polar form and then using Eq. (A17)],

$$g_N = g_{N-1} \pm \tan^{-1}(a_N/G_{N-1}). \qquad \text{(A21)}$$

Thus, with s_1 and g_1 arbitrary, we may use Eqs. (A18), (A19), and of course (A3) to generate *all* s_n, $n=1, 2, 3, \cdots$, up to any desired N and, in fact, for $N\to\infty$. However, as is seen by the arbitrary odd multiples of $\frac{1}{2}\pi$ added to each s_n, the phases are never uniquely determined QED.

Corollary. Since any real quadratic form of the type

$$G_N^2 = a_1'^2 + a_2'^2 + \cdots + a_N'^2 + 2a_1'a_2' + 2a_1'a_3' + \cdots + 2a_2'a_3' + \cdots + 2a_{N-1}'a_N$$

can always be reduced to the homogeneous form of Eq. (A3) by a suitable orthogonal transformation of the a_n' into the a_n, it is evident that the above theorem holds even without the restriction to *positive* a_n.

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The Gravitational Field of a Disk*

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The general solution of the static field equations of general relativity is given for a disk of "counter-rotating" dust particles. The only nonvanishing components of the energy-momentum tensor are $T_0{}^0$ and $T_\chi{}^\chi$, which are assumed to have δ-function singularities on the disk. Two representative families of solutions are considered, and it is shown that, for these solutions, physical considerations severely limit the strength of the gravitational potentials. The first family has surface density proportional to some power of $1-\rho^2$. The requirement that the velocity of the dust particles should not exceed c places a bound on the gravitational red-shift of $z=1.5803$ for these models. The second family is that of the uniformly rotating disks defined by $v^2=\rho^2\omega^2e^{-4\phi}$. Bardeen has pointed out that these disks can have arbitrarily large red-shifts without violating the velocity condition. However, it is shown that their red-shift cannot exceed 1.9015 before their binding energy becomes negative. This work suggests that the largest gravitational red-shift to which counter-rotating dust disks can give rise is of order of magnitude 1.

1. INTRODUCTION

IN order to study the physical implications of Einstein's theory of gravitation it is most desirable to have exact solutions of the static field equations everywhere in space with physically reasonable sources. Exact empty-space static solutions have been used by Bondi and Morgan[1] to study the transfer of energy by bodies slowly changing their shapes. The motivation behind this paper lies in the need for exact static solutions everywhere in space in order to extend this strong-field treatment of energy transfer. Spherically symmetric solutions have too high a degree of symmetry to be of very much use for this problem.

This paper has received much of its inspiration from a paper by Einstein.[2] Motivated by the problem of the physical significance of the Schwarzschild radius, he solved the static field equations for spheres of non-interacting dust particles moving in great circles. He showed that all the equilibrium configurations in which the particles had velocities less than the speed of light had radii larger than their Schwarzschild radius. In his conclusion the following paragraph appears.

"The essential result of this investigation is a clear understanding as to why the Schwarzschild singularities do not exist in physical reality. Although the theory given here treats only clusters whose particles move along circular paths it does not seem to be subject to reasonable doubt that more general cases will have analogous results. The 'Schwarzschild singularity' does not appear for the reason that matter cannot be concentrated arbitrarily. And this is due to the fact that otherwise the constituting particles would reach the velocity of light."

In Sec. 2, a method is given by which the gravitational field of a counter-rotating dust disk with prescribed "density" can be found. The method is based, essentially, on an analogous problem in electrostatics. In Sec. 3, the general solution is expressed, in a convenient coordinate system, in terms of an infinite series and a particularly simple family of solutions is studied. Section 4 turns to the problem of the excess energy of

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[1] H. Bondi and T. Morgan (unpublished).
[2] A. Einstein, Ann. Math. 40, 924 (1939).

tightly bound dust systems. Section 5 briefly describes uniformly counter-rotating dust disks which, as Bardeen has pointed out, can give rise to arbitrarily large red-shifts. However, those disks which have large red-shifts have a large excess of energy and are not physically allowed.

2. NEW SOLUTIONS

The simplest non-spherically-symmetric system that one can solve exactly is a disk of rotating dust in an asymptotically flat space. Consider particles moving in circles in a plane under the action of their collective gravitational field so that the space will be axially symmetric. We assume that there are as many particles circling to the right as to the left so that there is no net angular momentum and space-time will be static rather than stationary. With these assumptions the only nonvanishing components of the energy-momentum tensor are $T_0{}^0$ and $T_\chi{}^\chi$ and therefore it is possible to introduce locally a coordinate system with a line element of the form[3]

$$ds^2 = -e^{2\phi}dt^2 + e^{2(\sigma-\phi)}(d\rho^2+dz^2) + \rho^2 e^{-2\phi}d\chi^2\,, \qquad (1)$$

where ϕ and σ are functions only of the quasicylindrical coordinates ρ and z, and $c=1$. The exceptional simplicity of this form for the metric is due to the vanishing of the mechanical stresses and, in particular, the combination $T_\rho{}^\rho + T_z{}^z$. In the coordinate system defined by Eq. (1), Einstein's field equations reduce to[3]

$$\nabla^2\phi = -4\pi G e^{2(\sigma-\phi)}(T_0{}^0 - T_\chi{}^\chi)\,, \qquad (2)$$

$$\sigma_{,\rho} = \rho[(\phi_{,\rho})^2 - (\phi_{,z})^2]\,, \qquad (3a)$$

$$\sigma_{,z} = 2\rho\phi_{,\rho}\phi_{,z}\,, \qquad (3b)$$

$$\sigma_{,\rho\rho} + \sigma_{,zz} - \nabla^2\phi + (\phi_{,\rho})^2 + (\phi_{,z})^2 = 4\pi G e^{2(\sigma-\phi)}(T_0{}^0 + T_\chi{}^\chi)\,, \qquad (4)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian in cylindrical coordinates.

Given the source of the field $S(\rho)\delta(z) = -e^{2(\sigma-\phi)} \times (T_0{}^0 - T_\chi{}^\chi)$, it is a simple matter to solve these equations. In general Eq. (2) can be solved to yield the potential ϕ and then either Eq. (3a) or (3b) can be integrated directly to yield σ. As is well known from electrostatics, the vanishing of $S(\rho)\delta(z)$ everywhere except in the plane of the disk implies that the potential ϕ is continuous across the disk, and that $\phi_{,z}$ is discontinuous by an amount $4\pi G S(\rho)$. We note that $(\phi_{,z})^2$ is continuous[4] and therefore σ is also continuous, while $\sigma_{,z}$ is discontinuous by an amount $8\pi G\rho\phi_{,\rho}S(\rho)$.

The consistency of Eqs. (3) outside of the disk is ensured by $\nabla^2\phi=0$, [Eq. (2)] and inside the disk by the continuity of σ and $\sigma_{,\rho}$ as can be seen simply by comparing two paths of integration for σ, one passing through the disk and the other taken along the top surface of the disk to the rim and returning along the bottom surface. The last equation determines $T_\chi{}^\chi$ and corresponds to the equilibrium condition. It should be interpreted in the same manner as the first equation; that is, both sides of the equation should be integrated along an infinitesimal path passing through the disk in order to interpret the undefined function $\sigma_{,zz}$. It yields the requirement that

$$(T_0{}^0 - T_\chi{}^\chi)\phi_{,\rho} = (-1/\rho)T_\chi{}^\chi\,. \qquad (5)$$

In the Newtonian limit ϕ becomes the usual Newtonian potential, $T_0{}^0 - T_\chi{}^\chi$ is equal to minus the mass density, $T_\chi{}^\chi$ is the mass density times the square of the velocity of the dust particles, and thus Eq. (5) becomes the usual equilibrium condition for circular orbits.

In summary, given $S(\rho)$, the metric and the two nonvanishing components of the energy-momentum tensor, $T_0{}^0$ and $T_\chi{}^\chi$, are determined. The metric is continuous but its z derivatives are discontinuous because the source has a δ-function singularity. The condition that the velocity of the dust particles never exceed the velocity of light implies that $|T_0{}^0| \geq |T_\chi{}^\chi|$. This places a limitation on the choices of $S(\rho)$ and may severely limit the strength of the gravitation fields to which gravitating disks can give rise.

3. OBLATE ELLIPSOIDAL COORDINATES

The natural coordinate system to use for problems of disks is given by

$$\rho^2 = (1+\xi^2)(1-\eta^2) \quad (-1 \leqslant \eta \leqslant 1)\,,$$
$$z = \xi\eta \quad (0 \leqslant \xi < \infty)\,.$$

The ρ coordinate of the edge of the disk has been set equal to unity. (This is merely a definition of our unit of length, and it is a trivial matter to modify the formulas for a disk of coordinate radius d.) The disk has the coordinates $\xi=0$, $0 \leqslant \eta^2 \leqslant 1$. On crossing the disk η changes sign but does not change in absolute value. This singular behavior of the coordinate η implies that a polynomial in even powers of η is a continuous function everywhere but has a discontinuous η derivative at the disk. We shall use this property of oblate ellipsoidal coodinates to obtain a convenient expression for the general solution of Eq. (2). In fact, for an isolated disk[5]

$$\phi(\xi,\eta) = -G\sum_{m=0}^{\infty} C_{2m}P_{2m}(\eta)q_{2m}(\xi)\,, \qquad (6)$$

[3] J. L. Synge *Relativity: The General Theory* (North-Holland Publishing Co., Amsterdam, 1964), p. 312.

[4] We are making use of our assumption of asymptotically flat space here.

[5] H. Bateman, *Partial Differential Equations of Mathematical Physics* (Dover Publications, Inc., New York, 1944), p. 447.

where $P_{2m}(\eta)$ is the Legendre polynomial of order $2m$, and $q_{2m}(\xi)=i^{2m+1}\mathfrak{Q}_{2m}(i\xi)$, $\mathfrak{Q}_{2m}(z)$ being the Legendre function of the second kind.[6] The constants C_0, C_2, ... are determined by the source

$$S(\rho)=\frac{1}{2\pi(1-\rho^2)^{1/2}}\sum_{m=0}^{\infty}(2m+1)\times C_{2m}q_{2m+1}(0)P_{2m}((1-\rho^2)^{1/2}). \quad (7)$$

As it is possible to expand well-behaved functions in just the even Legendre polynomials over the half-interval [0,1], the solution given by Eq. (6) is the general solution for ϕ square integrable over this interval. The first coefficient C_0 can be given a simple physical interpretation. At great distances from the disk, the space is asymptotically flat and the Newtonian approximation is very good. The total gravitating mass M seen at infinity is then given by

$$M=\frac{1}{4\pi G}\int\phi_{,i}dS^i. \quad (8)$$

Converting this surface integral to a volume integral and using Eqs. (2), (6), and (7) we find $M=C_0$.

The well-known solution, $\phi=C_0\cot^{-1}\xi$, for a conducting disk has an infinite charge density on the rim. A more serious matter is that $\phi_{,\rho}$ is also singular on the rim. The simplest physically acceptable solution appears to be a linear combination of the first two terms in Eq. (7) chosen such that the charge density at the rim is nonsingular. This is

$$\phi=-GM\{\cot^{-1}\xi+\tfrac{1}{4}[(3\xi^2+1)\cot^{-1}\xi-3\xi](3\eta^2-1)\} \quad (9)$$

and the corresponding source is

$$S(\rho)=(3M/2\pi)(1-\rho^2)^{1/2}. \quad (10)$$

It is a simple matter to integrate Eqs. (3) to determine σ. First note that $\sigma=0$ on the z axis because of the factor ρ in Eq. (3b) and that each $\xi=$constant ellipsoidal surface intersects the z axis. It is therefore convenient to calculate the η derivative of σ using Eqs. (3) and (9) and then integrate σ from the z axis along a $\xi=$constant surface. One obtains

$$\sigma=9G^2\times\tfrac{1}{4}M^2\rho^2[\rho^2B^2-(1+\eta^2)A^2-2\xi(1-\eta^2)AB], \quad (11)$$

where

$$A=\xi\cot^{-1}\xi-1,\quad B=\tfrac{1}{2}(\xi/(1+\xi)-\cot^{-1}\xi).$$

ϕ and σ are continuous everywhere in space and vanish at infinity. The only singularities of their derivatives occur at the disk and are due to the singular character of the source. In the disk, Eqs. (9) and (11) simplify to

$$\phi=-\tfrac{3}{4}\pi GM(1-\tfrac{1}{2}\rho^2), \quad (12)$$

$$\sigma=(9/4)G^2M^2\rho^2[\rho^2(1+\tfrac{1}{16}\pi^2)-2]. \quad (13)$$

The equilibrium condition, Eq. (5) yields

$$\tfrac{3}{4}\pi GM\rho^2(T_0{}^0-T_\chi{}^\chi)=-T_\chi{}^\chi. \quad (14)$$

The magnitude of the velocity $v(\rho)$ of the dust particles moving in their equilibrium circular orbits is then given by

$$v^2=\frac{-T_\chi{}^\chi}{T_0{}^0}=\frac{\rho\phi_{,\rho}}{1-\rho\phi_{,\rho}}=\frac{3\pi MG\rho^2}{4-3\pi MG\rho^2}. \quad (15)$$

To complete the specification of the solution we note that

$$T_0{}^0=(3M/2\pi)e^{-2(\sigma-\phi)}(\tfrac{3}{4}\pi MG\rho^2-1)(1-\rho^2)^{1/2}\delta(z). \quad (16)$$

The magnitude of M is limited by the condition that the velocities of the dust cannot exceed that of light. The dust has its greatest velocity at the edge of the disk and we find that, in units where the radius of the disk is unity,

$$M\leqslant 2/3\pi G. \quad (17)$$

Therefore the largest gravitational red-shift to which a disk with a density distribution given by Eq. (10) can give rise, occurs for an emitter at the center of the disk when $M=2/3\pi G$. The maximum red-shift is 0.6487, which is of the same order of magnitude as the red-shifts found by Einstein for his various models of dust spheres.

The source density for the disk given in Eq. (10) can be obtained from an oblate ellipsoid of uniform density by a limiting process. The potential ϕ is thus identical in this limit to Rodriques' solution for the ellipsoid of uniform density. It is also the potential of a uniformly rotating disk, in the Newtonian theory, with angular velocity $(\frac{3}{4}\pi GM)^{1/2}$.

Not all solutions for the gravitational field of a disk are given in a convenient form by Eq. (6). For example, the disk for which $S(\rho)=$constant has as its potential an elliptic function which requires an infinite number of terms in this representation. Moreover, the disk of uniform density has a singularity in $\phi_{,\rho}$ at the rim. As is well known, this is due to the two-dimensional character of the density distribution. It is necessary that the surface density decrease at the rim to avoid this logarithmic singularity.

One of the important solutions of potential theory is that of the equipotential disk. By superimposing the solutions for equipotential disks of different radii[7] it is sometimes possible to obtain convenient expressions for the potential of a disk, given its density, or the density, given its potential. Equations (19) and (27) were obtained by us using this method.

[6] See W. Magnus and F. Oberhettinger *Functions of Mathematical Physics* (Chelsea Publishing Co., New York, 1954) for definitions of Legendre functions with complex arguments.

[7] I. Sneddon *Mixed Boundary Problems in Potential Theory* (North Holland Publishing Co., Amsterdam, 1966), p. 207.

In order to study disks with arbitrarily large central densities, we consider a one-parameter family of disks of mass M and unit coordinate radius, with sources

$$S(\rho)=[(2m+1)M/2\pi](1-\rho^2)^{m-1/2} \quad (m=1,2,\cdots). \quad (18)$$

The potential in the disk is given by

$$\phi(\rho)=\frac{-2GM}{\pi}(2m+1)A_m\int_0^{\pi/2}(1-\rho^2\cos^2\theta)^m d\theta,$$

where

$$A_m=[(1\times3\times\cdots\times2m-1)/(2\times4\times\cdots\times2m)]\tfrac{1}{2}\pi. \quad (19)$$

The velocity condition implies, that throughout the disk,

$$\rho\phi_{,\rho}\leqslant\tfrac{1}{2}. \quad (20)$$

The limit which this condition places on the mass M for each value of m can be determined by first finding the point ρ_0 at which $\rho\phi_{,\rho}$ takes on its maximum value and then setting $\rho\phi_{,\rho}=\frac{1}{2}$ at this point. The polynomial equation $(\rho\phi_{,\rho})_{,\rho}=0$ determines ρ_0. It is a simple matter to solve this equation numerically for large values of m as the higher-order terms tend to zero very rapidly. It is found that as $m\rightarrow\infty$, $m\rho_0^2\rightarrow1.58$ and that consequently $M\rightarrow0$ as $m^{-1/2}$. The maximum gravitational red-shift occurs for an emitter at the center of the disk and increases monotonically with m to a limit of 1.5803. Consequently the potentials ϕ and σ are at most of the order of magnitude 1.

4. EXCESS ENERGY OF TIGHTLY BOUND SYSTEMS

Einstein found an important general relativistic effect[2,8] in his study of contracting dust spheres. In Newtonian theory, the binding energy of a contracting system continuously increases. Einstein showed that for his models there was a maximum of the gravitational binding energy and that on further contraction the binding energy could become negative, giving rise to an energy excess for the bound system. Simply stated, the dust particles of a tightly bound system must have a very large KE, even larger than their potential energy. Systems with excess energy would be highly unstable and probably do not exist in nature. Indeed a system whose binding energy decreases on contraction requires an external energy source for further contraction. The existence of a maximum of the binding energy of a slowly contracting system would place a limit on the strength of the potentials which are physically allowable and thus on the gravitational red-shifts.

The energy-momentum tensor of dust[9]

$$T^{\mu\nu}=\sum_a m_a(-g)^{-1/2}\frac{dx^\mu}{ds}\frac{dx^\nu}{dt}\delta(x^i-x_a^i) \quad (21)$$

can be used to obtain the following expression for the total rest mass M_0 of the counter-rotating dust disks.

$$M_0=2\pi\int e^{-\phi}[(1-2\rho\phi_{,\rho})(1-\rho\phi_{,\rho})]^{1/2}S(\rho)\rho d\rho. \quad (22)$$

The binding energy is the difference between the total rest mass M_0 and the relativistic mass M.

With Brindeiro, we have calculated M_0 for disks whose densities are given by Eq. (18). In the case $m=1$ the density reduces to Eq. (10) and the maximum of the binding energy occurs for a mass 0.7012 times the maximum mass allowed by the velocity condition. The corresponding maximum gravitational red-shift from the center of the disk is 0.4927. In the case $m=2$ the binding energy actually becomes negative for a mass 0.9809 times the maximum mass of $4/5\pi G$ allowed by the velocity condition. The maximum binding energy occurs for a mass of 0.6107 times the maximum mass and gives a maximum red-shift of 0.5809 which can be compared with a red-shift of 1.1170 allowed by the velocity condition. We find that for $1<m\leqslant100$, the excess-energy condition places a more severe restriction on the maximum red-shift than does the velocity condition.

5. UNIFORM ROTATION

The limiting case of the velocity condition $\rho\phi_{,\rho}=\frac{1}{2}$ permits the potential to diverge logarithmically to $-\infty$ as $\rho\rightarrow0$. The gravitational red-shift z from the center would be infinitely great as $z=e^{-\phi}-1$. The corresponding potential and source are

$$\phi(\rho)=\tfrac{1}{2}\ln\rho-\tfrac{1}{2}\ln2, \quad (23)$$

$$S(\rho)=(2\pi^2G\rho)^{-1}\tan^{-1}[(1-\rho^2)^{1/2}/\rho]. \quad (24)$$

The singularity at $\rho=0$ precludes a direct treatment of this case of infinite red-shift. It is convenient to consider a family of disks whose potentials approach this singular case.

Bardeen has communicated to us the potential of a "uniformly counter-rotating" dust disk. The angular coordinate velocity $d\chi/dt$ is assumed to be a constant) ω, in the disk.[10] The magnitude of the velocity of the dust is therefore given by $v^2=\rho^2\omega^2e^{-4\phi}$. Equation (15, gives a first-order differential equation for ϕ whose solu-

[8] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965), p. 54.

[9] L. Landau and E. Lifshitz *The Classical Theory of Fields* (Addison Wesley Publishing Company, Inc., Reading, Mass., 1965), p. 368.

[10] See J. B. Hartle and D. H. Sharp, Astrophys. J. 147, 317 (1967) for a discussion of the physical significance of uniformly rotating configurations.

tion Bardeen has found and expressed in the form

$$e^{\phi}(1-v^2)^{1/2}=\text{const}. \tag{25}$$

As the velocity of the dust at the edge of the disk approaches the speed of light, e^{ϕ} evaluated at the center approaches zero and the red-shift from the center approaches infinity.

The maximum velocity μ occurs at the edge of the disk. Let $b=\mu/(1-\mu^2)$. The potential, given explicitly as a function of ρ, is

$$\phi=\tfrac{1}{2}\ln\{(\omega/2b)[1+(4b^2\rho^2)^{1/2}]\}. \tag{26}$$

The corresponding source is

$$S(\rho)=\frac{b}{\pi^2G}\frac{1}{(1+4b^2\rho^2)^{1/2}}\tan^{-1}\left(\frac{2b(1-\rho^2)^{1/2}}{(1+4b^2\rho^2)^{1/2}}\right). \tag{27}$$

We find that ω is given simply by

$$\omega=\mu/(1+\mu^2) \tag{28}$$

and the total mass of the disk is

$$M=(\pi G)^{-1}[1-(1/2b)\tan^{-1}2b]. \tag{29}$$

In the limit of small velocities one obtains the solution corresponding to Eqs. (10) and (12), which in the weak-field approximation is identical to the Newtonian uniformly rotating disk.

In the limit as $\mu\rightarrow 1$ Eqs. (26) and (27) become Eqs. (23) and (24), which explains why the uniformly rotating disks exhibit infinite red-shifts. However, when the binding energies of these disks are calculated, it is found that the maximum binding energy occurs for a value of μ close to 0.6937 which corresponds to a red-shift of only 0.6898. The ratio of the rest mass to the mass seen at infinity is only 1.0543 at this velocity. At a velocity of 0.8875 this ratio is 0.9983, which already corresponds to an excess of energy. The red-shift from the center is now 1.9015. The potential σ can be found in an analogous way to Eq. (11) and is of moderate size, as one can easily verify when ϕ and S are limited by these stability requirements. Consequently, uniformly rotating disks cannot give rise to large red-shifts and their potentials ϕ and σ cannot be more than moderately strong.

A bound can be placed on the value of the rest mass by overestimating the source, in particular by replacing the inverse tangent with its maximum value, $\frac{1}{2}\pi$. It then follows that, if the mass M is held fixed and $\mu\rightarrow 1$, the rest mass $M_0\rightarrow 0$ as $(2b)^{-1/2}$. This means that the excess energy approaches the total mass of the system.

Finally it should be made clear that the models discussed in this paper, even if suitably restricted so that their binding energies have not yet reached their maximum values, are not necessarily stable against small perturbations. For example, the uniformly rotating disk in Newtonian theory is unstable against radial oscillations.[11] Furthermore, it is extremely unlikely that in any actual physical system the particle orbits would be so regular as to admit counter-rotation.

6. CONCLUSION

The idealized models described in this paper are of importance because they can be used to examine the effects of potentials which are too strong to be treated by post-Newtonian methods. Previously this could be done only for systems with spherical symmetry. We find that the models which are physically allowable have potentials and red-shifts which are at most of the order of magnitude 1. These models are not in any way pathological and support the view that systems with reasonable equations of state do not give rise to physically unreasonable spaces. This work gives an indication that the potentials can become only moderately large, or, equivalently, space-time can become only moderately curved before the macroscopic bodies causing the curvature are unable to withstand the stresses and either explode or collapse.

It would be most interesting to find limits on the strength of the potentials of dust disks whose particles all rotate in the same direction.

Note added in proof. Recently we have been able to relax our condition $T_\rho{}^\rho=0$ and can, in principle, determine the gravitational field of the most general static disk.

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[11] C. Hunter, Monthly Notices Roy. Astron. Soc. **126**, 23 (1963).