# Simultaneous Measurement in the Bohm-Bub Hidden-Variable Theory

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A recently proposed theory of measurement for nonrelativistic quantum mechanics, due to Bohm and Bub, is generalized, and a mathematical analysis of the theory is presented. The fact that there are always theories of the Bohm-Bub type, given the standard formulation of quantum mechanics, is discussed. The phenomenological coefficient $\gamma$ of the original theory is defined to be the absolute value of the energy change of the system during measurement divided by $\hbar$. This definition leads to a Heisenberg-like relation between the average energy change of the system during the measurement process and the collapse time, but the new relation depends on the so-called hidden variables, and hence could lead to non-quantum-mechanical effects. The time dependence of the hidden variables is considered, and they are then assumed to be constant for each individual system. An experiment is proposed which would test this assumption. A positive result would disagree significantly with the quantum-mechanical prediction. The question of whether or not the new variables must remain unobservable is analyzed mathematically. A rigorous definition of simultaneous measurement is made, and the resulting interference is given a mathematical description.

## INTRODUCTION

In the past decade there have been several attempts to define and solve the problem of measurement in quantum mechanics.[1] It seems to this author that in order for a measurement theory to be successful it must incorporate extraclassical as well as extra-quantum-mechanical variables, which at present are hidden from us because of our lack of a well-structured concept of measurement phenomena in quantum mechanics. The existence of such a theory has long been thought an impossibility, but in 1966 an example of a hidden-variable theory of measurement[2] for nonrelativistic quantum mechanics was given by Bohm and Bub.[3] Though this theory is simple in that it deals only with the simplest types of measurements, it is sufficiently structured to produce a causal explanation for the collapse of the wave packet as well as an explanation of the quantum probabilities and the time-energy uncertainty relation.

The Bohm-Bub theory has several difficulties. It is nonlocal, nonlinear, and nonrelativistic. Whether these constitute problems or attributes will not be discussed here. The problems of observer independence, completeness, and simultaneous measurement, as well as the nature of $\gamma$ and the nature of the hidden variables themselves, will be considered.

To begin with, it seems that if several observations are possible, in the sense that it is up to an observer to activate a particular apparatus and not others, then the appropriate collapse equation is chosen by the observer rather than by the physically real situation. This state of affairs is problematic in that we would like our measurement equations to represent a physically real event and not just an observer's explanation of why the theory of quantum mechanics agrees with the data accumulated in the real laboratory.[4] In other words, an experiment should be a real experiment independent of any observer naming it an "experiment." Secondly, the collapse equations should take into account the fact that some of the observables may have degeneracies; hence a complete set of observables should be involved. Thirdly, some provision for the simultaneous measurement of both commuting and noncommuting observables should be made. Finally, the time dependence and physics of $\gamma$ and the hidden variables must be made more explicit if the theory is to be experimentally testable. The theory as given by Bohm and Bub is not testable as they seem to indicate, since $\gamma$, which determines the collapse time, is left practically arbitrary. The first three of these difficulties are easily resolved by the inclusion of additional terms in the collapse equations. A refinement of an earlier definition of $\gamma$ is made[5] and some new assumptions concerning the hidden variables are introduced and explored. Experimental tests of this theory are suggested.

## 1. GENERALIZED MEASUREMENT EQUATION

The collapse equation given by Bohm and Bub to describe a single measurement corresponding to an ob-

[1] Some of the more recent papers on this subject are: J. Bub, Brit. J. Phil. Sci. **19**, 185 (1968); S. Kochen and E. P. Specker, J. Math. & Mech. **17**, 59 (1967); J. L. Park and H. Margenau, Int. J. Theoret. Phys. **1**, 211 (1968); J. E. Turner, J. Math. Phys. **9**, 1411 (1968); N. Zierler and M. Schlessinger, Duke Math. J. **32**, 251 (1965); J. Bub, Int. J. Theoret. Phys. Report (unpublished); W. T. Scott, Ann. Phys. (N. Y.) **46**, 577 (1968); **47**, 489 (1968); J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley Publishing Co. Inc., Reading, Mass., 1968); J. Bub, Nuovo Cimento **57B**, 503 (1968); M. Bunge, *Foundations of Physics* (Springer-Verlag, Berlin, 1967); J. M. Jauch, E. P. Wigner, and M. M. Yanase, Nuovo Cimento **48B**, 144 (1967); C. Y. She and H. Heffner, Phys. Rev. **152**, 1103 (1966); A. Daneri, A. Loinger, and G. M. Prosperi, Nuovo Cimento **44B**, 119 (1966); Nucl. Phys. **33**, 297 (1962); L. Rosenfeld, Progr. Theoret. Phys. (Kyoto) Suppl., 222 (1965); E. Arthurs and J. L. Kelly, Bell System Technical Journal Briefs, 725 (1965); H. Margenau, Phil. Sci. **30**, 128 (1963); A. Shimony, Am. J. Phys. **31**, 755 (1963); E. Wigner, Am. J. Phys. **31**, 6 (1963); P. Suppes, Phil. Sci. **28**, 378 (1961); L. Durand, III, Phil. Sci. **27**, 115 (1960).

[2] The name "hidden variable" is a poor one, since it has so many meanings. I intend to use it only to refer to the new parameters introduced into quantum mechanics by Bohm and Bub in their 1966 paper.

[3] D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453 (1966).

[4] This point was discussed at the conference on Foundations of Quantum Theory, New Mexcio Tech., Socorro, N. M., 1968 (unpublished).

[5] J. H. Tutsch, Rev. Mod. Phys. **40**, 232 (1968).

servable with a finite, nondegenerate spectrum is

$$\frac{d\psi_i}{dt}=\sum_{j=1}^{n}\gamma(R_i-R_j)J_j\psi_i, \quad i=1,\ldots,n$$

where $J_j=|\psi_j|^2$, $R_j=J_j/|\xi_j|^2$, and the complex numbers $\xi_j$ are the components of the dual vector and are taken to be the hidden variables. The representation used in these equations is the one in which the observable in question is diagonal. The equation can be generalized as follows[6]:

$$\frac{d|\psi\rangle}{dt}=\sum_{i,j,A}\gamma^A(R_i^A-R_j^A)J_j^AP_i^A|\psi\rangle, \tag{1}$$

where $A$ ranges over the set of all physically possible experiments and corresponds in each case to the appropriate observable, which is represented by a Hermitian operator $\mathbf{A}$ which operates on the Hilbert-space vector of the system. It is not necessary to make the strong statement, which is usually made in quantum mechanics, that to each so-called observable there corresponds a measurement procedure and apparatus which produces the spectrum of the particular operator involved. In fact, it seems that any realistic measurement theory should explicitly avoid this strong statement.[7]

Margenau and Park[8] have recently questioned the converse of the foregoing assumption in the case of simultaneous measurement. They claim that there may be measurements for which there are no corresponding Hermitian operators. Indeed, if we are to have a testable hidden-variable theory, this would probably have to be the case, although I do not think this is the conclusion Margenau and Park wanted to reach. It is interesting to note that if $A$ is restricted to ordinary quantum-mechanical measurements, the generalized Bohm-Bub theory becomes a metameasurement theory in that it is intended to explain the measurement theory implicit in ordinary quantum mechanics. Processes which measure the hidden variables would not be expected to be of the quantum-mechanical type but instead completely new. Once this fact is realized, it is easy to see why the hidden variables have not yet been observed.

If $\mathbf{A}$ is degenerate, we can assume that there is a $\mathbf{B}$ which commutes with $\mathbf{A}$ and removes this degeneracy. If $\mathbf{B}$ does not remove all of the degeneracy, then assume that there is a $\mathbf{C}$ which commutes with both $\mathbf{A}$ and $\mathbf{B}$ which removes more of the degeneracy, etc. For simplicity, suppose that the complete set of commuting observables consists of $\mathbf{A}$ and $\mathbf{B}$. Let $|A_iB_j^i\rangle$ ($i=1,\cdots,n$; $j=1,\cdots,m_i$) form a complete orthonormal basis for the Hilbert space, with $m_i$ being the degeneracy of the eigenvalue $A_i$. For a measurement of the $A$ variable, take $\gamma^A\neq 0$ and $\gamma^B=0$, with

$$P_i^A=\sum_{j=1}^{m_i}|A_iB_j^i\rangle\langle A_iB_j^i|,$$

so that

$$P_i^A|\psi\rangle=\sum_{j=1}^{m_i}\langle A_iB_j^i|\psi\rangle|A_iB_j^i\rangle=\psi_i^A|A_i\rangle,$$

where $|A_i\rangle$ is taken to be the unique unit vector in the direction of the projection of $|\psi\rangle$ onto the subspace spanned by $|A_iB_j^i\rangle$, $j=1,\cdots,m_i$. The basis used here is not the diagonalizing basis but, with the definition of $\psi_i^A$ above the effect is the same in that there are $n$ possible outcomes for a measurement of $A$.

Similarly, for a measurement of the $B$ variable, take $\gamma^B\neq 0$ and $\gamma^A=0$, with

$$P_j^B=\sum_{i=1}^{n_j}|A_iB_j^i\rangle\langle A_iB_j^i|,$$

so that

$$P_j^B|\psi\rangle=\sum_{i=1}^{n_j}\langle A_iB_j^i|\psi\rangle|A_iB_j^i\rangle=\psi_j^B|B_j\rangle,$$

where $|B_j\rangle$ is the unique unit vector in the direction of the projection of $|\psi\rangle$ onto the subspace spanned by $|A_iB_j^i\rangle$, $i=1,\cdots,n_j$, with $n_j$ being the degeneracy of $B_j$, The number of collapse equations in this case need not be $n$ but, instead, corresponds to the number of distinct eigenvalues of $\mathbf{B}$. For a simultaneous measurement of $A$ and $B$, take both $\gamma^A\neq 0$ and $\gamma^B\neq 0$, with

$$P_{ij}^{AB}=|A_iB_j^i\rangle\langle A_iB_j^i|,$$

so that

$$P_{ij}^{AB}|\psi\rangle=\langle A_iB_j^i|\psi\rangle|A_iB_j^i\rangle\equiv\psi_{ij}^{AB}|A_iB_j^i\rangle,$$

where the $|A_iB_j^i\rangle$'s now span the entire ($\sum_{i=1}^n m_i=N$)-dimensional space. The sum $\gamma^A+\gamma^B$ can be taken to be $\gamma^{AB}$, which in turn produces a collapse time shorter than that for either process alone. The product $AB$ is used only to designate that an $A$ measurement and a $B$ measurement are being performed at the same time, and is not to be taken as the product of the operators $\mathbf{A}$ and $\mathbf{B}$. In the case discussed here, $[\mathbf{A},\mathbf{B}]=0$. In Sec. 4 of this paper, simultaneous measurement will be discussed for the more general case, $[\mathbf{A},\mathbf{B}]\neq 0$, in detail.

The other symbols in Eq. (1) are defined as follows:

$$J_j^A\equiv\langle\psi|P_j^A|\psi\rangle, \quad R_i^A\equiv J_i^A/\langle\xi|P_i^A|\xi\rangle,$$

with

$$\langle\xi|P_i^A|\xi\rangle\equiv|\xi_i^A|^2,$$

[6] The use of projection operators to make the equation representation-free is due to Bohm and Bub, while the summation over all experiments is due to the present author. The representation in which the observable being measured is diagonal produces the form first given. Any other representation produces a much more complicated set of collapse equations which still have the same collapse properties.

[7] W. C. Davidon and H. Ekstein, J. Math. Phys. 5, 1588 (1964).

[8] J. L. Park and H. Margenau, Int. J. Theoret. Phys. 1, 211 (1968).

where $\langle\xi|$ is the vector in the dual of the Hilbert space of the system whose components are taken to be the hidden variables. So there is one hidden variable for each distinct eigenvalue of the operator in question, which in this case is **A**. The $\xi_i^A$'s are assumed to have no space dependence. The $J_j^A$'s have no space dependence either, for it has been integrated out.

Writing the collapse equation in a representation-free form avoids the conclusion that Turner draws with respect to this theory.[9] Turner uses the weak form of a definition of a hidden-variable theory proposed by Zierler and Schlessinger.[10] Turner's definition is as follows: A hidden-variable theory (or Boolean embedding) is any embedding of the quantum logic $(S,P)$ into the distributive logic $(S',P')$ given by the pair of 1-1 maps $\tau: S\rightarrow S'$ and $\sigma: P\rightarrow P'$ such that the expectation values are preserved, i.e., $f\in S\Rightarrow f=[\tau(f)]\circ\sigma$. Here, $S$ is a set of quantum states, and $P$ is a set of experimental propositions. Zierler and Schlessinger add the condition that the quantum ordering of propositions is preserved by the embedding $\sigma: P\rightarrow P'$ and proceed to re-prove the von Neumann result that the embedding must be the trivial one.

Turner purports to give an example in which the Bohm-Bub theory does not preserve the quantum ordering of propositions. The example is incorrect because $P_{1'2'}$ (Turner's notation) corresponds to

$$\frac{|\psi_3'|^2}{|\xi_3'|^2}>\frac{|\psi_1'-\psi_2'\tan\theta|^2}{|\xi_1'-\xi_2'\tan\theta|^2},\quad \frac{|\psi_1'\tan\theta+\psi_2'|^2}{|\xi_1'\tan\theta+\xi_2'|^2} \tag{2}$$

and not to

$$\frac{|\psi_3|^2}{|\xi_3|^2}>\frac{|\psi_1+\psi_2\tan\theta|^2}{|\xi_1-\xi_2\tan\theta|^2},\quad \frac{|\psi_1\tan\theta-\psi_2|^2}{|\xi_1\tan+\xi_2|^2\theta}, \tag{3}$$

as given by Turner. This is so, since if we know

$$\frac{|\psi_3|^2}{|\xi_3|^2}>\frac{|\psi_1|^2}{|\xi_1|^2},\quad\frac{|\psi_2|^2}{|\xi_2|^2},$$

all we can conclude is (2), where $|\psi'\rangle\equiv R|\psi\rangle$ and $\langle\xi|R^\dagger\equiv\langle\xi'|$, with

$$R=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The equations in the new basis are not given by

$$\frac{d|\psi_i'|^2}{dt}=2\gamma|\psi_i'|^2\sum_{k=1}^{3}|\psi_k'|^2\left(\frac{|\psi_i'|^2}{|\xi_i'|^2}-\frac{|\psi_k'|^2}{|\xi_k'|^2}\right),\quad i=1,2,3$$

as assumed by Turner, but are just the unprimed equations with $|\psi_1|^2$ replaced by $|\psi_1'\cos\theta-\psi_2'\sin\theta|^2$, etc. Inequalities (2) together with the correct primed collapse equations produce the same result as in the unprimed case, and there is no reordering of the quantum propositions due to a change of basis.

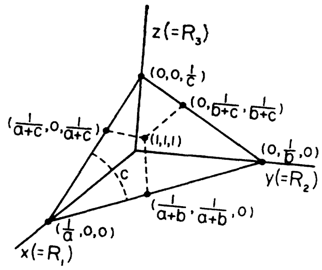


FIG. 1. Probability plane in the $n=3$ case: $ax+by+cz=|\xi_1|^2R_1+|\xi_2|^2R_2+|\xi_3|^2R_3=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2=1$.

The Bohm-Bub theory does achieve a reordering in a loose sense, but not in the sense of Zierler and Schlessinger, where $a\leq b$ means $f(a)\leq f(b)$. It is shown later in this paper that the Bohm-Bub theory produces the usual probabilities, so that the probability of finding $a$ to be true in the Bohm-Bub theory is just equal to $f(a)$ and similarly for $b$. Hence the inequality on states is preserved. It should be noted that the propositions (or projection operators) correspond to regions of asymptotic stability in the probability plane (see Fig. 1), whereas the states of the logicians correspond to normalized areas in the hidden-variable plane (see Fig. 2).

The heuristic sense in which the order is not preserved can best be seen in the following example: Let $B$ be a measurement of $S_z$ of orthohelium and $A$ a measurement that determines only if the spin is either up or down, or zero. Let

$$|\psi\rangle=\psi_1^B|S_z^0\rangle+\psi_2^B|S_z^+\rangle+\psi_3^B|S_z^-\rangle=\psi_1^A|S_z^0\rangle+\psi_2^A|S_z^\pm\rangle,$$

with

$$\psi_1^B=\psi_1^A,\quad \xi_1^B=\xi_1^A,\quad |\psi_2^A|^2=|\psi_2^B|^2+|\psi_3^B|^2,$$

and

$$|\xi_2^A|^2=|\xi_2^B|^2+|\xi_3^B|^2.$$

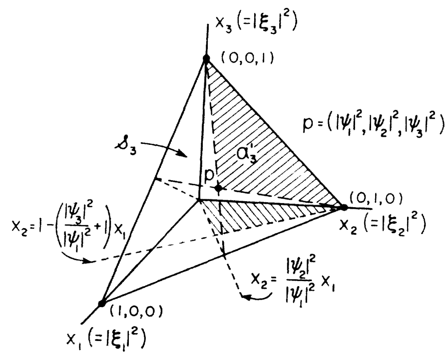


FIG. 2. Hidden-variable plane in the $n=3$ case: $x_1+x_2+x_3=|\xi_1|^2+|\xi_2|^2+|\xi_3|^2=1$.

[9] J. E. Turner, J. Math. Phys. **9**, 1411 (1968).

[10] N. Zierler and M. Schlessinger, Duke Math. J. **32**, 251 (1965).

Then the various regions of the probability plane correspond to the following propositions:

$R_1^B > R_2^B,\ R_3^B$ if and only if "$S_z$ is 0,"

$R_2^B > R_1^B,\ R_3^B$ if and only if "$S_z$ is 1,"

$R_3^B > R_1^B,\ R_2^B$ if and only if "$S_z$ is $-1$,"

$R_1^A > R_2^A$ if and only if "the spin is zero,"

$R_2^A > R_1^A$ if and only if "the spin is either up or down."

The generalized theory must be used so that $R_2^A$ is properly defined. Now to simplify the notation let

$$|\xi_1^B|^2 = a, \quad |\xi_2^B|^2 = b, \quad |\xi_3^B|^2 = c,$$

so

$$a+b+c=1$$

and

$$R_1^B = R_1, \quad R_2^B = R_2, \quad R_3^B = R_3,$$

so

$$aR_1 + bR_2 + cR_3 = 1.$$

It is easily seen that

$$R_1 > R_2,\ R_3 \Rightarrow R_1 > (R_2 b + R_3 c)/(b+c),$$

or that

$$R_1^B > R_2^B,\ R_3^B \Rightarrow R_1^A > R_2^A;$$

i.e., "$S_z$ is 0" $\Rightarrow$ "the spin is zero." To prove this, note that $R_1 > R_2 \Rightarrow bR_1 > bR_2$ and $R_1 > R_3 \Rightarrow cR_1 > cR_3$, so $(b+c)R_1 > R_2 b + R_3 c$. It is also easily seen that $R_1 > (R_2 b + R_3 c)/(b+c)$ does not imply $R_1 > R_2,\ R_3$, or that $R_1^A > R_2^A$ does not imply $R_1^B > R_2^B,\ R_3^B$; i.e., "the spin is zero" does not imply "$S_z$ is 0." An easy counterexample occurs when $a=b=c=\frac{1}{3}$, $R_1 = 5/4$, $R_2 = 0$, and $R_3 = 7/4$. We have $R_1 > (R_2 b + R_3 c)/(b+c)$ but $R_3 > R_1$.

The conclusion concerning the results of the previous two paragraphs may be that the definition of the order of propositions in terms of orderings of probabilities may be inadequate when the propositions refer to measurements made of different observables. On the other hand, if $a \leq b$ means $a \Rightarrow b$ in the ordinary heuristic sense of implication, then this result could mean that the propositions refer to properties of the system plus apparatus and, hence, different apparatus may produce different results. In any event, it is not clear what the import of the Zierler-Schlessinger paper is with respect to the Bohm-Bub theory.

It may also be the case that the definition of a hidden-variable theory in terms of a Boolean embedding is not a good definition. A new definition has been given by Gudder[11] which follows more closely the ideas proposed by Bohm and Bub (see Appendix A). Using this definition, Gudder is able to show that hidden-variable theories exist in great abundance. This result, which seems to contradict the conclusions of von Neumann and others, can be seen to be more reasonable when it is realized that the proponents of the quantum logic have, until now, implicitly built their measurement theory into their axiomatic system, as have the more orthodox physicists. The quantum logicians seem to combine the concepts of "simultaneous measurability," "deterministic theory," "Boolean algebra," and hidden-variable theory."[12] They succeed only in confusing themselves and their readers. Although the Bohm-Bub theory is deterministic in a certain sense which is to be discussed later, it does not say that all observables are simultaneously measurable. It is not true that Bohm and Bub have simply embedded the ordinary logic of quantum mechanics into a Boolean algebra. It would perhaps be better for the logicians to consider ordinary quantum mechanics as a physical theory which has already been embedded in a measurement theory.

Returning to the discussion of the generalized measurement equation, I would like to note that one of the fundamental assumptions of any physical science is that, in order to measure a physical property of any physical system, it is necessary to disturb the system; furthe-s more, I would like to assume that the disruption alwayr corresponds to an energy change in the system. The energy change will be assumed to correspond uniquely to the particular apparatus which is being used. I propose to take $\gamma^A \equiv |\Delta E|_A/\hbar$. Now, $\gamma^A$ will have some time dependence, which will depend on the apparatus involved, but this time dependence will not change the character of the solutions to the collapse equations, since $\gamma^A$ is always a multiplicative factor and is the same for each component of $|\psi\rangle$. The relation of the collapse time to the measurement time will be discussed in detail below, and it will be shown that this definition of $\gamma^A$ builds the uncertainty relation, for $\Delta E$ and $\Delta t$, into the collapse equation and offers a general explanation of this relation which cannot be rigorously derived in ordinary quantum mechanics.

The sum over all experiments together with the foregoing definition of $\gamma^A$ gives a more realistic measurement theory in that the correct set of terms is now turned on automatically, so to speak, by the $\gamma^A$ when the system interacts with apparatus $M^A$. We have no need for observers who turn on the apparatus or record numbers, since the system is now projected by the measurement equations into the observed eigenvector or subspace, as the case may be. Notice also that this theory does not say that the system either possesses or does not possess a particular value for a given observable before it is measured.

The collapse equation could be viewed as a preparation-of-state equation as well as a measurement equation (where the distinction is Margenau's).[13] For example, *if* a photon leaves a polarizer, then it is in a certain state of polarization by the operational definition of the statement "the photon has polarization $|S_1\rangle$." Whether anyone observes the photon or not is

[11] S. Gudder, J. Math. Phys. (to be published).

[12] See Ref. 10, p. 257.

[13] H. Margenau, Phil. Sci. 30, 1 (1963).

irrelevant, since the preparation statement is a conditional statement, conditioned in existence. A measurement, according to Margenau, must produce a recorded number, though the recording is not necessarily done in the mind of the observer. So if the idea of a collapse of the wave packet upon measurement is distasteful, one could instead consider the equation to be a preparation equation describing the collapse of the wave function upon preparation. I think the idea of a collapse during measurement is distasteful only if (a) no physical process or equation is involved, (b) it produces no new experimental results, (c) the individual system is confused with an ensemble, or (d) it produces no better explanation of quantum phenomena. None of these objections apply to the theory in question. It should also be noted that in the sense of Margenau, the Bohm-Bub theory is only a partial theory of measurement in that it does not take into account the final association of a number (pointer reading) with the state of the system. A theory like that due to Daneri, Loinger, and Prosperi[14] might be needed to complete it.

The time dependence of the hidden variables is the next point to consider. The theory could be generalized by giving the hidden variables some time dependence and introducing an equation of motion which produces a time randomization of the hidden variables, but such a generalization might destroy the ability of this theory to produce the usual quantum-mechanical probabilities if the randomization time turned out to be comparable to the collapse timel To make the frequency theory of probabilities in quantum mechanics viable, we must maintain that experiments are performed on ensembles. The idea of a time ensemble that uses the same particle over and over again is not a justifiable one, since quantum mechanics provides no way of testing to see if it indeed is the same particle. Hence, any realistic description of a time ensemble is a special case of a space ensemble which is spread out sufficiently in space to make the individual particles distinct. So we need only talk about space ensembles. Once this point is realized, it is no longer necessary to introduce a randomization time, as was done by Bohm and Bub, in order to produce the quantum probabilities. In fact, an experiment was performed by Papaliolios,[15] which showed the randomization time to be smaller than that conjectured by Bohm and Bub. The result of the Papaliolios experiment can be easily explained if one assumes the randomization time to be due to changes in the identity of the photon and not due to changes in the hidden variables of the initial photon. Indeed, in a polarizer the photon that enters is not the one which exists. For the apparatus in question, the interaction distance was about one-fifth of the thickness, which gives a time of about $1.5\times10^{-14}$ sec. Papaliolios concluded that the randomization time is $<2.4\times10^{-14}$ sec, as would be expected for a space randomization rather than a time randomization.

The hidden variables will be assumed to be constant in time. A simple experimental test of this assumption would be a double Stern-Gerlach experiment with spin-$\frac{1}{2}$ particles, where the first apparatus would project the system into eigenstates of $S_z$ and the second into eigenstates of $\hat{p}\cdot\mathbf{S}$, where $\hat{p}$ is a unit vector in the $yz$ plane that is not directed along either axis. The second measurement involves conditional probabilities in both the hidden-variable theory and the ordinary theory, but these theories predict different results. In both cases, the probability that $\hat{p}\cdot\mathbf{S}$ is $+\frac{1}{2}\hbar$ after a $S_z$ measurement is given by $P(\hat{p}\cdot\mathbf{S}=+\frac{1}{2}\hbar)=$ (the probability of being in the $+z$ beam)(the probability that $\hat{p}\cdot\mathbf{S}=+\frac{1}{2}\hbar$, given that the particle is in the $+z$ beam)+(the probability of being in the $-z$ beam)(the probability that $\hat{p}\cdot\mathbf{S}=+\frac{1}{2}\hbar$, given that the particle is in the $-z$ beam). If $|\psi\rangle=\psi_1|S_{z+}\rangle+\psi_2|S_{z-}\rangle$, then $|\psi'\rangle=R|\psi\rangle$ and $\langle\xi'|=\langle\xi|R^\dagger$, where

$$R=\begin{pmatrix}\cos\frac{1}{2}\theta & -i\sin\frac{1}{2}\theta\\ -i\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta\end{pmatrix}$$

and $\theta$ is the angle between $\hat{z}$ and $\hat{p}$. The probability that $\hat{p}\cdot\mathbf{S}=+\frac{1}{2}\hbar$ is given by

$$|\psi_1'|^2=|\psi_1|^2\cos^2(\tfrac{1}{2}\theta)+|\psi_2|^2\sin^2(\tfrac{1}{2}\theta)-|\psi_1||\psi_2|\sin(\phi_1-\phi_2)\sin\theta,$$

with

$$\psi_1=|\psi_1|e^{i\phi_1},\quad \psi_2=|\psi_2|e^{i\phi_2},\quad \xi_1=|\xi_1|e^{i\alpha_1},\quad \xi_2=|\xi_2|e^{i\alpha_2}.$$

Also,

$$|\xi_1'|^2=|\xi_1|^2\cos^2(\tfrac{1}{2}\theta)+|\xi_2|^2\sin^2(\tfrac{1}{2}\theta)+|\xi_1||\xi_2|\sin(\alpha_1-\alpha_2)\sin\theta.$$

In the $+z$ beam, $|\psi_1|^2=1$ and $|\psi_2|^2=0$, if the $S_z$ measurement is complete, and this is true for both theories. If the hidden variables are constant, then we know that for particles in the $+z$ beam $|\xi_1|^2<|\psi_1(t=0)|^2$. To see the difference in the predictions, suppose that the initial beam has been prepared so that $|\psi_1(t=0)|^2=\frac{1}{2}$ and that $\theta$ is chosen to be 45°. Then we can set

$$|\xi_1'|^2=\cos^2(45°/2)-\tfrac{1}{2}\sqrt{2}[(1-x^2)-xy(1-x^2)^{1/2}],$$

where $\frac{1}{2}<|\xi_2|^2=1-x^2<1$ and $-1<y=\sin(\alpha_1-\alpha_2)<1$. It is easy to see that if $0<|\xi_1|=x<\frac{1}{2}\sqrt{2}$, then $[(1-x^2)-xy(1-x^2)^{1/2}]>0$, and hence, $|\xi_1'|^2<\cos^2(45°/2)$. Now, if $|\xi_1'|^2<|\psi_1'(t=t_1)|^2=\cos^2(45°/2)$, where $t=t_1$ corresponds to the second measurement, then the $\hat{p}\cdot\mathbf{S}$ measurement, in the $+z$ beam, produces $+\frac{1}{2}\hbar$. So ordinary quantum mechanics predicts that 85.3% of the $+z$ beam will have $\hat{p}\cdot\mathbf{S}=+\frac{1}{2}\hbar$, while the Bohm-Bub theory predicts 100%.

As far as this author knows, this experiment has not been performed. It differs from an analogous double polarization experiment with light in that the identity

[14] A. Daneri, A. Loinger, and G. M. Prosperi, Nucl. Phys. 33, 297 (1962).

[15] C. Papaliolios, Phys. Rev. Letters 18, 622 (1967).

of the particle is preserved during the two measurements. A contradiction with ordinary quantum mechanics would be revolutionary and would preserve the assumption that the hidden variables are constant in time. On the other hand, if the result were the ordinary one, then we could set an upper bound on the randomization time, and if the randomization time came out close to the collapse time, the assumption that $\gamma^A = |\Delta E|_A/\hbar$ would have to be dropped and replaced by $\gamma^A = K|\Delta E|_A/\hbar$, where $K$ is some large number. It will be shown later that $\langle|\Delta E|_A\rangle_{\rm av}\tau_c \approx \frac{1}{2}\hbar$, provided a collapse takes place; so this would mean that the collapse time must always be unobservable if the time-energy Heisenberg relation is to be believed, which in turn would lead one to believe that the Bohm-Bub theory could have no observable consequences. This result would also be of interest in that one might then be tempted to show that a new class of hidden-variable theories must be trivial in that they can produce no new observable consequences. Another explanation of a negative result might be that the hidden variables are not just properties of the system but instead are relational and somehow depend on both the system and the apparatus. In this case, the second apparatus would be the cause of the necessary randomization and $\langle\xi'| = \langle\xi|R^\dagger$ would not hold.

In any event, if one does assume the hidden variables to be constant in time and properties of the individual system, then the quantum-mechanical probabilities are due to space ensembles of similarly prepared particles, similar in that they all have the same $|\psi\rangle$ vector, with randomly distributed $\langle\xi|$ vectors. The two ideas above are entirely consistent in that a preparation of state can only influence $|\psi\rangle$ if $\langle\xi|$ is assumed constant, and the preparation of state is described by the collapse equations. It is shown below that the assumption of space randomization for the hidden variables produces the usual probabilities when the hidden variables are taken to be constant properties of the system.

It is worth noting that there is only one dual vector introduced, but it of course has different components in various representations. As mentioned earlier, the number of hidden variables in each case is equal to the number of distinct eigenvalues of the observable corresponding to the measurement; so it is not the case that the results of this theory are obtained at the expense of introducing an arbitrary number of new parameters into the theory. Mathematically the dual space of any finite-dimensional Hilbert space is just the set of linear functionals from the space to the complex plane. It seems reasonable to suppose that these transformations have something to do with measurement, since a measurement should be a correspondence between the state vector $|\psi\rangle$ and the real numbers which appear on the data sheets. Note that neither the phases of the $\psi_i$'s nor the phases of the $\xi_i$'s influence the consequences of either ordinary quantum mechanics or the measurement theory under discussion. Indeed, the $\xi_i$'s are complex mainly because the $\psi$ s are. The process by which these hidden variables have been introduced does not lead to an infinite regress, since the dual space of the dual space is just the original finite-dimensional space to within isomorphism.

The case of a discrete, finite spectrum is the only one discussed for several reasons. First, the finite case is more realistic in that measurements yield only finite sets of finite real numbers. Secondly, going to limits may sometimes simplify the mathematics but should be avoided, since this approach often yields nonsensical physical conclusions while violating the basic implicit assumptions of quantum mechanics itself. In the theory under discussion, much can be learned about the general nature of the solutions to the differential equations by considering the $n=3$ case.

## 2. PROPERTIES OF THE SOLUTIONS IN THE SINGLE-MEASUREMENT CASE

For the case of a single measurement, Eq. (1) implies

$$\frac{dR_i}{dt} = 2\gamma R_i \sum_{k=1}^{n} |\xi_k|^2 R_k (R_i - R_k), \quad i = 1, \cdots, n$$

where the superscript denoting the measurement of $A$ has been suppressed for the sake of brevity and the $R_k$'s are used instead of the $\psi_k$'s so that the right-hand side becomes a polynomial in these variables. Without loss of generality assumed that $\mathbf{A}$ has a nondegenerate spectrum. As shown in a previous paper,[5] the system of equations has a unique solution which lies in the

$$\sum_{i=1}^{n} |\xi_i|^2 R_i = 1$$

plane in the first quadrant ($R_i \geq 0$ for all $i$). Assume, for the present, that $\gamma$ and $|\xi_k|^2$, $k = 1, \cdots, n$ are constant in time.

In order to show that this measurement theory produces the quantum probabilities, one must know the location of the points in $R_k$ space at which the right-hand side of this system is identically zero. These points are called the critical points of the system. The stability of the solution near a critical point is also of interest, since we would like to conclude that for any initial conditions and choice of hidden variables, one of the $R_i$'s grows to $1/|\xi_i|^2$ while the rest decay to zero. It is easy to show that any ordering of the $R_i$'s is preserved in time[5]; hence, the above conclusion is true, but this argument does not give a very good picture of the details of the collapse process. The problem will be discussed below in the context of Liapunov's direct method.[16]

[16] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications* (Academic Press Inc., New York, 1961).

By observation, it is evident that the points

$$(0,\cdots,0,1/|\xi_i|^2,0,\cdots,0)\,,\quad i=1,\cdots,n\,,$$

and $(1,1,1,\cdots,1)$ are critical points. It is shown below that the first $n$ of these points are asymptotically stable while the last is unstable. For the first set of points, it is sufficient to consider the point $(1/|\xi_1|^2,0,\cdots,0)$. To apply Liapunov's method, we must shift the origin. Let $R_1=\tilde{R}_1+1/|\xi_1|^2$ and $R_j=\tilde{R}_j$ for $j\neq 1$. So

$$\frac{dR_1}{dt}=\frac{d\tilde{R}_1}{dt}=2\gamma\left(\tilde{R}_1+\frac{1}{|\xi_1|^2}\right)\sum_{k=2}^{n}|\xi_k|^2\tilde{R}_k\left(\tilde{R}_1+\frac{1}{|\xi_1|^2}-\tilde{R}_k\right)$$

$$=\frac{2\gamma}{|\xi_1|^4}\Big(\sum_{k=2}^{n}|\xi_k|^2\tilde{R}_k\Big)+\text{higher-order terms}.$$

But

$$\sum_{k=2}^{n}|\xi_k|^2\tilde{R}_k=\sum_{k=2}^{n}|\xi_k|^2R_k=1-|\xi_1|^2R_1$$

$$=1-|\xi_1|^2(\tilde{R}_1+1/|\xi_1|^2)=-|\xi_1|^2\tilde{R}_1.$$

Therefore,

$$\frac{d\tilde{R}_1}{dt}=\frac{-2\gamma}{|\xi_1|^2}\tilde{R}_1+\cdots.$$

Hence,

$$\tilde{R}_1(t)\doteq\tilde{R}_1(0)e^{-2\gamma t/|\xi_1|^2}$$

or

$$R_1(t)\doteq R_1(0)e^{-2\gamma t/|\xi_1|^2}+(1/|\xi_1|^2)(1-e^{-2\gamma t/|\xi_1|^2}).$$

Now,

$$\frac{dR_j}{dt}=2\gamma R_j\sum_{k\neq j}^{n}|\xi_k|^2R_k(R_j-R_k)\,,\quad j\neq 1$$

or

$$\frac{d\tilde{R}_j}{dt}=2\gamma\tilde{R}_j\Bigg[|\xi_1|^2\left(\tilde{R}_1+\frac{1}{|\xi_1|^2}\right)\left(\tilde{R}_j-\tilde{R}_1-\frac{1}{|\xi_1|^2}\right)+\sum_{k=2;\,k\neq j}^{n}|\xi_k|^2\tilde{R}_k(\tilde{R}_j-\tilde{R}_k)\Bigg]$$

$$=-2\gamma\tilde{R}_j/|\xi_1|^2+\text{higher-order terms}.$$

Hence,

$$R_j(t)\doteq R_j(0)e^{-2\gamma t/|\xi_1|^2},\quad j\neq 1.$$

In general, we see that the collapse near a critical point of the form $(0,\cdots,0,1/|\xi_i|^2,0,\cdots,0)$ is exponential, with decay constant equal to $-2\gamma/|\xi_i|^2$. Away from these critical points, the collapse may be slower.

A general sufficiency condition for the origin of a nonlinear autonomous system to be asymptotically stable is that the characteristic roots of the linearized equations all have negative real parts. This state of affairs corresponds to exponential decay. If there is a characteristic root with positive real part, the origin is unstable.[17] Asymptotic stability means that the solution not only stays near the critical point but that it actually approaches the point as time increases. In the case above, the characteristic roots are given by

$$\begin{vmatrix}-2\gamma/|\xi_1|^2-\lambda & 0 & \cdots & 0\\ 0 & -2\gamma/|\xi_1|^2-\lambda & & \vdots\\ \vdots & & & \\ 0 & \cdots & \cdots & -2\gamma/|\xi_1|^2-\lambda\end{vmatrix}=0;$$

therefore,

$$(-1)^n(2\gamma/|\xi_1|^2+\lambda)^n=0\Rightarrow\gamma=-2\gamma/|\xi_1|^2<0.$$

So we, indeed, have asymptotic stability.

Next, consider the point $(1,1,\cdots,1)$. Let $R_j=\tilde{R}_j+1$, $j=1,\cdots n$. Now, $|\xi_k|^2R_k=|\xi_k|^2\tilde{R}_k+|\xi_k|^2$, so

$$\sum_{k=1}^{n}|\xi_k|^2R_k=1=\sum_{k=1}^{n}|\xi_k|^2\tilde{R}_k+\sum_{k=1}^{n}|\xi_k|^2=\sum_{k=1}^{n}|\xi_k|^2\tilde{R}_k+1\,,$$

or

$$\sum_{k=1}^{n}|\xi_k|^2\tilde{R}_k=0.$$

Therefore,

$$\frac{d\tilde{R}_j}{dt}=2\gamma(\tilde{R}_j+1)\sum_{k=1}^{n}|\xi_k|^2(\tilde{R}_k+1)(\tilde{R}_j-\tilde{R}_k)$$

$$=2\gamma\sum_{k\neq j}^{n}|\xi_k|^2(\tilde{R}_j-\tilde{R}_k)+\text{higher-order terms}$$

$$=2\gamma\tilde{R}_j\sum_{k\neq j}^{n}|\xi_k|^2-2\gamma\sum_{k\neq j}^{n}|\xi_k|^2\tilde{R}_k+\cdots$$

$$=2\gamma\tilde{R}_j-2\gamma\tilde{R}_j|\xi_k|^2-2\gamma\sum_{k\neq j}^{n}|\xi_k|^2\tilde{R}_k+\cdots$$

$$=2\gamma(\tilde{R}_j-\sum_{k=1}^{n}|\xi_k|^2\tilde{R}_k)+\cdots$$

$$=2\gamma\tilde{R}_j+\cdots.$$

So,

$$\begin{vmatrix}2\gamma-\lambda & 0 & \cdots & 0\\ 0 & 2\gamma-\lambda & & \\ \vdots & & & \vdots\\ 0 & \cdots & & 2\gamma-\lambda\end{vmatrix}=(2\gamma-\lambda)^n=0$$

implies that $\lambda=2\gamma>0$. Therefore, $(1,1,\cdots,1)$ is unstable.[16]

There are other critical points whose positions are not os obvious. To find these, it is best to consider the three-dimensional case in particular. To simply the notation further, let $R_1=x$, $R_2=y$, $R_3=z$ and $|\xi_1|^2=a$. $|\xi_2|^2=b$, $|\xi_3|^2=c$. So we have $a+b+c=1$, with $a$, $b$, and $c$ being constants while $ax(t)+by(t)+cz(t)=1$ for all $t$ (this is just conservation of probability, i.e., $|\psi_1|^2$

[17] See Ref. 16, p. 48.

$+|\psi_2|^2+|\psi_3|^2=1$) and $a, b, c, x, y, z>0$. The collapse equations now become

$$dx/dt=2\gamma x[by(x-y)+cz(x-z)],$$
$$dy/dt=2\gamma y[ax(y-x)+cz(y-z)],$$
$$dz/dt=2\gamma z[ax(z-x)+by(z-y)].$$

In Fig. 1, we see that the stable critical points are on the axis. On the lines joining stable critical points, we would expect to get unstable critical points. These critical points are

$$(1/(a+b),\ 1/(a+b),\ 0),\ (0,\ 1/(b+c),\ 1/(b+c)),$$

and $(1/(a+c), 0, 1/(a+c))$. That these make the right-hand sides of the above equations zero is readily observed. For example, to show that $(0, 1/(b+c), 1/(b+c))$ is unstable, let $1/(b+c)=D$, $x=\tilde{x}$, $y=\tilde{y}+D$, $z=\tilde{z}+D$, so $a\tilde{x}+b\tilde{y}+c\tilde{z}=0$. Then

$$d\tilde{x}/dt=2\gamma\tilde{x}[b(\tilde{y}+D)(\tilde{x}-\tilde{y}-D)+c(\tilde{z}+D)(\tilde{x}-\tilde{z}-D)]$$
$$=-2\gamma D\tilde{x}+\text{higher-order terms},$$
$$d\tilde{y}/dt=2\gamma(\tilde{y}+D)[a\tilde{x}(\tilde{y}+D-\tilde{x})+c(\tilde{z}+D)(\tilde{y}-\tilde{z})]$$
$$=2\gamma aD^2\tilde{x}+2\gamma cD^2\tilde{y}-2\gamma cD^2\tilde{z}+\text{higher-order terms},$$
$$d\tilde{z}/dt=2\gamma(\tilde{z}+D)[a\tilde{x}(\tilde{z}+D-\tilde{x})+b(\tilde{y}+D)(\tilde{z}-\tilde{y})]$$
$$=2\gamma aD^2\tilde{x}-2\gamma bD^2\tilde{y}+2\gamma bD^2\tilde{z}+\text{higher-order terms},$$

So,

$$\begin{vmatrix} -2\gamma D-\lambda & 0 & 0 \\ 2\gamma aD^2 & 2\gamma cD^2-\lambda & -2\gamma cD^2 \\ 2\gamma aD^2 & -2\gamma bD^2 & 2\gamma bD^2-\lambda \end{vmatrix} = -(2\gamma D+\lambda)\lambda(-2\gamma D+\lambda),$$

which is zero only if $\lambda=0$, $-2\gamma/(b+c)$, $2\gamma/(b+c)$. But $2\gamma/(b+c)>0$; hence, the point

$$(0,\ 1/(b+c),\ 1/(b+c))$$

is unstable. Similarly, $(1/(a+b), 1/(a+b), 0)$ and $(1/(a+c), 0, 1/(a+c))$ are unstable.

Now the question remains: are there other critical points? To see the answer, we only have to notice in how many ways $dx/dt$ can be zero. First, we can have $x=0$ and $[by(x-y)+cz(x-z)]\neq 0$. This case leads to the three critical points already discussed in the $yz$ plane. Secondly, we can have $x\neq 0$ and

$y=0,\ z=0,$ which gives $(1/a,0,0)$

$y=0,\ z=x,$ which gives $(1/(a+c),\ 0,\ 1/(a+c))$

$y=x,\ z=0,$ which gives $(1/(a+b),\ 1/(a+b),\ 0)$

$y=x,\ z=x,$ which gives $(1,1,1)$.

Finally, we can have $by(x-y)+cz(x-z)=0$ for arbitrary $x$, $y$, and $z$ in the $ax+by+cz=1$ plane. Fixing $x$ and solving the two simultaneously for $y$ and $z$ gives

$$y=\frac{(1-ax)\pm[(c/b)(1-ax)(x-1)]^{1/2}}{(b+c)}$$

and

$$z=\frac{(1-ax)\mp[(b/c)(1-ax)(x-1)]^{1/2}}{(b+c)}.$$

Since $y$ and $z$ are real, we must require $1/a\geq x\geq 1$. When $x=1/a$, then $y=z=0$. When $x=1$, then $y=z=1$. If $b=c$, then the two lines given by the above equations are symmetric about the line $y=z=(1-ax)/(b+c)$. In any event, $x\geq y$ and $x\geq z$, since $x\geq 1$, and we are in the plane $ax+by+cz=1$. If $x\neq 1/a$ or 1, then $y\neq z$. So $dy/dt$ and $dz/dt$ cannot both be zero, and we cannot have a critical point. Thus, there are only seven critical points for the $n=3$ case.

In the general case, there will be $2^n-1$ critical points. This is just the total number of ways in which the $n$ $R_i$'s can be chosen to be zero or nonzero, consistent with

$$\sum_{i=1}^{n}|\xi_i|^2R_i=1.$$

To see what happens in the $n$-dimensional case, let $R_n=0$. $dR_n/dt$ is then zero, and we have a system of $n-1$ equations as before but with

$$\sum_{i=1}^{n-1}|\xi_i|^2=1-|\xi_n|^2.$$

It has already been shown that $(0,\cdots,0,1/|\xi_i|^2,0,\cdots,0)$ is always stable while $(1,1,\cdots,1)$ is always unstable. Now consider $R_n\neq 0$ but $R_j=0$ for some $j$ for each point. Again we are back to the $n-1$ case. If one tests the new critical points, one finds that the new determinant is essentially the old one but with one more factor, and possibly a sign change; hence, the positive characteristic root is still present, and so the old unstable critical points become new unstable critical points. For $R_n=0$, we get $N_{n-1}$ critical points, where $N_{n-1}$ is the number of critical points in the $(n-1)$-dimensional case. For $R_n\neq 0$, we get $N_{n-1}+1$ critical points, so $N_n=2N_{n-1}+1$. The solution to this equation is seen to be $N_n=2^n-1$. The discussion above is made more explicit for the $n=4$ case in Appendix B.

Returning to the $n=3$ case, we note that since the ordering with respect to size of $x$, $y$, and $z$ is preserved by the collapse equations,[5] the dashed lines in Fig. 1 divide the probability plane into three regions of asymptotic stability in that any solution which starts in a particular region goes to the axis in that region. Notice further that since we can write the equations as

$$\frac{dx}{dy}=\frac{x}{y}\left(\frac{bx(x-y)+cz(x-z)}{ax(y-x)+cz(y-z)}\right),$$

and

$$\frac{dy}{dz}=\frac{y}{z}\left(\frac{ax(y-x)+cz(y-z)}{ax(z-x)+by(z-y)}\right),$$

the orbits in the probability plane are independent of any time dependence in $\gamma$ but do depend on the time dependence of the hidden variables since changes in the hidden variables will change the slopes of the orbits and even cause them to shift from one region of asymptotic stability to another. By looking at the orbits in the probability plane, we see that existence, uniqueness, and stability of the solutions will be preserved by time-dependent $\gamma$'s but that uniqueness and stability will not be preserved by time-dependent hidden variables.

The probabilistic results of ordinary quantum mechanics can be obtained from this measurement theory if the probabilities are interpreted as being due to measurements made on ensembles of individual systems with the same $|\psi\rangle$ vector but different $\langle\xi|$ vectors. Indeed, this seems to be the only way of saying that the systems are the same yet different. These ensembles are to be considered as space ensembles, as discussed earlier. The result, which is demonstrated below in general, was demonstrated for the simplest case in the original paper by Bohm and Bub and is proven indirectly in a new paper by Bub.[18] In the construction given here, it is easy to envision the source of the quantum probabilities as ratios of planar areas.

First consider the case $n=3$. In particular, let us calculate the probability of getting $A_1$ with a given $|\psi\rangle$ but random $\langle\xi|$. The wave function for an individual system will collapse to $|A_1\rangle$ if and only if the following two conditions are met: $R_1>R_2$ and $R_1>R_3$, which in turn imply $x_1<(|\psi_1|^2/|\psi_2|^2)x_2$ and $x_1<(|\psi_1|^2/|\psi_3|^2)x_3 = (|\psi_1|^2/|\psi_3|^2)(1-x_1-x_2)$, where $x_i=|\xi_i|^2$ and $x_1+x_2+x_3=1$ is the hidden-variable plane given in Fig. 2. The conditions are met for the part of the plane labeled $\mathcal{A}_3{}^1$. Hence the probability of getting $A_1$ for the result is equal to $\mathcal{A}_3{}^1/\mathcal{S}_3$, where $\mathcal{S}_3$ is the total area of the plane. The general formula for the area of a surface gives the following:

$$\mathcal{S}_3=\int_0^1\int_0^{1-x_1}\sqrt{3}\,dx_2dx_1=\sqrt{3}\int_0^1(1-x_1)dx_1=\frac{\sqrt{3}}{2!}.$$

$$\mathcal{A}_3{}^1=\int_0^{|\psi_1|^2}\int_{(|\psi_2|^2/|\psi_1|^2)x_1}^{1-[(|\psi_3|^2/|\psi_1|^2)+1]x_1}\sqrt{3}\,dx_2dx_1$$

$$=\sqrt{3}\int_0^{|\psi_1|^2}\left[1-\left(\frac{|\psi_3|^2}{|\psi_1|^2}+\frac{|\psi_2|^2}{|\psi_1|^2}+1\right)x_1\right]dx_1$$

$$=\sqrt{3}\left[|\psi_1|^2-\left(\frac{|\psi_3|^2}{|\psi_1|^2}+\frac{|\psi_2|^2}{|\psi_1|^2}+\frac{|\psi_1|^2}{|\psi_1|^2}\right)\tfrac{1}{2}|\psi_1|^4\right]$$

$$=\sqrt{3}|\psi_1|^2(1-\tfrac{1}{2})=(\sqrt{3}/2!)|\psi_1|^2;$$

therefore,

$$\mathcal{A}_3{}^1/\mathcal{S}_3=|\psi_1|^2.$$

the other two segments of the triangle ($R_2>R_1$, $R_2>R_3$ and $R_3>R_1$, $R_3>R_2$) give $|\psi_2|^2$ and $|\psi_3|^2$, respectively, when normalized by $\mathcal{S}_3$.

Now consider the general $n$-dimensional case. Let $\mathcal{A}_n{}^1$ be the region of the hyperplane given by the conditions $R_1>R_j$, $j=2,3,\cdots,n$ or, equivalently, $x_1<(|\psi_1|^2/|\psi_j|^2)x_j$, $j=2,\cdots,n$, with

$$\sum_{i=1}^n x_i=1=\sum_{i=1}^n|\psi_i|^2.$$

In terms of limits on the integrals for $\mathcal{A}_n{}^1$, these conditions are

$$(|\psi_{n-1}|^2/|\psi_1|^2)x_1<x_{n-1}<1-(|\psi_n|^2/|\psi_1|^2+1)x_1-x_2-\cdots-x_{n-2},$$
$$(|\psi_{n-2}|^2/|\psi_1|^2)x_1<x_{n-2}<1-(|\psi_n|^2/|\psi_1|^2+|\psi_{n-1}|^2/|\psi_1|^2+1)x_1-\cdots-x_{n-3},$$
$$\cdots$$
$$(|\psi_2|^2/|\psi_1|^2)x_1<x_2<1-(|\psi_n|^2/|\psi_1|^2+\cdots+|\psi_3|^2/|\psi_1|^2+1)x_1,$$
$$0<x_1<|\psi_1|^2.$$

Now make the following change of variables:

$$x_1=|\psi_1|^2y_1+0+0+\cdots+0,$$
$$x_2=|\psi_2|^2y_1+y_2+0+\cdots+0,$$
$$\cdots$$
$$x_{n-1}=|\psi_{n-1}|^2y_1+0+0+\cdots+y_{n-1}.$$

Then,

$$J=\left|\frac{\partial(x_1,\cdots,x_{n-1})}{\partial(y_1,\cdots,y_{n-1})}\right|=\begin{vmatrix}|\psi_1|^2 & 0 & 0 & \cdots & 0\\ |\psi_2|^2 & 1 & 0 & \cdots & 0\\ \cdots & & & & \\ |\psi_{n-1}|^2 & 0 & 0 & \cdots & 1\end{vmatrix}=|\psi_1|^2,$$

and the new limits are

$$0<y_{n-1}<1-y_1-y_2-\tfrac{1}{2}\cdots-y_{n-2},$$
$$0<y_{n-2}<1-y_1-y_2-\cdots-y_{n-3},$$
$$\vdots$$
$$0<y_1<1,$$

which, if we replace $x_k$ by $y_k$, are the same as the limits on the integral for $\mathcal{S}_n$. Hence,

$$\frac{\mathcal{A}_n{}^1}{\mathcal{S}_n}=n^{1/2}\int_0^1\int_0^{1-y_1}\cdots\int_0^{1-y_1-\cdots-y_{n-2}}|\psi_1|^2dy_{n-1}dy_{n-2}\cdots dy_1\Big/ n^{1/2}\int_0^1\int_0^{1-y_1}\cdots\int_0^{1-y_1-\cdots-y_{n-2}}dy_{n-1}dy_{n-2}\cdots dy_1=|\psi_1|^2; \quad (4)$$

[18] J. Bub, Int. J. Theoret. Phys. (to be published).

or, in general, $\mathcal{A}_n{}^i/\mathcal{S}_n = |\psi_i|^2$ [aside: $\mathcal{S}_n = n^{1/2}/(n-1)!$]. Hence, $\langle \mathbf{A}\rangle$ in the Bohm-Bub theory$=\sum_1^n A_i$ (probability of getting $A_i$)$=\sum_1^n A_i|\psi_i|^2 = \langle \mathbf{A}\rangle$ in ordinary quantum mechanics. Also, $(\Delta\mathbf{A})^2$ in the Bohm-Bub theory is equal to $(\Delta\mathbf{A})^2$ in ordinary quantum mechanics.

## 3. COLLAPSE TIME AND THE TIME-ENERGY HEISENBERG RELATION

In a previous paper, the problem of finding a collapse time was considered under the assumption that $\gamma$ was a simple step function in time. This assumption is unrealistic if $\gamma$ is to be related to an energy change in the system due to measurement. The collapse time is important in that it seems to be the only physical manifestation of this hidden-variable theory at present. The problem will be reconsidered for arbitrary $\gamma(t)$.

It is difficult to define the idea of a collapse time because each orbit is associated with a different time. Since any solution which starts at a critical point will stay there (whether it is stable or not), it is not meaningful to ask for the longest collapse time. Without loss of generality, consider the region of the probability plane in Fig. 1 which contains the point $(1/a,0,0)$. Let

$$\epsilon = e^{-5}(3-2/a+1/a^2)^{1/2},$$

and exclude from consideration an $\epsilon$ neighborhood of each critical point. Denote the arc with center at $(1/a,0,0)$ by $C$ and the straight line between $(1,1,1)$ and $(1/a,0,0)$ by $L$. Any orbit which starts on $L$ will stay on $L$,[5] so the orbit of smallest length which starts on $C$ will be the one which also starts on $L$. Now on the average, the ends of arc $C$ will be closer to the critical points than will the center of $C$. Let $\tau_1=$ (the average collapse time for orbits starting on an arbitrary arc $C$), $\tau_2=$ (the collapse time for the orbit starting on $C$ and also on $L$), and $\tau_3=$ (the collapse time for the orbit starting on $C$ and on $L$, where the linearized collapse equation is used). It seems reasonable to conjecture that $\tau_1 \geq \tau_2$, since $\tau_1$ involves longer orbits which start slower on the average.

It is easy to show that $\tau_2 \geq \tau_3$. To do this, let $R_1 = \tilde{R}_1 + 1/a$, $R_2 = \tilde{R}_2$, and $R_3 = \tilde{R}_3$, where $a = |\xi_1|^2$. Also, $R_1 > R_2 = R_3 = (1-|\xi_1|^2 R_1)/(1-|\xi_1|^2)$. So

$$\frac{d\tilde{R}_1}{dt} = -\frac{2\gamma\tilde{R}_1}{|\xi_1|^2}\left[(1+|\xi_1|^2\tilde{R}_1)\Big/\left(1+\frac{|\xi_1|^2\tilde{R}_1}{1-|\xi_1|^2}\right)\right]$$

(true collapse equation)

and $|\xi_1|^2-1 \leq |\xi_1|^2 R_1 - 1 = |\xi_1|^2\tilde{R}_1 \leq 0$. We know that $d\tilde{R}_1/dt = dR_1/dt > 0$, so we have

$$0 \leq (1+|\xi_1|^2\tilde{R}_1)\left(1+\frac{|\xi_1|^2\tilde{R}_1}{1-|\xi_1|^2}\right) \leq 1.$$

Hence, the true equation gives a smaller $dR_1/dt$ or larger collapse time than does the linearized equation, $d\tilde{R}_1/dt = -2\gamma\tilde{R}_1/|\xi_1|^2$. For the linearized equation, we get

$$\tilde{R}_1(t) = \tilde{R}_1(0)\exp\left[\frac{-2}{|\xi_1|^2}\int_0^t \gamma(t)\,dt\right].$$

Define $\tau_3$ to be the time it takes $\tilde{R}_1(t)$ to go to $\tilde{R}_1(0)e^{-5}$; then,

$$\int_0^{\tau_3} \gamma(t)\,dt \equiv \tfrac{5}{2}|\xi_1|^2,$$

or $\tau_3 = \frac{5}{2}|\xi_1|^2/\langle\gamma\rangle_{av}$, Hence, $\tau_1 \geq \tau_3 = \frac{5}{2}|\xi_1|^2/\langle\gamma\rangle_{av}$ or, in the general case $\langle|\Delta E|_{A_i}\rangle_{av}\tau_i \geq \frac{5}{2}\hbar|\xi_i|^2$, where $\tau_i = \tau_1$ is the average collapse time for a measurement of $\mathbf{A}$ which produces the eigenvalue $A_i$ and

$$\langle|\Delta E|_{A_i}\rangle_{av} \equiv \frac{\hbar}{\tau_3}\int_0^{\tau_3}\gamma(t)\,dt$$

with $R_i = \max\{R_j\}$.

Since $0 < |\xi_i|^2 < 1$, this uncertainty relation, which has been derived in a measurement-theory context, may or may not agree with the heuristic Heisenberg result. In particular, since $|\xi_i|^2$ is most likely to be close to $1/n$ if $n$ is large, the right-hand side is most likely to be smaller than $\frac{1}{2}\hbar$. In the limit of many degrees of freedom, we can thus get the classical result without letting $\hbar$ go to zero. If the assumption that $\gamma^A = |\Delta E|_A/\hbar$ is dropped, the relation becomes $\langle\gamma_i{}^A\rangle_{av}\tau_i \geq \frac{5}{2}|\xi_i|^2$, so the $i$th collapse time still depends on the value of the $i$th hidden variable. This result lends itself to possible experimental verification.

## 4. SIMULTANEOUS MEASUREMENT IN THE GENERALIZED BOHM-BUB THEORY

In the ordinary formulation of quantum mechanics, the idea of noncommeasurability is expressed algebraically by noncommutativity of the operators involved, whereas in the axiomatic formulation we have a loss of distributivity of the experimental propositions, leading to the use of a nonclassical logic. Noncommutativity gives rise to the Heisenberg uncertainty relations, which are generally assumed to place limits on the accuracy of simultaneous measurements, and are used to justify the use of nondistributive logics. Recently this interpretation of the uncertainty relations has been questioned by several authors,[19,20] who have pointed out that the time-energy relation is of a different nature than the general operator relation involving the commutator and that neither is a measurement-theoretic statement, as the measurement theory of ordinary quantum mechanics is implicit, not explicit.

In the generalized Bohm-Bub theory of measurement, it is possible to give the idea of simultaneous measurement an exact mathematical definition. When the observables are not commeasurable, the resulting inter-

[19] W. T. Scott, Ann. Phys. (N. Y.) 46, 577 (1968); 47, 489 (1968).
[20] M. Bunge, Am. J. Phys. 24, 272 (1956).

ference between the measurements, often discussed only heurisitically by physicists, can be given mathematical reality via a coupling of the collapse equations for the two representations involved. It is shown below that this interference may make the classical idea of a joint probability distribution meaningless, that the operator-measurement correspondence used in classical quantum mechanics may break down and, consequently, that the idea of simultaneous measurement cannot be meaningfully formulated without going outside of quantum mechanics itself.

The original paper by Bohm and Bub only alludes to the idea of a simultaneous measurement. Indeed, since Bohm and Bub were only trying to dispel the idea that hidden-variable theories are impossible, they were not concerned with giving a complete theory. To derive the coupled collapse equations which define a simultaneous measurement, the generalized Bohm-Bub equations must be used. Whether these equations are correct or not is not of large concern, since I am only trying to show that the concept of a simultaneous measurement can be well defined only in such a theoretical context. Gudder[11] has shown that many other particular hidden-variable theories are possible, so that the Bohm-Bub theory may not be the best one to use.

To simplify the notation, let

$$\left.\frac{d\psi_i}{dt}\right|_A=\sum_{j=1}^{n}\gamma^A(R_i^A-R_j^A)J_j^A\psi_i^A,\quad i=1,\cdots,n.$$

This is the time rate of change of $\psi_i$ due to an $A$ measurement alone. So $d\psi_i/dt|_{AB}$ represents a simultaneous measurement of the observables $A$ and $B$. Now if $[\mathbf{A},\mathbf{B}]\neq 0$ and $\gamma^A\neq 0$, $\gamma^B\neq 0$, but $\gamma^C=0$ for all $C\neq A$, $B$, then

$$\left.\frac{d\psi_i}{dt}\right|_{AB}=\left.\frac{d\psi_i}{dt}\right|_A+\frac{d}{dt}\Big(\sum_{j=1}^{n}\phi_j\langle A_i|B_j\rangle\Big),\tag{5}$$

where $|\psi\rangle=\sum^n\psi_i|A_i\rangle=\sum^n\phi_j|B_j\rangle$ or

$$\psi_i=\sum_{j=1}^{n}\phi_j\langle A_i|B_j\rangle.$$

To show that this is true, one must make use of the generalized collapse equation. By (1),

$$\frac{d|\psi\rangle}{dt}=\sum_{i,j}\gamma^A(R_i^A-R_j^A)J_j^AP_i^A|\psi\rangle+\sum_{i,j}\gamma^B(R_i^B-R_j^B)J_j^BP_i^B|\psi\rangle.$$

Now,

$$\frac{d}{dt}\sum_{l=1}^{n}P_l^A|\psi\rangle=\frac{d\psi_1}{dt}|A_1\rangle+\cdots+\frac{d\psi_n}{dt}|A_n\rangle=\frac{d|\psi\rangle}{dt}.$$

So

$$\begin{aligned}\frac{d|\psi\rangle}{dt}&=\gamma^A\psi_1\sum_j(R_1^A-R_j^A)J_j^A|A_1\rangle+\cdots+\gamma^A\psi_n\sum_j(R_n^A-R_j^A)J_j^A|A_n\rangle+\sum_{i,j}\gamma^B(R_i^B-R_j^B)J_j^BP_i^B|\psi\rangle,\\&=\gamma^A\psi_1\sum_j(R_1^A-R_j^A)J_j^A|A_1\rangle+\cdots+\sum_i\sum_{j,k}\gamma^B(R_j^B-R_k^B)J_k^B\phi_j|A_i\rangle\langle A_i|B_j\rangle,\\&=\gamma^A\psi_1\sum_j(R_1^A-R_j^A)J_j^A|A_1\rangle+\cdots+\sum_{j,k}\gamma^B(R_j^B-R_k^B)J_k^B\phi_j\langle A_1|B_j\rangle|A_1\rangle+\cdots,\end{aligned}$$

where

$$\sum_{i=1}^{n}|A_i\rangle\langle A_i|=1$$

has been used. Hence,

$$\frac{d\psi_i}{dt}=\gamma^A\psi_i\sum_j(R_i^A-R_j^A)J_j^A+\sum_{j,k}\gamma^B(R_j^B-R_k^B)J_k^B\phi_j\langle A_i|B_j\rangle.\tag{6}$$

The last sum can be rewritten as

$$\sum_j\Big(\gamma^B\phi_j\sum_k(R_j^B-R_k^B)J_k^B\Big)\langle A_i|B_j\rangle$$

and

$$\left.\frac{d\phi_j}{dt}\right|_B=\gamma^B\phi_j\sum_k(R_j^B-R_k^B)J_k^B.$$

So

$$\left.\frac{d\psi_i}{dt}\right|_{AB}=\left.\frac{d\psi_i}{dt}\right|_A+\frac{d}{dt}\Big(\sum_j\phi_j\langle A_i|B_j\rangle\Big)$$

as desired. Starting from the **B** representation we get the corresponding equations for the $\phi_i$'s:

$$\left.\frac{d\phi_i}{dt}\right|_{AB}=\left.\frac{d\phi_i}{dt}\right|_B+\frac{d}{dt}\Big(\sum_j\psi_j\langle B_i|A\rangle_j\Big).$$

It should be noted that

$$\left.\frac{d\psi_i}{dt}\right|_A\neq\frac{d}{dt}\Big(\sum_j\phi_j\langle A_i|B_j\rangle\Big)$$

even though $\sum_j \phi_j\langle A_i|B_j\rangle=\psi_i$, since the $\psi_i$'s and $\phi_j$'s correspond to different measurement processes and the $|_A$ signals a particular process.

Equation (6) shows the coupling of the two sets of collapse equations explicitly. In the case where $[\mathbf{A},\mathbf{B}]\neq 0$ but the two measurements are made at different times, it is easy to see that the results depend on the order of the measurements. Suppose that at $t=0$, $R_1{}^A>R_j{}^A$, $R_1{}^B>R_j{}^B$, $j=2,\cdots,n$. If an $A$ measurement is performed, then $|\psi\rangle\to|A_1\rangle\neq|\psi(0)\rangle$. Now let $t_1>0$. So

$$|\psi(t_1)\rangle=\sum_{i=1}\langle B_i|A_1\rangle|B_i\rangle=\sum_1^n\phi_i(t_1)|B_i\rangle \neq\sum_1^n\langle B_i|\psi(0)\rangle|B_i\rangle=\sum_1^n\phi_i(0)|B_i\rangle.$$

Consequently, the inequalities $R_1{}^B>R_j{}^B$, $j=2,\cdots,n$, may be changed at $t=t_1$, so a $B$ measurement at $t=t_1$ would not produce the same result as a $B$ measurement at $t=0$.

On the other hand, if $[\mathbf{A},\mathbf{B}]=0$ and $\gamma^A\neq 0$, $\gamma^B\neq 0$, but $\gamma^C=0$ for all $C\neq A$, $B$, then

$$\left.\frac{d\psi_i}{dt}\right|_{AB}=\frac{\gamma^A+\gamma^B}{\gamma^A}\left.\frac{d\psi_i}{dt}\right|_A=\frac{\gamma^A+\gamma^B}{\gamma^B}\left.\frac{d\phi_i}{dt}\right|_B=\left.\frac{d\phi_i}{dt}\right|_{AB}.$$

Thus there is then only one set of collapse equations, but with a new collapse constant $\gamma^{AB}=\gamma^A+\gamma^B$. To see this, notice that $[\mathbf{A},\mathbf{B}]=0$ implies that there is a common eigenbasis; hence, $\psi_i=\phi_i$ and

$$(1)\Rightarrow\frac{d|\psi\rangle}{dt}=(\gamma^A+\gamma^B)\sum_{i,j}(R_i{}^A-R_j{}^B)J_j{}^AP_i{}^A|\psi\rangle,$$

or

$$\left.\frac{d\psi_i}{dt}\right|_{AB}=(\gamma^A+\gamma^B)\psi_i\sum_j(R_i{}^A-R_j{}^A)J_j{}^A=\frac{(\gamma^A+\gamma^B)}{\gamma^A}\left.\frac{d\psi_i}{dt}\right|_A.$$

Now,

$$\gamma^{AB}=\gamma^A+\gamma^B=\frac{|\Delta E|_A+|\Delta E|_B}{\hbar}>\frac{|\Delta E|_A}{\hbar},\ \frac{|\Delta E|_B}{\hbar},$$

or $\tau_{AB}<\tau_A$, $\tau_B$, so the collapse is faster for a simultaneous measurement if $[\mathbf{A},\mathbf{B}]=0$. The simultaneity discussed here violates special relativity for any realistic pair of apparatus in that they would have to be in almost exactly the same point in space if the interference propagates at a speed near that of light. This result is to be expected in this theory, since the original Bohm-Bub theory is nonrelativistic due to the integration over all space explicitly contained in the collapse equations. The relationship of relativity to quantum mechanics is a difficult philosophical and mathematical problem in itself; hence, this difficulty will be overlooked here.[21]

In order to study the interference term, it is necessary to calculate $dJ_i{}^A/dt|_{AB}$. Let $\tilde{\psi}_i=\sum_j\phi_j\langle A_i|B_j\rangle$, and suppose that $[\mathbf{A},\mathbf{B}]\neq 0$, $\gamma^A\neq 0$, $\gamma^B\neq 0$, but $\gamma^C=0$ for all $C\neq A$, $B$. Then by (5), we have

$$\left.\frac{d\psi_i}{dt}\right|_{AB}=\left.\frac{d\psi_i}{dt}\right|_A+\frac{d\tilde{\psi}_i}{dt}.$$

Now,

$$\left.\frac{dJ_i{}^A}{dt}\right|_{AB}=\left.\frac{d}{dt}\right|_{AB}\int\psi_i\psi_i{}^*d^3x$$
$$=\int\left.\frac{d\psi_i}{dt}\right|_{AB}\psi_i{}^*d^3x+\int\psi_i\left.\frac{d\psi_i}{dt}\right|_{AB}d^3x$$
$$=\int\left(\left.\frac{d\psi_i}{dt}\right|_A+\frac{d\tilde{\psi}_i}{dt}\right)\psi_i{}^*d^3x+\int\psi_i\left(\left.\frac{d\psi_i{}^*}{dt}\right|_A+\frac{d\tilde{\psi}_i{}^*}{dt}\right)d^3x$$
$$=\int\left.\frac{d\psi_i}{dt}\right|_A\psi_i{}^*d^3x+\int\psi_i\left.\frac{d\psi_i{}^*}{dt}\right|_A d^3x+\int\frac{d\tilde{\psi}_i}{dt}\psi_i{}^*d^3x+\int\psi_i\frac{d\tilde{\psi}_i{}^*}{dt}d^3x.$$

But $\psi_i{}^*=\tilde{\psi}_i{}^*$ and $\psi_i=\tilde{\psi}_i$, even though $d\psi_i/dt|_A\neq d\tilde{\psi}_i/dt$, since the $|_A$ denotes a particular collapse process. Hence,

$$\left.\frac{dJ_i{}^A}{dt}\right|_{AB}=\left.\frac{dJ_i{}^A}{dt}\right|_A+\frac{d\tilde{J}_i{}^A}{dt}\tag{7}$$

as expected, where

$$\tilde{J}_i{}^A=\int\tilde{\psi}_i\tilde{\psi}_i{}^*d^3x.$$

To get the equations in terms of the $R_i$'s, let $R_i{}^A=R_i$, $R_i{}^B=S_i$ and let the hidden variables in the dual space of the $\mathbf{A}$ representation be $|\xi_i|^2$ and those in the dual space of the $\mathbf{B}$ representation $|\rho_i|^2$. So

$$|\psi\rangle=\sum_{i=1}^n\psi_i|A_i\rangle=\sum_{j=1}^n\phi_j|B_j\rangle,\quad \tilde{R}_i\equiv\frac{\tilde{J}_i}{|\xi_i|^2},$$

and

$$\frac{d\tilde{R}_i}{dt}=\frac{1}{|\xi_i|^2}\frac{d\tilde{J}_i}{dt}.$$

[21] For a discussion see D. Bohm and D. L. Schumacher (unpublished).

Making use of (7), we have

$$|\xi_i|^2\frac{d\tilde{R}_i}{dt}=\frac{d}{dt}\int\sum_j \phi_j\langle A_i|B_j\rangle)(\sum_k \phi_k^*\langle A_i|B_k\rangle^*)d^3x$$

$$=\int\left(\sum_j \frac{d\phi_j}{dt}\langle A_i|B_j\rangle\right)\psi_i^*+\int\psi_i\left(\sum_k\frac{d\phi_k^*}{dt}\langle A_i|B_k\rangle^*\right)d^3x$$

$$=\int\sum_{j,l}[\psi_i^*\langle A_i|B_j\rangle\gamma^B\phi_j|\phi_l|^2(S_j-S_l)+\psi_i\langle A_i|B_j\rangle^*\gamma^B\phi_j^*|\phi_l|^2(S_j-S_l)]d^3x$$

$$=\gamma^B\sum_{j,l}|\phi_l|^2(S_j-S_l)\int(\psi_i\phi_j^*\langle A_i|B_j\rangle^*+\psi_i^*\phi_j\langle A_i|B_j\rangle)d^3x.$$

Now, let

$$Z=\psi_i\phi_j^*\langle A_i|B_j\rangle^*=|z|\exp[i(\arg\psi_i-\arg\phi_j-\arg\langle A_i|B_j\rangle)].$$

So $Z+Z^*=2\,\mathrm{Re}(Z)=2|Z|\cos\theta_{ij}$, where $\theta_{ij}=(\arg\psi_i-\arg\phi_j-\arg\langle A_i|B_j\rangle)$ and $|Z|=|\psi_i||\phi_j||\langle A_i|B_j\rangle|$. The absolute value must not be confused with the $L^2(d^3x)$ norm. Hence,

$$|\xi_i|^2\frac{d\tilde{R}_i}{dt}=2\gamma^B\sum_{j,l}|\phi_l|^2(S_j-S_l)\cos\theta_{ij}|\langle A_i|B_j\rangle|\int|\psi_i||\phi_j|d^3x,$$

and the coupled collapse equations are

$$dR_i/dt=2\gamma^A R_i\sum_j^n|\xi_j|^2R_j(R_i-R_j)+\frac{2\gamma^B}{|\xi_i|^2}\sum_{j,l}^n|\rho_l|^2S_l(S_j-S_l)|\langle A_i|B_j\rangle|\cos\theta_{ij}\int|\psi_i||\phi_j|d^3x,$$
$$dS_i/dt=2\gamma^B S_i\sum_j^n|\rho_j|^2S_j(S_i-S_j)+\frac{2\gamma^A}{|\rho_i|^2}\sum_{j,l}^n|\xi_l|^2R_l(R_j-R_l)|\langle A_i|B_j\rangle|\cos\theta_{ij}\int|\phi_i||\psi_j|d^3x,\quad i=1,2,\cdots,n. \tag{8}$$

The first term in each equation is, as it was in the single-measurement case, simply a third-degree polynomial in the $R$'s (or $S$'s), whereas the interference term represents an integral coupling of all the components of the wave function in one representation to those in the other representation. This coupling greatly complicates the collapse process.

Though the coupled collapse equations are complicated, it is easily shown that they preserve the total probability and consequently that there exist unique solutions. To see that

$$\sum_1^n|\xi_i|^2R_i=\sum_1^n|\rho_j|^2S_j=1 \text{ for all } t,$$

consider Eq. (7):

$$\left.\frac{dR_i}{dt}\right|_{AB}=\left.\frac{dR_i}{dt}\right|_A+\frac{d\tilde{R}_i}{dt}.$$

So

$$\left.\frac{d}{dt}\right|_{AB}(\sum_1^n|\xi_i|^2R_i)=\sum_1^n|\xi_i|^2\left.\frac{dR_i}{dt}\right|_A+\sum_1^n|\xi_i|^2\frac{d\tilde{R}_i}{dt}.$$

The first term on the right is

$$\sum_1^n|\xi_i|^2\left.\frac{dR_i}{dt}\right|_A=2\gamma^A\sum_{j,l}|\psi_j|^2|\psi_l|^2(R_j-R_l)$$
$$=0,\quad\text{by symmetry.}$$

Also,

$$\sum_1^n |\xi_i|^2 \frac{d\tilde{R}_i}{dt} = \gamma^B \sum_{j,l} |\phi_l|^2 (S_j - S_l) \int \Big[\sum_i (\psi_i^* \langle A_i|B_j\rangle \varphi_j + \psi_i \langle B_j|A_i\rangle \phi_j^*)\Big] d^3x$$

$$= \gamma^B \sum_{j,l} |\phi_l|^2 (S_j - S_l) \int \Big(\sum_i \sum_k \phi_k^* \langle B_k|A_i\rangle\langle A_i|B_j\rangle \phi_j + \sum_i \sum_k \phi_k \langle A_i|B_k\rangle\langle B_j|A_i\rangle \phi_j^*\Big) d^3x$$

$$= \gamma^B \sum_{j,l} |\phi_l|^2 (S_j - S_l) \int \Big[\sum_k (\phi_k^* \langle B_k|B_j\rangle \phi_j + \phi_k \langle B_j|B_k\rangle \phi_j^*)\Big] d^3x$$

$$= \gamma^B \sum_{j,l} |\phi_l|^2 (S_j - S_l) \int 2\phi_j \phi_j^* d^3x$$

$$= 2\gamma^B \sum_{j,l} |\phi_j|^2 |\phi_l|^2 (S_j - S_l)$$

$$= 0, \quad \text{by symmetry.}$$

Hence,

$$\frac{d}{dt}\Big|_{AB} \sum |\xi_i|^2 R_i = 0;$$

therefore $\sum |\xi_i|^2 R_i = 1$ for all $t$. Also, $\sum |\psi_i|^2 = \sum |\phi_j|^2$, so $\sum |\xi_i|^2 R_i = \sum |\rho_j|^2 S_j = 1$. This boundedness insures the existence of a unique solution to this new system of equations.

In the case of a single measurement, all of the $2^n - 1$ critical points could be found, and the $n$ stable critical points could be identified with the $n$ possible outcomes $A_i$, $i = 1, \cdots, n$, The system of equations given by (8) does not easily yield its critical points. The elements of the product set of single-measurement stable critical points are again critical points. This fact can be seen as follows: Consider the point in $R \times S$ space given by

$$(0,0,\cdots,0,1/|\xi_k|^2,0,\cdots,0) \times (0,0,\cdots,0,1/|\rho_m|^2,0,\cdots,0).$$

$$R_i = 0, \quad i \neq k \Rightarrow |\psi_i|_2{}^2 = \int |\psi_i|^2 d^3x = 0 \Rightarrow |\psi_i| = 0$$

and

$$S_j = 0, \quad j \neq m \Rightarrow |\phi_j|_2{}^2 = \int |\phi_j|^2 d^3x = 0 \Rightarrow |\phi_j| = 0$$

Therefore,

$$\int |\psi_i|\,|\phi_j| d^3x = 0 \text{ unless } i = k \text{ and } j = m,$$

and

$$\int |\phi_i|\,|\psi_j| d^3x = 0 \text{ unless } i = j \text{ and } j = k.$$

So, $dR_i/dt = 0$ for $i \neq k$, trivially, and

$$\frac{dR_k}{dt} = 0 + \frac{2\gamma^2}{|\xi_k|^2} \sum_l |\rho_l|^2 S_l (S_m - S_l) \times |\langle A_k|B_m\rangle| \cos\theta_{km} \int |\psi_k|\,|\phi_m|.$$

But $S_l = 0$ unless $l = m$ and then $(S_m - S_l) = 0$. Hence, the second term in $dR_k/dt$ is also zero. [The first term is zero, since $(0,0,\cdots,0,1/|\xi_k|^2,0,\cdots,0)$ is a critical point for the $A$ measurement above.] Likewise, it is easy to see that $dS_i/dt$, as given by Eq. (8), is also zero. The point $(1,1,\cdots,1)$ in $R \times S$ is also easily seen to be a critical point.

The coupled collapse equations are not easily linearized; hence, the stability near a critical point is difficult to decide. It can be shown that critical points of the first kind, discussed above, are stable if and only if $|A_k\rangle = |B_m\rangle$ (in this case relabel the $|B_i\rangle$'s so that $|B_m\rangle$ is called $|B_k\rangle$). To do so, first suppose $|A_k\rangle = |B_k\rangle$ so that $\langle A_k|B_j\rangle = \langle B_k|B_j\rangle = \delta_{kj}$. Then,

$$\frac{dR_k}{dt} = 2\gamma^A R_k \sum_l^n |\xi_l|^2 R_l (R_k - R_l) + \frac{2\gamma^B}{|\xi_k|^2} \sum_l^n |\rho_l|^2 S_l (S_k - S_l) \cos\theta_{kk} \int |\psi_k|\,|\phi_k| d^3x$$

and

$$\frac{dS_k}{dt} = 2\gamma^B S_k \sum_l^n |\rho_l|^2 S_l (S_k - S_l) + \frac{2\gamma^A}{|\rho_k|^2} \sum_l^n |\xi_l|^2 R_l (R_k - R_l) \cos\theta_{kk} \int |\phi_k|\,|\psi_k| d^3x.$$

$|A_k\rangle = |B_k\rangle \Rightarrow \psi_k = \phi_k$, and so $\theta_{kk} = -\arg\langle A_k|A_k\rangle$. But $\arg\langle A_k|A_k\rangle = 0$, since $\langle A_k|A_k\rangle$ is real, and using the fact that near the critical point $R_k = \max\{R_i\}$, and $S_k = \max\{S_i\}$, we see that both $dR_k/dt$ and $dS_k/dt$ are positive. Since the normalization condition holds, $R_k \to 1/|\xi_k|^2$ and $S_k \to 1/|\rho_k|^2$, while $R_i \to 0$, and $S_i \to 0$ for all $i \neq k$. On the other hand, if the point is asymptotically stable then $|\psi\rangle \to |A_k\rangle$ and $|\psi\rangle \to |B_m\rangle$, but by uniqueness, we must have $|A_k\rangle = |B_m\rangle$. Consequently, the result above is established.

In the event that $|A_i\rangle \neq |B_j\rangle$ for all $i$ and $j$, one would expect stable critical points to correspond to vectors between basis vectors. The number of these points should depend on $n$, but their exact locations could involve the hidden variables. The relevant probabilities should be defined on a completely new sample space, and the idea of a joint distribution defined on the product spaces of the old sample spaces, as is done classically, would be meaningless. Also, the idea that a Hermitian operator should correspond to the simultaneous measurement becomes somewhat dubious. In any event, it seems that approaching the problem of simultaneous measurement in quantum mechanics from the standpoint of a hidden-variable theory of measurement is conceptually advantageous.

## 5. CONCLUSIONS

The problems raised by hidden-variable theories are of great philosophical importance to our understanding and interpretation of quantum mechanics and to the direction which the theory of quantum mechanics may take in the future, but the mathematical details must be worked out if we are to get beyond philosophizing. With this objective in mind, I have tried to explore the mathematics of a particular hidden-variable theory of measurement in quantum mechanics. Whether or not the new variables must remain hidden will be settled by mathematics and experimental physics, not by philosophy. Two experimentally testable results have been given along with a mathematical model for simultaneous measurement. The impossibility of formulating the concept of simultaneous measurement in the context of orthodox quantum mechanics is a problem of central importance. It is hoped that the coupled collapse equations may resolve some of the difficulties or at least lead to a better understanding of the problem.

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## APPENDIX A: DISCUSSION OF A NEW DEFINITION OF THE TERM "HIDDEN-VARIABLE (HV) THEORY" PROPOSED BY S. GUDDER

Gudder introduces the following definition: "A quantum proposition system $(L,M)$ *admits* an HV *theory* if there is a probability space $(\Omega,F,\mu)$ with the property that for any maximal Boolean sub-$\sigma$-algebra $B\subseteq L$ there is a map $H_B$ from $M\times\Omega$ onto $M_B$ such that:

(1) $\omega \to H_B(m,\omega)(a)$ is measurable for every $m\in M$, $a\in B$;

(2) $\int_\Omega H_B(m,\omega)(a)d\mu(\omega) = m(a)$ for every $m\in M$, $a\in B$."[11] In the Hilbert-space formulation, the complete sets of commuting observables correspond to different maximal Boolean sub-$\sigma$-algebras, the states to the set $M$, and the hidden variables to the set $\Omega$. The elements of $L$ are the experimental propositions and $H_B(m,\omega)(a)$ corresponds roughly to the collapse equation.

The generalized Bohm-Bub theory fits this definition if the following changes in notation are made; Choose a basis in the Hilbert space which diagonalizes the observable in question (call it **A**). Then let

$$M = \{(|\psi_1^A|^2,\cdots,|\psi_n^A|^2) \mid \sum_{i=1}^n |\psi_i^A|^2 = 1\},$$

where $n$ is the number of eigenvalues of **A**. Assume for simplicity that **A** is nondegenerate. Call the elements of $M$, $m$. Let $B$ be the maximal Boolean sub-$\sigma$-algebra corresponding to the observable **A**. Take

$$\Omega = \{(|\xi_1^A|^2,\cdots,|\xi_n^A|^2) \mid \sum_{i=1}^n |\xi_i^A|^2 = 1\}.$$

This is the hidden-variable plane given in Fig. 2. Denote the elements of $\Omega$ by $\omega$. Let

$$R^A = \{(R_1^A,\cdots,R_n^A) \mid \sum_{i=1}^n |\xi_i^A|^2 R_i = 1\}.$$

This is the probability plane given in Fig. 1. Let

$$M_B = \{(0,\cdots,0,1/|\xi_i^A|^2,0,\cdots,0)\in R^A,\quad i=1,\cdots,n\},$$

and define $G: M\times\Omega \to R^A$ by

$$G(|\psi_1^A|^2,\cdots,|\psi_n^A|^2,|\xi_1^A|^2\cdots,|\xi_n^A|^2) = (R_1^A,\cdots,R_n^A)$$

and $C: R^A \to M_B$ by

$$C(R_1^A,\cdots,R_n^A) = (0,\cdots,0,1/|\xi_i^A|^2,0,\cdots,0),$$

where $R_i^A = \max\{R_j^A\}$. $C$ corresponds to the collapse equation in the Bohm-Bub theory. Now, let $H_B: M\times\Omega \to M_B$ be given by $H_B = C\circ G$. Gudder shows that there is a unique minimal HV theory in the same sense that $H_B$ could be 1-1.

It is obvious that the Bohm-Bub theory is not minimal in this sense and hence, the collapse equations are not unique. Take the proposition $a_i$ to be "the measured value of **A** is $A_i$." Let

$$(a_i) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ |\xi_i^A|^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

so that

$$H_B(m,\omega)(a_i)=1\,,\quad \text{if } R_i^A=\max\{R_j^A\}$$
$$=0\,,\quad \text{otherwise}$$

where the operation of $H_B(m,\omega)$ on $(a_i)$ is the ordinary inner product in $E^n$. This defines the map: $\omega \rightarrow H_B(m,\omega)$ and it is measurable as required by (1). Now,

$$\int_\Omega H_B(m,\omega)(a_i)d\mu(\omega)=m(a_i)$$

if

$$d\mu(\omega)=[(n-1)!/\sqrt{n}]d(|\xi_n^A|^2)\cdots d(|\xi_n^A|^2)\,,$$

with $m(a_i)=|\psi_i^A|^2$. This integral is given by Eq. (4) in Sec. 3 of this paper.

It is worth noting that Gudder's results are so general that no information is given with respect to the nature of the hidden variables, the nature of the "collapse," or the kind of average that must be used to get the usual quantum-mechanical probabilities. Since the concept of time does not occur, it is also not clear what a "simultaneous" measurement would mean.

## APPENDIX B: DISCUSSION OF THE CRITICAL POINTS FOR THE $n=4$ CASE

The collapse equations are

$$dx/dt=2\gamma x[by(x-y)+cz(x-z)+dw(x-w)]\,,$$
$$dy/dt=2\gamma y[ax(y-x)+cz(y-z)+dw(y-w)]\,,$$
$$dz/dt=2\gamma z[ax(z-x)+by(z-y)+dw(z-w)]\,,$$
$$dw/dt=2\gamma w[ax(w-x)+by(w-y)+cz(w-z)]\,,$$

with $ax+by+cz+dw=1=a+b+c+d$. The critical points are

| | | |
|---|---|---|
| $w=0$: | I | $(1/a,0,0,0)$ |
| | | $(0,1/b,0,0)$ |
| | | $(0,0,1/c,0)$ |
| | | $(1/(a+b),\ 1/(a+b),\ 0,\ 0)$ |
| | | $(1/(a+c),\ 0,\ 1/(a+c),\ 0)$ |
| | III | $(0,\ 1/(b+c),\ 1/(b+c),\ 0)$ |
| | IV | $(1/(1-d),\ 1/(1-d),\ 1/(1-d),\ 0)$ |
| $w\neq 0$: | | $(0,0,0,1/d)$ |
| | | $(0,\ 1/(1-a),\ 1/(1-a),\ 1/(1-a))$ |
| | | $(1/(1-c),\ 1/(1-c),\ 0,\ 1/(1-c))$ |
| | | $(1/(1-b),\ 0,\ 1/(1-b),\ 1/(1-b))$ |
| | | $(0,\ 0,\ 1/(c+d),\ 1/(c+d))$ |
| | | $(0,\ 1/(b+d),\ 0,\ 1/(b+d))$ |
| | | $(1/(a+d),\ 0,\ 0,\ 1/(a+d))$ |
| | II | $(1,1,1,1)$. |

Points of type I and II have been discussed in general. Consider points of type III. If we let $x=\tilde{x}$, $y=\tilde{y}+D$, $z=\tilde{z}+D$, and $w=\tilde{w}$ for the characteristic roots, we get $(2\gamma D+\lambda)^2\lambda(-2\gamma D+\lambda)=0 \Rightarrow \lambda=2\gamma D>0$; therefore, the point is unstable. Now consider points of type IV. Let $x=\tilde{x}+E$, $y=\tilde{y}+E$, $z={}_z+E$, and $w=\tilde{w}$. We get $(2\gamma E+\lambda)(2\gamma E-\lambda)^3=0\Rightarrow \lambda=2\gamma E>0$; therefore, the point is unstable. How to carry out a general inductive argument seems evident but not worth the effort. The fact that there are only $2^n-1$ critical points can be seen by induction also.