success of the dominant $\Delta I=\frac{1}{2}$ rule in $K\to 2\pi$ and other nonleptonic strange-particle decays obviously suggests the former.

The data on the slopes [Eqs. (4) and (5) of Table V] imply that $\mathrm{Re}(c_{ch}/a_{ch})$ equals 0.029 ± 0.005. If the ratios of the $I=1$ parameters are defined as

$$U=(a_3/a_1),\quad V=(b_1/a_1),\quad W=(b_3/a_1),$$

relations (4), (5), and (6) yield for V and W, for the value of $U=0.032\pm0.006$ determined from the rates, the values

$$V=0.218\pm0.006,\quad W=0.015\pm0.006.$$

In conclusion, the current data on $K\to 3\pi$ exhibit the following isospin properties:

(a) A comparison of τ and τ' rates and a comparison of $K_2{}^0\to\pi^+\pi^-\pi^0$ with $K_2{}^0\to 3\pi^0$ show that $I=3$ final states are not required.

(b) A $\Delta I=\frac{3}{2}$, $I=1$ amplitude ($a_3/a_1=0.032\pm0.006$) is indicated by the $K_2{}^0\to\pi^+\pi^-\pi^0$ decay rate's being too low by four standard deviations with respect to the rate for $K^\pm\to\pi^\pm\pi^+\pi^-$ to be consistent with the $\Delta I=\frac{1}{2}$ rule.

(c) An $I=2$ ($\Delta I=\frac{3}{2}$ or $\frac{5}{2}$) amplitude ($c_{ch}/a_{ch}=+0.029\pm0.005$) arises from the τ' slope's being 6.5 standard deviations larger than is expected from the τ slope and the $\Delta I=\frac{1}{2}$ rule.

(d) The slopes of the three Dalitz plots show the presence of an $I=1$ state of mixed symmetry. From comparison of the τ and τ' slopes with the $K_2{}^0$ slope, this state is found to come predominantly from $\Delta I=\frac{1}{2}$ ($b_1/a_1=+0.218\pm0.006$), with a small admixture of $\Delta I=\frac{3}{2}$ ($b_3/a_1=+0.015\pm0.006$).

It is important to reemphasize that the above quantitative results are based on the neglect of final-state interactions, which do give rise to imaginary parts to the amplitudes. In addition, the prescriptions we have used to account for Coulomb effects and mass differences are subject to theoretical uncertainty.

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Unitarity and the Veneziano Representation*

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The conjecture that simple Veneziano-type formulas represent the zeroth-order approximation to certain strong-interaction amplitudes is investigated. With $\mathrm{Im}\alpha$ as a small parameter, the question of incorporating unitarity to next order is studied, and a simple example illustrating some of the features is constructed. The problem is posed in terms of partial-wave amplitudes; therefore, the partial-wave projection of the Veneziano formula is also discussed in some detail.

I. INTRODUCTION

THE function $B(1-\alpha(t),1-\alpha(s))$ originally proposed by Veneziano[1] in connection with the process $\pi\pi\to\pi\omega$ has provided the theory of strong interactions with an example of a crossing-symmetric amplitude with Regge behavior and zero-width resonances. The application of Veneziano-type models to a few other reactions has also led to some agreement with experiment. For example, there seems to be evidence for the existence of an s-wave large-width resonance ϵ in the $\pi\pi\to\pi\pi$ amplitude at a mass close to the ρ meson. A simple Veneziano-type model for this process, first proposed by Lovelace,[2] contains such a resonance. The ratio of the widths predicted by this model is quite sensitive to the exact value of the ρ intercept and slope, but the qualitative agreement is encouraging and conducive to further investigations of the extent of the validity of the proposed models.

On the other hand, incorporating unitarity into the Veneziano model has proved to be an extremely difficult task. The functions used in the existing models have exact Regge behavior only when the trajectory functions are linear, so that it is difficult to introduce a cut in $\alpha(t)$ without ruining the simple Regge-pole behavior. This is, of course, connected with the fact that when a

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[1] G. Veneziano, Nuovo Cimento **57**, 190 (1968).

[2] C. Lovelace, Phys. Letters **28B**, 264 (1968); J. Shapiro and J. Yellin, Report No. UCRL-18500 (unpublished).

trajectory passes through an integer n, in addition to the resonances with angular momentum less than or equal to n, the Veneziano formula with nonlinear trajectories also contains resonances in the higher partial waves. Moreover, the addition of an imaginary part to $\alpha(t)$ in simple Veneziano-type models predicts equal widths for equal-mass resonances, a fact which is not borne out by experiment. The behavior of Regge trajectories near threshold also indicates that the $\alpha(t)$ in a Veneziano formula can only be an approximation to the true trajectory function. From elastic unitarity and the assumption of simple poles, one can prove that the elastic cut of $\alpha(t)$, in general, is not a square-root cut unless the value of $\alpha(t)$ at threshold is exactly an integer or a half-integer. The amplitude, of course, contains a square-root cut at the first threshold of the t channel.

In this paper we intend to investigate the conjecture that in the presence of a small parameter, simple Veneziano-type formulas, such as the $\pi\pi$ model of Lovelace, only represent the zeroth-order approximation to strong-interaction amplitudes, while the next-order solution need not be easily expressible in terms of Veneziano-type functions. The quantity $\text{Im}\alpha(t)$ is, of course, a good candidate for the small parameter of the problem. For the values of energy where $\text{Im}\alpha$ has been measured through its relation to the width and the mass of resonances ($\text{Im}\alpha=\alpha' m\Gamma$), this quantity has proved to be small compared to unity. For example, for the ρ trajectory $\text{Im}\alpha\approx 0.1$ at $t=m_\rho^2$.

Since elastic unitarity has its simplest form when expressed in terms of states with definite angular momentum, we begin with the partial-wave amplitude corresponding to the Veneziano form and assume that it is the zeroth-order approximation to the true partial wave. A perturbation of this amplitude would presumably change the position of the nth Regge pole at $\alpha_0(t)-n+1$ by a small independent complex quantity so that the new trajectories would not be integer-spaced and parallel.[3] [$\alpha_0(t)$ denotes the zeroth-order linear trajectory of the Veneziano form.] Ideally, we would like the perturbed amplitude to be unitary and continue to satisfy the requirements of crossing. However, it is difficult to impose the latter condition to all orders. Therefore, in this paper we will be content with crossing symmetry to zeroth order and investigate the question of elastic unitarity. Although a complete solution to the problem was not obtained, we will discuss a model which contains many of the desired features of unitarity, analyticity, and Regge behavior.

[3] Actually, since the residue of the trajectories in simple Veneziano models does not correspond to the residues of Freedman-Wang–type daughter trajectories at $t=0$, we may expect that the nth pole is n-fold degenerate and that the small perturbation breaks this degeneracy. However, in this paper we have neglected this splitting of poles and considered only simple displacements of their position.

Before delving into a detailed discussion of unitarity, we should mention the difficulty with the threshold behavior of the partial-wave amplitude corresponding to a Veneziano formula in the left-hand part of the complex l plane. In potential scattering with Regge poles alone, an infinite number of poles approach the line $\text{Re}l=-\frac{1}{2}$, leading to the correct behavior at threshold in the entire l plane. It is difficult to see how a small perturbation of the Veneziano formula will produce this infinite set of poles at the required positions, since the poles of the zeroth-order solution lie along the real axis. The example that we present in the next section overcomes this difficulty at threshold by introducing cuts associated with each pole of the zeroth-order solution. As we will discuss in the next section, the introduction of cuts seems to be the simplest way to overcome the difficulty with threshold behavior.

We have given the details of the partial-wave projection of the $\pi\pi\rightarrow\pi\pi$ amplitude for the Lovelace-Veneziano model, and the continuation into the l plane, in the Appendix of this paper. Irrespective of the question of unitarity, the reader may find the content of this Appendix useful for other investigations of the Veneziano model. In fact, if the reader is not very familiar with such models, he may find it helpful to read the Appendix first.

II. SIMPLE EXAMPLE OF A SMALL PERTURBATION

We consider the partial-wave amplitude corresponding to a Veneziano formula as the zeroth-order approximation to the real amplitude and denote it by $a_0(l,t)$. (We suppress the isospin indices of the $\pi\pi$ problem.) $a_0(l,t)$ depends on an over-all constant β and the zeroth-order trajectory function $\alpha_0(t)$. As shown explicitly in the Appendix, $a_0(l,t)$ has poles at $\alpha_0(t)$, $\alpha_0(t)-1$, $\cdots$, and decreases exponentially as l goes to infinity in the right-hand plane. The residue of the pole at a given α behaves like $(q_t^2)^{\alpha(q_t=0)}$ near threshold, which is the correct threshold behavior for the poles lying to the right of the $\text{Re}l=-\frac{1}{2}$ axis. The standard arguments of unitarity plus the assumption of simple poles through the introduction of an infinite number of poles approaching the $\text{Re}l=-\frac{1}{2}$ axis lead to the behavior $(q_t^2)^{-(\alpha+1)}$ for the residue of the poles in the left-hand plane.[4] Since no small perturbation can change a $(q^2)^\alpha$ into a $(q^2)^{-(\alpha+1)}$ behavior, the perturbation of $a_0(l,t)$ should introduce effects which will invalidate the standard arguments, and allow the $(q_t^2)^{\alpha(q_t=0)}$ behavior even for $\text{Re}\alpha<-\frac{1}{2}$. The simplest possibility is to introduce weak cuts in the l plane whose position coincides with that of the poles at the elastic threshold. This is, in fact, the result obtained

[4] A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962); V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962).

from the simple construction that we present in the following discussion.

We first construct a function $a_1(l,t)$ which is approximately equal to $a_0(l,t)$ almost everywhere except near the position of the poles. Moreover, the positions of these poles are shifted by a small amount into the complex plane. This procedure is nonunique, but we are not concerned with the question of uniqueness here and wish to investigate the possibility of the conjecture of this paper by constructing a simple example. We thus define a function $c(l,t)$ by

$$c(l,t)=\frac{[l-\alpha_0(t)][l-\alpha_0(t)+1]\cdots}{[l-\alpha_0(0)-\sigma_1(t)][l-\alpha_0(t)-\sigma_2(t)+1]\cdots}. \quad (2.1)$$

The functions $\sigma_i(t)$ are numerically small functions with cuts at the t-channel normal thresholds. For values of t above the elastic threshold we denote the real and imaginary part of these functions by $\delta_i(t)$ and $\epsilon_i(t)$, so that $\epsilon_i(t)$ is the imaginary part of the ith trajectory function. The over-all scale of the functions $\sigma_i(t)$ which is the scale of the imaginary part of the trajectories is the small parameter of the expansion we are investigating in this paper.

We define a function $a_1(l,t)$ by

$$a_1(l,t)\equiv c(l,t)a_0(l,t). \quad (2.2)$$

Since for small $\sigma_i(t)$, $c(l,t)$ is close to unity everywhere in the l plane except at the position of its poles and zeros, $a_1(l,t)$ is in fact approximately equal to $a_0(l,t)$, except that the position of each pole is shifted by a small amount $\sigma_i(t)$. Moreover, we require that $a_1(l,t)$ as a function of t be real-analytic with a square-root branch cut at the elastic threshold. This can be most easily achieved if these properties are satisfied by the $\sigma_i(t)$ individually. We then have to investigate the possibility of unitarizing $a_1(l,t)$ with a small perturbation consistent with the small magnitude, the nature of the elastic cut, and the real-analyticity of $\sigma_i(t)$.

For integer l and values of t in the elastic region, the unitarity relation is written as

$$\mathrm{Im}a_l(t)=\rho(t)a_l(t)a_l^*(t), \quad (2.3)$$

where $\rho(t)=q_t/t^{1/2}$. (Throughout this paper, when l is restricted to integer values it will be written as a subscript, otherwise as the argument of a function.) For fixed values of t in this region Eq. (2.3) can be continued into the entire l plane:

$$a(l,t)-a^*(l^*,t)=2i\rho a(l,t)a^*(l^*,t). \quad (2.4)$$

For real-analytic functions $a_l(t)$, Eq. (2.3) can also be written as a discontinuity relation and continued in the t variable:

$$a_l(t)-a_l(t_2)=2i\rho(t)a_l(t)a_l(t_2). \quad (2.5)$$

Let us first restrict ourselves to fixed values of t in the elastic region and consider the function $a(l,t)$ defined by

$$a(l,t)=\{a_1(l,t)a_1^*(l^*,t)-\rho^2[a_1(l,t)a_1^*(l^*,t)]^2\}^{1/2}+i\rho a_1(l,t)a_1^*(l^*,t). \quad (2.6)$$

In the region $\mathrm{Re}l\gg\alpha_0(t)$, $a_0(l,t)$ is much smaller than unity and $c(l,t)$ is close to unity. The function $a_1(l,t)\times a_1^*(l^*,t)$ is real-analytic in l so that for real l in this region the function under the square-root sign is in fact positive. The function $a(l,t)$ then satisfies Eq. (2.4) in the region $\mathrm{Re}l\gg\alpha_0(t)$ and therefore by analytic continuation is unitary for l in the entire complex plane (t is still fixed in the elastic region). We now show that a certain number of the parameters $\sigma_i(t)$ can be so chosen as to make $a(l,t)$ approximately equal to $a_0(l,t)$ for most values of l.

Since $a_0(l,t)$ is very small compared to unity for $\mathrm{Re}l\gg\alpha_0(t)$, $a_0^2(l,t)$ can be neglected compared to $a_0(l,t)$, so that in this region $a(l,t)$ and $a_0(l,t)$ are in fact approximately equal. As we approach the point $\alpha_0(t)$—say, along the real axis—we reach a point where $a_1(l,t)a_1^*(l^*,t)$ becomes equal to $1/\rho(t)$. At this point the expression under the square-root sign vanishes. This value of l is then the beginning of the cut corresponding to the first Regge pole $\alpha_1(t)$. When the residue of the pole at $\alpha_0(t)$ is small, which is actually the case with the $\pi\pi$ model discussed in the Appendix, this cut occurs very close to the pole and, in fact, the position of the two singularities coincides if t is at the elastic threshold.

As we now go further to the left approaching the poles of $a_1(l,t)a_1^*(l^*,t)$ at $\alpha_0(t)+\sigma_1(t)$ and $\alpha_0(t)+\sigma_1^*(t)$, the second term under the square-root sign becomes large compared to the first one and we can write

$$a(l,t)=\pm i\rho a_1(l,t)a_1^*(l^*,t)+i\rho a_1(l,t)a_1^*(l^*,t)+\text{regular terms}. \quad (2.7)$$

The $\pm$ correspond to values of l on the upper or the lower side of the cut, so that we find a pole on the first sheet at $\alpha_0(t)+\delta_1(t)+i\epsilon_1(t)$, while the pole at $\alpha_0(t)+\delta_1(t)-i\epsilon_1(t)$ is under the cut on the second sheet. In order to find the residue we note that if $\sigma_i(t)$ are very small, then near this pole

$$a_1(l,t)\approx\frac{(\delta_1+i\epsilon_1)a_0(l=\alpha_0+\delta_1+i\epsilon_1,\ t)}{l-\alpha_0-\delta_1-i\epsilon}. \quad (2.8)$$

Since $\lim_{\delta_1+i\epsilon_1\to0}(\delta_1+i\epsilon_1)a_0(l=\alpha_0+\delta_1+i\epsilon_1,\ t)=\beta_1$, where β_1 is the residue of the original function $a_0(l,t)$ at $l=\alpha_0(t)$, we can write

$$a_1(l,t)\approx\frac{\beta_1}{l-\alpha_0-\delta_1-i\epsilon_1}. \quad (2.9)$$

Similarly,

$$a_1^*(l^*,t)\approx\frac{(\delta_1-i\epsilon_1)a_0^*(l=\alpha_0^*+\delta_1-i\epsilon_1,\ t)}{2i\epsilon_1}\approx\frac{\beta_1}{2i\epsilon_1}. \quad (2.10)$$

Therefore, the residue of the pole of $a(l,t)$ is given by

$$r_1 \approx \frac{2i\rho(\delta_1^2+\epsilon_1^2)a_0(l=\alpha_0+\delta_1+i\epsilon_1,\, t)a_0^*(l=\alpha_0+\delta_1-i\epsilon_1,\, t)}{2i\epsilon_1}$$
$$\approx \rho\beta_1^2/\epsilon_1. \quad (2.11)$$

Thus, if we choose ϵ_1 to be equal to $\rho\beta_1$ in first order, the residue of the pole of $a(l,t)$ will be almost equal to the residue of the pole of $a_0(l,t)$ at $\alpha_0(t)$. The same argument holds for all the Regge poles to the right of the line $\mathrm{Re}l=-\frac{1}{2}$, so that the imaginary part of these trajectories is determined, at least to first order, to be $\rho\beta_i$, where β_i is the residue of the ith pole in the original zeroth-order solution. As shown in the Appendix, the $\rho\beta_i$ behave like $(q_t^2)^{\alpha_i(q_t=0)+1/2}$ near threshold. For the $\pi\pi$ problem with the over-all scale β chosen to fit the width of the ρ meson, in the region of t under consideration, the $\rho\beta_i$ are small compared to unity. The function $\epsilon_i(t)$ are therefore small and $a(l,t)$ is in fact approximately equal to $a_0(l,t)$ in all of the half-plane to the right of the $\mathrm{Re}l=-\frac{1}{2}$ axis. Note that for small residues $a_0(l,t)$ rapidly decreases as l goes further to the left of, say, $\alpha_0(t)$, before it increases again near $\alpha_0(t)-1$. Therefore, the square-root expression vanishes again to the left of $\alpha_0(t)$ and the cut can be drawn to extend over a finite region of the real l axis.

Before discussing the left-hand part of the l plane, we should first consider the behavior in the variable t. In order for Eqs. (2.4) and (2.5) to be equivalent, as a function of t, $a_l(t)$ must be real-analytic. In that case for integer l, $a^*(l^*,t)$ can be replaced by $a_l(t^*)\equiv a_l(t_2)$ and continued in the t plane. Note that although the quantity under the square-root sign vanishes at $q_t^2=0$, for fixed integer $l>0$ the vanishing is like $(q_t^2)^{2l}$, so that the square root introduces no extra cuts at threshold. The cut of $a(l,t)$ at $q_t^2=0$ is due to the phase-space factor ρ and the functions $\sigma_i(t)$. If we require some of the $\epsilon_i(t)$ to be approximately equal to $\rho\beta_i(t)$, then the threshold behavior of $\epsilon_i(t)$ is given by $(q_t^2)^{[\alpha_0(q_t=0)-i+1]+1/2}$. In order for the cut of $\sigma_i(t)$ to be of the square-root type, a property which will ensure the real-analyticity of $\sigma_i(t)$ both on the first and the second sheets of the elastic cut, $\alpha_0(q_t=0)$ must be an integer or a half-integer. Thus, for the $\pi\pi$ example of the Appendix we would require $\alpha_{0\rho}(q_t=0)$ to be equal to $\frac{1}{2}$. Note that $\alpha_{0\rho}(q_t=0)$ does not represent the true value of the ρ trajectory at threshold, since this quantity will be perturbed by the value of $\delta_1(t)$ at $q_t^2=0$. With this choice of $\alpha_0(q_t^2=0)$, we now write $a(l,t)$ for integer l in a form continuable in the entire t plane:

$$a_l(t)=\{a_{1l}(t)a_{1l}(t_2)-\rho^2[a_{1l}(t)a_{1l}(t_2)]^2\}^{1/2} + i\rho a_{1l}(t)a_{1l}(t_2). \quad (2.12)$$

As mentioned above, although the quantity under the square-root sign vanishes at threshold, there is no extra cut introduced there. As we increase t from its value at threshold, we reach a point at which $a_{1l}(t)a_{1l}(t_2) = |a_{1l}(t)|^2$ is equal to $1/\rho$. This is the beginning of a cut in t associated with the first resonance of $a_l(t)$. This cut is, of course, the image of the cut in the l plane. For small-width resonances, the resonance pole lies slightly to the right of this cut in the complex plane. Although numerically this combination of a cut (probably of finite extent) and a pole is not very different from a pole alone, these cuts in the t plane constitute a bad feature of our example.

Finally, we briefly discuss the behavior to the left of the $\mathrm{Re}l=-\frac{1}{2}$ axis. The residues of the poles of $a(l,t)$ in this half-plane behave like $(q_t^2)^{\alpha(q_t=0)}$, but due to the existence of the cuts, this is not in contradiction with unitarity since $a(l,t)$ is manifestly unitary. However, in regions of t where a given $\beta_i(t)$ is large due to threshold behavior, we cannot require the relation $\epsilon_i\approx\rho\beta_i$ to hold since without the condition $|\epsilon_i(t)|\ll 1$ our perturbation would have no meaning. Therefore, for poles in the left-hand plane we have to abandon the condition $\epsilon_i(t)\approx\rho\beta_i(t)$ near threshold and choose a function $\epsilon_i(t)$ which stays small even near $q_t^2=0$. It is interesting to note that, as discussed in the Appendix, if the value of $\alpha_0(q_t=0)$ is a half-integer, the function $a_0(l,t)$ near threshold has a zero near each pole in the left-hand plane. Thus, small values of ϵ_i and δ_i can still be chosen such that the residue of the pole at $\alpha_i=\alpha_0-i+1+\sigma_i(t)$, i.e., the quantity $\rho(\delta_i^2+\epsilon_i^2)a_0(l=\alpha_i,\, t)a_0^* \times(l=\alpha_i^*,\, t)/\epsilon_i$ of Eq. (2.11), is almost equal to β_i. It is thus possible to make the function $a(l,t)$ and $a_0(l,t)$ almost equal even near threshold and near the poles to the left of the $\mathrm{Re}l<-\frac{1}{2}$ axis.

In addition to the cuts associated with poles in the left-hand plane, there also exists a cut caused by the threshold behavior of $a_0(l,t)$. This is due to the fact that irrespective of poles the $(q_t^2)^l$ behavior of $a_0(l,t)$ causes the square-root expression to vanish for some l to the left of $-\frac{1}{2}$ if q_t^2 is close to zero. Therefore, this cut begins at $l=-\frac{1}{2}$ for $q_t^2=0$ and recedes to the left as t moves away from threshold. In fact, it is due to the existence of this cut that unitarity and the $(q_t^2)^{2l+1/2}$ behavior of the imaginary part of the amplitude in the left-hand plane are not in contradiction.

We have thus given an example of a unitary function $a(l,t)$ with Regge poles as well as cuts associated with each pole. The function $a(l,t)$ in some sense differs from the partial wave corresponding to the Veneziano form only to first order in $\mathrm{Im}\alpha$, a quantity which experimentally seems to be small compared to unity. This approximate equality of $a(l,t)$ and $a_0(l,t)$ is not as valid for $\mathrm{Re}l<-\frac{1}{2}$ and t near threshold, due to the threshold behavior of $a_0(l,t)$. The parameters ϵ_i corresponding to the imaginary parts of the trajectories seem to be determined at least to first order through the constraint of unitarity and the requirement that the difference of $a(l,t)$ and $a_0(l,t)$ be a small quantity. Since in this model the trajectories do not intersect each other, one can write a simple dispersion relation for each $\alpha_i(t)$.

Thus the parameters $\delta_i(t)$ are not independent of $\epsilon_i(t)$ and in a sense they are also determined. However, the over-all scale of the widths—namely, the constant β multiplying the entire amplitude—is not determined in the model presented here. The only requirement is that β should be small enough to allow both the smallness of the quantities ϵ_i and the approximate relation $\epsilon_i \approx \rho\beta_i$ for the first few trajectories. The question of uniqueness was neglected in this paper, but one may expect that some of the features of the example presented here will be present in any other perturbation of $a_0(l,t)$ consistent with unitarity.

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APPENDIX

The function $V(t,s)$ is defined as

$$V(t,s)=\frac{\Gamma(1-\alpha(t))\Gamma(1-\alpha(s))}{\Gamma(1-\alpha(t)-\alpha(s))}, \tag{A1}$$

where α refers to the ρ trajectory function. One possible set of crossing-symmetric amplitudes with definite isospin in the t channel is[2]

$$\begin{aligned} A^2(t,s)&=2\beta V(s,u),\\ A^1(t,s)&=2\beta[V(t,s)-V(t,u)],\\ A^0(t,s)&=3\beta[V(t,s)+V(t,u)]-\beta B(s,u). \end{aligned} \tag{A2}$$

For a linear $\alpha(t)$, the asymptotic behavior of $A^1(t,s)$ and $A^0(t,s)$ correspond to exchange-degenerate ρ and f^0 Regge trajectories. The asymptotic behavior of $A^2(t,s)$ corresponds to that of the background integral in the Sommerfeld-Watson transform, since there are no Regge poles, and therefore no resonances, in the partial-wave amplitudes of this channel. When $\alpha(t)$ passes through an integer n, $A^1(t,s)$ $[A^0(t,s)]$ contains zero-width resonances in the odd (even) partial-wave amplitudes with angular momentum less than or equal to n.

For the partial-wave projection, we need to define the quantities a and b as

$$\begin{aligned} a&=1-\alpha(0)+2\alpha' q_t^2,\\ b&=\alpha' q_t^2, \end{aligned} \tag{A3}$$

so that

$$\begin{aligned} 1-\alpha(s)&=a-2bz,\\ 1-\alpha(u)&=a+2bz. \end{aligned} \tag{A4}$$

The quantity $z_n=(a+n)/2b$ denotes the position of the poles of $\Gamma(a-2bz)$ at negative integers with a residue $(-1)^{n+1}/2bn!$.

The partial-wave amplitude $g_l(t)$ is first defined for integer l,

$$\begin{aligned} g_l(t)&=\frac{1}{2}\int_{-1}^{1} V(t,s(z))P_l(z)dz\\ &=\frac{1}{2\pi i}\int_c V(t,s(z))Q_l(z)dz. \end{aligned} \tag{A5}$$

We take contour c to be the unit circle centered around the origin with a clockwise sense. The radius of this circle can now be increased to infinity and the contribution of each pole in the z plane calculated. For $l>\alpha(t)$ the contribution of the contour at infinity vanishes and $g_l(t)$ can be written as a sum of the contributions of the poles of $V(t,s(z))$:

$$g_l(t)=-\frac{1}{2b}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{\Gamma(1-\alpha_t)}{\Gamma(-\alpha_t-n)}Q_l(z_n). \tag{A6}$$

Similarly, the function $h_l(t)$ defined by

$$h_l(t)=\frac{1}{2}\int_{-1}^{1} V(s(t,z),u(t,z))P_l(z)dz \tag{A7}$$

can be written as

$$\begin{aligned} h_l(t)&=0, && l \text{ odd}\\ &=-\frac{1}{b}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{\Gamma(n+2a)}{\Gamma(2a-1)}Q_l(z_n), && l \text{ even}. \end{aligned} \tag{A8}$$

The sum representing $g_l(t)$ converges for all $l>\alpha(t)$ and is correct in its present form for l in the complex plane to the right of the line $\mathrm{Re}\,l=\alpha(t)$. Therefore, for the $\pi\pi$ problem, taking the symmetry of the amplitudes in the z variable into account, we can write the signatured amplitudes

$$\begin{aligned} a_2^-(l,t)&=a_1^+(l,t)=a_0^-(l,t)\equiv 0,\\ a_2^+(l,t)&=2h(l,t),\\ a_1^-(l,t)&=4g(l,t),\\ a_0^+(l,t)&=6g(l,t)-h(l,t). \end{aligned} \tag{A9}$$

We now consider the problem of the continuation of $g(l,t)$ and $h(l,t)$ into the complex l plane to the left of the line $\mathrm{Re}\,l=\alpha(t)$. We first restrict ourselves to values of t above threshold for which $0\le\alpha(t)\le 1$ and $a(t)>0$. The quantities z_n are then greater than unity, so that we can expand $Q_l(z_n)$ in powers of $1/z_n$:

$$Q_l(z_n)=\sum_k (2b)^{l+1+2k}f_k(l)\frac{1}{(n+a)^{l+1+2k}}, \tag{A10}$$

where

$$f_k(l)=\frac{1}{2}\frac{\Gamma(1+\frac{1}{2}l+k)\Gamma(\frac{1}{2}+\frac{1}{2}l+k)}{\Gamma(l+\frac{3}{2}+k)k!}. \tag{A11}$$

For integer l, Eq. (A10) can be written as

$$Q_l(z_n)=\sum_{k=0}^{\infty}(2b)^{l+1+2k}f_k(l)\frac{(-1)^{l+2k}}{\Gamma(l+1+2k)}\times\frac{d^{l+2k}}{da^{l+2k}}\frac{1}{a+n}. \quad (A12)$$

For integer $l>\alpha$ we interchange the summation over k and n in (A6) and use the following expansion for $B(a,1-\alpha_t)$ good for $\alpha_t<1$:

$$(a-\alpha_t)B(a,1-\alpha_t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(-\alpha_t)\times(-\alpha_t-1)\cdots(-\alpha_t-n)\frac{1}{a+n}. \quad (A13)$$

The function $g_l(t)$ can then be represented by the sum

$$g_l(t)=\sum_{k=0}^{\infty}(2b)^{l+2k}f_k(l)\frac{(-1)^{l+2k}}{\Gamma(l+1+2k)}\frac{d^{l+2k}}{da^{l+2k}}\times[(a-\alpha_t)B(a,1-\alpha_t)]. \quad (A14)$$

We now write an integral representation for $B(a,1-\alpha_t)$ good for $a>0$ and $\alpha_t\leq 1$,

$$B(a,1-\alpha_t)=\int_0^1 x^{a-1}(1-x)^{-\alpha_t}\,dx. \quad (A15)$$

Therefore,

$$\frac{d^m}{da^m}(a-\alpha_t)B(a,1-\alpha_t)=\int_0^1 x^{a-1}(1-x)^{-\alpha}(\ln x)^{m-1}\times[m+(a-\alpha_t)\ln x]dx. \quad (A16)$$

Let us define the function $R(a,\alpha,\nu)$ for $a>0$ and $\mathrm{Re}\nu>\alpha$ by the integral

$$R(a,\alpha,\nu)=\int_0^1 x^{a-1}(1-x)^{-\alpha}\left(\ln\frac{1}{x}\right)^{\nu-1}dx=\int_0^{\infty}e^{-ax}(1-e^{-x})^{-\alpha}x^{\nu-1}\,dx. \quad (A17)$$

The partial-wave amplitude $g(l,t)$ is then given by the sum

$$g(l,t)=\tfrac{1}{2}\pi^{1/2}\sum_k b^{l+2k}\frac{1}{k!\Gamma(l+\frac{3}{2}+k)}\times[-(l+2k)R(a,\alpha,l+2k)+(a-\alpha_t)R(a,\alpha,l+2k+1)]. \quad (A18)$$

The sum over k is now convergent even for $\mathrm{Re}l<\alpha$, since the convergence depends on high values of k for which $l+2k$ can be greater than α. Therefore, this representation for $g(l,t)$ is good in the entire l plane except possibly at $\mathrm{Re}l=-\infty$. The function $R(a,\alpha,\nu)$, of course, has to be continued to the region $\mathrm{Re}\nu<\alpha$. For special values of α the function $R(a,\alpha,\nu)$ can be related to known functions. For example, for $\alpha=1$, $R(a,\alpha,\nu)=\Gamma(\nu)\zeta(\nu,a)$, where $\zeta(\nu,a)$ is the generalized zeta function. In general, we note from the integral representation (A17) that $R(a,\alpha,s)$ has poles at $\nu=a$, $\nu=\alpha-1,\cdots$. If we denote the function $e^{-ax}(1-e^{-x})^{-\alpha}x^{\alpha}$ by $f(x)$, then the residues of the poles at $\alpha,\alpha-1,\cdots$, are easily obtained from the Taylor expansion of $f(x)$ around the point $x=0$. Thus, the nth pole at $l=\alpha_n(t)=\alpha(t)-n+1$ occurs in a finite number of the terms in (A18) with $k=0,1,\cdots,\frac{1}{2}n-\frac{1}{2}$ (or $\frac{1}{2}n-1$). The dominant behavior near threshold will always come from the $k=0$ term, so that the threshold behavior of $\beta_n(t)$ is given by $(\alpha' q_t^2)^{\alpha_n(q=0)}$. Note that the factor $1/\Gamma(l+\frac{3}{2}+k)$ introduces zeros at $l=-\frac{3}{2}-n$, where n is an integer. Thus, if $\alpha(q^2=0)$ is equal to $\frac{1}{2}$, the terms in which $(q_t^2)^{\alpha_n}$ become infinite near threshold also have a zero near the position of their poles. This point was utilized in the discussion of Sec. II of this paper.

Finally, we consider the quantity $h(l,t)$. From the asymptotic behavior of $V(s,u)$ we already know that $h(l,t)$ does not have any moving poles. However, fixed poles at negative odd integers are allowed since they would not contribute to the asymptotic behavior of $V(s,u)$. The explicit continuation of $h(l,t)$ into the entire l plane leads to a more complicated representation than (A18). However, for special values of t the representation becomes rather simple and demonstrates the existence of the fixed poles at negative odd integers. Let us choose a value of $t=t_1$ for which $a=1$. We then have

$$h(l,t)=-\frac{1}{b}\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)Q_l(z_n). \quad (A19)$$

Expanding $Q_l(z_n)$ in powers of $1/z_n$ and interchanging the sums, we find

$$h(l,t_1)=2\sum_{k=0}^{\infty}(2b)^{l+2k}f_k(l)(1-2^{1-l-2k})\zeta(l+2k), \quad (A20)$$

where $\zeta(\nu)$ is the Riemann zeta function. The only poles of $h(l,t)$ are the poles of $f_k(l)$ at negative integers. For a given k, $f_k(l)$ has poles at $l=-(2k+1),-(2k+2),\cdots$. However, $\zeta(l+2k)$ has zeros when $l+2k$ is an even negative integer. Thus, the function $h(l,t_1)$ has fixed poles at all negative odd integers. Note that in the $\pi\pi$ model these fixed poles are additive since they do not affect the residues of the moving poles in $g(l,t)$.