tion as determined from our angular distribution assuming set I for the $T=1$ phase shifts.

All solutions listed in Table I have been obtained for the moving-neutron angular distribution in the form-factor impulse approximation. We have verified the existence of similar solutions for the moving- and stationary-neutron angular distributions with the closure and form-factor impulse approximations. In all solutions for the moving-neutron angular distributions, a better χ^2 is obtained with the form-factor method. To allow for all the uncertainties caused by the deuteron, the errors estimated from χ^2 minimization have been doubled.

We conclude: Both the standard and CD methods of analysis give equivalent results and establish a Yang-type phase-shift solution for the $T=0$ channel. The data are consistent with S- and P-wave K-nucleon scattering in both the $T=0$ and 1 channels. When fitting the angular distribution alone, the CD method usually gives fewer solutions for a given number of parameters; in particular, the CD method gives no acceptable $T=0$ Fermi solution. The $T=0$ Yang solution is also consistent with the ratio $K_S{}^0p/(\Lambda\pi^+ + 2\Sigma^0\pi^+)$ observed in $K_L{}^0p$ scattering at comparable energies.[15]

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[15] J. A. Kadyk, Y. Oren, G. Goldhaber, S. Goldhaber, and G. H. Trilling, Phys. Rev. Letters **17**, 599 (1966); J. K. Kim, *ibid.* **19**, 1074 (1967).

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Dilatation Covariance and Exact Solutions in Local Relativistic Field Theories*

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Consideration is given to the invariance of field equations under space-time dilatations with (induced) transformation of constant parameters therein. Analysis of such "dilatation covariance" facilitates the determination of rigorous closed-form singularity-free solutions to essentially nonlinear classical field equations. More over, for essentially nonlinear quantized (boson) field theories, dilatation-covariance considerations facilitate the determination of rigorous closed-form solutions to the Schrödinger stationary-state equation.

BECAUSE it has been customary to regard mass constants and coupling constants that appear in Lagrangians as being absolutely fixed, space-time dilatation (scale) transformations have played a minor role in the analysis of local relativistic field theories. Only special classical field theories feature dilatation-transformation invariance with absolutely constant physical parameters in their Lagrangians, and such theories do not ordinarily retain dilatation invariance when subject to quantization.[1] On the other hand, dilatation *covariance* on both the classical and the quantum level is a property of all local relativistic field theories with the appropriate (induced) dilatation transformation of parameters in the field equations. Our purpose in this article is to show that dilatation-covariance considerations not only facilitate the determination of rigorous closed-form solutions to classical field equations, but also facilitate the determination of rigorous closed-form solutions to the Schrödinger stationary-state equation in local relativistic quantum field theories.

We illustrate the general notion and utility of dilatation covariance by discussing a specific model theory, the self-interacting complex scalar field theory based on the Lagrangian density

$$\mathcal{L} = -(\partial\psi^*/\partial x^\mu)(\partial\psi/\partial x_\mu) - m^2\psi^*\psi + g\psi^*\psi \ln\psi^*\psi, \quad (1)$$

where m^2 and g are real physical constants and a system of convenient physical units is employed. The derived classical field equation

$$\partial^2\psi/\partial x^\mu\partial x_\mu - m^2\psi + g(1+\ln\psi^*\psi)\psi = 0 \quad (2)$$

admits solutions $\psi=\psi(x;m^2,g)$ that depend parametrically on m^2 and g with $x=(x^0,x^1,x^2,x^3)$ a point in space-time. For distinct pairs of values of m^2 and g, the manifolds of solutions[2] to Eq. (2) are homologous, the

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[1] For the implications of dilatation invariance in a special nonlinear field model, see H. Mitter, Nuovo Cimento **32**, 1789 (1964).

[2] Suitable regularity conditions associated with the existence of global solutions must supplement Eq. (1) for definition of the manifold of classical (c-number) solutions; the existence of global solutions to the generic equation $\partial^2\psi/\partial x^\mu\partial x_\mu - U'(\psi^*\psi)\psi = 0$ has been the subject of numerous recent investigations: K. Jörgens, Math. Z. **77**, 295 (1961); I. Segal, Proc. Symp. Appl. Math. **17**, 210 (1965); C. S. Morawetz, Proc. Roy. Soc. (London) **306A**, 291 (1968).

correspondence of solutions being given by

$$\psi(x;m^2,g)=\lambda\psi(\xi x;\xi^{-2}(m^2-2g\ln\lambda),\xi^{-2}g), \quad (3)$$

with λ and ξ arbitrary positive constant parameters. Alternatively, the content of Eq. (3) is expressed by the covariance of Eq. (2) under the space-time dilatation transformations

$$\begin{aligned}\psi(x)&\to\lambda\psi(\xi x),\\ m^2&\to\xi^{-2}(m^2-2g\ln\lambda),\\ g&\to\xi^{-2}g.\end{aligned} \quad (4)$$

Notwithstanding the fact that Eq. (3) involves transformed values of m^2 and g on the right-hand side, the general group-theoretic method[3] for obtaining self-similar solutions to Eq. (2) is applicable; it leads, for example, to rigorous, spatially localized, singularity-free solutions of the form

$$\psi(x;m^2,g)=\exp\{ik_\mu x^\mu-\tfrac{1}{2}g[x_\mu x^\mu-(k_\mu x^\mu)^2/(k_\nu k^\nu)]+\tfrac{1}{2}g^{-1}(m^2+k_\mu k^\mu)+1\}, \quad (5)$$

where k_μ is a real timelike four-vector constant of integration related to the solution's canonical energy-momentum four-vector[4]

$$p_\mu=(\pi/g)^{3/2}[2(-k_\nu k^\nu)^{1/2}+g(-k_\nu k^\nu)^{-1/2}]\times\{\exp[g^{-1}(m^2+k_\nu k^\nu)+2]\}k_\mu. \quad (6)$$

Quantization of a scalar (boson) field that satisfies an essentially nonlinear dynamical equation like (1) is achieved by evoking the field-diagonal representation.[5] Then we have the equal-time canonical commutation relations

$$[\psi(\mathbf{x}),\pi(\mathbf{y})]=[\psi^*(\mathbf{x}),\pi^*(\mathbf{y})]=i\delta(\mathbf{x}-\mathbf{y}) \quad (7)$$

satisfied, with $\psi(\mathbf{x})$ and $\psi^*(\mathbf{x})$ acting effectively as c-number fields and the associated momentum densities prescribed as the (commuting) functional differential operators

$$\pi(\mathbf{x})=-i\delta/\delta\psi(\mathbf{x}),\quad \pi^*(\mathbf{x})=-i\delta/\delta\psi^*(\mathbf{x}). \quad (8)$$

We obtain the operator form of Eq. (1) by combining the Heisenberg equations of motion for $\psi(\mathbf{x})$ and $\pi(\mathbf{x})$, in which the canonical Hamiltonian operator appears as

$$H=\int[\pi^*(\mathbf{x})\pi(\mathbf{x})+\nabla\psi^*(\mathbf{x})\cdot\nabla\psi(\mathbf{x})+m^2\psi^*(\mathbf{x})\psi(\mathbf{x})-g\psi^*(\mathbf{x})\psi(\mathbf{x})\ln\psi^*(\mathbf{x})\psi(\mathbf{x})-u]d^3x, \quad (9)$$

with the disposable constant u inserted to compensate the zero-point quantum energy density. Stationary states are complex-valued functionals of $\psi(\mathbf{x})$ and $\psi^*(\mathbf{x})$ in the field-diagonal representation

$$\Psi=\Psi[\psi(\mathbf{x}),\psi^*(\mathbf{x});m^2,g,u,E], \quad (10)$$

satisfying the Schrödinger functional differential equation $H\Psi=E\Psi$. The Hamiltonian operator (9) is homologous with respect to the one-parameter subgroup of dilatation transformations (4) for which $\lambda=\xi$:

$$\begin{aligned}\psi(\mathbf{x})&\to\xi\psi(\xi\mathbf{x}),\\ \pi(\mathbf{x})&\to\xi^2\pi(\xi\mathbf{x}),\\ m^2&\to\xi^{-2}(m^2-2g\ln\xi),\\ g&\to\xi^{-2}g.\end{aligned} \quad (11)$$

It follows from the Schrödinger equation that the correspondence of stationary states is given by

$$\Psi[\psi(\mathbf{x}),\psi^*(\mathbf{x});m^2,g,u,E]=\xi^s\Psi[\xi^{-1}\psi(\xi^{-1}\mathbf{x}),\xi^{-1}\psi^*(\xi^{-1}\mathbf{x});\xi^{-2}(m^2-2g\ln\xi),\xi^{-2}g,\xi^{-4}u,\xi^{-1}E]. \quad (12)$$

In Eq. (12), s is a constant related to the state functional normalization with the form of the prefactor ξ^s on the right-hand side dictated by the subgroup composition law. For the vacuum state with energy eigenvalue $E=0$, Eq. (12) implies dilatation invariance; this generalizes the manifest dilatation invariance of the "bare" vacuum-state functional

$$\Psi[\psi(\mathbf{x}),\psi^*(\mathbf{x});m^2,0,u,0]=\exp\left(-\int\psi^*(\mathbf{x})(-\nabla^2+m^2)^{1/2}\psi(\mathbf{x})d^3x\right), \quad (13)$$

in which

$$u=[(-\nabla^2+m^2)^{1/2}\delta(\mathbf{x})]_{\mathbf{x}=0}\quad\text{for}\quad g=0.$$

The "dressed" vacuum state satisfies the functional differential equation $H\Psi=0$ with the Hamiltonian given by (8) and (9), and we solve for it by making the dilatation-invariant ansatz

$$\Psi=\Psi[\psi(\mathbf{x}),\psi^*(\mathbf{x});m^2,g,u,0]=\exp\left\{-\int\left[\psi^*(\mathbf{x})(-\nabla^2+\hat{m}^2)^{1/2}\psi(\mathbf{x})+\tfrac{1}{4}\epsilon g(\psi^*(\mathbf{x})\psi(\mathbf{x}))^2\times\left(a-1+\ln\frac{\psi^*(\mathbf{x})\psi(\mathbf{x})}{g}\right)\right]d^3x\right\}, \quad (14)$$

[3] G. Birkhoff, *Hydrodynamics* (Princeton University Press, Princeton, N. J., 1960), Chaps. 4 and 5; A. J. A. Morgan, Quart. J. Math. (Oxford) **2**, 250 (1952).

[4] It should be noted that the rigorous solution (5) is formally invariant under the dilatation transformations (4) if $k_\mu\to\xi^{-1}k_\mu$, or, equivalently, if $p_\mu\to\xi\lambda^{-2}p_\mu$; in general, the constants of integration in a solution acquire specific dilatation transformation character if one requires invariance of the solution.

[5] Recent applications of the field-diagonal representation have been presented: G. Rosen, Phys. Rev. **173**, 1680 (1968), and works cited therein. Although a representation of the field in terms of creation and annihilation operators diagonalizes the energy in the case of a linear field theory and provides a practical basis for perturbation calculations in the case of a quasilinear field theory, it is expedient to work with the field-diagonal representation for an essentially nonlinear (boson) field theory, just as it is expedient to work with the coordinate-diagonal representation (and the Schrödinger equation) for a particle system involving a potential energy that is not simply quadratic in the particle coordinates.

where the "dressed" mass $\hat{m}$ must transform under dilatations (11) $\hat{m}\rightarrow\xi^{-1}\hat{m}$, which implies that

$$\hat{m}^2=m^2-g\ln g+ag\,, \tag{15}$$

with a an absolute constant; the limit $\epsilon\rightarrow 0$ is understood to be taken in (14) in such a way that[6]

$$\lim_{\epsilon\rightarrow 0}\{\epsilon[\delta(\mathbf{x})]_{\mathbf{x}=0}\}=1\,,$$

implying the transformation character $\epsilon\rightarrow\xi^3\epsilon$ under dilatations (11). To verify that (14) is the exact[7] solution to the vacuum-state equation $H\Psi=0$, one simply computes

$$-\frac{\delta^2\Psi}{\delta\psi^*(\mathbf{x})\delta\psi(\mathbf{x})}=\{[(-\nabla^2+\hat{m}^2)^{1/2}\delta(\mathbf{x})]_{\mathbf{x}=0}+\epsilon[\delta(\mathbf{x})]_{\mathbf{x}=0}\{ag\psi^*(\mathbf{x})\psi(\mathbf{x})+g\psi^*(\mathbf{x})\psi(\mathbf{x})\ln[\psi^*(\mathbf{x})\psi(\mathbf{x})/g]\}-[(-\nabla^2+\hat{m}^2)^{1/2}\psi^*(\mathbf{x})][(-\nabla^2+\hat{m}^2)^{1/2}\psi(\mathbf{x})]+O(\epsilon)\}\Psi \tag{16}$$

[6] An immediate way to secure this relation is to introduce the wave-number cutoff-limit representation

$$\delta(\mathbf{x})=\lim_{K\rightarrow\infty}\int_{|\mathbf{k}|\leq K}e^{i\mathbf{k}\cdot\mathbf{x}}d^3k/(2\pi)^3,$$

for which $\epsilon=6\pi^2/K^3$.

[7] To be precise, we have

$$\lim_{\epsilon\rightarrow 0}(H_\epsilon\Psi)=0,\quad\text{where}\quad H_\epsilon\equiv H-\int\Lambda_\epsilon(\mathbf{x})d^3x,$$

with

$$\Lambda_\epsilon(\mathbf{x})\equiv-\tfrac{1}{2}\epsilon g\psi^*(\mathbf{x})\times\{(-\nabla^2+\hat{m}^2)^{1/2},\psi^*(\mathbf{x})\psi(\mathbf{x})[a-\tfrac{1}{2}+\ln\psi^*(\mathbf{x})\psi(\mathbf{x})/g]\}\psi(\mathbf{x})-\tfrac{1}{4}\epsilon^2g^2[\psi^*(\mathbf{x})\psi(\mathbf{x})]^3[a-\tfrac{1}{2}+\ln\psi^*(\mathbf{x})\psi(\mathbf{x})/g]^2,$$

i.e., the terms represented by $O(\epsilon)$ in (16). Such a modification of the Hamiltonian definition $H\rightarrow\lim_{\epsilon\rightarrow 0}H_\epsilon$ is necessary because $\Lambda_\epsilon(\mathbf{x})$ is not uniformly convergent to zero for all $\psi^*(\mathbf{x})$, $\psi(\mathbf{x})$ as $\epsilon\rightarrow 0$.

and puts $u=[(-\nabla^2+\hat{m}^2)^{1/2}\delta(\mathbf{x})]_{\mathbf{x}=0}$. The constant a in (14) is fixed by the functional-integral normalization condition[5] $\langle\Psi|\Psi\rangle=1$, and thus a is related to mass renormalization with $\hat{m}^2$ required to be finite and non-negative.

It is easy to abstract the analysis presented here for the specific model theory based on (1), and similar dilatation-covariance considerations can be applied to any other local relativistic field theory. For the generic self-interacting complex scalar field theory with the Hamiltonian operator

$$H=\int[\pi^*(\mathbf{x})\pi(\mathbf{x})+\nabla\psi^*(\mathbf{x})\cdot\nabla\psi(\mathbf{x})+U(\psi^*(\mathbf{x})\psi(\mathbf{x}))-u]d^3x, \tag{17}$$

the vacuum-state functional satisfying $H\Psi=0$ is given by[8]

$$\Psi=\exp-\int[\psi^*(\mathbf{x})(-\nabla^2+\hat{m}^2)^{1/2}\psi(\mathbf{x})+\epsilon F(\psi^*(\mathbf{x})\psi(\mathbf{x}))]d^3x\,, \tag{18}$$

where

$$F(\rho)\equiv\int_0^\rho[U(\tau)-\hat{m}^2\tau]\ln(\tau/\rho)d\tau\,, \tag{19}$$

with the "dressed" mass $\hat{m}$ prescribed in terms of physical constants in $U(\psi^*(\mathbf{x})\psi(\mathbf{x}))$ by dilatation covariance.

[8] In order to make this formal solution rigorous for $F(\rho)$ such that $\lim_{\rho\rightarrow\infty}F(\rho)/\rho\neq 0$, one must add to the Hamiltonian appropriate terms proportional to ϵ and ϵ^2. If the self-interaction energy density is such that $\lim_{\rho\rightarrow\infty}F(\rho)/\rho=0$, the vacuum-state solution (18) is a specialized form of the author's generic result in an unpublished report.