2i\epsilon^{abc}\tau^c$.

[16] Terms in $q_a\cdot q_b$ are required in order to agree with the soft-pion limits in this case. A more general form, including terms in q^2, is discussed in Appendix B.

$$(\Pi\text{-pole term})^{1/2}=\tfrac{1}{2}F_\pi^{-3}[A_{1/2}M_K^3+B_{1/2}M_K k\cdot(q_a+q_b+q_c)][(q_a+q_b+q_c)^2-m_\pi^2]^{-1}\{\delta^{ab}\bar{s}_{1/2}\tau^c K^n[(q_a+q_b)^2-m_\pi^2]+\delta^{bc}\bar{s}_{1/2}\tau^a K^n[(q_b+q_c)^2-m_\pi^2]+\delta^{ca}\bar{s}_{1/2}\tau^b K^n[(q_c+q_a)^2-m_\pi^2]\}, \quad (12c)$$

$$(\Pi\text{-pole term})^{3/2}=\tfrac{1}{2}F_\pi^{-3}B_{3/2}M_K k\cdot(q_a+q_b+q_c)[(q_a+q_b+q_c)^2-m_\pi^2]^{-1}\{\delta^{ab}\bar{s}_{3/2}{}^c K^n[(q_a+q_b)^2-m_\pi^2]+\delta^{bc}\bar{s}_{3/2}{}^a K^n[(q_b+q_c)^2-m_\pi^2]+\delta^{ca}\bar{s}_{3/2}{}^b K^n[(q_c+q_a)^2-m_\pi^2]\}. \quad (12d)$$

Demanding agreement with the soft-pion limit,

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b\pi_{q_c}{}^c|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\xrightarrow[q_c\to 0]{} -F_\pi^{-1}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|[F_c{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle,$$

with π^a and π^b on their mass shells, then yields

$$\begin{aligned}
{}^{1/2}A_n{}^{abc}&=-(A_{1/2}M_K{}^3/8F_\pi{}^3)[\delta^{ab}\bar{s}_{1/2}\tau^c K^n+\delta^{bc}\bar{s}_{1/2}\tau^a K^n+\delta^{ca}\bar{s}_{1/2}\tau^b K^n],\\
{}^{1/2}B_n{}^{abc}&=(B_{1/2}M_K/8F_\pi{}^3)[\delta^{bc}\bar{s}_{1/2}\tau^a K^n-2\delta^{ac}\bar{s}_{1/2}\tau^b K^n-2\delta^{ab}\bar{s}_{1/2}\tau^c K^n],\\
{}^{1/2}D_n{}^{abc}&=(B_{1/2}M_K/8F_\pi{}^3)[-2\delta^b{}_d\bar{s}_{1/2}\tau^a K^n+\delta^{ac}\bar{s}_{1/2}\tau^b K^n+\delta^{ab}\bar{s}_{1/2}\tau^c K^n],\\
{}^{3/2}A_n{}^{abc}&=0,\\
{}^{3/2}B_n{}^{abc}&=(B_{3/2}M_K/8F_\pi{}^3)[5\delta^{bc}\bar{s}_{3/2}{}^a K^n-4\delta^{ac}\bar{s}_{3/2}{}^b K^n-4\delta^{ab}\bar{s}_{3/2}{}^c K^n+2i\epsilon^{cad}\bar{s}_{3/2}{}^d\tau^b K^n+2i\epsilon^{bad}\bar{s}_{3/2}{}^d\tau^c K^n],\\
{}^{3/2}D_n{}^{abc}&=(B_{3/2}M_K/8F_\pi{}^3)[i\epsilon^{acd}\bar{s}_{3/2}{}^b\tau^d K^n+i\epsilon^{abd}\bar{s}_{3/2}{}^c\tau^d K^n].
\end{aligned} \quad (13)$$

If we set $B_{1/2}=B_{3/2}=0$, we recover the results of McNamee.

III. COMPARISON WITH PREVIOUS WORK

In order to contrast our model with previous work or with experimental results, we need to know the form of the "physical" decay amplitudes (i.e., all particles on their mass shell and over-all four-momentum conservation imposed).[17] We find

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^{1/2}(0)|K_k{}^n\rangle_{\text{phy}}=\delta^{ab}\bar{s}_{1/2}K^n\frac{(A_{1/2}+B_{1/2})M_K{}^3}{4F_\pi{}^2}, \quad (14a)$$

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^{3/2}(0)|K_k{}^n\rangle_{\text{phy}}=[\bar{s}_{3/2}{}^a\tau^b K^n+\bar{s}_{3/2}{}^b\tau^a K^n]\frac{B_{3/2}M_K{}^3}{8F_\pi{}^2}, \quad (14b)$$

$$\begin{aligned}
&\langle\pi_{q_a}{}^a\pi_{q_b}{}^b\pi_{q_c}{}^c|\mathcal{H}_w{}^{1/2}(0)|K_k{}^n\rangle_{\text{phy}}\\
&=-\frac{(A_{1/2}+B_{1/2})M_K{}^3}{8F_\pi{}^3}[\delta^{ab}\bar{s}_{1/2}\tau^c K^n+\delta^{ac}\bar{s}_{1/2}\tau^b K^n+\delta^{bc}\bar{s}_{1/2}\tau^a K^n]+\frac{(A_{1/2}+B_{1/2})M_K{}^3}{2F_\pi{}^3}\frac{1}{M_K{}^2-m_\pi{}^2}\\
&\quad\times[\delta^{ab}\bar{s}_{1/2}\tau^c K^n(s^c-m_\pi{}^2)+\delta^{ac}\bar{s}_{1/2}\tau^b K^n(s^b-m_\pi{}^2)+\delta^{bc}\bar{s}_{1/2}\tau^a K^n(s^a-m_\pi{}^2)]\\
&\quad-\tfrac{3}{8}(B_{1/2}M_K/F_\pi{}^3)[\delta^{ab}\bar{s}_{1/2}\tau^c K^n(s^c-s_0)+\delta^{ac}\bar{s}_{1/2}\tau^b(s^b-s_0)+\delta^{bc}\bar{s}_{1/2}\tau^a K^n(s^a-s_0)]\\
&\quad+\frac{(A_{1/2}M_K{}^3+B_{1/2}M_K m_\pi{}^2)}{8F_\pi{}^3}\frac{1}{M_K{}^2-m_\pi{}^2}[i\epsilon^{abd}\bar{s}_{1/2}\tau^c\tau^d K^n(s^a-s^b)+i\epsilon^{bcd}\bar{s}_{1/2}\tau^a\tau^d K^n(s^b-s^c)\\
&\quad+i\epsilon^{cad}\bar{s}_{1/2}\tau^b\tau^d K^n(s^c-s^a)],
\end{aligned} \quad (14c)$$

$$\begin{aligned}
&\langle\pi_{q_a}{}^a\pi_{q_b}{}^b\pi_{q_c}{}^c|\mathcal{H}_w{}^{3/2}(0)|K_k{}^n\rangle_{\text{phy}}\\
&=(B_{3/2}M_K{}^3/16F_\pi{}^3)[\delta^{ab}\bar{s}_{3/2}{}^c K^n+\delta^{ac}\bar{s}_{3/2}{}^b K^n+\delta^{bc}\bar{s}_{3/2}{}^a K^n]+(B_{3/2}M_K/8F_\pi{}^3)[i\epsilon^{abd}\bar{s}_{3/2}{}^d\tau^c K^n(s^a-s^b)\\
&\quad+i\epsilon^{bcd}\bar{s}_{3/2}{}^d\tau^a K^n(s^b-s^c)+i\epsilon^{cad}\bar{s}_{3/2}{}^d\tau^b K^n(s^c-s^a)]+\frac{B_{3/2}M_K}{16F_\pi{}^3}\left(\frac{-M_K{}^2+9m_\pi{}^2}{M_K{}^2-m_\pi{}^2}\right)[\delta^{ab}\bar{s}_{3/2}{}^c K^n(s^c-m_\pi{}^2)\\
&\quad+\delta^{ac}\bar{s}_{3/2}{}^b K^n(s^b-m_\pi{}^2)+\delta^{bc}\bar{s}_{3/2}{}^a K^n(s^a-m_\pi{}^2)]+\frac{B_{3/2}M_K}{16F_\pi{}^3}\left(\frac{M_K{}^2+m_\pi{}^2}{M_K{}^2-m_\pi{}^2}\right)[i\epsilon^{abd}\bar{s}_{3/2}{}^c\tau^d K^n(s^a-s^b)\\
&\quad+i\epsilon^{bcd}\bar{s}_{3/2}{}^a K^n(s^b-s^c)+i\epsilon^{cad}\bar{s}_{3/2}{}^b K^n(s^c-s^a)],
\end{aligned} \quad (14d)$$

where s_i is the Lorentz-invariant variable $(k-q_i)^2$.

[17] We neglect electromagnetic mass splittings in the pion and kaon multiplets in calculating decay amplitudes.

We may now compare the predictions of this model with those of Nambu and Hara[6] and Bouchiat and Meyer.[7] We discussed their work in a previous communication,[18] in which we parametrized the $K\to 2\pi$ amplitude as

$$\begin{aligned}
&\mathrm{Amp}(K^+\to\pi^+\pi^0)=(\sqrt{\tfrac{3}{10}})f_3\,,\\
&\mathrm{Amp}(K^0\to\pi^+\pi^-)=(\sqrt{\tfrac{1}{3}})f_1+[\sqrt{(1/15)}]f_3\,,\\
&\mathrm{Amp}(K^0\to\pi^0\pi^0)=-(\sqrt{\tfrac{1}{3}})f_1+[2\sqrt{(1/15)}]f_3\,,
\end{aligned}\tag{15}$$

and the $K\to 3\pi$ amplitude as

$$\begin{aligned}
&\mathrm{Amp}(K^+\to\pi^+\pi^0\pi^0)\\
&\quad=-\tfrac{1}{3}\sqrt{2}[(\alpha_1-(\sqrt{\tfrac{1}{2}})\alpha_3)+(\beta_1-(\sqrt{\tfrac{1}{2}})\beta_3)E_+]\\
&\qquad-\tfrac{3}{2}(\sqrt{\tfrac{1}{5}})\gamma_3(E_+-\tfrac{1}{3}M_K)\,,\\
&\mathrm{Amp}(K^+\to\pi^+\pi^+\pi^-)\\
&\quad=\tfrac{1}{3}\sqrt{2}[(2\alpha_1-\sqrt{2}\alpha_3)+(\beta_1-(\sqrt{\tfrac{1}{2}})\beta_3)(M_K-E_-)]\\
&\qquad+\tfrac{3}{2}(\sqrt{\tfrac{1}{5}})\gamma_3(E_--\tfrac{1}{3}M_K)\,,
\end{aligned}\tag{16}$$

$$\begin{aligned}
&\mathrm{Amp}(K^0\to\pi^+\pi^-\pi^0)\\
&\quad=\tfrac{1}{3}[(\alpha_1+\sqrt{2}\alpha_3)+(\beta_1+\sqrt{2}\beta_3)E_0]\\
&\qquad+(\sqrt{\tfrac{1}{10}})\gamma_3(E_--E_+)\,,\\
&\mathrm{Amp}(K^0\to\pi^0\pi^0\pi^0)=-\tfrac{1}{3}[(\alpha_1+\sqrt{2}\alpha_3)+(\beta_1+\sqrt{2}\beta_3)M_K]\,.
\end{aligned}$$

In this formalism, the model of Nambu and Hara yields

$$\alpha_1=-\tfrac{1}{2}\sqrt{3}f_1/F_\pi\,,\quad M_K\beta_1=\sqrt{3}f_1/F_\pi\,,\tag{17a}$$

while that of Bouchiat and Meyer yields

$$\begin{aligned}
&\alpha_3=-\tfrac{1}{2}(\sqrt{\tfrac{3}{10}})f_3/F_\pi\,,\quad M_K\beta_3=\tfrac{5}{2}(\sqrt{\tfrac{3}{10}})f_3/F_\pi\,,\\
&M_K\gamma_3=\tfrac{3}{2}(\sqrt{6})f_3/F_\pi\,.
\end{aligned}\tag{17b}$$

Comparison of our model with this parametrization yields for $K\to 2\pi$

$$\begin{aligned}
f_1&=-\tfrac{1}{4}\sqrt{3}(M_K{}^3/F_\pi{}^2)(A_{1/2}+B_{1/2})\\
&=-\tfrac{1}{4}\sqrt{3}(A_{1/2}M_K{}^3/F_\pi{}^2)(1+2x)\,,\\
f_3&=\tfrac{1}{8}(\sqrt{15})(M_K{}^3/F_\pi{}^2)B_{3/2}\\
&=\tfrac{1}{4}(\sqrt{15})(A_{1/2}M_K{}^3/F_\pi{}^2)y\,,
\end{aligned}\tag{18}$$

with $x=\tfrac{1}{2}B_{1/2}/A_{1/2}$ and $y=\tfrac{1}{2}B_{3/2}/A_{1/2}$. For the $K\to 3\pi$-decay amplitude, we have

$$\begin{aligned}
&\alpha_1=-\tfrac{1}{2}\sqrt{3}\frac{f_1}{F_\pi}\frac{1+\eta}{1-\eta}\,,\quad M_K\beta_1=\sqrt{3}\frac{f_1}{F_\pi}\frac{1}{1-\eta}\,,\\
&\alpha_3=-\left(\sqrt{\frac{3}{10}}\right)\frac{1}{2}\frac{f_3}{F_\pi}\frac{1-11\eta}{1-\eta}\,,\quad M_K\beta_3=\left(\sqrt{\frac{3}{10}}\right)\frac{5}{2}\frac{f_3}{F_\pi}\frac{1-(27/5)\eta}{1-\eta}\,,\quad M_K\gamma_3=(\sqrt{6})\frac{3}{2}\frac{f_3}{F_\pi}\frac{1-\frac{1}{3}\eta}{1-\eta}\,,
\end{aligned}\tag{19}$$

where $\eta=m_\pi{}^2/M_K{}^2$.

The $\Delta I=\tfrac{1}{2}$ results are just those of McNamee, and both the $\Delta I=\tfrac{1}{2}$ and $\Delta I=\tfrac{3}{2}$ predictions are the same as those of Nambu and Hara and of Bouchiat and Meyer [Eq. (17)] up to terms of order η.[19] Of course, terms of this order can have a significant effect, as in the case of α_3, which is much smaller in our model than in that of Bouchiat and Meyer. Nonetheless, agreement to order η confirms the work of Nambu and Hara and of Bouchiat and Meyer. This is especially reassuring for the latters' calculation, which we questioned in Sec. II.

IV. COMPARISON WITH EXPERIMENT

We now compare predictions of our model with experimental findings. For $K\to 2\pi$ decay, the existence of the mode $K^+\to\pi^+\pi^0$ provides evidence for violation of the $\Delta I=\tfrac{1}{2}$ rule[20] and provides a measure[21] of $|f_3/f_1|$:

$$(\sqrt{\tfrac{1}{5}})|f_3/f_1|=0.032\pm0.001\,.\tag{20a}$$

The deviation of $\Gamma(K_s\to\pi^+\pi^-)/\Gamma(K_s\to\pi^0\pi^0)$ from unity yields[22] the sign of f_3/f_1:

$$\begin{aligned}
(\sqrt{\tfrac{1}{5}})\ \mathrm{Re}(f_3/f_1)&=(\sqrt{\tfrac{1}{5}})|f_3/f_1|\cos(\delta_2-\delta_0)\\
&=+0.023\pm0.005\,.
\end{aligned}\tag{20b}$$

Comparison of these two measurements yields information about the strong-interaction phase shifts. In terms of our parameters, we find that such measurements provide use with an empirical measure of the parameter $y/(1+2x)$.

$$(\sqrt{\tfrac{1}{5}})f_3/f_1=-y'\,,\quad\text{where}\quad y'=y/(1+2x)\,.$$

In our discussion of $K\to 3\pi$, there arises the very difficult problem of the strong-interaction phase shifts for the 3π system. We shall, in our discussion, generally neglect such effects. If we had chosen not to, we should have included three separate average strong-interaction phase shifts—one (δ_s) associated with the totally sym-

[18] B. R. Holstein, Phys. Rev. **177**, 2417 (1969).

[19] One might wonder why these results should differ at all, since in both cases one has an expansion of the $K\to 3\pi$ decay amplitude which agrees with the soft-pion limit. The difference is that while Nambu-Hara parametrized the τ' amplitude as

$$\mathrm{Amp}^{1/2}(K^+\to\pi^+\pi^0\pi^0)=-\tfrac{1}{3}\sqrt{2}(\alpha_1+\beta_1E_+),$$

and then demanded agreement with the soft-pion limit, one can reproduce McNamee's results by parametrizing the τ' amplitude as

$$\begin{aligned}
&\mathrm{Amp}^{1/2}(K^+\to\pi^+\pi^0\pi^0)\\
&\quad=-\tfrac{1}{3}\sqrt{2}\left[\alpha_1\left(\frac{s_++s_{01}+s_{02}-2m_\pi{}^2}{M_K{}^2+m_\pi{}^2}\right)+\beta_1\left(\frac{M_K{}^2+m_\pi{}^2-s_+}{2M_K}\right)\right],
\end{aligned}$$

which agrees with Nambu and Hara for the physical-decay amplitude but differs in the soft-pion limit by terms $\sim m_\pi{}^2/M_K{}^2$.

[20] Although some authors still tend to think that such a decay is electromagnetic in origin, with the $\Delta I=\tfrac{1}{2}$ mode suppressed by $SU(3)$ considerations, we shall consider such an amplitude to arise due to $\mathcal{H}_w{}^{3/2}$. As emphasized later on, however, such electromagnetic effects are most likely not unimportant.

[21] G. Trilling, Argonne National Laboratory Report No. ANL-7130, 1965 (unpublished).

[22] B. Gobbi *et al.*, Phys. Rev. Letters **22**, 682 (1969).

metric $I=1$ 3π state,[23] another (δ_m) with the $I=1$ state of mixed symmetry, and a third (δ_2) with the $I=2$ state of mixed symmetry.[24] Our rate predictions are independent of these phase shifts; but the theoretical slopes should include some linear combination of $\cos(\delta_m-\delta_s)$ and $\cos(\delta_2-\delta_s)$.[25] Since we expect these phase shifts to be relatively small because of the low Q value involved in $K\rightarrow 3\pi$ decays, we shall not include them. It then seems consistent to use for y' the values

$$y'=-0.032\pm0.001 \quad \text{or} \quad y'\simeq-1/30 \qquad (21)$$

obtained from Eq. (20a).[26]

One notes that in our isospin analysis of $K\rightarrow 2\pi$, the three amplitudes are described in terms of just two parameters—f_1 and f_3. We can then find a well-known relation between these amplitudes:

$$\text{Amp}(K^0\rightarrow\pi^+\pi^-)+\text{Amp}(K^0\rightarrow\pi^0\pi^0) =\sqrt{2}\,\text{Amp}(K^+\rightarrow\pi^+\pi^0)\,,$$

which provides a test for possible $\Delta I=\frac{5}{2}$ terms. Such terms might be expected if the violation of the $\Delta I=\frac{1}{2}$ rule is electromagnetic in origin. Experimentally, however, this relation is satisfied within errors.[27]

Moving on to consider the $K\rightarrow 3\pi$ system, here also we have a simple test for the possible presence of $\Delta I>\frac{3}{2}$ terms by measurement of[28]

$$\Gamma_{++-}/4\Gamma_{+00}=3\Gamma_{+-0}/2\Gamma_{000}=1 \quad \text{if no } \Delta I>\tfrac{3}{2}\,. \quad (22)$$

Our model has only $\Delta I=\frac{1}{2}$ and $\frac{3}{2}$ and thus predicts both ratios to be unity. Experimentally, we find[4]

$$\Gamma_{++-}/4\Gamma_{+00}=1.00\pm0.03\,,\quad 3\Gamma_{+-0}/2\Gamma_{000}=0.97\pm0.10\,. \quad (23)$$

Thus, there is no evidence for $\Delta I=\frac{5}{2}$ and $\frac{7}{2}$. On the other hand, a test for $\Delta I=\frac{3}{2}$ is provided by the relations

$$\frac{1}{2}\frac{\Gamma_{+-0}}{\Gamma_{+00}}=\frac{\Gamma_{+-0}+\Gamma_{000}}{\Gamma_{+00}+\Gamma_{++-}}=1 \text{ if no } \Delta I>\tfrac{1}{2}\,, \qquad (24)$$

whereas our model predicts

$$\frac{1}{2}\frac{\Gamma_{+-0}}{\Gamma_{+00}}=\frac{\Gamma_{+-0}+\Gamma_{000}}{\Gamma_{+00}+\Gamma_{++-}}=\left(\frac{1+2y'}{1-y'}\right)^2, \qquad (25)$$

which is identical with the Bouchiat-Meyer prediction[29] and in agreement with the experimental numbers[4,30,31] given in Table I. Such agreement as well as other experimental comparisons which we shall discuss must be considered as interesting but hardly compelling, since we have omitted any discussion of the difficult problem of electromagnetic corrections.

TABLE I. Summary of predicted versus experimental $\Delta I=\frac{3}{2}$ effects ($y'=-1/30$).

Quantity	Experimental value	Theoretical value This model	Theoretical value NHBM model
$\frac{1}{2}\Gamma_{+-0}/\Gamma_{+00}-1$	-0.184 ± 0.034[a]	-0.185	-0.185
$(\Gamma_{+-0}+\Gamma_{000})/(\Gamma_{++-}+\Gamma_{+00})-1$	-0.175 ± 0.031[b]	-0.185	-0.185
$-\frac{1}{2}\lambda_{+00}/\lambda_{++-}-1$	$0.34\ \pm0.20$[c]	0.44	0.45
$\lambda_{+-0}/\lambda_{+00}-1$	$-0.15\ \pm0.09$[c]	-0.05	0

[a] See Ref. 4.
[b] See Ref. 30.
[c] See Ref. 31.

$K\rightarrow 3\pi$ amplitudes are often written phenomenologically as

$$|\text{Amp}(K\rightarrow 3\pi)|^2=|A|^2\left(1-2\lambda\frac{s_i-s_0}{m_\pi^2}\right),$$

with

$$s_0=\tfrac{1}{3}\sum_i s_i=\tfrac{1}{3}M_K^2+m_\pi^2\,,$$

where i represents the odd pion (π^+ in τ' decay, π^- in τ decay, π^0 in $K_L\rightarrow\pi^+\pi^-\pi^0$). The parameter λ is the slope; its measurement provides another test for the presence of $\Delta I=\frac{3}{2}$ terms. We find

$$-\tfrac{1}{2}\lambda_{+00}/\lambda_{++-}=\lambda_{+-0}/\lambda_{+00}=1 \text{ if no } \Delta I>\tfrac{1}{2}\,, \quad (26)$$

while our model predicts

$$\lambda_{+-0}/\lambda_{+00}\cong1+18y'\eta\,,\quad \text{(B-M predict 1)}$$
$$-\tfrac{1}{2}\lambda_{+00}/\lambda_{++-}\cong1-(27/2)(1-\tfrac{1}{3}\eta)y'\,, \qquad (27)$$
$$[\text{B-M predict } 1-(27/2)y'].$$

The Bouchiat-Meyer predictions are given in parentheses. Table I shows both our model and that of Bouchiat and Meyer to be in fair agreement with experi-

[23] By symmetry, we mean symmetry under interchange of isospin and momentum indices of two of the pions. See L. Wolfenstein, Erice Summer School Proceedings, 1968 (unpublished).

[24] In Ref. 18, we employed the suggestion of B. Barrett and T. Truong [Phys. Rev. Letters 17, 880 (1966)], that only δ_s is sizable. However, we now feel that this is unrealistic, and that indeed δ_m or δ_2 could be of order of, or larger than, δ_s. See, e.g., B. Ya. Zeldo'vich, Yadern. Fiz. 6, 840 (1967) [English transl.: Soviet J. Nucl. Phys. 6, 611 (1968)].

[25] It is possible to find combinations of slopes which are independent of either $\cos(\delta_2-\delta_s)$ or $\cos(\delta_m-\delta_s)$. For example,

$$\lambda_{+00}+2\lambda_{++-}\propto\cos(\delta_2-\delta_s),$$
$$\lambda_{+00}-2\lambda_{++-}\propto\cos(\delta_m-\delta_s),$$
$$\lambda_{+-0}\propto\cos(\delta_m-\delta_s).$$

Thus, our model's prediction of $\lambda_{+-0}/(\lambda_{+00}-2\lambda_{++-})$ should be independent of the effects of final-state phase shifts.

[26] Bouchiat and Meyer (Ref. 3) suggest that the difference between $|f_3/f_1|$ and $\text{Re}f_3/f_1=|f_3/f_1|\cos(\delta_2-\delta_0)$ should be considered as a measure of the theoretical uncertainty of our predictions. However, it seems to us that, because of the low Q value involved in $K\rightarrow 3\pi$, the 2π phase shift $\delta_2-\delta_0$ when $s=M_K^2$ should have little bearing on the $K\rightarrow 3\pi$ results. Thus, we use $|f_3/f_1|$ to evaluate y.

[27] Y. Chiu, J. Schechter, and Y. Ueda, Phys. Rev. 157, 1317 (1967).

[28] By the symbol Γ we mean the experimental width divided by the conventional phase-space factors.

[29] Actually, the Bouchiat-Meyer result for these ratios is $[(1+2y)/(1-y)]^2$. However, in their case $-y=(\sqrt{\frac{1}{2}})f_3/f_1$, while in our case $-y/(1+2x)=(\sqrt{\frac{1}{2}})f_3/f_1$; so that the predictions of both models are actually the same.

[30] T. Devlin, Phys. Rev. Letters 20, 683 (1968).

[31] For λ_{++-} and λ_{+00} we use the values given by G. Trilling (Ref. 21). However, for λ_{+-0}, we use $\lambda_{+-0}=-0.22\pm0.01$ given as the current world average by J. Cronin, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968).

ment. Thus, we conclude that, over all, the predicted deviations from the $\Delta I=\frac{1}{2}$ rule seem to go in the right direction and are of the proper order of magnitude.

V. HARD-PION MODEL

In the preceding sections, we have discussed a model for nonleptonic kaon decay which agrees with current algebraic constraints and with experimental results. The relative amount of $\Delta I=\frac{1}{2}$ rule violation was treated somewhat phenomenologically in that it was taken from an isospin analysis of $K\rightarrow 2\pi$ decay, rather than predicted from a purely theoretical standpoint.

In order to remedy this, and also to justify our original ansatz [Eq. (4)], we derive a more or less formal expression for the corrections to the soft-pion limit in the K-π matrix element patterned after the hard-pion methods of Weinberg and Schnitzer.[8] These corrections are then estimated in a simple model. A similar method is applied to the $K\rightarrow 2\pi$ matrix element in Appendix C.

We work with the K-π matrix element. One defines the amplitudes

$$\begin{aligned} {}^iN^{an}{}_\mu(k,q_a)&=\int d^4x\, e^{iq_a\cdot x}\langle 0|T(A^a{}_\mu(x)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle \\ &=G_{A_1}\Delta^{A_1}{}_{\mu\nu}(q_a){}^i\Gamma_{an}{}^\nu(k,q_a) \\ &\qquad -F_\pi q_{a\mu}\Delta_\pi(q_a){}^i\Gamma_{an}(k,q_a)\,, \\ {}^iN^{an}(k,q_a)&=\int d^4x\, e^{iq_a\cdot x}\langle 0|T(\partial^\mu A^a{}_\mu(x)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle \\ &=iF_\pi m_\pi{}^2\Delta_\pi(q_a){}^i\Gamma_{an}(k,q_a)\,, \end{aligned} \tag{28}$$

where we have assumed the spin-1 part of the axial current to be dominated by the A_1 meson and the spin-zero portion by the pion. $\Delta_\pi(q)=i/(q^2-m_\pi{}^2)$ is the pion propagator, while

$$\Delta^{A_1}{}_{\mu\nu}(q)=\frac{i}{q^2-m_{A_1}{}^2}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_{A_1}{}^2}\right)$$

is the A_1 propagator. The coupling constant G_{A_1} is defined by

$$\langle 0|A_\mu{}^a(0)|A_{1,q}{}^b\rangle=G_{A_1}\epsilon_\mu\delta^{ab}. \tag{29}$$

It is easy to see that ${}^i\Gamma_{an}(k,q_a)$ is just the K-π matrix element, in which our interest lies, while $\epsilon_\nu{}^i\Gamma_{an}{}^\nu(k,q_a)$ is the K-A_1 matrix element.

We now employ a Ward identity:

$$\begin{aligned} &iq_a{}^{\mu i}N^{an}{}_\mu(k,q_a) \\ &=-iF_\pi q_a{}^2\Delta_\pi(q_a){}^i\Gamma^{an}(k,q_a) \\ &\qquad -(G_{A_1}/m_{A_1}{}^2)q_a{}^{\mu i}\Gamma^{an}{}_\mu(k,q_a) \\ &=-{}^iN^{an}(k,q_a)-\langle 0|[F_a{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle \\ &=-iF_\pi m_\pi{}^2\Delta_\pi(q_a){}^i\Gamma^{an}(k,q_a)+\tfrac{1}{2}A_{1/2}M_K{}^3\mathfrak{s}_{1/2}\tau^aK^n. \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned} {}^i\Gamma_{an}(k,q_a)=&(A_{1/2}M_K{}^3/2F_\pi)\mathfrak{s}_{1/2}\tau^aK^n \\ &+(G_{A_1}/F_\pi M_{A_1}{}^2)q_a{}^{\lambda i}\Gamma_{an}{}^\lambda(k,q_a). \end{aligned} \tag{31}$$

The first term on the right-hand side is the soft-pion result. The second term is the correction term and is directly related to the K-A_1 matrix element.

From Lorentz invariance we can write

$${}^i\Gamma_{an}{}^\lambda(k,q_a)={}^iB_n{}^a(k\cdot q_a,q_a{}^2)k^\lambda+{}^iC_n{}^a(k\cdot q_a,q_a{}^2)q_a{}^\lambda. \tag{32}$$

In analogy to Weinberg and Schnitzer's assumption of the simplest possible form for "primitive" matrix elements, we assume ${}^iB_n{}^a$ and ${}^iC_n{}^a$ to be constants.[32] This is equivalent to the forms used in Eqs. (4) (if ${}^iC_n{}^a=0$) and (B1) for the K-π matrix element.

To gain information about B and C we need a model. As an order-of-magnitude estimate, we assume we may evaluate a matrix element of the type

$$\langle A_{1,q_a}{}^a|\{\mathcal{J}^a{}_\mu(0),\mathcal{J}^n{}_\mu(0)\}|K_k{}^n\rangle$$

by including only the vacuum intermediate state[33]

$$\langle A_{1,q_a}{}^a|\mathcal{J}^a{}_\mu(0)|0\rangle\langle 0|\mathcal{J}^n{}_\mu(0)|K_k{}^n\rangle.$$

The accuracy of such an approximation will be discussed at a later stage. At present, we examine its consequences. We find that

$$\begin{aligned} \langle A_{1,q}{}^0|\mathcal{H}_w{}^{1/2}(0)|K_k{}^0\rangle&=-\langle A_{1,q}{}^0|\mathcal{H}_w{}^{3/2}(0)|K_k{}^0\rangle \\ &\simeq\tfrac{1}{3}\sqrt{2}(G_V/\sqrt{2})\cos\theta\sin\theta\langle A_{1,q}{}^0|A_{\pi^0\mu}|0\rangle\langle 0|A_{\bar{K}^0\mu}(0)|K_k{}^0\rangle \\ &\simeq+\epsilon_{A_1}\cdot k\tfrac{1}{3}G_{A_1}F_KG_V\cos\theta\sin\theta. \end{aligned} \tag{33}$$

Hence, in such a model, terms in q^2 are absent (i.e., ${}^iC_n{}^a=0$), as previously claimed.

Phenomenologically, we had defined in Eq. (6)

$$\begin{aligned} (G_{A_1}/F_\pi M_{A_1}{}^2)^{1/2}\Gamma_{an}{}^\lambda(k,q_a)&=(B_{1/2}M_K/2F_\pi)\mathfrak{s}_{1/2}\tau^aK^nk^\lambda, \\ (G_{A_1}/F_\pi M_{A_1}{}^2)^{3/2}\Gamma_{an}{}^\lambda(k,q_a)&=(B_{3/2}M_K/2F_\pi)\mathfrak{s}_{3/2}{}^aK^nk^\lambda. \end{aligned} \tag{34}$$

Comparison with our model yields

$$\begin{aligned} B_{1/2}M_K&=-B_{3/2}M_K \\ &=-(G_{A_1}{}^2/M_{A_1}{}^2)F_K\tfrac{2}{3}G_V\cos\theta\sin\theta. \end{aligned} \tag{35}$$

Using the Weinberg[34] sum rule $G_{A_1}=G_\rho$, the Kawarabayashi-Suzuki-Fayyazuddin-Riazuddin[35] sum rule $G_\rho{}^2=-2F_\pi{}^2m_\rho{}^2$, and the relation $m_{A_1}{}^2=2m_\rho{}^2$, we find

$$G_{A_1}{}^2/M_{A_1}{}^2=-F_\pi{}^2.$$

Since we are after an order-of-magnitude result, we shall not worry about the accuracy of such sum rules. We have, then,

$$B_{1/2}M_K=-B_{3/2}M_K=+F_\pi{}^2F_K\tfrac{2}{3}G_V\cos\theta\sin\theta. \tag{36}$$

We take $A_{1/2}$ from experiment, assuming it dominates

[32] Actually, Weinberg and Schnitzer make this assumption about their so-called "primitive" functions. However, we have gone as far as we can go without such an assumption.

[33] Such an approximation has been used recently by D. F. Greenberg, Phys. Rev. **178**, 2190 (1969).

[34] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

[35] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 225 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

the $\Delta I=\frac{1}{2}$ part of the K-2π–decay amplitude:

$$|A_{1/2}|M_K{}^3=|\tfrac{4}{3}F_\pi{}^2(\langle\pi^0\pi^0|\mathcal{H}_w(0)|K^0\rangle -2\langle\pi^+\pi^-|\mathcal{H}_w(0)|K^0\rangle)| \simeq -4F_\pi{}^2\times5.5\times10^{-7}M_K. \quad (37)$$

Thus

$$y^{\text{theo}}=-x^{\text{theo}}=-\frac{|F_K|M_K}{12}\frac{G_V\cos\theta\sin\theta}{5.5\times10^{-7}} \simeq -2.0\times10^{-2}\frac{F_K}{F_\pi}, \quad (38)$$

where we have used the experimental ($\sim$94 MeV) value for $|F_\pi|$. Using $F_K/F_\pi=1.28$, we find

$$-y^{\text{theor}}=x^{\text{theor}}=2.6\times10^{-2},$$

whereas from Eq. (21),

$$-y^{\text{expt}}\simeq-y'^{\text{expt}}\simeq3.2\times10^{-2}.$$

This good agreement should not be taken seriously, since we have made a drastic assumption by including only the vacuum intermediate state in our calculation of B_i. On the other hand, such agreement does seem to lend some credibility to our procedure.

A treatment of $\Delta I=\frac{3}{2}$ effects in nonleptonic K decay invoking such a vacuum-intermediate-state approximation has been given previously by Pati and Oneda[36] and also by Schwinger.[37] They pointed out that the $\Delta I=\frac{3}{2}$ amplitude was indeed quite well predicted in such a model. To explain the experimental predominance of $\Delta I=\frac{1}{2}$ they were forced to assume octet dominance.[38] We do not necessarily have to invoke such a hypothesis, since the $K-A_1$ amplitude is added to a term proportional to the K-vac amplitude which can *only* be $\Delta I=\frac{1}{2}$. If this latter amplitude is much larger than the former, we have a natural explanation of the $\Delta I=\frac{1}{2}$ rule. Indeed, one can estimate $A_{1/2}$ in the soft-kaon limit by using the convergent intermediate-vector-boson model of Glashow, Schnitzer, and Weinberg,[39] with the $SU(3)\times SU(3)$ sum rules.[40] These authors show that a good value for $A_{1/2}$ is obtained for an intermediate-vector-boson mass of about 8 BeV.[41]

VI. CONCLUSIONS

We have constructed a semiphenomenological model for CP-conserving nonleptonic kaon decay in which we assume (i) all single soft-pion limits are satisfied; (ii) aside from pole terms, amplitudes can be expanded as a power series in the pion momenta; and (iii) in particular, in the K-π matrix element only zeroth- and first-order terms in pion momenta are included. We then find that all such K-decay amplitudes are described in terms of two parameters—one of which (f_1) satisfies the $\Delta I=\frac{1}{2}$ rule and the other of which (f_3) violates it.

Our $\Delta I=\frac{3}{2}$ predictions were found to agree with the model of Bouchiat and Meyer up to terms of order $m_\pi{}^2/M_K{}^2$. Although particular parameters (e.g., α_3 and β_3) were found to deviate significantly from the corresponding Bouchiat-Meyer results, cancellations made the two models nearly the same as regards their predictions for the amplitude and slope in $K\to3\pi$. With f_3/f_1 determined from $K\to2\pi$ data, the $\Delta I=\frac{3}{2}$ effects predicted for the $K\to3\pi$ system were consistent with experimental findings, as shown in Table I.

For the $\Delta I=\frac{1}{2}$ amplitude McNamee has already given a model, dependent only upon the parameter $A_{1/2}$, which is consistent with current-algebraic constraints, and which agrees with the model of Nambu and Hara to order $m_\pi{}^2/M_K{}^2$. When we extend his model to include first-order terms in the pion momenta, although we introduced an additional term $B_{1/2}$, these parameters always appear in our results in the combination $A_{1/2}+B_{1/2}$. Thus, there is effectively only one parameter, which we may choose to be f_1, and our $\Delta I=\frac{1}{2}$ predictions are found to be identical to those of McNamee.

To check on these predictions one can calculate the amplitude and slope for the $K\to3\pi$ decays, which are predominantly $\Delta I=\frac{1}{2}$ effects. We find in our model that

$$|A_{+-0}|\simeq\frac{1}{3\sqrt{6}}\left|\frac{f_1}{F_\pi}\right|\frac{1+3\eta}{1-\eta}(1+2y'), \quad \left[\text{BMNH predict }\frac{1}{3\sqrt{6}}\left|\frac{f_1}{F_\pi}\right|(1+2y')\right],$$

$$\tfrac{1}{2}|A_{++-}|\simeq\frac{1}{3\sqrt{6}}\left|\frac{f_1}{F_\pi}\right|\frac{1+3\eta}{1-\eta}(1-y'), \quad \left[\text{BMNH predict }\frac{1}{3\sqrt{6}}\left|\frac{f_1}{F_\pi}\right|(1-y')\right],$$

$$\lambda_{+00}\simeq-\frac{3\eta}{1+3\eta}[1-\tfrac{9}{2}y'(1+\eta)], \quad [\text{BMNH predict }-3\eta(1-\tfrac{9}{2}y')],$$

[36] J. C. Pati and S. Oneda, Phys. Rev. **140**, B1351 (1965). See also J. C. Pati and S. Oneda, *ibid.* **136**, B1064 (1964).

[37] J. Schwinger, Phys. Rev. Letters **12**, 630 (1964).

[38] See, e.g., R. Dashen, S. Frautschi, M. Gell-Mann, and Y. Hara, in *The Eightfold Way*, edited by M. Gell-Mann and Y. Ne'eman (W. A. Benjamin, Inc., New York, 1965).

[39] S. Glashow, H. J. Schnitzer, and S. Weinberg, Phys. Rev. Letters **19**, 205 (1967).

[40] S. Glashow, H. J. Schnitzer, and S. Weinberg, Phys. Rev. Letters **19**, 139 (1967). As pointed out by J. J. Sakurai and others, the second Weinberg sum rule is of questionable validity applied to $SU(3)\times SU(3)$. Nevertheless, we are after an order-of-magnitude estimate and shall use it.

[41] Since the mass of the intermediate vector boson appears primarily in a logarithmic factor, roughly the same value for $A_{1/2}$ will be obtained for a wide variety of M_B.

$$\lambda_{+-0} \cong -\frac{3\eta}{1+3\eta}[1-\tfrac{9}{2}y'(1-3\eta)], \quad [\text{BMNH predict } -3\eta(1-\tfrac{9}{2}y')],$$

$$\lambda_{++-} \cong \frac{\frac{3}{2}\eta}{1+3\eta}[1+9y'(1-\eta)], \quad [\text{BMNH predict } \tfrac{3}{2}\eta(1+9y')].$$

Our predictions reduce to those of McNamee if we set $y'=0$. The experimental agreement of the three models is checked in Table II.

One possible uncertainty in our work involves the ever-present uncertainty in any current-algebra calculation involving more than one pion due to σ terms.[42] We appear to have avoided discussion of these quantities by always taking single soft-pion limits with the remaining pions on the mass shell. However, assumptions concerning the σ term are present in the forms used for the π-π and K-π scattering amplitudes. (For further discussion of this point see Ref. 5.) In fact, Cronin[43] and Griffith[44] have suggested that the scattering amplitude contains an additional piece proportional to $\delta^{ab}\delta^{mn}(2M_K^2+2m_\pi^2-2t-s-u)$. We have, in Appendix B, introduced such a term with arbitrary coefficient λ into our model and have studied its effect upon our results.

One may also wonder about uncertainty due to terms which our model neglected. To study this question, we have, in Appendix B, extended our model to include $O(q^2)$ terms. This introduced four new parameters, two of which (s and t) violate the $\Delta I=\frac{1}{2}$ rule and two of which (w and z) do not. We then have a seven-parameter fit to the decay amplitudes. Corrections to the $\Delta I=\frac{3}{2}$ results are $[1+\eta(C_1s+C_2t+C_3\lambda)]$ with $\eta = m_\pi^2/M_K^3$ and C_1, C_2, C_3, $\sim 10^0$. Since we expect s, t, and λ to be of the order of, or smaller than, unity, corrections to $\Delta I=\frac{3}{2}$ terms could conceivably have a fairly significant effect on our predictions, although we have shown in particular models that $s=t=\lambda=0$.

An additional source of theoretical uncertainty arises because we have not included effects due to electromagnetic corrections. These, calculated for $K\to 3\pi$ in a simple model by Neveu and Scherk,[45] could be as large as several percent. However, we feel that they are still too uncertain to be included.

Finally uncertainties might result from the effects of the strong interactions upon our work. Although we have tried in Sec. IV to indicate qualitatively how our results might be affected by the presence of final-state-interaction phase shifts, the over-all influence of the strong interactions upon our calculations certainly needs further study.

TABLE II. Summary of predicted versus experimental amplitudes and slopes.[a]

Quantity	Experimental value	Theoretical value: This model	McNamee model	NHBM model
$\lvert A_{+-0}\rvert$	$(8.6\pm0.1)\times10^{-7}$[b]	8.5×10^{-7}	9.1×10^{-7}	6.4×10^{-7}
$\frac{1}{2}\lvert A_{++-}\rvert$	$(9.6\pm0.1)\times10^{-7}$[c]	9.4×10^{-7}	9.1×10^{-7}	7.0×10^{-7}
λ_{+00}	-0.25 ± 0.02[d]	-0.233	-0.192	-0.272
λ_{+-0}	-0.22 ± 0.01[d]	-0.218	-0.192	-0.272
λ_{++-}	0.093 ± 0.011[d]	0.070	0.096	0.083

[a] We have used $y'=-1/30$ and the experimental ($\lvert F_\pi\rvert\simeq 94$ MeV) for F_π.
[b] See Ref. 4.
[c] See Ref. 5.
[d] See Ref. 31.

ACKNOWLEDGMENTS

I am grateful to Professor L. Wolfenstein for suggesting this topic and for many discussions which helped to clarify several of the concepts involved. I also wish to thank Dr. David F. Greenberg for a critical reading of the manuscript.

APPENDIX A

We use Dirac matrices given by Bjorken and Drell,[46] except that $\gamma_5=-i\gamma_0\gamma_1\gamma_2\gamma_3$. Our metric is $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$. All matrix elements are defined without the usual $(2\omega)^{-1/2}$ factors. Our phase convention for octet states is

$$\varphi_{\pi^+}{}^\dagger=-(\sqrt{\tfrac{1}{2}})(\varphi_1+i\varphi_2),$$
$$\varphi_{\pi^0}{}^\dagger=\varphi_3, \quad \varphi_{\pi^-}{}^\dagger=(\sqrt{\tfrac{1}{2}})(\varphi_1-i\varphi_2),$$
$$\varphi_{K^+}{}^\dagger=-(\sqrt{\tfrac{1}{2}})(\varphi_4+i\varphi_5), \quad \varphi_{K^0}{}^\dagger=-(\sqrt{\tfrac{1}{2}})(\varphi_6+i\varphi_7),$$
$$\varphi_{K^-}{}^\dagger=(\sqrt{\tfrac{1}{2}})(\varphi_4-i\varphi_5), \quad \varphi_{\bar K^0}{}^\dagger=(\sqrt{\tfrac{1}{2}})(\varphi_6-i\varphi_7).$$

Thus (π^+, π^0, and π^-) form an isotriplet, while (K^+,K^0) and ($\bar K^0$, $-K^-$) form isodoublets. The notation $V_{\pi^+}{}^\mu(x)$ means, in the quark model, $-\bar\psi(x)\frac{1}{2}(\sqrt{\frac{1}{2}})(\lambda_1+i\lambda_2)\gamma_\mu\psi(x)$.

APPENDIX B

We give here a more general treatment of the $K\to n\pi$ matrix elements including terms in q^2 and in $q_1\cdot q_2$. Also, we shall employ the more general form for the K-π scattering amplitude,

$$\langle \pi_{q_b}{}^b K_{k'}{}^m | T | \pi_{q_a}{}^a K_k{}^n \rangle = (2\pi)^4\delta^4(k'+q_b-k-q_a) \times -(4F_\pi^2)^{-1}[\lambda\delta^{ab}\delta^{mn}(2M_K^2+2m_\pi^2-2t-s-u) + i\epsilon^{abc}\tau^c{}_{mn}(s-u)]. \quad \text{(B1)}$$

This is identical to the form used in the main body of the text and to that given by a minimal chiral $SU(2)\times SU(2)$ Lagrangian model by Zumino[47] if $\lambda=0$, and

[42] S. Weinberg, Phys. Rev. Letters **17**, 336 (1966); L. Kisslinger, *ibid.* **18**, 861 (1967).
[43] J. Cronin, Phys. Rev. **161**, 1483 (1967).
[44] R. W. Griffith, Phys. Rev. **176**, 1705 (1968).
[45] A. Neveu and J. Scherk, Phys. Letters **27B**, 384 (1968).
[46] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964).
[47] B. Zumino, Phys. Letters **25B**, 349 (1967).

is equivalent to the forms given by Cronin[43] and Griffith,[44] based on certain models of $SU(3)\times SU(3)$ symmetry breaking, if $\lambda=1$. Without a definite model of how $SU(3)\times SU(3)$ is broken, however, the value of λ can be arbitrary. Thus, we shall keep it variable. Such a scattering amplitude yields the matrix element of the σ between two kaon states

$$\langle K_{k'}{}^m|\sigma(0)|K_k{}^n\rangle=-i\tfrac{1}{2}\lambda m_\pi{}^2\delta^{mn}\,, \tag{B2}$$

while the matrix element between two pions can be found from our form for the π-π scattering:

$$\langle\pi_{q_b}{}^b|\sigma(0)|\pi_{q_a}{}^a\rangle=-im_\pi{}^2\delta^{ab}. \tag{B3}$$

With these two assumptions we have also derived the results of this Appendix by means of a hard-pion model.[48] However, it is easier to use the procedures of the main text with one slight change, as we shall note. Here we shall give only the results of such a calculation.

Our results for the K-vacuum matrix element are unchanged. For the K-π matrix element, we define

$$\langle\pi_{q_a}{}^a|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle={}^iA_n{}^a+{}^iB_n{}^ak\cdot q_a+{}^iC_n{}^aq_a{}^2. \tag{B4}$$

Then ${}^iA_n{}^a$ and ${}^iB_n{}^a$ are given in Eq. (6), while for ${}^iC_n{}^a$ we define

$$\begin{aligned}{}^{1/2}C_n{}^a&=(C_{1/2}M_K/2F_\pi)\bar{s}_{1/2}\tau^aK^n\,,\\ {}^{3/2}C_n{}^a&=(C_{3/2}M_K/2F_\pi)\bar{s}_{3/2}{}^aK^n.\end{aligned} \tag{B5}$$

Moving to the $K\rightarrow 2\pi$ matrix element, we define

$$\begin{aligned}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle&={}^iA_n{}^{ab}+{}^iB_n{}^{ab}k\cdot q_a+{}^iB_n{}^{ba}k\cdot q_b\\ &\quad+{}^iC_n{}^{ab}q_a{}^2+{}^iC_n{}^{ba}q_b{}^2+(K\text{-pole term})^i,\end{aligned} \tag{B6}$$

where now (K-pole term)^i uses the K-π scattering amplitude given in Eq. (B1). In order to evaluate the q^2 dependence we must alter our previous procedure. When we take the soft-pion limit $q_b\rightarrow 0$, we must no longer require π^a to be on its mass shell. Then we find that

$$\begin{aligned}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\xrightarrow[q_b\rightarrow 0]{}\\ -F_\pi{}^{-1}\langle\pi_{q_a}{}^a|[F_b{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle\\ +i\delta^{ab}[\Delta_\pi{}^{-1}(q_a)/F_\pi{}^2m_\pi{}^2]\,{}^i\Sigma^n(k,q_a)\,,\end{aligned} \tag{B7}$$

where

$${}^i\Sigma^n(k,q_a)=\int d^4x\, e^{iq_a\cdot x}\langle 0|T(\sigma(x)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle$$

and includes a kaon-pole term if $\lambda\neq 0$. We see that we have the usual result

$$\langle\pi_{q_b}{}^a\pi_{q_a}{}^b|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\xrightarrow[q_b\rightarrow 0]{} -F_\pi{}^{-1}\langle\pi_{q_a}{}^a|[F_b{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle\,,$$

if π^a is on its mass shell. We then find ${}^iB_n{}^{ab}$ as given in Eq. (8), while for ${}^iA_n{}^{ab}$ and ${}^iC_n{}^{ab}$ we find that

$$\begin{aligned}{}^{1/2}C_n{}^{ab}&=(C_{1/2}/B_{1/2})B_n{}^{ab}\,,\quad {}^{3/2}C_n{}^{ab}=(C_{3/2}/B_{3/2})B_n{}^{ab}\,,\\ {}^{1/2}A_n{}^{ab}&=(A_{1/2}M_K{}^3/4F_\pi{}^2)(1-\lambda)\delta^{ab}\bar{s}_{1/2}K^n\,,\\ {}^{3/2}A_n{}^{ab}&=0.\end{aligned} \tag{B8}$$

The soft-pion limit can yield no information about ${}^iD_n{}^{ab}$; so for it we define

$$\begin{aligned}{}^{1/2}D_n{}^{ab}&=(D_{1/2}M_K/4F_\pi{}^2)\delta^{ab}\bar{s}_{1/2}K^n\,,\\ {}^{3/2}D_n{}^{ab}&=(D_{3/2}M_K/4F_\pi{}^2)[\bar{s}_{3/2}{}^a\tau^bK^n+\bar{s}_{3/2}{}^b\tau^aK^n].\end{aligned} \tag{B9}$$

For $K\rightarrow 3\pi$ we define

$$\begin{aligned}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b\pi_{q_c}{}^c|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\\ ={}^iA_n{}^{abc}+{}^iB_n{}^{abc}k\cdot q_a+{}^iB_n{}^{bca}k\cdot q_b+{}^iB_n{}^{cab}k\cdot q_c\\ +{}^iC_n{}^{abc}q_a{}^2+{}^iC_n{}^{bca}q_b{}^2+{}^iC_n{}^{cab}q_c{}^2+{}^iD_n{}^{abc}q_b\cdot q_c\\ +{}^iD_n{}^{bca}q_c\cdot q_a+{}^iD_n{}^{cab}q_a\cdot q_b+(K\text{-pole term})^i\\ +(\Pi\text{-pole term})^i,\end{aligned} \tag{B10}$$

where again (K-pole term)^i employs the general K-π scattering amplitude given in Eq. (B1). Here again, since we are trying to determine the q^2 dependence, when the soft-pion limit $q_c\rightarrow 0$ is taken, we no longer require π^a, π^b to be on their mass shells. We find that

$$\begin{aligned}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b\pi_{q_c}{}^c|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\xrightarrow[q_c\rightarrow 0]{}\\ -F_\pi{}^{-1}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|[F_c{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle\\ +i\delta^{ac}[\Delta_\pi{}^{-1}(q_b)/F_\pi{}^2m_\pi{}^2]\,{}^i\Lambda^{bn}(k,q_b,q_a)\\ +i\delta^{bc}[\Delta_\pi{}^{-1}(q_a)/F_\pi{}^2m_\pi{}^2]\,{}^i\Lambda^{an}(k,q_a,q_b)\,,\end{aligned} \tag{B11}$$

where

$${}^i\Lambda^{bn}(k,q_b,q_a)=\int d^4x\, e^{iq_a\cdot x}\langle\pi_{q_b}{}^b|T(\sigma(x)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle$$

and includes both π and, if $\lambda\neq 0$, K poles. If π^a, π^b are on the mass shell, (B11) reduces to the conventional result

$$\langle\pi_{q_a}{}^a\pi_{q_b}{}^b\pi_{q_c}{}^c|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle\xrightarrow[q_c\rightarrow 0]{} -F_\pi{}^{-1}\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|[F_c{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle.$$

We then find that

$$\begin{aligned}{}^{1/2}A_n{}^{abc}&=-(A_{1/2}M_K{}^3/8F_\pi{}^3)(1+\lambda)[\delta^{ab}\bar{s}_{1/2}\tau^cK^n+\delta^{ac}\bar{s}_{1/2}\tau^bK^n+\delta^{bc}\bar{s}_{1/2}\tau^aK^n]\,,\quad {}^{3/2}A_n{}^{abc}=0\,,\\ {}^{1/2}B_n{}^{abc}&=(B_{1/2}M_K/8F_\pi{}^3)[(1-\lambda)\delta^{bc}\bar{s}_{1/2}\tau^aK^n-2\delta^{ab}\bar{s}_{1/2}\tau^cK^n-2\delta^{ac}\bar{s}_{1/2}\tau^bK^n]\,,\\ {}^{3/2}B_n{}^{abc}&=(B_{3/2}M_K/8F_\pi{}^3)[(5-\lambda)\delta^{bc}\bar{s}_{3/2}{}^aK^n-4\delta^{ac}\bar{s}_{3/2}{}^bK^n-4\delta^{ab}\bar{s}_{3/2}{}^cK^n-2i\epsilon^{abd}\bar{s}_{3/2}{}^d\tau^cK^n-2i\epsilon^{acd}\bar{s}_{3/2}{}^d\tau^bK^n]\,,\\ &\qquad {}^{1/2}C_n{}^{abc}=(C_{1/2}/B_{1/2})\,{}^{1/2}B_n{}^{abc}\,,\quad {}^{3/2}C_n{}^{abc}=(C_{3/2}/B_{3/2})\,{}^{3/2}B_n{}^{abc}\,,\\ {}^{1/2}D_n{}^{abc}&=(D_{1/2}M_K/8F_\pi{}^3)\delta^{bc}\bar{s}_{1/2}\tau^aK^n-(C_{1/2}M_K/F_\pi{}^3)\delta^{bc}\bar{s}_{1/2}\tau^aK^n+(B_{1/2}M_K/8F_\pi{}^3)\\ &\qquad\times[-2\delta^{bc}\bar{s}_{1/2}\tau^aK^n+(1+\lambda)\delta^{ac}\bar{s}_{1/2}\tau^bK^n+(1+\lambda)\delta^{ab}\bar{s}_{1/2}\tau^cK^n]\,,\\ {}^{3/2}D_n{}^{abc}&=(D_{3/2}M_K/8F_\pi{}^3)[\delta^{ac}\bar{s}_{3/2}{}^bK^n+\delta^{ab}\bar{s}_{3/2}{}^cK^n+i\epsilon^{abd}(2\bar{s}_{3/2}{}^d\tau^cK^n+\bar{s}_{3/2}{}^c\tau^dK^n)+i\epsilon^{acd}(2\bar{s}_{3/2}{}^d\tau^bK^n+\bar{s}_{3/2}{}^b\tau^dK^n)]\\ &\quad-(C_{3/2}M_K/F_\pi{}^3)\delta^{bc}\bar{s}_{3/2}{}^aK^n+(B_{3/2}M_K/8F_\pi{}^3)[\lambda(\delta^{ac}\bar{s}_{3/2}{}^bK^n+\delta^{ab}\bar{s}_{3/2}{}^cK^n)\\ &\qquad+i\epsilon^{acd}\bar{s}_{3/2}{}^b\tau^dK^n+i\epsilon^{abd}\bar{s}_{3/2}{}^c\tau^dK^n]\,,\end{aligned} \tag{B12}$$

[48] B. R. Holstein, Ph.D. thesis, Carnegie-Mellon University, 1969 (unpublished).

and in terms of the parameters defined in Eq. (16) we find

$$\begin{aligned}
\alpha_1 &= -\tfrac{1}{2}\sqrt{3}\frac{f_1}{F_\pi}\frac{1}{1-\eta}\left[\frac{(1-\lambda)(1+\eta)+2x(1-\lambda\eta)(1+\eta)+2w\eta(3-\lambda-\eta-\lambda\eta)+z(1-\eta)^2}{1-\lambda+2x+4w\eta+z(1-2\eta)}\right],\\
M_K\beta_1 &= \sqrt{3}\frac{f_1}{F_\pi}\frac{1}{1-\eta}\left[\frac{1-\lambda+2x(1-\lambda\eta)+2w\eta(1-\lambda)+z(1-\eta)}{1-\lambda+2x+4w\eta+z(1-2\eta)}\right],\\
\alpha_3 &= -\left(\frac{1}{2}\sqrt{\frac{3}{10}}\right)\frac{f_3}{F_\pi}\frac{1}{1-\eta}\left[\frac{1-11\eta+2\lambda\eta(1+\eta)-2s\eta[6-\eta-\lambda(1+\eta)]+t(1-\eta)(1+2\eta)}{1+2s\eta+t(1-2\eta)}\right],\\
M_K\beta_3 &= \left(\frac{5}{2}\sqrt{\frac{3}{10}}\right)\frac{f_3}{F_\pi}\frac{1}{1-\eta}\left[\frac{1-(27/5)\eta+\tfrac{4}{5}\lambda\eta-\tfrac{2}{5}s\eta(11-2\lambda)+t(1-\eta)}{1+2s\eta+t(1-2\eta)}\right],\\
M_K\gamma_3 &= \tfrac{3}{2}(\sqrt{6})\frac{f_3}{F_\pi}\frac{1}{1-\eta}\left[\frac{1-\tfrac{1}{3}\eta+\tfrac{2}{3}s\eta+t(1-\eta)}{1+2s\eta+t(1-2\eta)}\right],
\end{aligned} \tag{B13}$$

where $x=B_{1/2}/2A_{1/2}$, $w=C_{1/2}/2A_{1/2}$, $z=D_{1/2}/2A_{1/2}$, $s=C_{3/2}/B_{3/2}$, and $t=D_{3/2}/B_{3/2}$, and we expect x, w, and $z\sim 10^{-2}$ while s and $t\sim 10^0$. We see that for suitable choices of s and t, results quite different from those of Bouchiat and Meyer may be obtained, although our models given in Sec. V and Appendix C predict $s=t=0$. For the $\Delta I=\frac{1}{2}$ amplitudes, we distinguish two cases. For $|\lambda|\ll 1$ our previous results are reproduced up to corrections $1+\eta(C_1w+C_2x+C_3z)$, which are $1+C_4\times 10^{-3}$ since w, x, and z are expected to be $\sim 10^{-2}$ and C_1, C_2, C_3, $C_4\sim 10^0$. On the other hand, if $\lambda\simeq 1$, our previous results are also reproduced but now up to corrections $1+\eta(C_1'\lambda+C_2'w/x+C_3'z/x)$ which are $1+C_4'\times 10^{-1}$ and could conceivably have a sizable effect upon our predictions.

Perhaps the most interesting observation is that if $\lambda=1$, as given by Cronin and Griffith, $A_{1/2}$ does not contribute to the physical $K\to 2\pi$ or $K\to 3\pi$ matrix elements. Thus, the leading terms for the $\Delta I=\frac{1}{2}$ component will be those in $B_{1/2}$ and $D_{1/2}$, while the leading terms in the $\Delta I=\frac{3}{2}$ component will be those in $B_{3/2}$ and $D_{3/2}$. Since we expect these to be of the same order of magnitude, we lose our natural explanation for the $\Delta I=\frac{1}{2}$ rule and, in order to fit the experimental results either we must postulate some type of octet enhancement or an accidental cancellation between $B_{3/2}$ and $D_{3/2}$ to account for the suppression of the $\Delta I=\frac{3}{2}$ amplitude, or we must introduce a new nonleptonic Hamiltonian, which predominantly $\Delta I=\frac{1}{2}$. It is much more appealing to suggest that λ may in fact be zero (or at least very small) so that the suppression of $\Delta I=\frac{3}{2}$ terms arises naturally from the usual current-current nonleptonic Hamiltonian, Eq. (1).

APPENDIX C

Here we apply hard-pion methods to the $K\to 2\pi$ system. We define

$$\begin{aligned}
{}^iM^{abn}{}_{\mu\nu}(k,q,p) &= \int d^4x d^4y\, e^{iq\cdot x+ip\cdot y}\langle 0|T(A^a{}_\mu(x)A^b{}_\nu(y)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle\\
&= F_\pi{}^2 q_\mu p_\nu \Delta_\pi(q)\Delta_\pi(p)\,{}^i\Gamma_{abn}(k,q,p) - F_\pi G_{A_1} q_\mu \Delta^{A_1}{}_{\nu\eta}(p)\Delta_\pi(q)\,{}^i\Gamma_{ban}{}^\eta(k,p,q) - F_\pi G_{A_1} p_\nu \Delta^{A_1}{}_{\mu\lambda}(q)\Delta_\pi(p)\\
&\qquad \times {}^i\Gamma_{abn}{}^\lambda(k,q,p) + G_{A_1}{}^2\Delta^{A_1}{}_{\mu\lambda}(q)\Delta^{A_1}{}_{\nu\eta}(p)\,{}^i\Gamma_{abn}{}^{\lambda\eta}(k,q,p)\,,\\
{}^iM^{abn}{}_\mu(k,q,p) &= \int d^4x d^4y\, e^{iq\cdot x+ip\cdot y}\langle 0|T(A^a{}_\mu(x)\partial^\nu A^b{}_\nu(y)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle\\
&= -iF_\pi{}^2 m_\pi{}^2 q_\mu \Delta_\pi(q)\Delta_\pi(p)\,{}^i\Gamma_{abn}(k,q,p) + iF_\pi m_\pi{}^2 G_{A_1}\Delta_\pi(p)\Delta^{A_1}{}_{\mu\lambda}(q)\,{}^i\Gamma_{abn}{}^\lambda(k,q,p)\,,\\
{}^iM^{abn}(k,q,p) &= \int d^4x d^4y\, e^{iq\cdot x+ip\cdot y}\langle 0|T(\partial^\mu A^a{}_\mu(x)\partial^\nu A^b{}_\nu(y)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle\\
&= -F_\pi{}^2 m_\pi{}^4 \Delta_\pi(q)\Delta_\pi(p)\,{}^i\Gamma_{abn}(k,q,p)\,,\\
{}^i\Sigma^n(k,q) &= \int d^4x\, e^{iq\cdot x}\langle 0|T(\sigma(x)\mathcal{H}_w{}^i(0))|K_k{}^n\rangle\,,
\end{aligned} \tag{C1}$$

where $\sigma(x)$ is the σ field and arises from the commutation relation

$$\delta(x^0-y^0)[A_a{}^0(x),\partial_\mu A_b{}^\mu(y)]=\delta^{ab}\sigma(x)\delta^4(x-y).$$

We then apply Ward identities as in Sec. V and eventually find the result

$$
\begin{aligned}
{}^i\Gamma_{abn}(k,q,p)=&\frac{G_{A_1}{}^2}{m_{A_1}{}^4}\frac{q_\lambda p_\eta}{F_\pi{}^2}\,{}^i\Gamma_{abn}{}^{\lambda\eta}(k,q,p)-\frac{m_\pi{}^2-q^2-p^2}{F_\pi{}^2m_\pi{}^2}\delta^{ab}\,{}^i\Sigma^n(k,\,q+p)-F_\pi{}^{-1}\langle\pi_q{}^a|[F_b{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle\\
&-F_\pi{}^{-1}\langle\pi_p{}^b|[F_a{}^5(0),\mathcal{H}_w{}^i(0)]|K_k{}^n\rangle+(K\text{-pole term})^i\\
&-(2F_\pi{}^2)^{-1}\langle 0|[F_a{}^5(0),[F_b{}^5(0),\mathcal{H}_w{}^i(0)]]+[F_b{}^5(0),[F_a{}^5(0),\mathcal{H}_w{}^i(0)]]|K_k{}^n\rangle. \quad (\text{C2})
\end{aligned}
$$

Now it is easy to see that

$${}^i\Gamma_{abn}(k,q_a,q_b)=\langle\pi_{q_a}{}^a\pi_{q_b}{}^b|\mathcal{H}_w{}^i(0)|K_k{}^n\rangle$$

is just the amplitude we want. Use of our previously derived K-π and K-vac matrix elements yields

$$
\begin{aligned}
\langle\pi_q{}^a\pi_p{}^b|\mathcal{H}_w{}^{1/2}(0)|K_k{}^n\rangle=&(G_{A_1}{}^2/m_{A_1}{}^4)F_\pi{}^{-2}q_\lambda p_\eta{}^{1/2}\Gamma_{abn}{}^{\lambda\eta}(k,q,p)+\delta^{ab}\mathfrak{s}_{1/2}K^n(4F_\pi{}^2)^{-1}[A_{1/2}M_K{}^3+B_{1/2}M_Kk\cdot(q+p)\\
&+C_{1/2}M_K(q^2+p^2)]+i\epsilon^{abc}\mathfrak{s}_{1/2}\tau^cK^n[B_{1/2}M_Kk\cdot(p-q)+C_{1/2}M_K(p^2-q^2)]\\
&+(K\text{-pole term})^{1/2}-\delta^{ab}\frac{m_\pi{}^2-q^2-p^2}{F_\pi{}^2m_\pi{}^2}{}^{1/2}\Sigma^n(k,\,q+p). \quad (\text{C3})
\end{aligned}
$$

In accordance with Weinberg's assumption, we drop the Σ term.[42] The vertex function ${}^{1/2}\Gamma_{abn}{}^{\lambda\eta}(k,q,p)$ is just the amplitude for K^n to go to $A_1{}^a$, $A_1{}^b$ via $\mathcal{H}_w{}^{1/2}$. If we make the simplest possible assumption concerning it,[49]

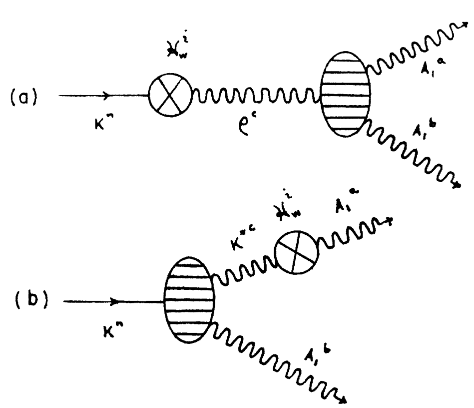

FIG 3. Diagrams contributing to $K\to A_1{}^aA_1{}^b$.

then

$$(G_{A_1}{}^2/m_{A_1}{}^4)^{1/2}\Gamma_{abn}{}^{\lambda\eta}(k,q,p)=\tfrac{1}{4}D_{1/2}M_K\delta^{ab}\mathfrak{s}_{1/2}K^ng^{\lambda\eta}. \quad (\text{C4})$$

Then we just reproduce the general form for the K-2π matrix element given in Appendix B. A similar result is obtained for $\mathcal{H}_w{}^{3/2}$.

As an estimate of the order of magnitude of the K-A_1-A_1 vertex, we have calculated it using the diagrams in Fig. 3 and using the vacuum intermediate-state approximation for $\mathcal{H}_w{}^i$.[50] We find that in this model the simple form in Eq. (B12) is unacceptable—that D_i must be momentum-dependent. However, the calculated result vanishes for the physical $K\to\pi\pi$ amplitude and, in any case, it is too small to make any significant contribution. Thus we felt justified in dropping it in our discussion in Sec. II.

[49] We assume, consistent with our assumption $\lambda=0$, that the $K-A_1$ scattering amplitude vanishes. Otherwise we would have a contribution from a graph as in Fig. 1 with π's replaced by A_1's.

[50] For the strong-interaction vertices, we have employed Cronin's phenomenological Lagrangian extended to include vector and axial-vector mesons. See Ref. 48.