Applications of the Algebra of Current Densities to Third-Order Weak Amplitudes*

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The techniques developed in previous work for obtaining infinite-momentum sum rules for second-order weak amplitudes from current commutation relations are generalized so that they are applicable to third-order weak amplitudes. Sum rules are then obtained for a particular third-order current matrix element. We show that current-algebra sum rules for the corresponding second-order weak amplitudes can be obtained from our sum rules as special cases. On the basis of the sum rules, j-plane analyticity of third-order weak amplitudes is considered; its relation to j-plane analyticity of lower-order weak amplitude is pointed out. The Bjorken limit for third-order weak amplitudes is obtained and applied to discuss the radiative corrections to leptonic decays of pseudoscalar mesons.

I. INTRODUCTION

THE local commutation relations of current densities proposed by Gell-Mann[1] have been extensively applied recently[2] with great success, in particular, to processes involving two weak or electromagnetic currents of hadrons. It is the purpose of the present paper to develop techniques for systematically exploring the consequences of the algebra of current densities for third-order weak or electromagnetic processes, and to apply these techniques to obtain sum rules for these amplitudes.

The techniques that we shall develop are generalizations of those used in the second-order case.[3] We recall that in that case the function $M_{\mu\nu}{}^{\alpha\beta}$ defined by

$$M_{\mu\nu}{}^{\alpha\beta}=\int d^4x\, e^{ikx}\langle p|[J_\mu{}^\alpha(x),J_\nu{}^\beta(0)]|p'\rangle \quad (1.1)$$

plays a crucial role. The reason for this is that $M_{\mu\nu}{}^{\alpha\beta}$ is the discontinuity in k_0 of the second-order weak amplitude $T_{\mu\nu}{}^{\alpha\beta}$ given by

$$T_{\mu\nu}{}^{\alpha\beta}=i\int d^4x\, e^{ikx}\langle p|T\{J_\mu{}^\alpha(x),J_\nu{}^\beta(0)\}|p'\rangle. \quad (1.2)$$

In order to explore the consequences of local current algebra for the third-order weak amplitude given by

$$T^{\alpha\beta\gamma}=\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)} \times\langle 0|T\{J^\alpha(x_1),J^\beta(x_2),J^\gamma(x_3)\}|p'\rangle, \quad (1.3)$$

we need to obtain discontinuity expressions which bear the same relation to $T^{\alpha\beta\gamma}$ as $M_{\mu\nu}{}^{\alpha\beta}$ does to $T_{\mu\nu}{}^{\alpha\beta}$. In contrast to the second-order case, this turns out to be a nontrivial task. The difficulty is that the usual derivation of the T-product representation for the third-order weak-amplitude equation (1.3) by the Lehmann-Symanzik-Zimmermann (LSZ) reduction technique sheds no light on the discontinuity structure of the T-product representation. To overcome this difficulty, we assume in Sec. II that the third-order weak amplitude satisfies an unsubtracted Mandelstam representation for suitably fixed values of the energy variables of the currents. We then observe that there is a one-to-one correspondence between the various terms in the T-product representation and those in the Mandelstam representation. Thus, there is a very close connection between the time-ordering concept and crossing. Moreover, this correspondence allows us to prove several theorems on the discontinuity formulas for the T-product representation. In Sec. III, we obtain three discontinuity functions $F^{\alpha\beta\gamma}$, $G^{\alpha\beta\gamma}$, and $H^{\alpha\beta\gamma}$ of the third-order weak amplitude which bear the same relation to $T^{\alpha\beta\gamma}$ as $M_{\mu\nu}{}^{\alpha\beta}$ does to $T_{\mu\nu}{}^{\alpha\beta}$. It is, of course, crucial for the application of current algebra to the third-order weak amplitude that we can define discontinuity functions in terms of current operators with the desired properties. It may well be that our results are, in fact, more general and are independent of the no-subtraction hypothesis for Mandelstam representation. We do not pursue this point further in the present paper, since in order to obtain sum rules we have to assume unsubtracted dispersion relations in all three channels, and an unsubtracted Mandelstam representation is the simplest way to guarantee this. Thus, our statement is that if an unsubtracted Mandelstam representation holds, then our results follow.

Having obtained the discontinuity functions and elucidated their properties, it is straightforward using the infinite-momentum techniques to derive sum rules for third-order amplitudes from assumed current commutation relations. We do this in Sec. IV for a particular third-order matrix element. An important feature of our sum rules is immediately evident. That is, if we take suitable pole residues in the energy variables of currents, we can show that the third-order sum rules contain all the sum rules for the corresponding second-order weak amplitudes—those amplitudes that can be obtained by taking pole residues of the third-order amplitude—as special cases. This situation is, of course, very similar to those

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[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[2] S. L. Adler and R. F. Dashen, *Current Algebra and Applications to Particle Theory* (W. A. Benjamin Inc., New York, 1967); B. Renner, *Current Algebras and their Applications* (Pergamon Press Ltd., London, 1967).
[3] M. O. Taha, Nuovo Cimento **60A**, 651, 663 (1969); Phys. Rev. **162**, 1694 (1967). See also E. Y. C. Lu, *ibid.* **169**, 1308 (1968).

existing between the Fubini–Dashen–Gell-Mann sum rule and the strong-interaction superconvergence relations. This correspondence will have consequences on the j-plane analyticity of the third-order weak amplitudes.

Having obtained sum rules for the third-order amplitude, we may consider their implications for its j-plane analyticity. We find in Sec. V that, in general, fixed poles are present in its s-channel and u-channel helicity amplitudes as well as in its t-channel helicity amplitudes. The presence of these fixed poles is intimately related to the fact that the third-order amplitude must contain all the fixed poles present in its corresponding second-order amplitudes. In fact, current algebra requires just these fixed poles and no others.

In Sec. VI, we give a derivation of the Bjorken[4] limit for the third-order weak amplitude which agrees with the results of Olesen.[5] Our derivation, however, is closer in spirit to the original Bjorken derivation for second-order weak amplitude. We then consider the divergent part of the radiative corrections to the leptonic decays of pseudoscalar mesons on the basis of the Bjorken limit. We show that although one may obtain finite radiative corrections to vector β decay and finite electromagnetic mass difference of hadrons by specifying certain model-dependent commutators, these same commutators lead to radiative corrections to the leptonic decays of pseudoscalar mesons that are logarithmically divergent. Thus, it appears that logarithmic divergences are unavoidable in conventional quantum electrodynamics.

II. MANDELSTAM REPRESENTATION AND T PRODUCT

We assume for suitably fixed values of p^2, k^2, and k'^2 that the connected part of the third-order weak amplitude $T^{\alpha\beta\gamma}$ satisfies an unsubtracted Mandelstam representation[6]

$$\begin{aligned}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\langle 0|T\{J^\alpha(x_1),J^\beta(x_2),J^\gamma(x_3)\}|p'\rangle = -\frac{1}{\pi^2}\int_0^\infty ds'\int_{-\infty}^0 \frac{du'A_{12}(s',u')}{(s'-s-i\epsilon)(u'-u+i\epsilon)}\\ -\frac{1}{\pi^2}\int_0^\infty dt'\int_{-\infty}^0\frac{du'A_{12}(t',u')}{(t'-t-i\epsilon)(u'-u+i\epsilon)}-\frac{1}{\pi^2}\int_0^\infty du'\int_{-\infty}^0\frac{ds'A_{23}(u',s')}{(u'-u-i\epsilon)(s'-s+i\epsilon)}-\frac{1}{\pi^2}\int_0^\infty dt'\int_{-\infty}^0\frac{ds'A_{23}(t',s')}{(t'-t-i\epsilon)(s'-s+i\epsilon)}\\ -\frac{1}{\pi^2}\int_0^\infty ds'\int_{-\infty}^0\frac{dt'A_{13}(s',t')}{(s'-s-i\epsilon)(t'-t+i\epsilon)}-\frac{1}{\pi^2}\int_0^\infty du'\int_{-\infty}^0\frac{dt'A_{13}(u',t')}{(u'-u-i\epsilon)(t'-t+i\epsilon)},\quad (2.1)\end{aligned}$$

with

$$s=(p+k)^2=(p'+k')^2,\quad t=(p-p')^2=(k-k')^2,$$
$$u=(p-k')^2=(p'-k)^2,$$

where we have suppressed the dependence of the double-spectral functions on the energy variables of the currents and have rearranged terms using partial fraction for reasons which will become clear shortly. It is now crucial to observe that there is a one-to-one correspondence between the terms on the left-hand side of Eq. (2.1) and those on the right-hand side. To see this, let us consider a particular term in the T product, say

$$J_1\equiv\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times\langle 0|\theta(t_2-t_1)\theta(t_1-t_3)J^\beta(x_2)J^\alpha(x_1)J^\gamma(x_3)|p'\rangle.\quad (2.2)$$

Using energy-momentum conservation to rewrite the exponential factor, we have

$$J_1=\int d^4x_1d^4x_2d^4x_3\, e^{ip'x_1+ik(x_2-x_1)-ik'(x_3-x_1)}$$
$$\times\langle 0|\theta(t_2-t_1)\theta(t_1-t_3)J^\beta(x_2)J^\alpha(x_1)J^\gamma(x_3)|p'\rangle,\quad (2.3)$$

[4] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).
[5] P. Olesen, Phys. Rev. **175**, 2165 (1968).

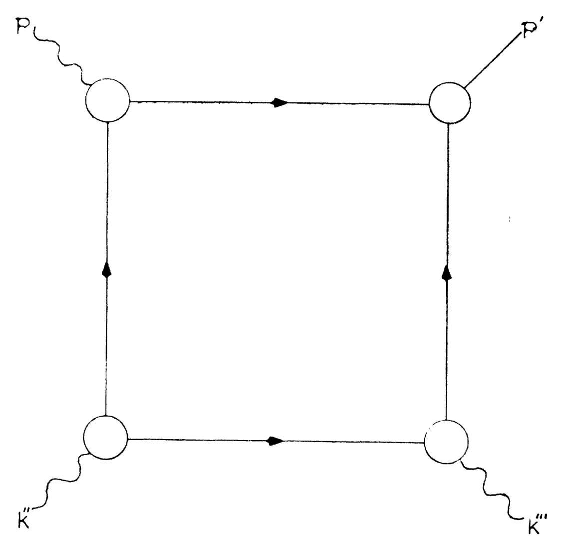

FIG. 1. Landau-Cutkosky diagram giving the nonzero boundary of double-spectral function in J_1. The internal lines are on the mass shell.

[6] S. Mandelstam, Phys. Rev. **112**, 1344 (1958). We have for simplicity assumed that the amplitude has no pole terms in s, t, or u. This simplification does not affect the generality of our result since these poles terms can always be explicitly separated out. Further, we fix the masses such that there is no anomalous threshold on the physical sheet. Hence, the unpleasant singularities such as acnodes and crunodes do not arise.

which may be rewritten further as

$$J_1=\frac{-1}{\pi^2}\int\frac{dq_0dq_0'}{(q_0-i\epsilon)(q_0'-i\epsilon)}\int d^4x_1d^4x_2d^4x_3\, e^{ip'x_1+ik''(x_2-x_1)-ik'''(x_3-x_1)}\langle 0|J^\beta(x_2)J^\alpha(x_1)J^\gamma(x_3)|p'\rangle$$
$$\equiv\frac{-1}{\pi^2}\int\frac{dq_0dq_0'}{(q_0-i\epsilon)(q_0'-i\epsilon)}F_1(p,k'',p',k'''),\tag{2.4}$$

where $k''\equiv(k_0+q_0,\mathbf{k})$, $k'''=(k_0'+q_0',\mathbf{k}')$.

Now F_1 in Eq. (2.4) may be considered as functions of $[(p'+k''')^2, (p'-k'')^2, k''^2, k'''^2, p^2]$. If we now make a change of variable from q_0 to u' and q_0' to s', where

$$u'=(p'-k')^2-2q_0(p-k)_0\equiv u-2q_0(p'-k)_0,\tag{2.5}$$

$$s'=(p'+k')^2-2q_0'(p'+k')_0\equiv s+2q_0'(p'+k')_0,\tag{2.6}$$

we obtain

$$J_1=\frac{-1}{\pi^2}\int\frac{ds'}{(s'-s-i\epsilon)}\int\frac{du'}{(u'-u-i\epsilon)}F_1\left(s'+\left(\frac{s'-s}{2(p'+k')_0}\right)^2,\; u'+\left(\frac{u'-u}{2(p'-k)_0}\right)^2,\right.$$
$$\left.k^2+\frac{(u'-u)k_0}{p_0'-k_0}+\frac{(u'-u)^2}{4(p_0'-k_0)^2},\; k'^2+\frac{(s'-s)k_0'}{p_0'+k_0'}+\frac{(s'-s)^2}{4(p_0'+k_0')^2},\; p^2\right).\tag{2.7}$$

We now employ the infinite-momentum trick[2,3] and assume that the limit $p_{0,z}'\to\infty$ can be taken inside the integral to obtain

$$\lim_{p_{0,z}'\to\infty}J_1=\frac{-1}{\pi^2}\int\frac{ds'}{(s'-s-i\epsilon)}\int\frac{du'}{(u'-u+i\epsilon)}F_1(s',u',k^2,k'^2,p^2).\tag{2.8}$$

Equation (2.8), as it stands, is a Bergmann-Weil representation for J_1, which is considered as a function of two complex variables s and u. Our next task is to find the nonzero regions of $F_1(s,u)$. It can be shown that the boundary of the nonvanishing regions of $F_1(s,u)$ is given by the four-cornered box diagram, as shown in Fig. 1. It is well known that such a Landau-Cutkosky diagram is nonvanishing only for $s>4m^2$ and $t>4m^2$. Thus, the integration region is Eq. (2.7) is over positive s and negative u.

Performing similar manipulations on the remaining terms in the T product, we find that each term may be rewritten as a double-spectral integral as follows:

$$\lim_{p_{0,z}'\to\infty}J_2\equiv\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\langle 0|\theta(t_2-t_3)\theta(t_3-t_1)J^\beta(x_2)J^\gamma(x_3)J^\alpha(x_1)|p'\rangle$$
$$=\frac{-1}{\pi^2}\int_0^\infty\frac{dt'}{(t'-t-i\epsilon)}\int_{-\infty}^0\frac{du'}{(u'-u+i\epsilon)}F_2(t',u'),\tag{2.9}$$

$$\lim_{p_{0,z}'\to\infty}J_3\equiv\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\langle 0|\theta(t_3-t_1)\theta(t_1-t_2)J^\gamma(x_3)J^\alpha(x_1)J^\beta(x_2)|p'\rangle$$
$$=\frac{-1}{\pi^2}\int_0^\infty\frac{du'}{(u'-u-i\epsilon)}\int_{-\infty}^0\frac{ds'}{(s'-s+i\epsilon)}F_3(u',s'),\tag{2.10}$$

$$\lim_{p_{0,z}'\to\infty}J_4\equiv\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\langle 0|\theta(t_3-t_2)\theta(t_2-t_1)J^\gamma(x_3)J^\beta(x_2)J^\alpha(x_1)|p'\rangle$$
$$=\frac{-1}{\pi^2}\int_0^\infty\frac{dt'}{(t'-t-i\epsilon)}\int_{-\infty}^0\frac{ds'}{(s'-s+i\epsilon)}F_4(t',s'),\tag{2.11}$$

$$\lim_{p_{0,z}'\to\infty}J_5\equiv\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\langle 0|\theta(t_1-t_2)\theta(t_2-t_3)J^\alpha(x_1)J^\beta(x_2)J^\gamma(x_3)|p'\rangle$$
$$=\frac{-1}{\pi^2}\int_0^\infty\frac{ds'}{(s'-s-i\epsilon)}\int_{-\infty}^0\frac{dt'}{(t'-t+i\epsilon)}F_5(s',t'),\tag{2.12}$$

$$\lim_{p_0, s' \to \infty} J_6 \equiv \int d^4x_1 d^4x_2 d^4x_3\, e^{i(px_1+kx_2-k'x_3)} \langle 0|\theta(t_1-t_3)\theta(t_3-t_2)J^\alpha(x_1)J^\gamma(x_3)J^\beta(x_2)|p'\rangle$$
$$= \frac{-1}{\pi^2}\int_0^\infty \frac{du'}{(u'-u-i\epsilon)}\int_{-\infty}^0 \frac{dt'}{(t'-t+i\epsilon)} F_6(u',t'), \tag{2.13}$$

where the F_i's are just the corresponding J_i's with the θ functions omitted. Comparing Eqs. (2.8)–(2.13) with Eq. (2.1), we see that

$$F_1 = F_2 = A_{12}, \quad F_3 = F_4 = A_{23}, \quad F_5 = F_6 = A_{13}.$$

Thus, there is a one-to-one correspondence between the terms in the T-product representation and those in the Mandelstam representation. This allows us to obtain discontinuity expressions for the third-order current elements entirely in terms of first-order current operators, as we proceed to do in Sec. III.

Note added in proof. We have succeeded in constructing the T product representation for three- and four-point functions of current starting from an unsubtracted Mandelstam representation. See E. Y. C. Lu, Nuovo Cimento **61A**, 249 (1969).

III. DISCONTINUITY FUNCTIONS

In this section, we shall derive the discontinuity functions which are crucial for later applications. We first prove the following lemmas.

Lemma 1. If we denote the discontinuity of a function in s for fixed t by $[\;\;]_{s,t}$, then

$$[J_1]_{\nu_1,t} = [J_3]_{\nu_1,t} = 0 \quad \text{for } t < 4m^2, \tag{3.1}$$

where $\nu_1 \equiv \frac{1}{2}(k+k')\cdot(p+p')$.

Proof. From the double-spectral representation of J_3 in Eq. (2.10), we have

$$[J_3]_{\nu_1,t} = \frac{-1}{\pi^2}\left(\int_0^\infty du' \int_{-\infty}^0 \frac{ds' A_{23}(s',u')}{(s'-s+i\epsilon)(u'-u-i\epsilon)} - \int_0^\infty du' \int_{-\infty}^0 \frac{ds' A_{23}(s',u')}{(u'-u+i\epsilon)(s'-s-i\epsilon)}\right)$$
$$= \frac{-i}{\pi}\left(\int_{-\infty}^0 \frac{ds' A_{23}(s',u')}{(s'-s-i\epsilon)} + \int_0^\infty \frac{du' A_{23}(s',u')}{(u'-u-i\epsilon)}\right)$$
$$= A_{23}(s,u). \tag{3.2}$$

Hence, $[J_3]_{\nu_1,t}$ vanishes for $t<4m^2$, since $A_{23}(s,u)$ vanishes for $t<4m^2$.

Similarly, one can show from the double-spectral representation of J_1 in Eq. (2.7) that

$$[J_1]_{\nu_1,t} = A_{12}(u,s)$$
$$= 0 \quad \text{for } t < 4m^2, \tag{3.3}$$

since A_{12} is nonzero only for $t > 4m^2$. Hence, we have proved Lemma 1.

In exactly the same way, we can prove the following two lemmas.

Lemma 2.

$$[J_2]_{\nu_2,s} = [J_6]_{\nu_2,s} = 0 \quad \text{for } s < 4m^2, \tag{3.4}$$

where $\nu_2 \equiv \frac{1}{2}(k-p)\cdot(k'-p')$.

Lemma 3.

$$[J_4]_{\nu_3,u} = [J_5]_{\nu_3,u} = 0 \quad \text{for } u < 4m^2, \tag{3.5}$$

where $\nu_3 \equiv -\frac{1}{2}(k+p')\cdot(k'+p)$. With these lemmas we are now ready to prove the following theorems:

Theorem 1.

$$F^{\alpha\beta\gamma} \equiv [T^{\alpha\beta\gamma}]_{\nu_1,t} = \tfrac{1}{2}\int d^4x_1 d^4x_2 d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times \langle 0|T\{J^\alpha(x_1),[J^\beta(x_2),J^\gamma(x_3)]\}|p'\rangle \text{ for } t<4m^2, \tag{3.6}$$

where the time-ordering operator is defined as

$$T\{J^\alpha(x_1),[J^\beta(x_2),J^\gamma(x_3)]\} \equiv \{\theta(t_1-t_2)\theta(t_1-t_3) \times [J^\beta(x_2)J^\gamma(x_3) - J^\gamma(x_3)J^\beta(x_2)] + \theta(t_2-t_1)\theta(t_3-t_1) \times [J^\beta(x_2)J^\gamma(x_3) - J^\gamma(x_3)J^\beta(x_2)]J^\alpha(x_1)\}. \tag{3.7}$$

Proof. From Lemma 1, we have immediately

$$[T^{\alpha\beta\gamma}]_{\nu_1,t} = \sum_i [J_i]_{\nu_1,t} \quad \text{for } t<4m^2, \tag{3.8}$$

where $i=2, 4, 5, 6$. The discontinuities in Eq. (3.8) are easily evaluated by simply replacing the $\theta(t_2-t_3)$ with $\frac{1}{2}$ and the $\theta(t_3-t_2)$ with $-\frac{1}{2}$. Hence, we have proved Theorem 1.

Corollary (1).

$$[T^{\alpha\beta\gamma}]_{k_0,(t,p,p',\mathbf{k})} = F^{\alpha\beta\gamma} \quad \text{for } t<4m^2. \tag{3.9}$$

This follows immediately from Theorem 1, since for fixed t, p, p', and $\mathbf{k}$, ν_1 is linearly related to k_0, and a change of variable from ν_1 to k_0 leads us from Eq. (3.6) to Eq. (3.9).

In exactly the same way, we can prove the following theorems and corollaries:

Theorem 2.

$$G^{\alpha\beta\gamma} \equiv [T^{\alpha\beta\gamma}]_{\nu_2,s} = \tfrac{1}{2}\int d^4x_1 d^4x_2 d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times \langle 0|T\{J^\gamma(x_3),[J^\alpha(x_1),J^\beta(x_2)]\}|p'\rangle \text{ for } s<4m^2. \tag{3.10}$$

Corollary 2.

$$G^{\alpha\beta\gamma} = [T^{\alpha\beta\gamma}]_{p_0,(s,k',p',\mathbf{p})}. \tag{3.11}$$

Theorem 3.

$$H^{\alpha\beta\gamma}\equiv[T^{\alpha\beta\gamma}]_{\nu_3,u}=\tfrac{1}{2}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times\langle 0|T\{J^\beta(x_2),[J^\gamma(x_3),J^\alpha(x_1)]\}|p'\rangle$$
$$\text{for } u<4m^2. \quad (3.12)$$

Corollary 3.

$$H^{\alpha\beta\gamma}=[T^{\alpha\beta\gamma}]_{k_0',(u,p',k,k')}. \quad (3.13)$$

Theorems 1–3 together with their corollaries prove that the functions $F^{\alpha\beta\gamma}$, $G^{\alpha\beta\gamma}$, and $H^{\alpha\beta\gamma}$ bear the same relation to $J^{\alpha\beta\gamma}$ as $M_{\mu\nu}{}^{\alpha\beta}$ does to $T_{\mu\nu}{}^{\alpha\beta}$. In fact, there is a self-consistency check of our results. We notice that $T_{\mu\nu}{}^{\alpha\beta}$ may be obtained from $M_{\mu\nu}{}^{\alpha\beta}$ if we assume for $T_{\mu\nu}{}^{\alpha\beta}$ an unsubtracted dispersion relation in s for fixed t.

If the functions $F^{\alpha\beta\gamma}$, $G^{\alpha\beta\gamma}$, and $H^{\alpha\beta\gamma}$ are those functions which Theorems 1–3 claim them to be, we must be able to recover $T^{\alpha\beta\gamma}$ from neither one of $F^{\alpha\beta\gamma}$, $G^{\alpha\beta\gamma}$, and $H^{\alpha\beta\gamma}$ by writing suitable dispersion relations. This can, in fact, be done, although we shall not go into details here.

IV. SUM RULES

The results of Sec. III together with the infinite-momentum techniques allows us, in principle, to obtain sum rules for any general third-order weak amplitude. In this section, we shall, however, consider detailed sum rules only for one particular matrix element, namely, $T_{\mu\nu}{}^{\alpha\beta\gamma}$ defined by

$$T_{\mu\nu}{}^{\alpha\beta\gamma}\equiv\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times\langle 0|T\{J^\alpha(x_1),J_\mu{}^\beta(x_2),J_\nu{}^\gamma(x_3)\}|p'\rangle, \quad (4.1)$$

where J^α is a pseudoscalar current, $J_\mu{}^\beta J_\nu{}^\gamma$ is a vector or axial-vector current, and $|p'\rangle$ is a pseudoscalar meson state.

Let us define the following functions:

$$R_{\mu\nu}{}^1=\tfrac{1}{2}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\delta(t_2-t_3)$$
$$\times\langle 0|T\{J^\alpha(x_1),[J_\mu{}^\beta(x_2),J_\nu{}^\gamma(x_3)]\}|p'\rangle, \quad (4.2)$$

$$R_{\mu\nu}{}^2=\tfrac{1}{2}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\delta(t_1-t_2)$$
$$\times\langle 0|T\{J_\nu{}^\gamma(x_3),[J^\alpha(x_1),J_\mu{}^\beta(x_2)]\}|p'\rangle, \quad (4.3)$$

$$R_{\mu\nu}{}^3=\tfrac{1}{2}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}\delta(t_3-t_1)$$
$$\times\langle 0|T\{J_\mu{}^\beta(x_2),[J_\nu{}^\gamma(x_3),J^\alpha(x_1)]\}|p'\rangle, \quad (4.4)$$

$$F_{\mu\nu}{}^{\alpha\beta\gamma}=\tfrac{1}{2}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times\langle 0|T\{J^\alpha(x_1),[J_\mu{}^\beta(x_2),J_\nu{}^\gamma(x_3)]\}|p'\rangle, \quad (4.5)$$

$$G_{\mu\nu}{}^{\alpha\beta\gamma}=\tfrac{1}{2}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times\langle 0|T\{J_\nu{}^\gamma(x_3),[J^\alpha(x_1),J_\mu{}^\beta(x_2)]\}|p'\rangle, \quad (4.6)$$

$$H_{\mu\nu}{}^{\alpha\beta\gamma}=\tfrac{1}{2}\int d^4x_1d^4x_2d^4x_3\, e^{i(px_1+kx_2-k'x_3)}$$
$$\times\langle 0|T\{J_\mu{}^\beta(x_2),[J_\nu{}^\gamma(x_3),J^\alpha(x_1)]\}|p'\rangle. \quad (4.7)$$

Clearly, $F_{\mu\nu}{}^{\alpha\beta\gamma}$, $G_{\mu\nu}{}^{\alpha\beta\gamma}$, and $H_{\mu\nu}{}^{\alpha\beta\gamma}$ are the analogs of $F^{\alpha\beta\gamma}$, etc., for the matrix element of Eq. (4.1).

We now recall some useful properties of the functions $F_{\mu\nu}{}^{\alpha\beta\gamma}$, $G_{\mu\nu}{}^{\alpha\beta\gamma}$, and $H_{\mu\nu}{}^{\alpha\beta\gamma}$ in the infinite-momentum limits. Consider first the limit $(p+p')_{0,3}\to\infty$ such that all the scalar invariants ν_1, t, $u_1=k+k'^2$, $v_1=k^2-k'^2$, p^2 are held fixed at finite values. In order to do so, we must keep $k_0=k_3$ and $k_0'=k_3'$. Hence, u_1 and v_1, in the $(p=p')_{0,3}\to\infty$ limit, no longer depends on k_0. The dependence of $T_{\mu\nu}{}^{\alpha\beta\gamma}(\nu_1,t,u_1,v_1,p^2)$ on k_0 is entirely contained in its dependence on ν_1, which is linearly related to k_0. Hence, we have

$$\lim_{(p+p')0,3\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{k_0,t}=[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{\nu_1,t}=F_{\mu\nu}{}^{\alpha\beta\gamma}. \quad (4.8)$$

Exactly the same reasoning leads to the following conclusions:

$$\lim_{(p-p')_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{k_0,t}=[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{v_1,t}=F_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.9)$$

$$\lim_{(k+k')_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{k_0,t}=[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{u_1,t}=F_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.10)$$

$$\lim_{(k'-p')_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{\nu_2,s}=G_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.11)$$

$$\lim_{(p+k)_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{u_2,s}=G_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.12)$$

$$\lim_{(p-k)_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{v_2,s}=G_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.13)$$

$$\lim_{(k+p')_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{\nu_3,u}=H_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.14)$$

$$\lim_{(p+k')_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{u_3,u}=H_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.15)$$

$$\lim_{(p-k')_{0,3}\to\infty}[T_{\mu\nu}{}^{\alpha\beta\gamma}]_{v_3,u}=H_{\mu\nu}{}^{\alpha\beta\gamma}, \quad (4.16)$$

where $u_2\equiv p^2+k^2$, $v_2\equiv p^2-k^2$, $u_3\equiv p^2+k'^2$, $v_3\equiv p^2-k'^2$.

We now employ the techniques of Ref. 3 to obtain the following sum rules:

$$\lim_{(p+p')_{0,3}\to\infty}R_{0\nu}{}'(t,p^2)$$
$$=\int\frac{d\nu_1}{(p+p')_0}F_{0\nu}{}^{\alpha\beta\gamma}(\nu_1,t,u_1,v_1,p^2)$$
$$=\int\frac{d\nu_1}{(p+p')_0}\,\mathrm{Im}T_{0\nu}{}^{\alpha\beta\gamma}(\nu_1,t,u_1,v_1,p^2), \quad (4.17)$$

$$\lim_{(k+k')_{0,3}\to\infty}R_{0\nu}{}'(t,p^2)$$
$$=\int\frac{du_1}{(k+k')_0}F_{0\nu}{}^{\alpha\beta\gamma}(\nu_1,t,u_1,v_1,p^2)$$
$$=\int\frac{du_1}{(k+k')_0}\,\mathrm{Im}T_{0\nu}{}^{\alpha\beta\gamma}(\nu_1,t,u_1,v_1,p^2), \quad (4.18)$$

$$\lim_{(p-p')_{0,3}\to\infty} R_{0\nu}'(t,p^2) = \int \frac{d\nu_1}{(p-p')_0} F_{0\nu}{}^{\alpha\beta\gamma}(\nu_1,t,u_1,v_1,p^2) = \int \frac{d\nu_1}{(p-p')_0} \operatorname{Im}T_{0\nu}{}^{\alpha\beta\gamma}(\nu_1,t,u_1,v_1,p^2)\,, \quad (4.19)$$

$$\lim_{(k'-p')_{0,3}\to\infty} R_{0\nu}{}^2(s,k'^2) = \int \frac{d\nu_2}{(k'-p')_0} G_{0\nu}{}^{\alpha\beta\gamma}(\nu_2,s,u_2,v_2,k'^2) = \int \frac{d\nu_2}{(k'-p')_0} \operatorname{Im}T_{0\nu}{}^{\alpha\beta\gamma}(\nu_2,s,u_2,v_2,k'^2)\,, \quad (4.20)$$

$$\lim_{(p+k)_{0,3}\to\infty} R_{0\nu}{}^2(s,k'^2) = \int \frac{du_2}{(p+k)_0} G_{0\nu}{}^{\alpha\beta\gamma}(\nu_2,s,u_2,v_2,k'^2) = \int \frac{du_2}{(p+k_0)} \operatorname{Im}T_{0\nu}{}^{\alpha\beta\gamma}(\nu_2,s,u_2,v_2,k'^2)\,, \quad (4.21)$$

$$\lim_{(p-k)_{0,3}\to\infty} R_{0\nu}{}^2(s,k'^2) = \int \frac{dv_2}{(p-k)_0} G_{0\nu}{}^{\alpha\beta\gamma}(\nu_2,s,u_2,v_2,k'^2) = \int \frac{dv_2}{(p-k)_0} \operatorname{Im}T_{0\nu}{}^{\alpha\beta\gamma}(\nu_2,s,u_2,v_2,k'^2)\,, \quad (4.22)$$

$$\lim_{(k+p')_{0,3}\to\infty} R_{\mu0}{}^3(u,k^2) = \int \frac{d\nu_3}{(k+p')_0} H_{\mu0}{}^{\alpha\beta\gamma}(\nu_3,u,u_3,v_3,k^2) = \int \frac{d\nu_3}{(k+p')_0} \operatorname{Im}T_{\mu0}{}^{\alpha\beta\gamma}(\nu_3,u,u_3,v_3,k^2)\,, \quad (4.23)$$

$$\lim_{(p+k')_{0,3}\to\infty} R_{\mu0}{}^3(u,k^2) = \int \frac{du_3'}{(p+k')_0} H_{\mu0}(\nu_3,u,u_3,v_3,k^2) = \int \frac{du_3}{(p+k')_0} \operatorname{Im}T_{\mu0}{}^{\alpha\beta\gamma}(\nu_3,u,u_3,v_3,k^2)\,, \quad (4.24)$$

$$\lim_{(p-k')_{0,3}\to\infty} R_{\mu0}{}^3(u,k^2) = \int \frac{dv_3}{(p-k')_0} H_{\mu0}(\nu_3,u,u_3,v_3,k^2) = \int \frac{dv_3}{(p-k')_0} \operatorname{Im}T_{\mu0}{}^{\alpha\beta\gamma}(\nu_3,u,u_3,v_3,k^2)\,, \quad (4.25)$$

where the last equalities in the above equations follow from the fact that according to Eqs. (4.9)–(4.16) the integrands are the respective discontinuity functions of the $T_{0\nu}{}^{\alpha\beta\gamma}$ in the integration variables.

Sum-rules equations (4.17)–(4.25) are the most general sum rules we can obtain for the third-order matrix element. They contain, as special cases, three sets of sum rules for three second-order matrix elements. For example, if we take poles in p^2 of Eqs. (4.17)–(4.19), we obtain sum rules for the second-order matrix element $M_{\mu\nu}{}^{\alpha\beta}$. If we take pole residues in k'^2 at the mass of a spin-1 particles of Eqs. (4.20)–(4.22), we obtain sum rules for $M_\mu{}^{\alpha\beta}$ defined by

$$M_\mu{}^{\alpha\beta} \equiv i\int d^4x_1 d^4x_2\, e^{i(px_1+kx_2)} \times\langle 0|T\{J^\alpha(x_1),J_\mu{}^\beta(x_2)\}|k',p'\rangle\,. \quad (4.26)$$

At the pole residues in k^2 of a spin-1 particle, Eqs. (4.23)–(4.25) give sum rules for $M_\nu{}^{\alpha\gamma}$ defined by

$$M_\nu{}^{\alpha\gamma} \equiv i\int d^4x_1 d^4x_3\, e^{i(px_1-kx_3)} \times\langle k|T\{J^\alpha(x_1),J_\nu{}^\gamma(x_3)\}|p'\rangle\,. \quad (4.27)$$

It is clear that current-algebra sum-rule constraints are stronger for the third-order amplitude than for second-order amplitudes, since all three sets of sum rules must hold simultaneously for the third-order amplitude. In Sec. V, we examine the implication of sum rules on the j-plane analyticity of third-order amplitudes.

V. FIXED POLES

In this section, we shall show that the sum rules that we obtained in Sec. IV imply the existence of fixed singularities in the angular momentum plane of the third-order weak amplitudes just as the Fubini–Dashen-Gell-Mann sum rule does in the case of second-order weak amplitudes. In the latter case, a fixed pole was found[7] at the highest nonsense point in the t-channel helicity flip-2 amplitude. We shall show that such a fixed pole persists in the third-order amplitude. Furthermore, fixed poles are found to be present in the s-channel and u-channel helicity amplitudes as well. The occurrence of these fixed poles is intimately connected with the fact, observed in Sec. IV, that the third-order sum rules contain, as special cases, sum rules for three different second-order weak amplitudes. Thus, the fixed poles that are present in each of these second-order weak amplitudes are simulatneously present in the third-order amplitude.

Let us first expand the tensor amplitude $\operatorname{Im}T_{\mu\nu}{}^{\alpha\beta\gamma}$ in terms of three different bases of the space spanned by the momentum vectors.

$$\operatorname{Im}T_{\mu\nu}{}^{\alpha\beta\gamma} = \sum_i I_{\mu\nu}{}^i a_i\,, \quad (5.1)$$

[7] V. Singh, Phys. Rev. Letters **18**, 36 (1967); J. Bronzan *et al.*, *ibid.* **18**, 32 (1967).

$$=\sum_i I_{\mu\nu}'^{i} b_i, \tag{5.2}$$

$$=\sum_i I_{\mu\nu}''^{i} c_i \tag{5.3}$$

where

$$I_{\mu\nu}^{i}\equiv[(p+p')_\mu(p+p')_\nu,\cdots], \tag{5.4}$$

$$\begin{aligned}I_{\mu\nu}'^{i}\equiv[&(k'-p')_\mu(k'-p')_\nu,\quad (k'-p')_\mu(k-p)_\nu,\\ &(k'-p')_\mu(k+p)_\nu,\quad (k-p_\mu(k'-p')_\nu,\\ &(k-p)_\mu(k-p)_\nu,\quad (k-p)_\mu(k+p)_\nu,\\ &(k+p)_\mu(k'-p')_\nu,\quad (k+p)_\mu(k-p)_\nu,\\ &(k+p)_\mu(k+p)_\nu,\quad \delta_{\mu\nu}],\end{aligned} \tag{5.5}$$

$$I_{\mu\nu}''^{i}\equiv I_{\mu\nu}'^{i}(k'\leftrightarrow -k). \tag{5.6}$$

The R^i are similarly expanded as

$$R_{0\nu}^{1}=(p+p')_\nu F_1(t,p^2)+(p-p')_\nu F_2(t,p^2), \tag{5.7}$$

$$R_{0\nu}^{2}=(k'-p')_\nu G_1(s,k'^2)+(k'+p')_\nu G_2(s,k'^2), \tag{5.8}$$

$$R_{\mu 0}^{3}=(k'+p)_\mu H_1(u,k^2)+(k'-p)_\mu H_2(u,k^2). \tag{5.9}$$

Substituting Eqs. (5.1)–(5.9) into Eqs. (4.17), (4.20), and (4.23), we obtain

$$\int d\nu_1\; a_1(\nu_1,t,u_1,v_1,p^2)=F_1(t,p^2), \tag{5.10}$$

$$\int d\nu_2\, b_1(\nu_2,s,u_2,v_2,k'^2)=G_1(s,k'^2), \tag{5.11}$$

$$\int d\nu_2\, b_2(\nu_2,s,u_2,v_2,k'^2)=0, \tag{5.12}$$

$$\int d\nu_2\, b_3(\nu_2,s,u_2,v_2,k'^2)=G_2(s,k'^2), \tag{5.13}$$

$$\int d\nu_3\; c_1(\nu_3,u,u_3,v_3,k^2)=H_1(u,k^2), \tag{5.14}$$

$$\int d\nu_3\; c_4(\nu_3,u,u_3,v_3,k^2)=0, \tag{5.15}$$

$$\int d\nu_3\; c_7(\nu_3,u,u_3,v_3,k^2)=H_2(u,k^2). \tag{5.16}$$

Let us also define t-, s-, and u-channel c.m. helicity amplitudes by

$$T_{\lambda_1\lambda_2,00}{}^{t}=(\epsilon_t)_\mu{}^{\lambda_1}(\epsilon_t)_\nu{}^{\lambda_2}T_{\mu\nu}{}^{\alpha\beta\gamma}, \tag{5.17}$$

$$T_{\lambda_1 0,\lambda_2 0}{}^{s}=(\epsilon_s)_\mu{}^{\lambda_1}(\epsilon_s)_\nu{}^{\lambda_2}T_{\mu\nu}{}^{\alpha\beta\gamma}, \tag{5.18}$$

$$T_{\lambda_2 0,\lambda_1 0}{}^{u}=(\epsilon_u)_\mu{}^{\lambda_1}(\epsilon_u)_\nu{}^{\lambda_2}T_{\mu\nu}{}^{\alpha\beta\gamma}, \tag{5.19}$$

where the ϵ_t, ϵ_u, and ϵ_s are, respectively, the t-, u-, and s-channel c.m. helicity vectors for the vector or axial-vector currents. They satisfy the following transversality conditions:

$$\epsilon_t{}^{\pm}\cdot\mathbf{k}=\epsilon_t{}^{\pm}\cdot\mathbf{k}'=0, \tag{5.20}$$

$$\epsilon_s{}'^{\pm}\cdot\mathbf{k}=\epsilon_s{}'^{\pm}\cdot\mathbf{p}=\epsilon_s{}^{2\pm}\cdot\mathbf{k}'=\epsilon_s{}^{2\pm}\cdot\mathbf{p}'=0, \tag{5.21}$$

$$\epsilon_u{}'^{\pm}\cdot\mathbf{k}=\epsilon_u{}'^{\pm}\cdot\mathbf{p}'=\epsilon_u{}^{2\pm}\cdot\mathbf{k}'=\epsilon_u{}^{2\pm}\cdot\mathbf{p}=0. \tag{5.22}$$

Using these definitions for helicity amplitudes and asymptotic behavior implied by the sum-rules equations (5.10)–(5.16), we obtain the following conclusions:

(a) The t-channel helicity flip-2 amplitude $T_{1-1;00}{}^{t}$ has a fixed pole at its highest nonsense point $J=1$, with residue proportional to $F_1(t,p^2)$.

(b) The s-channel helicity amplitude $T_{10;10}{}^{s}$, which receives contribution from B_2 and B_{10}, is regular at its sense-sense point $J=1$, unless B_{10} requires more than one subtraction. Similar conclusions hold for $T_{10;10}{}^{u}$.

(c) The s-channel helicity amplitude $T_{10;00}{}^{s}$, which receives contribution from B_1, B_2, and B_3 will, in general have a fixed pole at its nonsense-sense point $J=0$, because of the fixed asymptotic behavior in B_1 and B_3 implied by sum-rules equations (5.11) and (5.13), unless B_2 contains fixed asymptotic behavior not required by current algebra which somehow exactly cancels those of B_1 and B_3. Similar conclusions hold for $T_{10;00}{}^{u}$.

(d) $T_{11,00}{}^{t}$, $T_{00;00}{}^{s}$, and $T_{00;00}{}^{u}$ will, in general, have Kronecker deltas[8] at its sense-sense point $J=0$, because of the contributions they receive from A_1; (B_1,B_3) and (C_1,C_7), respectively.

VI. BJORKEN LIMIT

In this section, we derive the Bjorken limit for third-order weak amplitudes $T_{\mu\nu\lambda}{}^{\alpha\beta\gamma}$ following closely the original approach of Bjorken.[4] We first rewrite $T_{\mu\nu\lambda}{}^{\alpha\beta\gamma}$ as

$$\begin{aligned}T_{\mu\nu\lambda}{}^{\alpha\beta\gamma}=&\frac{-1}{\pi^2}\int\frac{dk_0'''}{k_0'''-k_0'-i\epsilon}\int\frac{dk_0''F_{\mu\nu\lambda}{}^{1}}{k_0''-k_0-i\epsilon}+\frac{1}{\pi^2}\int\frac{dp_0''dk_0''F_{\mu\nu\lambda}{}^{2}}{(p_0''-p_0+i\epsilon)(k_0''-k_0-i\epsilon)}\\ &-\frac{1}{\pi^2}\int\frac{dk_0'''dk_0''F_{\mu\nu\lambda}{}^{3}}{(k_0'''-k_0'+i\epsilon)(k_0''-k_0+i\epsilon)}-\frac{1}{\pi^2}\int\frac{dp_0''dk_0'''F_{\mu\nu\lambda}{}^{4}}{(p_0''-p_0+i\epsilon)(k_0'''-k_0'+i\epsilon)}\\ &-\frac{1}{\pi^2}\int\frac{dk_0'''dp_0''F_{\mu\nu\lambda}{}^{5}}{(k_0'''-k_0'-i\epsilon)(p_0''-p_0-i\epsilon)}+\frac{1}{\pi^2}\int\frac{dk_0''dp_0''F_{\mu\nu\lambda}{}^{6}}{(k_0''-k_0+i\epsilon)(p_0''-p_0-i\epsilon)}.\end{aligned} \tag{6.1}$$

[8] D. Gross and H. Pagels, Phys. Rev. Letters **20**, 961 (1968); H. Dosch and D. Gordon, Nuovo Cimento **57A**, 82 (1968).

If the intermediate state sums in $F_{\mu\nu\lambda}{}^{i}$ can be truncated, then in the limit k_0, p_0, $k_0'\to\infty$ with p_0' kept finite, $T_{\mu\nu\lambda}{}^{\alpha\beta\gamma}$ behaves as

$$\lim_{k_0,p_0,k_0'\to\infty} T_{\mu\nu\lambda}{}^{\alpha\beta\gamma}\sim\frac{-1}{\pi^2}\left[\frac{1}{k_0k_0'}\int dk_0'''dk_0''(F_{\mu\nu\lambda}{}^1+F_{\mu\nu\lambda}{}^3)-\frac{1}{k_0p_0}\int dk_0''dp_0''(F_{\mu\nu\lambda}{}^2+F_{\mu\nu\lambda}{}^6)\right.$$
$$\left.+\frac{1}{k_0'p_0}\int dk_0'''dp_0''(F_{\mu\nu\lambda}{}^4+F_{\mu\nu\lambda}{}^5)\right]. \quad (6.2)$$

Using the identity

$$1/k_0(k_0+p_0)=1/k_0p_0-1/p_0(k_0+p_0)\,,$$

Equation (6.2) may be rewritten as

$$\lim_{k_0,p_0,k_0'\to\infty} T_{\mu\nu\lambda}{}^{\alpha\beta\gamma}\sim\frac{1}{k_0p_0}\int d^4x_1d^4x_2d^4x_3\,e^{i(px_1+kx_2-k'x_3)}\langle 0|\,\delta(t_1-t_3)\delta(t_2-t_1)[J_\nu{}^\beta(x_2),[J_\mu{}^\alpha(x_1),J_\lambda{}^\gamma(x_3)]]\,|p'\rangle$$
$$+\frac{1}{(k_0+p_0)p_0}\int d^4x_1d^4x_2d^4x_3\,e^{i(px_1+kx_2-k'x_3)}\langle 0|\,\delta(t_1-t_2)\delta(t_2-t_1)[[J_\mu{}^\alpha(x_1),J_\nu{}^\beta(x_2)],J_\lambda{}^\gamma(x_3)]\,|p'\rangle. \quad (6.3)$$

The result in Eq. (6.3) is in agreement with those obtained by Olesen using a different method. In the limit $k_0\to\infty$ with p_0, p_0' held fixed, we obtain from Eq. (6.1), assuming again that intermediate state sums may be truncated,

$$\lim_{k_0\to\infty;\ p_0\text{ fixed}} T_{\mu\nu\lambda}{}^{\alpha\beta\gamma}\sim\frac{1}{k_0}\int d^4x_1d^4x_2d^4x_3\,e^{i(px_1+kx_2-k'x_3)}\delta(t_2-t_3)\langle 0|\,T\{J_\mu{}^\alpha(x_1),[J_\nu{}^\beta(x_2),J_\lambda{}^\gamma(x_3)]\}\,|p'\rangle. \quad (6.4)$$

The next-higher order term is given by multiple commutators:

$$\frac{1}{k_0{}^2}\int d^4x_1d^4x_2d^4x_3\,e^{i(px_1+kx_2-k'x_3)}\{\delta(t_2-t_3)\langle 0|\,T\{J_\mu{}^\alpha(x_1),[\partial_0J_\nu{}^\beta(x_2),J_\lambda{}^\gamma(x_3)]\}\,|p'\rangle$$
$$+\delta(t_2-t_1)\delta(t_1-t_3)\langle 0|\,[[J_\nu{}^\beta(x_2),J_\mu{}^\alpha(x_1)],J_\lambda{}^\gamma(x_3)]\,|p'\rangle\}. \quad (6.5)$$

Let us now consider the implications of these asymptotic behaviors on radiative corrections to weak decays. It can be shown[9] that the divergent or possibly divergent contribution to the radiative corrections of the weak decays $A\to B+l+\nu$ is given by

$$M_{\rm div}=\frac{G}{\sqrt{2}}e^2\bar{u}(p_l)\gamma_\lambda(1+\gamma_5)v_\nu(p_\nu)\left[\tfrac{3}{2}\langle B|J_\lambda{}^w(0)|A\rangle\frac{1}{(2\pi)^4}\int\frac{d^4k}{k^4}+\epsilon_{\lambda\rho\sigma\mu}\frac{1}{(2\pi)^4}\int\frac{d^4k}{k^4}k_\rho T_{\sigma\mu}(k,\Delta)\right.$$
$$\left.-\frac{1}{2(2\pi)^4}\int\frac{d^4k}{k^2}\frac{\partial}{\partial\Delta_\lambda}M_{\mu\mu}(k,\Delta)+\frac{\Delta\rho}{2(2\pi)^4}\int\frac{d^4k}{k^2}\frac{\partial}{\partial\Delta_\lambda}T_{\rho\mu\mu}(k,\Delta)\right], \quad (6.6)$$

where

$$T_{\sigma\mu}(k,\Delta)\equiv i\int d^4x_1d^4x_2\,e^{i(\Delta x_1+kx_2)}\langle B|\,T\{J_\sigma{}^w(x_1),J_\mu{}^{\rm em}(x_2)\}\,|A\rangle, \quad (6.7)$$

$$M_{\mu\nu}(k,\Delta)\equiv i\int d^4x_1d^4x_2d^4x_3\,e^{i(\Delta x_1+kx_2-kx_3)}\langle B|\,T\{\partial_\sigma J_\sigma{}^w(x_1),J_\mu{}^{\rm em}(x_2),J_\nu{}^{\rm em}(x_3)\}\,|A\rangle, \quad (6.8)$$

$$T_{\rho\mu\nu}(k,\Delta)\equiv\int d^4x_1d^4x_2d^4x_3\,e^{i(\Delta x_1+kx_2-kx_3)}\langle B|\,T\{J_\rho{}^w(x_1),J_\mu{}^{\rm em}(x_2),J_\nu{}^{\rm em}(x_3)\}\,|A\rangle. \quad (6.9)$$

In the case of vector β decay, the third and fourth terms do not contribute to order α. In the case of $A=\pi$, $B=|0\rangle$, there is also no divergent contribution from the third term, because the coefficient of the $1/k^2$ term of $M_{\mu\mu}$ is, according to Eq. (6.5),

$$\int d^4x_1d^4x_2d^4x_3\,e^{i(\Delta x_1+kx_2-kx_3)}[\delta(t_2-t_3)\langle 0|\,T\{\partial_\sigma J_\sigma{}^w(x_1),[\partial_0J_\mu{}^{\rm em}(x_2),J_\mu{}^{\rm em}(x_3)]\}\,|\pi\rangle$$
$$+\delta(t_2-t_1)\delta(t_1-t_3)\langle 0|\,[[J_\mu{}^{\rm em}(x_2),\partial_\sigma J_\sigma{}^w(x_1)],J_\mu{}^{\rm em}(x_3)]\,|\pi\rangle]. \quad (6.10)$$

[9] E. Abers, D. Dicus, R. Norton, and H. Quinn, Phys. Rev. **167**, 1461 (1968).

It is clear that Eq. (6.10) can only depend on the vector Δ, and since it is a scalar function, it only depends on $\Delta^2=m_\pi^2$. Hence, Eq. (6.10) vanishes upon differentiation with respect to Δ_λ. The coefficient of the divergent contribution from the fourth term is

$$\Delta_\rho \frac{\partial}{\partial\Delta_\lambda} \int d^4x_1 d^4x_2 d^4x_3\, e^{i(\Delta x_1+kx_2-kx_3)} \delta(t_2-t_3)\langle 0|T\{J_\rho{}^w(x_1),[\partial_0 J_\mu{}^{\mathrm{em}}(x_2),J_\mu{}^{\mathrm{em}}(x_3)]\}|\pi\rangle \\ +\delta(t_2-t_1)\delta(t_1-t_3)\langle 0|[[J_\mu{}^{\mathrm{em}}(x_2),J_\rho{}^w(x_1)],J_\mu{}^{\mathrm{em}}l(x_3)]|\pi\rangle. \quad (6.11)$$

Both of the commutators in Eq. (6.11), as well as the commutator which determines the divergent contribution of the second term on the right-hand side of Eq. (6.6), are model-dependent. It has been suggested[10] that the model-dependent commutator in the second term be arranged so that it cancels the divergence of the first term and hence renders the radiative corrections to vector β decay finite. The question we wish to ask is: Can we arrange the model-dependent commutators so that to lowest order in α, the radiative corrections to vector and to axial-vector β decay, as well as the electromagnetic mass differences of the hadrons, become finite? We shall show that within the framework of Bjorken limits the answer to the above question is in the negative. The reason is as follows: For finite radiative corrections to vector β decay, we must have[10]

$$\int d^3x[J_\mu{}^{\mathrm{em}}(\mathbf{x},0),J_\sigma{}^w(0)]=i\epsilon_{\mu\sigma\lambda}J_\lambda{}^w. \quad (6.12)$$

It is easy to see that the equal-time commutator equation (6.12) also causes a cancellation of divergence in first and second terms in the case of leptonic decay of π and K mesons. Furthermore, it allows an evaluation of the second commutator on the right-hand side of Eq. (6.11),

$$\int d^3x d^3y[[J_\mu{}^{\mathrm{em}}(\mathbf{x},0),J_\rho{}^w(\mathbf{y},0)],J_\lambda{}^{\mathrm{em}}(0)] \\ =\epsilon_{\mu\rho\rho'}\epsilon_{\rho''\lambda\lambda'}J_{\lambda'}{}^w. \quad (6.13)$$

If we further require that the electromagnetic mass difference of hadrons be finite, this can be achieved,[4] within the framework of the Bjorken limit, by setting

$$\int d^3x[\partial_0 J_\mu{}^{\mathrm{em}}(\mathbf{x},0),J_\nu{}^{\mathrm{em}}(0)]=\delta_{\mu\nu}C. \quad (6.14)$$

Substituting Eqs. (6.13) and (6.14) into Eq. (6.11), we have, apart from disconnected parts,

$$\Delta_\rho \frac{\partial}{\partial\Delta_\lambda}\langle 0|J_\rho{}^w|\pi\rangle=\Delta_\lambda m_\pi{}^2F_\pi\neq 0, \quad (6.15)$$

which results in a logarithmic divergent contribution to the radiative correction of π leptonic decay. Thus, it appears that logarithmic divergences are unavoidable with conventional quantum electrodynamics independent of the models one may assume for equal-time commutators of hadronic currents.

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