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Quantized Fields and Particle Creation in Expanding Universes. I

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Spin-0 fields of arbitrary mass and massless fields of arbitrary spin are considered. The equations governing the fields are the covariant generalizations of the special-relativistic free-field equations. The metric, which is not quantized, is that of a universe with an expanding (or contracting) Euclidean 3-space. The spin-0 field of arbitrary mass is quantized in the expanding universe by the canonical procedure. The quantization is consistent with the time development dictated by the equation of motion only when the boson commutation relations are imposed. This consistency requirement provides a new proof of the connection between spin and statistics. We show that the particle number is an adiabatic invariant, but not a strict constant of the motion. We obtain an expression for the average particle density as a function of the time, and show that particle creation occurs in pairs. The canonical creation and annihilation operators corresponding to physical particles during the expansion are specified. Thus, we do not use an S-matrix approach. We show that in a universe with flat 3-space containing only massless particles in equilibrium, there will be precisely no creation of massless particles as a result of the expansion, provided the Einstein field equations without the cosmological term are correct. Furthermore, in a dust-filled universe with flat 3-space there will be precisely no creation of massive spin-0 particles in the limit of infinite mass, again provided that the Einstein field equations are correct. Conversely, without assuming any particular equations, such as the Einstein equations, as governing the expansion of the universe, we obtain the familiar Friedmann expansions for the radiation-filled and the dust-filled universes with flat 3-space. We only make a very general and natural hypothesis connecting the particle creation rate with the macroscopic expansion of the universe. In one derivation, we assume that in an expansion of the universe in which a particular type of particle is predominant, the type of expansion approached after a long time will be such as to minimize the average creation rate of that particle. In another derivation, we use the assumption that the reaction of the particle creation back on the gravitational field will modify the expansion in such a way as to reduce, if possible, the creation rate. This connection between the particle creation and the Einstein equations is surprising because the Einstein equations themselves played no part at all in the derivation of the equations governing the particle creation. Finally, on the basis of a so-called infinite-mass approximation, we argue that in the present predominantly dust-filled universe, only massless particles of zero spin might possibly be produced in significant amounts by the present expansion. In this connection, we show that massless particles of arbitrary nonzero spin, such as photons or gravitons, are not created by the expansion, regardless of its form.

IN a previous paper,[1] the results of an investigation of particle creation in expanding universes were summarized. The present paper is the first of several in which the previously summarized results will be derived.[2] The considerations in this paper will be restricted to (1) spin-0 particles of arbitrary mass and (2) particles of arbitrary spin but zero mass. The class of metrics considered here have the form

$$ds^2=-dt^2+R(t)^2(dx^2+dy^2+dz^2)\,, \tag{1}$$

where $R(t)$ is an unspecified positive function of t. We will refer to a universe with such a metric conveniently as an expanding universe, although $R(t)$ need not be increasing with time. The equations governing the fields are the covariant generalizations of the special-relativistic free-field equations. The gravitational metric is treated as an unquantized external field. No additional interactions are included.

[1] L. Parker, Phys. Rev. Letters **21**, 562 (1968).

[2] These articles will be based mainly on the author's thesis: L. Parker, Ph.D. thesis, Harvard University, 1966 (unpublished). Other related articles were cited in Ref. 1. A relevant article that the author was not previously aware of is Y. Takahashi and H. Umezawa, Nuovo Cimento **6**, 1324 (1957). This is the earliest article we know of dealing with quantized particle creation in expanding systems. It treats a problem corresponding to a sudden expansion of the universe.

A. CANONICAL QUANTIZATION OF THE SPIN-0 FIELD

The equation governing the spin-0 field is taken to be the Klein-Gordon equation with covariant derivatives ∇_j in place of ordinary derivatives[3]:

$$(g^{jk}\nabla_j\nabla_k - m^2)\phi = 0. \quad (2)$$

This is the simplest manifestly covariant equation which reduces to the Klein-Gordon equation in normal coordinates near any given point. The possible addition of a term to m^2 proportional to the scalar curvature will be considered when mass-zero fields are discussed. Equation (2) follows in the usual way from the scalar Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\sqrt{-g})(g^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2). \quad (3)$$

With the metric of (1), Eqs. (2) and (3) become

$$\ddot{\phi} + 3[\dot{R}(t)/R(t)]\dot{\phi} - R(t)^{-2}\sum_{j=1}^{3}\partial_j^2\phi + m^2\phi = 0 \quad (4)$$

and

$$\mathcal{L} = \tfrac{1}{2}R(t)^3[\dot{\phi}^2 - R(t)^{-2}\sum_{j=1}^{3}(\partial_j\phi)^2 - m^2\phi^2]. \quad (5)$$

It is convenient to impose the periodic boundary condition that $\phi(\mathbf{x}+\mathbf{n}L, t) = \phi(\mathbf{x},t)$, where $\mathbf{n}$ is a vector with integer Cartesian components. The limit $L \to \infty$ is to be taken after the physically significant quantities have been calculated. This is a purely mathematical device with no physical influence on the results. The general Hermitian solution of Eq. (4) can be expanded in terms of Fourier components in the form

$$\phi(\mathbf{x},t) = [LR(t)]^{-3/2}\sum_{\mathbf{k}}[2W(k,t)]^{-1/2} \times \left\{a_{\mathbf{k}}(t)\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \int_{t_0}^{t}W(k,t')dt'\right)\right] + \text{H.c.}\right\}. \quad (6)$$

The function $W(k,t)$ is an essentially arbitrary[4] real function of $k = |\mathbf{k}|$ and t. Conditions will be imposed on W in later sections. The constant t_0 is an arbitrary time, and H.c. denotes the Hermitian conjugate of the first term in brackets. Our reasons for writing ϕ in this way will become clear as the formalism is developed. The quantity $LR(t)$ appearing in (6) is the metric length of the periodicity cube. The form of $a_{\mathbf{k}}(t)$ of course depends on $W(k,t)$. No approximation is being made, although the appearance of $W^{-1/2}\exp(\int^t W dt')$ in (6) was originally motivated by a WKB-like approximation.

The field can be quantized canonically in a manner consistent with the equation of motion as follows: The momentum conjugate to ϕ is

$$\pi = \partial\mathcal{L}/\partial\dot{\phi} = (\sqrt{-g})\dot{\phi}. \quad (7)$$

A somewhat lengthy but straightforward calculation[5] shows that the commutation rules

$$[\phi(\mathbf{x},t),\phi(\mathbf{x}',t')] = 0, \quad [\pi(\mathbf{x},t),\pi(\mathbf{x}',t)] = 0,$$
and
$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \quad (8)$$

imply the commutation relations

$$[a_{\mathbf{k}}(t),a_{\mathbf{k}'}(t)] = 0, \quad [a_{\mathbf{k}}^\dagger(t),a_{\mathbf{k}'}^\dagger(t)] = 0,$$
and
$$[a_{\mathbf{k}}(t),a_{\mathbf{k}'}^\dagger(t)] = \delta_{\mathbf{k},\mathbf{k}'}, \quad (9)$$

provided that the following condition holds:

$$\dot{\phi}(\mathbf{x},t) = \sum_{\mathbf{k}}\left(a_{\mathbf{k}}(t)\frac{d}{dt}\left\{\frac{[LR(t)]^{-3/2}}{[2W(k,t)]^{1/2}} \times \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \int_{t_0}^{t}W dt'\right)\right]\right\} + \text{H.c.}\right). \quad (10)$$

If the $a_{\mathbf{k}}(t)$ are given at a particular time t_1, then because the time-dependent Fourier amplitudes of ϕ obey a second-order differential equation, it follows that Eqs. (6) and (10) at the particular time t_1 determine ϕ, and thus the $a_{\mathbf{k}}(t)$,[6] uniquely for all time. It should be emphasized that it does not necessarily follow that Eq. (10) will continue to be satisfied at times other than t_1. However, we will show that as a consequence of the equation of motion (4) the relation (10) does continue to be satisfied for all t.

To do that, we assume that we are given

$$a_{\mathbf{k}}(t_1) = A_{\mathbf{k}}, \quad (11)$$

and that Eq. (10) holds at the time t_1. The equal-time commutators (8) at time t_1 imply that

$$[A_{\mathbf{k}},A_{\mathbf{k}'}] = [A_{\mathbf{k}}^\dagger,A_{\mathbf{k}'}^\dagger] = 0 \quad \text{and} \quad [A_{\mathbf{k}},A_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'}. \quad (12)$$

We make the ansatz

$$a_{\mathbf{k}}(t) = \alpha(k,t)^* A_{\mathbf{k}} + \beta(k,t)A_{-\mathbf{k}}^\dagger, \quad (13)$$

where $\alpha(k,t)$ and $\beta(k,t)$ are complex c-number functions of k and t possessing at least first and second derivatives with respect to t. We will now obtain the integral equations for α and β. Although we will not need to do so here, those equations can be solved in terms of convergent infinite series for α and β.[7] Thus the unique solution of Eq. (4) satisfying the present boundary conditions can be found explicitly, so that the ansatz (13) is justified.

[3] Roman indices run from 0 to 3; $x^0 = t$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. We consider the neutral Hermitian spin-0 field here. The same procedure can be extended to the charged field [given in Parker (Ref. 2)]. We work throughout in the Heisenberg picture, in which the physical state vectors are independent of time. In our units $\hbar = c = 1$.

[4] For the present developments we will require that expressions like (6) be well defined, and that the first and second time derivatives of W exist.

[5] More details are in Parker (Ref. 2), pp. 20-22. For example, in the $[\pi,\pi]$ commutator certain terms cancel one another, rather than vanishing individually.

[6] The $a_{\mathbf{k}}(t)$ are defined through Eq. (6) for any given $W(k,t)$.

[7] The solutions are given in L. Parker, Nuovo Cimento **40B**, 99 (1965).

Here we only use the integral equations to show that Eq. (10) and the commutation relations (8) and (9) continue to be satisfied for all t, so that the quantization procedure is consistent with the time development dictated by Eq. (4).

Substituting (13) into (6) and regrouping, we find that

$$\phi(\mathbf{x},t)=[LR(t)]^{-3/2}\sum_{\mathbf{k}} 2^{-1/2}[A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}h(k,t)^*+\text{H.c.}], \quad (14)$$

where

$$h(k,t)=W(k,t)^{-1/2}\left[\alpha(k,t)\exp\left(i\int_{t_0}^{t}W(k,t')dt'\right)+\beta(k,t)\exp\left(-i\int_{t_0}^{t}W(k,t')dt'\right)\right]. \quad (15)$$

Using (12), (14), and the fact that $[A_{\mathbf{k}},\phi(\mathbf{x},t)]$ satisfies Eq. (4), one finds that $h(k,t)$ satisfies the equation

$$\ddot{h}(k,t)+\left[\frac{k^2}{R(t)^2}+m^2-\frac{3}{4}\left(\frac{\dot{R}(t)}{R(t)}\right)^2-\frac{3}{2}\frac{\ddot{R}(t)}{R(t)}\right]h(k,t)=0. \quad (16)$$

The boundary conditions are obtained from Eqs. (11) and (10) at time t_1. Thus, from (11) it follows that

$$\alpha(k,t_1)=1, \quad \beta(k,t_1)=0,$$

or

$$h(k,t_1)=W(k,t_1)^{-1/2}\exp\left(i\int_{t_0}^{t_1}W(k,t')dt'\right). \quad (17)$$

From (12), (14), and Eq. (10) at time t_1, it follows that

$$\dot{h}(k,t_1)=\left\{\frac{d}{dt}\left[W(k,t)^{-1/2}\exp\left(i\int_{t_0}^{t}W(k,t')dt'\right)\right]\right\}. \quad (18)$$

According to the theory of ordinary differential equations, a unique solution of Eqs. (16)–(18) does exist, so that the ansatz (13) was justified.

We obtain an integral equation for $h(k,t)$ by comparing Eq. (16) with the second-order equation satisfied exactly by the function

$$h_0(k,t)\equiv W(k,t)^{-1/2}\exp\left(i\int_{t_0}^{t}W(k,t')dt'\right), \quad (19)$$

namely, the equation

$$\ddot{h}_0+[W^2-W^{1/2}(d^2/dt^2)W^{-1/2}]h_0=0. \quad (20)$$

Equation (16) can be written

$$\ddot{h}+[W^2-W^{1/2}(d^2/dt^2)W^{-1/2}]h=2WSh, \quad (21)$$

where $S(k,t)$ is defined by

$$2WS=W^2-W^{1/2}\frac{d^2}{dt^2}W^{-1/2}-\frac{k^2}{R^2}-m^2+\frac{3}{4}\left(\frac{\dot{R}}{R}\right)^2+\frac{3}{2}\frac{\ddot{R}}{R}. \quad (22)$$

[Note that if the function S were to vanish for all t, then the $a_{\mathbf{k}}(t)$ appearing in (6) would be independent of the time.] The function

$$G(k,t,t')\equiv(2i)^{-1}[W(k,t)W(k,t')]^{-1/2}\times\left[\exp\left(i\int_{t'}^{t}Wdt''\right)+\exp\left(-i\int_{t'}^{t}Wdt''\right)\right] \quad (23)$$

is a solution, for fixed t', of Eq. (20), and satisfies the equation

$$G(k,t,t)=0, \quad \frac{\partial}{\partial t}G(k,t,t')|_{t=t'}=1. \quad (24)$$

Therefore, we can write the integral equation

$$h(k,t)=W(k,t)^{-1/2}\exp\left(i\int_{t_0}^{t}Wdt'\right)+\int_{t_1}^{t}G(k,t,t')\times 2W(k,t')S(k,t')h(k,t')dt'. \quad (25)$$

It is easily verified that (25) is a solution of Eq. (21) and satisfies the boundary conditions (17) and (18).

Substituting Eq. (15) for $h(k,t)$ on both sides of Eq. (25), we find that we can write

$$\alpha(t)=1-i\int_{t_1}^{t}dt'S(t')\times\left[\alpha(t')+\beta(t')\exp\left(-2i\int_{t_0}^{t'}Wdt''\right)\right]$$

and

$$\beta(t)=i\int_{t_1}^{t}dt'S(t')\times\left[\beta(t')+\alpha(t')\exp\left(2i\int_{t_0}^{t'}Wdt''\right)\right]. \quad (26)$$

(Since the present considerations refer to a given value of k, we do not write the k dependence explicitly.) It follows immediately that

$$\beta(k,t)=-\dot{\alpha}(k,t)\exp\left(2i\int_{t_0}^{t}W(k,t')dt'\right). \quad (27)$$

We also find from the time derivative of Eqs. (26) and the reality of $S(t)$ that

$$\alpha\dot{\alpha}^*+\alpha^*\dot{\alpha}-\beta^*\dot{\beta}-\beta\dot{\beta}^*=\frac{d}{dt}(|\alpha|^2-|\beta|^2)=0.$$

It then follows, using (17), that

$$|\alpha(k,t)|^2-|\beta(k,t)|^2=1. \quad (28)$$

Equation (28) could also have been derived from the fact that $h^*\dot{h}-h\dot{h}^*$ is a constant of the motion, for it follows from (27) that $h^*\dot{h}-h\dot{h}^*=2i(|\alpha|^2-|\beta|^2)$.

Using Eqs. (27) and (13), we obtain

$$\dot{a}_{\mathbf{k}}(t)=-\dot{a}_{-\mathbf{k}}^{\dagger}(t)\exp\left(2i\int_{t_0}^{t}W(k,t')dt'\right). \quad (29)$$

Hence

$$\sum_{\mathbf{k}}(2W)^{-1/2}\dot{a}_{\mathbf{k}}(t)\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\int_{t_0}^{t}Wdt'\right)\right]$$
$$=-\sum_{\mathbf{k}}(2W)^{-1/2}\dot{a}_{-\mathbf{k}}^{\dagger}(t)\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}+\int_{t_0}^{t}Wdt'\right)\right]$$
$$=-\sum_{\mathbf{k}}(2W)^{-1/2}\dot{a}_{\mathbf{k}}^{\dagger}(t)\exp\left[-i\left(\mathbf{k}\cdot\mathbf{x}-\int_{t_0}^{t}Wdt'\right)\right]. \quad (30)$$

It is now easily verified, using Eq. (30), that Eq. (10) does indeed continue to hold for all t. Consequently, the equal-time commutators (8) do continue to imply the commutation relations (9) for all t.

One may also confirm that the commutation relations (9) are consistent for all t. For example, using (12), (13), and (28), we obtain

$$[a_{\mathbf{k}}(t),a_{\mathbf{k}'}^{\dagger}(t)]=[|\alpha(k,t)|^2-|\beta(k,t)|^2]\delta_{\mathbf{k},\mathbf{k}'}=\delta_{\mathbf{k},\mathbf{k}'}, \quad (31)$$

in agreement with (9). Note that the calculation in (31) and the analogous calculations for the other commutators in (9) imply that if Eqs. (9) hold at any particular time t, then they hold at all times t. It then follows, using Eq. (10), that the commutation relations (8) also hold for all t. Therefore, Eqs. (8) and (9) are perfectly consistent with the equation of motion (4). In fact, Eqs. (8) and (9) are propagated unchanged by the equation of motion if they hold at any particular time.

In particular, if $R(t)$ is constant during a given period, then the familiar special-relativistic quantization scheme in which Eqs. (8) and (9) do hold [with $W=(k^2/R^2+m^2)^{1/2}$ and $a_{\mathbf{k}}$ time-independent during that period] will imply that Eqs. (8) and (9) continue to hold at all times, provided W is a function which reduces to $(k^2/R^2+m^2)^{1/2}$ when $R(t)$ is constant. Therefore, we require that if the $a_{\mathbf{k}}(t)$ are to be the annihilation operators for the physically observable particles during the expansion, when $R(t)$ is not constant, then we can write the corresponding $W(k,t)$ in the form

$$W(k,t)=\omega(k,t)+\lambda(k,t), \quad (32)$$

where

$$\omega(k,t)=[k^2/R(t)^2+m^2]^{1/2}$$

and

$$\lambda(k,t)=0 \text{ when } R(t) \text{ is constant.}$$

There still remains much freedom in the choice of W when $R(t)$ is not constant. We assume that for some particular choice of W the corresponding $a_{\mathbf{k}}(t)$ correspond to the physical particles during the expansion. We take (32) as one restriction on that function W. Further requirements for the $a_{\mathbf{k}}(t)$ to correspond to physical particles during the expansion will be imposed later in this paper. Meanwhile, our considerations are valid for a wide class of W, and in particular for the W corresponding to the physical $a_{\mathbf{k}}(t)$.

To complete the canonical formalism, we define the time-dependent Hamiltonian operator in the usual way by

$$H(t)=\int d^3x\,(\pi\dot{\phi}-\mathcal{L}). \quad (33)$$

A lengthy but straightforward calculation using (7), (8), and the equation of motion then shows that

$$[F,H(t)]+i\partial F/\partial t=idF/dt, \quad (34)$$

where F is any function which can be written as a power series in terms of ϕ, $\partial_1\phi$, $\partial_2\phi$, $\partial_3\phi$, and π.

Because of the translational invariance of the theory in 3-space, we would expect a quantity analogous to linear momentum to be conserved. That quantity is

$$K_j=\tfrac{1}{2}\int d^3x\,(\pi\partial_j\phi+\partial_j\phi\pi), \quad j=1,2,3. \quad (35)$$

It follows from (6), (7), and (9) that

$$K_j=\tfrac{1}{2}\sum_{\mathbf{k}}k_j\{a_{\mathbf{k}}(t)a_{\mathbf{k}}^{\dagger}(t)+a_{\mathbf{k}}^{\dagger}(t)a_{\mathbf{k}}(t)\}. \quad (36)$$

Then (13) and (28) imply that K_j can be written in the clearly time-independent form

$$K_j=\tfrac{1}{2}\sum_{\mathbf{k}}k_j\{A_{\mathbf{k}}A_{\mathbf{k}}^{\dagger}+A_{\mathbf{k}}^{\dagger}A_{\mathbf{k}}\}. \quad (37)$$

It follows from (35) and from the commutation relations, just as in special relativity, that

$$[K_j,F]=idF/dx^j. \quad (38)$$

Since a coordinate translation by dx^j corresponds to a translation by the physical length $R(t)dx^j$, the physical translation operator is not K_j itself, but $K_j/R(t)$. We identify this with the physical momentum. It follows from (36) that $a_{\mathbf{k}}(t)$ is the annihilation operator for a particle of momentum $\mathbf{k}/R(t)$. Thus, as the universe expands, the momentum of a particle in a given mode will be attenuated just as in classical general relativity.[8] At this point we will interrupt the further development of the theory to discuss the bearing of the previous results on the connection between spin and commutators in special relativity.

B. SPIN AND STATISTICS

A new and independent proof that boson rather than fermion commutators should be used for the spin-0 creation and annihilation operators in special relativity

[8] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, New York, 1934), p. 385, Eq. (153.9). The momentum is measured by an observer whose coordinates do not change.

can be based on the previous results. We showed in Sec. A that if the boson commutators (9) are imposed at a particular time t_1, then they continue to hold for all t. If we were to impose, for example, the fermion anticommutation relation at t_1

$$\{A_{\mathbf{k}},A_{\mathbf{k}'}^{\dagger}\}=\delta_{\mathbf{k},\mathbf{k}'}\,,$$

then Eq. (13) would give

$$\begin{aligned}\{a_{\mathbf{k}}(t),a_{\mathbf{k}'}^{\dagger}(t)\}&=[|\alpha(k,t)|^2+|\beta(k,t)|^2]\delta_{\mathbf{k},\mathbf{k}'}\\&=[1+2|\beta(k,t)|^2]\delta_{\mathbf{k},\mathbf{k}'}\,.\end{aligned}$$

As will become clear in the section on the statically bounded expansion, the function $\beta(k,t)$ generally cannot be made to vanish for all k and t by means of any choice of $\lambda(k,t)$ in (32).[9] Therefore, the fermion anticommutators are generally not propagated in time by the equation of motion and cannot consistently be required to hold at all times.

It now follows from the assumption of physical continuity[10] that the boson commutation relations, as opposed to the fermion relations, must hold for the creation and annihilation operators of the spin-0 field even in special relativity, that is, even when $R(t)$ is constant. For suppose that $R(t)$ is a function which is constant only during a limited period of time. Since only the boson commutators are consistent with the generally covariant equation of motion when $R(t)$ is not constant, physical continutiy demands that the boson commutators should continue to hold during the period when $R(t)$ is constant, even though neither commutation relation is ruled out solely by the special-relativistic equation of motion which holds when $R(t)$ is constant.

Alternatively, one could consider a sequence of functions for $R(t)$, namely, $R_n(t)=1+\epsilon_n(t)$, where $\epsilon_n(t)\to 0$ as $n\to\infty$, and where each $\epsilon_n(t)$ is not constant. For each value of n only the boson commutators can be consistently imposed. Continuity then implies that for the case $R(t)=1$ (special relativity), the boson commutation relations are correct.

Note that our derivation differs from other methods of obtaining the connection between spin and statistics, in that only the conditions of consistency with a slightly generalized equation of motion and continuity are used. A similar derivation of the connection between spin and statistics has been worked out for the case of spin $\frac{1}{2}$ and will be presented in a later paper of this series. It would be surprising if the method cannot be extended to arbitrary spin.

[9] There are a few particular $R(t)$ for which such a $\lambda(k,t)$ can be found, but that does not affect our argument.

[10] Namely, the requirement that the commutation relations should not suddenly jump from the fermion relations when $R(t)$ is constant to the boson relations when $R(t)$ becomes slightly time-dependent.

C. UNITARITY, PAIR CREATION, PARTICLE DENSITY, AND SEPARABILITY

In this section we will discuss some aspects and consequences of the formalism developed in Sec. A. Our considerations are for general $W(k,t)$, and hold in particular for whatever form of W corresponds to the physical particles. Equation (13) together with Eq. (28) have the form of a Bogoliubov transformation.[11] The transformation is unitary if and only if the $A_{\mathbf{k}}$ obey the boson commutation relations (12). In that case we can write, according to (28),

$$\begin{aligned}\alpha(k,t)&=e^{-i\gamma_\alpha(k,t)}\cosh\theta(k,t)\,,\\ \beta(k,t)&=e^{i\gamma_\beta(k,t)}\sinh\theta(k,t)\,.\end{aligned}\tag{39}$$

Then Eq. (13) can be written

$$a_{\mathbf{k}}(t)=U(t)A_{\mathbf{k}}U^{-1}(t)\,,\tag{40}$$

where $U^{-1}(t)=U^{\dagger}(t)$ and $U(t_1)=1$. The explicit form of U is

$$U(t)=\exp[\tfrac{1}{2}\sum_k\theta(k,t)(e^{-i\gamma(k,t)}A_{\mathbf{k}}A_{-\mathbf{k}}-e^{i\gamma(k,t)}A_{\mathbf{k}}^{\dagger}A_{-\mathbf{k}}^{\dagger})]\times\exp[i\sum_k\gamma_\alpha(k,t)A_{\mathbf{k}}^{\dagger}A_{\mathbf{k}}],\tag{41}$$

where $\gamma(k,t)=\gamma_\alpha(k,t)+\gamma_\beta(k,t)$. The relations (40) and (41) are only correct if the $A_{\mathbf{k}}$ and $A_{\mathbf{k}}^{\dagger}$ obey the boson commutators. Otherwise, as we have seen, the commutators obeyed by the $a_{\mathbf{k}}(t)$ at t_1 are not propagated in the same form by the equation of motion, so that a relation like (40) could not be valid.

If we define

$$\phi_A(\mathbf{x},t)\equiv[LR(t)]^{-3/2}\sum_{\mathbf{k}}[2W(k,t)]^{-1/2}\times\left\{A_{\mathbf{k}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\int_{t_0}^{t}Wdt'\right)\right]+\text{H.c.}\right\},\tag{42}$$

it follows from (6) and (40) that

$$\phi(\mathbf{x},t)=U(t)\phi_A(\mathbf{x},t)U^{-1}(t)\,.\tag{43}$$

If we define

$$\pi_A(\mathbf{x},t)\equiv(\sqrt{-g})\dot{\phi}_A(\mathbf{x},t)\,,\tag{44}$$

then Eqs. (10) and (40) yield the equation

$$\pi(\mathbf{x},t)=U(t)\pi_A(x,t)U^{-1}(t)\,.\tag{45}$$

The time development of $\phi_A(\mathbf{x},t)$ is not unitary, because of the factors $R(t)^{-3/2}$ and $W(k,t)^{-1/2}$ in (42). However, its time development is canonical, in the sense that the canonical commutators (8) for ϕ_A and π_A do imply the boson commutators (12), as a consequence of (43) and (45). We only mention the fields ϕ_A and π_A in passing and will not use them further in this paper.

Working in the Heisenberg picture, as before, we

[11] N. N. Bogoliubov, Zh. Eksperim. i Teor. Fiz. 34, 73 (1958) [English transl.: Soviet Phys.—JETP 7, 51 (1958)].

define the state $|0\rangle$ which contains no particles at t_1 by

$$A_{\mathbf{k}}|0\rangle=0, \quad \text{for all } \mathbf{k}. \tag{46}$$

Then, according to (40), we have at each time t

$$a_{\mathbf{k}}(t)\ |0\rangle_t=0, \quad \text{for all } \mathbf{k} \tag{47}$$

where

$$|0\rangle_t=U(t)|0\rangle.$$

The state $|0\rangle_t$ is the state containing no particles at the particular time t.

From (41) it follows that

$$|\langle 0|0\rangle_t| = |\langle 0|\exp[\tfrac{1}{2}\sum_{\mathbf{k}} \theta(e^{-i\gamma}A_{\mathbf{k}}A_{-\mathbf{k}}-e^{i\gamma}A_{\mathbf{k}}^\dagger A_{-\mathbf{k}}^\dagger)]|0\rangle|. \tag{48}$$

The quantity in Eq. (48) has been evaluated in another context by Kamefuchi and Umezawa,[12] with the result that

$$|\langle 0|0\rangle_t|=\exp[-\sum_{\mathbf{k}} \ln|\alpha(k,t)|]. \tag{49}$$

Because of (28) the above quantity is generally less than unity.

If the state is $|0\rangle$, then the square of Eq. (49) gives the probability of finding, at the time t, zero particles corresponding to the $a_{\mathbf{k}}(t)$. Using (28), the square of (49) can be written

$$|\langle 0|0\rangle_t|^2=\prod_{\mathbf{k}}[1+|\beta(k,t)|^2]^{-1} =\prod_{\mathbf{k}}\left(1-\frac{|\beta(k,t)|^2}{1+|\beta(k,t)|^2}\right),$$

from which we conclude that if the state is $|0\rangle$, then the probability P of observing a nonzero number in the mode $\mathbf{k}$ at time t is

$$P=\frac{|\beta(k,t)|^2}{1+|\beta(k,t)|^2}=\left|\frac{\beta(k,t)}{\alpha(k,t)}\right|^2. \tag{50}$$

Conservation of K_j implies that in the state $|0\rangle$, as many particles are present in mode $-\mathbf{k}$ as in mode $\mathbf{k}$ at all times. Therefore, the particles must be created in pairs of the net momentum zero.

A rather lengthy calculation which we do not give here[13] shows that the probability of observing $n_{\mathbf{k}}$ pairs present at time t in the state $|0\rangle$, where one particle is in the mode $\mathbf{k}$ and the other in the mode $-\mathbf{k}$ for some set of occupied modes $\{\mathbf{k}\}$, is

$$|_t\langle\{2n_{\mathbf{k}}\}|0\rangle|^2=\prod_{\mathbf{k}}\left[\left(\left|\frac{\beta(k,t)}{\alpha(k,t)}\right|^2\right)^{n_{\mathbf{k}}}\frac{1}{|\alpha(k,t)|^2}\right]. \tag{51}$$

[12] S. Kamefuchi and H. Umezawa, Nuovo Cimento 31, 492 (1964).

[13] The calculation is in Parker (Ref. 2), pp. 53–56.

Equation (50) can be confirmed by summing the quantity in square brackets in (51) over $n_{\mathbf{k}}=1, 2, \cdots$, and by using Eq. (28). Note that the probability of observing two pairs in the same mode is generally greater than the square of the probability of finding one pair in that mode, indicating that the presence of one pair tends to favor somewhat the creation of an identical pair.

The average number of particles present in the state $|0\rangle$ in the mode $\mathbf{k}$ and volume $[LR(t)]^3$ at time t is[14]

$$\langle N_{\mathbf{k}}(t)\rangle_0=\sum_{n_{\mathbf{k}}=1}^{\infty} n_{\mathbf{k}}\left(\left|\frac{\beta(k,t)}{\alpha(k,t)}\right|\right)^{n_{\mathbf{k}}}\frac{1}{|\alpha(k,t)|^2}=|\beta(k,t)|^2,$$

where we have used (28). This agrees with

$$\langle N_{\mathbf{k}}(t)\rangle_0=\langle 0|a_{\mathbf{k}}^\dagger(t)a_{\mathbf{k}}(t)|0\rangle=|\beta(k,t)|^2, \tag{52}$$

which can be directly obtained by using Eq. (13).

Suppose the state of the universe is described by a statistical mixture of pure states, each of which contains a definite number of particles at t_1. Then the statistical density matrix ρ is diagonal in the representation whose basis consists of the eigenstates of the operators $A_{\mathbf{k}}^\dagger A_{\mathbf{k}}$ (the t_1 representation), and the operator ρ must contain an equal number of operators $A_{\mathbf{k}}$ and $A_{\mathbf{k}}^\dagger$. For example, if $R(t)$ were constant before t_1, so that the Hamiltonian were $\frac{1}{2}\sum_{\mathbf{k}}\omega(k,t_1)\{A_{\mathbf{k}}^\dagger,A_{\mathbf{k}}\}$ before t_1, and equilibrium had been reached by t_1, one might expect ρ to be a function of the initial Hamiltonian, which is diagonal in the t_1 representation. For a ρ which is diagonal in the t_1 representation, a straightforward calculation using (12), (13), and (28) shows that the average number of particles in mode $\mathbf{k}$ and volume $[LR(t)]^2$ present at time t is

$$\begin{aligned}\langle N_{\mathbf{k}}(t)\rangle&=\mathrm{Tr}\rho a_{\mathbf{k}}^\dagger(t)a_{\mathbf{k}}(t)\\&=\langle N_{\mathbf{k}}(t_1)\rangle+|\beta(k,t)|^2[1+2\langle N_{\mathbf{k}}(t_1)\rangle],\end{aligned} \tag{53}$$

where $\langle N_{\mathbf{k}}(t_1)\rangle=\mathrm{Tr}\rho A_{\mathbf{k}}^\dagger A_{\mathbf{k}}$. Comparison with (52) shows that the initial presence of bosons tends to increase the number of bosons created by the expansion of the universe between the times t_1 and t. As we shall show in a later paper, the situation is reversed for fermions.

The average total number of particles present in the state $|0\rangle$ in the volume $[LR(t)]^3$ at time t is

$$\langle N(t)\rangle_0=\sum_{\mathbf{k}}|\beta(k,t)|^2. \tag{54}$$

The average particle density in the limit $L\to\infty$, when $\sum_{\mathbf{k}}\to(L/2\pi)^3\int d^3k$, is[15]

$$\lim_{L\to\infty}[LR(t)]^{-3}\langle N(t)\rangle_0 =[2\pi^2R(t)^3]^{-1}\int_0^\infty dk\, k^2|\beta(k,t)|^2. \tag{55}$$

[14] $\sum_n nx^n=x(d/dx)\sum_n x^n=x(1-x)^{-2}$.

[15] When a closed universe with a hyperspherical 3-space is considered, then the quantity $2\pi^2R(t)^3$ appears naturally as the volume of the universe, as shown in Parker (Ref. 2).

A similar expression follows from (53) when the state is described by a density operator ρ.

Finally, consider the Hilbert space spanned by the operators $a_{\mathbf{k}}{}^{\dagger}(t)a_{\mathbf{k}}(t)$ for all $\mathbf{k}$ and t. When $|\beta(k,t)|^2 \ll 1$, it follows from Eqs. (28) and (49) that

$$|\langle 0|0\rangle_t|^2 \cong \exp[-\sum_{\mathbf{k}} |\beta(k,t)|^2]. \quad (56)$$

Therefore, when the average total number of particles in the volume $[LR(t)]^3$ in the state $|0\rangle$, as given by (54), is finite, the quantity $|\langle 0|0\rangle_t|$ does not vanish. However, when $\int |\beta(k,t)|^2 d^3k$ is nonzero, then the total particle number approaches infinity as $L \to \infty$, and $|\langle 0|0\rangle_t|$ vanishes in the limit of infinite volume. Since $|_t\langle\{2n_{\mathbf{k}}\}|0\rangle$ in (51) is proportional to $|\langle 0|0\rangle_t|$, it also vanishes as $L \to \infty$. Consequently, the state $|0\rangle$ can not be expressed as a superposition of the eigenstates of $a_{\mathbf{k}}{}^{\dagger}(t)a_{\mathbf{k}}(t)$ for $t \neq t_1$ in the limit $L \to \infty$. Thus, the Hilbert space becomes nonseparable in the limit of infinite volume. This is not surprising, since any finite particle density implies an infinite number of particles in an infinite volume. No difficulties are encountered if one works with L finite and takes the limit $L \to \infty$ after the physically significant quantities have been deduced.[16]

D. STATICALLY BOUNDED EXPANSION AND ADIABATIC INVARIANCE

In order to show that the particle number in any given mode $\mathbf{k}$ is an adiabatic invariant, as well as to assure ourselves that a change in the particle number must generally take place during an expansion, we consider a statically bounded expansion. An expansion is by definition statically bounded if the function $R(t)$ in (1) satisfies the conditions

$$R(t) \to R_{\pm} \text{ and } d^nR(t)/dt^n \to 0 \ (n \geq 1) \text{ as } t \to \pm\infty. \quad (57)$$

The equation of motion (4) becomes the ordinary Klein-Gordon equation, with scale factor $R_{\pm}$, as $t \to \pm\infty$. We require that the $W(k,t)$ in (6) which corresponds to the physical creation and annihilation operators should satisfy the following conditions when $R(t)$ satisfies (57):

$$W(k,t) \to \omega_{\pm}(k) \quad \text{and} \quad d^nW(k,t)/dt^n \to 0 \ \ (n \geq 1) \quad \text{as } t \to \pm\infty, \quad (58)$$

where $\omega_{\pm}(k) = (k^2/R_{\pm} + m^2)^{1/2}$. The requirement of (58) is a slight extension of (32) and is implied by (32) for statically bounded expansions in which $R(t)$ is constant for $|t|$ greater than some finite time.

In the limit $t \to \pm\infty$, the theory becomes like that in special relativity, with $R_{\pm}dx$ corresponding to an element of physical length. In that limit, the operators $a_{\mathbf{k}}(t)$ unambiguously correspond to the observable particles. To establish the adiabatic invariance of the observable particle number in a given mode, we note that Eq. (16) has the same form as the equation of motion of an oscillator, with a time-dependent angular frequency given by

$$\Omega(k,t) = \left[\frac{k^2}{R(t)^2} + m^2 - \frac{3}{4}\left(\frac{\dot{R}(t)}{R(t)}\right)^2 - \frac{3}{2}\frac{\ddot{R}(t)}{R(t)}\right]^{1/2}. \quad (59)$$

The real or imaginary part of $h(k,t)$ represents the displacement of the oscillator at time t. Because of (57), the angular frequency Ω approaches $\omega_{\pm}(k)$ as $t \to \pm\infty$, and the time derivatives of Ω vanish in that limit. It follows from (15) and (58) that the total energy of the fictitious oscillator associated with Eq. (16), namely, $E(k,t) = \frac{1}{2}[|\dot{h}(k,t)|^2 + \Omega(k,t)^2|h(k,t)|^2]$, approaches the constant value $E_{\pm}(k) = \omega_{\pm}(k)[|\alpha_{\pm}(k)|^2 + |\beta_{\pm}(k)|^2] = \omega_{\pm}(k)[1 + 2|\beta_{\pm}(k)|^2]$ as $t \to \pm\infty$. Here $\alpha_{\pm}(k)$ and $\beta_{\pm}(k)$ are the limits of $\alpha(k,t)$ and $\beta(k,t)$ as $t \to \pm\infty$. The well-known adiabatic invariance of the energy divided by the frequency of such an oscillator implies the following[17]: If $d^n\Omega(k,t)/dt^n$ is bounded ($n \geq 1$), and the maximum of $|d^n\Omega(k,t)/dt^n|$ over all t is proportional to $\epsilon(k)^n$, where $\epsilon(k)$ is positive, then

$$\frac{E_+(k)/\omega_+(k)}{E_-(k)/\omega_-(k)} = \frac{1+2|\beta_+(k)|^2}{1+2|\beta_-(k)|^2} \to 1 \quad \text{as} \quad \epsilon(k) \to 0$$

or, equivalently,

$$|\beta_+(k)| \to |\beta_-(k)| \quad \text{as} \quad \epsilon(k) \to 0. \quad (60)$$

This remains valid even if the change in $R(t)$ or $\Omega(k,t)$ is large. If we suppose that the state before the statically bounded expansion is known, so that t_1 in (11) and (17) is $-\infty$, then $\beta_-(k) = 0$. It then follows from (60), (52), and (53) that in the limit of an infinitely slow expansion of the universe, the average particle number in each mode is the same before and after the expansion, even if the total change in $R(t)$ is large. Thus, the average particle number is an adiabatic invariant.[17]

It is known that in general the quantity $(E_+/\omega_+)/(E_-/\omega_-)$ is not precisely equal to unity, even when the rate of change of $\Omega(k,t)$ is small.[18] Therefore, $|\beta_+(k)|$ and $|\beta_-(k)|$ are generally not equal, so that a change in the average particle number in each mode does occur in a statically bounded expansion. Since the $a_{\mathbf{k}}(t)$ unambiguously correspond to the observable particles before and after such an expansion, the average observable particle number while the universe is expanding must unquestionably change with time. We

[16] Nonseparability can be avoided by working in a closed universe with $ds^2 = -dt^2 + R(t)^2(d\psi^2 + \sin^2\psi d\theta^2 + \cos^2\psi d\chi^2)$. However, for the present stage of the expansion the magnitude of the particle creation is essentially the same as for the metric of (1), as shown in Parker (Ref. 2), Appendix AII.

[17] See, for example, S. Chandrasekhar, in *The Plasma in a Magnetic Field*, edited by R. K. M. Landshoff (Stanford University Press, Stanford, Calif., 1958), p. 9.

[18] See Ref. 17, or G. Backus, A. Lenard, and R. Kulsrud, Z. Naturforsch. **15a**, 1007 (1960), where some specific cases are treated exactly.

can conclude that the average particle number is generally not a constant of the motion during an expansion, even if the expansion is not statically bounded. We now turn to the definition of the creation and annihilation operators which correspond to the observable particles during an expansion. The considerations in the following sections are not restricted to a statically bounded expansion.

E. PHYSICAL CREATION AND ANNIHILATION OPERATORS DURING AN EXPANSION

When the particle number is not a constant of the motion, as during an expansion of the universe, the definition of the particle number at a given time t is in principle somewhat fuzzy from a physical standpoint. This can be seen from the following heuristic argument. Suppose that a measurement of the particle number in a given coordinate volume takes a time interval Δt. In a realistic measurement there will be interactions which create particles when the time interval Δt becomes too small. In analogy with the time-energy uncertainty relation, these disturbances resulting from the measurement process will produce an uncertainty in the particle number of order $\Delta N_1 \approx (m\Delta t)^{-1}$. Simultaneously, during the interval Δt, the particle number is changing as a consequence of the expansion of the universe. The change caused by the expansion implies an uncertainty in particle number of order $\Delta N_2 \approx |A|\Delta t$, where A is the average creation rate in the given volume during the interval Δt. Thus we can write, for the total uncertainty in the particle number measured in the time interval Δt,

$$\Delta N \gtrsim (m\Delta t)^{-1} + |A|\Delta t. \qquad (61)$$

When $\Delta t = (m|A|)^{-1/2}$, the right-hand side of (61) is minimized and has the value $2(|A|/m)^{1/2}$. Since no choice of Δt will make ΔN vanish, arbitrary accuracy in the measurement of the particle number in a given volume is generally not possible in principle (unless A vanishes or $m \to \infty$).

Therefore, there is no reason why a precisely defined operator should correspond to the physical particle number when A does not vanish. (Similar arguments can be made about the physically measurable energy.) Nevertheless, if $|A|$ is small, as we would expect it to be at the present time, there should be a range of Δt for which the right-hand side of (61) is sufficiently small for any practical purpose, without being precisely zero. Thus, in the present stage of the expansion, it should be possible to define ,with reasonable accuracy, an operator corresponding to the particle number which would be measured in a time interval Δt for which the right-hand side of (61) is small.

We accomplish this essentially by treating the field $\phi(\mathbf{x},t)$ as a free field in a static universe during the interval Δt of the measurement. The particle number obtained by such a procedure will clearly possess an uncertainty of the order of the second term on the right of (61). Since our theory does not take into account explicitly the interactions which occur in the measurement process, the first term on the right-hand side of (61) will not appear as a direct consequence of the theory. For that reason Δt need not appear explicitly in the expression for the particle number operator.

We are further guided in defining the physical particle number operator $N_{\mathbf{k}}(t)$, corresponding to the number of particles in the mode $\mathbf{k}$ at the time t, by the following three fundamental requirements: (a) It is Hermitian, its eigenvalues being the non-negative integers (since a direct measurement of the particle number generally involves a counting procedure, which can only yield a non-negative integer result). (b) When the expansion is stopped slowly, the operator becomes the well-defined particle number operator for the static universe. (c) It should be measured in the slowly expanding universe by essentially the same apparatus as in the static case. (We call such an apparatus a staticlike apparatus.) Requirements (a) and (b) can be satisfied by putting

$$N_{\mathbf{k}}(t) = a_{\mathbf{k}}^{\dagger}(t)a_{\mathbf{k}}(t), \qquad (62)$$

where $a_{\mathbf{k}}(t)$ appears in (6) with a $W(k,t)$ satisfying (32), for then (a) follows from the commutation relations (9), while (b) follows as a result of (32). Evidently condition (c) can be satisfied by requiring that $N_{\mathbf{k}}(t)$ vary with time as slowly as possible, so that the Fock space corresponding to the $a_{\mathbf{k}}(t)$ will resemble as nearly as possible the Fock space of a static universe. It seems plausible that the staticlike apparatus would measure the $N_{\mathbf{k}}(t)$ in (62) which corresponds most closely to the particle number operator in a static universe, i.e., which varies most slowly with time. We therefore require that $W(k,t)$ in (6) be chosen so that,[19] first of all, for each $\mathbf{k}$, $|dN_{\mathbf{k}}(t)/dt|$ is minimized with respect to variation of $W(k,t)$. If some ambiguity in the choice of $W(k,t)$ still remains, then we require further that $|d^2N_{\mathbf{k}}(t)/dt^2|$ be minimized with respect to variation in $W(k,t)$, and so on. In general, *$W(k,t)$ is determined, for each k, by minimizing $|d^nN_{\mathbf{k}}(t)/dt^n|$ with respect to variation in $W(k,t)$ in the order $n=1, 2, 3, \cdots$, until all ambiguity in $W(k,t)$ is removed.* We call this the minimization postulate. In applying the minimization postulate, we require that $N_{\mathbf{k}}(t)$ be given by (62), and that $W(k,t)$ satisfy (32).

By taking $N_{\mathbf{k}}(t)$ in the form (62) we are essentially treating the field $\phi(\mathbf{x},t)$ in (6) as though it were a free field in a static universe. Such a procedure appears to be more justified when $W(k,t)$ satisfies the minimization postulate. That postulate also means that in a statically bounded smooth expansion of the universe $\langle N_{\mathbf{k}}(t)\rangle$ will

[19] We also assume that $N_{\mathbf{k}}(t)$ and $W(k,t)$ are continuous and possess continuous derivatives when $R(t)$ and its derivatives are continuous. When we say that $|d^nN_{\mathbf{k}}(t)/dt^n|$ is minimized, we mean that its expectation value is minimized for the state of the universe. For states of the type leading to Eqs. (53) and (54), the minimization will reduce to conditions on $\beta(k,t)$ and its time derivatives which are independent of the particular state of the universe.

vary as gradually as possible between its unambiguous initial and final values. Thus, as nearly as possible, the observable particles created during an expansion will be just those particles which remain after the expansion has been very gradually stopped. Physically, this is very plausible behavior for the observable particle number.

Instead of defining the physical particle number operator in the above manner, one might instead diagonalize the Hamiltonian $H(t)$ of (33) by means of a Bogoliubov transformation,[20] and try to identify the creation and annihilation operators which diagonalize $H(t)$ as the ones from which $N_{\mathbf{k}}(t)$ should be formed. When $H(t)$ is constant, this procedure is justified, and leads to the same result as the procedure we have adopted. However, when $H(t)$ varies with time, the system can no longer be treated as though it were an isolated closed system. Rather, it is interacting with an external time-dependent gravitational field. That interaction should be represented by time-dependent terms in $H(t)$ which are not necessarily diagonal in the physical particle number representation. Therefore, diagonalization of $H(t)$ is not the correct method of obtaining the physical creation and annihilation operators during the expansion of the universe. In fact, it can be shown by means of a lengthy calculation that if the diagonalization procedure is carried out in detail, it will yield an expression for $N_{\mathbf{k}}(t)$ such that, in general, $\int d^3k\langle 0|N_{\mathbf{k}}(t)|0\rangle$ diverges, so that the particle density corresponding to (55) will be infinite. We base the definition of the physical particle number on (32), (62), and on the minimization postulate as described above, and not on diagonalization of $H(t)$.

Since using the physical $a_{\mathbf{k}}(t)$ in (62) minimizes $|\langle dN_{\mathbf{k}}(t)/dt\rangle|$, the use of an $a_{\mathbf{k}}(t)$ in (62) corresponding to any $W(k,t)$ in (6) which does not satisfy the minimization postulate will yield a larger value for $|\langle dN_{\mathbf{k}}(t)/dt\rangle|$. This latter value can serve as an upper bound on the absolute value of the average creation rate. However, unless great care is taken in the choice of the $W(k,t)$ for the upper bound, the result obtained diverges when summed over all modes, and is therefore useless as an upper bound. We do not go into such matters here, but reserve them for a later paper of this series. In this paper, we will discuss some special but important cases in which the choice of $W(k,t)$ which corresponds to the physical particles during the expansion (i.e., satisfies the minimization postulate) can be obtained by relatively simple considerations.

F. CONNECTION WITH EINSTEIN'S FIELD EQUATIONS

Given a particular function $R(t)$ in (1), it is generally not an easy matter to find the $W(k,t)$ satisfying the minimization postulate and (32). However, there are two particularly simple cases which we wish to discuss here. As we shall see, these two cases have a remarkable connection with the Friedmann solutions of the Einstein field equations with a metric of the form (1). This connection, by no means purely fortuitous, seems to point to some deeper relationship between the creation of spin-0 particles and the solutions of the Einstein field equations. The connection is all the more unexpected because the Einstein field equations have had no bearing whatever upon the development of the theory in this paper. We did use general covariance and the equivalence principle to help determine the equation governing the spin-0 field, but the function $R(t)$, which in general relativity is determined by the gravitational field equations, has been left unspecified.

We get at the results mentioned above by looking for cases in which the function $W(k,t)$ satisfying the minimization postulate has the simplest form allowed by (32), namely, the cases in which

$$W(k,t)=\omega(k,t)=[k^2/R(t)^2+m^2]^{1/2}. \quad (63)$$

It will be recalled that when the function $S(k,t)$, which appears in Eq. (21) and is defined by Eq. (22), vanishes, then the corresponding $a_{\mathbf{k}}(t)$ appearing in (6) are independent of the time t. Time-independent $a_{\mathbf{k}}(t)$ obviously satisfy the minimization postulate. Therefore, we substitute (63) into the right-hand side of Eq. (22), and we look for cases in which S vanishes identically. Upon substitution of (63) and some calculation, Eq. (22) becomes

$$2\omega S=C_1(k,t)[\dot{R}(t)/R(t)]^2+C_2(k,t)\ddot{R}(t)/R(t), \quad (64)$$

with

$$C_1(k,t)=\frac{k^4+3m^2R(t)^2k^2+\frac{3}{4}m^4R(t)^4}{[k^4+m^2R(t)^2]^2}$$

and

$$C_2(k,t)=\frac{k^2+\frac{3}{2}m^2R(t)^2}{k^2+m^2R(t)^2}.$$

The functions C_1 and C_2 must be independent of k before it will be possible for S to vanish for all k and t. There are just two choices of the mass m for which C_1 and C_2 become independent of k, namely, when $m=0$ and in the limit that $m\to\infty$. When $m=0$, we find that $C_1=C_2=1$, and

$$2\omega S=[\dot{R}(t)/R(t)]^2+\ddot{R}(t)/R(t), \quad m=0. \quad (65)$$

When $m\to\infty$, we find that $C_1=\frac{3}{4}$ and $C_2=\frac{3}{2}$, or

$$2\omega S=\tfrac{3}{4}[\dot{R}(t)/R(t)]^2+\tfrac{3}{2}\ddot{R}(t)/R(t), \quad m\to\infty. \quad (66)$$

As we shall argue in Sec. G, as far as particle creation at the present time is concerned, all known particles of nonzero mass can be treated as though their mass were infinite to very good approximation.

It follows from Eq. (65), that if $R(t)$ satisfies the equation

$$[\dot{R}(t)/R(t)]^2+\ddot{R}(t)/R(t)=0, \quad (67)$$

[20] See, for example, T. Imamura, Phys. Rev. 118, 1430 (1960).

then the $W(k,t)$ corresponding to the physical spin-0 particles of vanishing mass ($m=0$) is given by Eq. (63), and the corresponding $a_{\mathbf{k}}(t)$ are independent of time. The constancy of the $a_{\mathbf{k}}(t)$ means that there is precisely no creation of spin-0 massless particles by the expansion when $R(t)$ satisfies Eq. (67). This result remains valid even when the equation of motion (4) contains a term proportional to the contracted Riemann tensor added on to m^2, for the contracted Riemann tensor corresponding to the metric of (1) is $\frac{1}{6}[(\dot{R}/R)^2+\ddot{R}/R]$, which vanishes when Eq. (67) is satisfied.

According to Eq. (66), if $R(t)$ satisfies the equation

$$\tfrac{3}{4}[\dot{R}(t)/R(t)]^2+\tfrac{3}{2}\ddot{R}(t)/R(t)=0\,, \qquad (68)$$

then S vanishes, so that the $W(k,t)$ corresponding to the physical spin-0 particles in the limit of infinite mass ($m\to\infty$) is also given by Eq. (63), and the corresponding $a_{\mathbf{k}}(t)$ are independent of time. Hence, there is precisely no creation of spin-0 particles in the limit of infinite mass when $R(t)$ satisfies Eq. (68).

Equation (68) is one of the Einstein field equations (without the cosmological term) for a universe of metric (1) filled with dust particles having negligible random velocities (or a gas of noninteracting elementary particles with random velocities small compared to the speed of light).[21] Thus, Eq. (68) yields the well-known Friedmann expansion $R(t)\propto t^{2/3}$. The other member of the Einstein field equations for the dust-filled universe relates the average matter density to $\dot{R}(t)/R(t)$, and is not relevant to our discussion. Therefore, we may conclude that *in a dust-filled Friedmann universe with flat 3-space there is precisely no creation of spin-0 particles in the limit of infinite mass.*

Similarly, Eq. (67) is one of the Einstein field equations (without the cosmological term) for a universe of metric (1) filled with radiation.[22] Thus, Eq. (67) yields the Friedmann expansion $R(t)\propto t^{1/2}$. Therefore, *in a Friedmann universe with flat 3-space containing only massless particles in equilibrium, there will be precisely no creation of massless spin-0 particles.*

The last two italicized results support the following hypothesis: *In an expansion of the universe in which a particular type of particle is predominant, the expansion achieved after a long time will be such as to minimize the average creation rate of that particle.* We call this hypothesis (A). *Without making any assumption as to the particular equations governing the macroscopic evolution of the universe,* we can assume hypothesis (A), and thereby derive the Friedmann expansions for the dust-filled and radiation-filled universes having metric (1), for in a dust-filled universe the predominant particles have effectively infinite mass, and the creation of such particles is minimized, in fact extinguished, when $R(t)$ satisfies Eq. (68), which yields the corresponding Friedmann expansion. Similarly, in a universe filled with massless particles, the creation of such particles is extinguished when $R(t)$ satisfies Eq. (67), which again yields the corresponding Friedmann expansion. As we shall show in Sec. H. massless particles of nonzero spin are not created in an expansion regardless of the form of $R(t)$, so that in the argument concerning massless particles, we need not assume that only spin-0 particles are present, although it is the spin-0 particles which determine the particular form of $R(t)$ which is approached. Similarly, we will show in a later paper that in the infinite-mass limit there is no creation of spin-$\frac{1}{2}$ particles regardless of the form of $R(t)$. This last result is probably valid also for higher spins, so that in our argument involving very massive particles, it is probably not necessary to assume that only spin-0 particles are present, although it is again the spin-0 particles which determine the form approached by $R(t)$.

Another slightly different way of getting at the equation satisfied by $R(t)$ through consideration of the particle creation, without assuming the Einstein field equation or any other macroscopic equations, is by means of the following *gedanken experiments.* First consider a hypothetical universe in which only massless particles exist and can be created. Suppose that $R(t)$ is initially increasing with time in some rapid manner which does not necessarily satisfy Eq. (67). Then the massless spin-0 particles governed by Eq. (4) will be created (or annihilated if the density is high enough). We now make the natural assumption, which we will call hypothesis (B), that *the reaction of the particle creation (or annihilation) back on the gravitational field will modify the expansion in such a way as to reduce the creation rate.* This assumption is analogous to Lenz's law, according to which an effect acts in such a way as to oppose its cause. Clearly, hypothesis (B) is closely related to hypothesis (A). If hypothesis (B) is true, then eventually a type of expansion should be approached in which the creation of massless spin-0 particles is minimized or, if possible, extinguished. Therefore, the function $R(t)$ which is approached, and corresponds to the form of $R(t)$ for a universe filled with massless particles in equilibrium, must satisfy Eq. (67). This is in agreement with the Einstein field equations, although no hypothesis as to the particular equations governing the expansion has been made.

Next consider a universe in which only very massive particles of spin-0 exist or can be created. Exactly the same argument based on hypothesis (B) as was used above for massless particles now leads to the result that $R(t)$, for a universe filled with very massive particles of negligible thermal energies in equilibrium, must satisfy Eq. (68), in agreement again with the Einstein field equations. The connections found in this section between the creation of spin-0 particles in accordance with

[21] See, for example, A. Einstein, *Meaning of Relativity* (Princeton University Press, Princeton, N. J., 1955), 5th ed., p. 118, Eq. (5). The first of his equations is Eq. (68) multiplied by $\frac{4}{3}$ (when the 3-space is flat).

[22] See, for example, R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill Book Co., New York, 1965), p. 362, Eqs. (12.18a) and (12.18b) with $k=0$ and $p=\frac{1}{3}\rho c^2$. Elimination of ρ gives Eq. (67).

quantum field theory and the Einstein field equations governing the large-scale expansion of the universe are apparently examples of the far-reaching consistency of nature.

G. INFINITE-MASS APPROXIMATION

We can make use of the previous considerations to give a rough argument that the only particles which might be created in significant quantities at the present time, as a result of the actual expansion of the universe, are massless spin-0 particles. Consider the creation of particles of mass m by the expansion of the universe taking place at the present time. One can argue roughly that significant particle creation should only be possible in the modes $\mathbf{k}$ for which the corresponding wavelength $[k/R(t)]^{-1}$ is at least of the order of $H^{-1}\approx 10^{27}$ cm, where H is Hubble's constant. Such wavelengths would presumably be the most affected by the expansion because of the large differences in the relative expansion velocities of points lying within a single wavelength (just as a rapid change in the potential energy of a particle over a single wavelength invalidates the WKB approximation in quantum mechanics). If we assume this rough argument is correct, then we would expect significant particle creation at the present time only in the modes satisfying the inequality $k/R(t)\lesssim 10^{-27}$ cm^{-1}. It follows that for all known particles of nonvanishing mass, one may ignore $k^2/R(t)^2$ with respect to m^2 to extremely good approximation in considering the particle creation at the present time. We will call this approximation the infinite-mass approximation. For π mesons, for example, we have $(H/m)^2\approx 10^{-80}$, so that the infinite-mass approximation will be extremely good, provided the rough argument leading to it is valid.

The observational evidence seems to indicate that the expansion at the present time is a Friedmann expansion corresponding to a dust-filled universe. Observation has not yet determined whether or not the 3-space is flat. Assuming that the metric is of the form (1), with $R(t)$ corresponding to the Friedmann expansion of a dust-filled universe, it follows that $R(t)$ satisfies Eq. (68). But in the infinite-mass approximation, Eq. (68) is the condition that there be no creation of the spin-0 particles. Thus, if the assumptions in this admittedly rough argument are correct, there is no creation of spin-0 particles of nonzero mass to extremely good approximation. As will be shown in a future paper, in the infinite-mass approximation there is no creation of spin-$\frac{1}{2}$ particles of known nonzero mass regardless of the form of $R(t)$. This result is probably valid for higher-spin particles of nonzero mass also. Therefore, we may conclude that to extremely good approximation there is no creation of particles of nonzero mass as a result of the present expansion of the universe (assuming it corresponds to a dust-filled Friedmann universe with flat 3-space).

Next consider the creation of massless particles by an expansion of the universe. As we will show in Sec. H, there is precisely no creation of massless particles of spin greater than zero, regardless of the form of $R(t)$. Thus, on the basis of the rough heuristic argument given in this section, we can conclude that if the present expansion is that of a dust-filled Friedmann universe with flat 3-space, then to extremely good approximation there are no particles created by the present expansion, with the possible exception of massless spin-0 particles.

H. MASSLESS PARTICLES OF NONZERO SPIN

In this section, we show that there is no creation of massless particles of spin greater than zero in a universe with metric (1), regardless of the form of $R(t)$. The proof depends on the notion of conformal invariance. A conformal transformation is a transformation of the metric g_{jk}, such that

$$g_{jk}\longrightarrow \tilde{g}_{jk}=\Omega^{-2}g_{jk},\quad j,k=0,\,1,\,2\,,3 \tag{69}$$

where Ω is a scalar function of the coordinates. It corresponds to a stretching of the interval at each point ($ds=\Omega d\tilde{s}$). The equation governing the field of a given spin is said to be conformally invariant, if under the conformal transformation (69), together with a transformation of the field (involving multiplication by a suitable power of Ω), the equation governing the transformed field has the same form as the original equation. The simplest generally covariant equations governing the massless fields of nonzero spin are all conformally invariant.[23]

The simplest generally covariant equation for the spin-0 massless field—namely, Eq. (2) with $m=0$—is not conformally invariant. Our previous considerations were based on Eq. (2). However, just to illustrate the method of proof in the context of a familiar notation, we will consider here the covariant and conformally invariant generalization of the massless Klein-Gordon equation. That equation is[23]

$$(g^{jk}\nabla_j\nabla_k+\tfrac{1}{6}g^{jk}R_{jk})\phi=0\,, \tag{70}$$

where $g^{jk}R_{jk}$ is the scalar curvature, and ∇_j denotes covariant differentiation, as in Eq. (2). Under a conformal transformation (69), it can be shown that Eq. (70) leads to the equation

$$(\tilde{g}^{jk}\tilde{\nabla}_j\tilde{\nabla}_k+\tfrac{1}{6}\tilde{g}^{jk}\tilde{R}_{jk})\tilde{\phi}=0\,, \tag{71}$$

where $\tilde{g}^{jk}\tilde{R}_{jk}$ and $\tilde{\nabla}_j$ are the scalar curvature and covariant derivative operator, respectively, in the conformally transformed space, and

$$\tilde{\phi}=\Omega\phi\,. \tag{72}$$

As before, we are considering the metric of (1). It can be

[23] R. Penrose, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, Science Publishers, Inc., New York, 1964), pp. 565 and 566.

transformed to

$$d\tilde{s}^2 = d\tau^2 - dx^2 - dy^2 - dz^2 \qquad (73)$$

by the conformal transformation (69) with $\Omega = R(t)$. In Eq. (73), $\tau = \int_{t_0}^{t} R(t')^{-1} dt'$. The conformally transformed equation (71) is just the special-relativistic Klein-Gordon equation for massless mesons in the metric of (73). Therefore, using (72), we have

$$\tilde{\phi} \propto e^{\pm i(\mathbf{k}\cdot\mathbf{x} - k\tau)}$$

or

$$\phi \propto R(t)^{-1} \exp\left[\pm i\left(\mathbf{k}\cdot\mathbf{x} - \int_{t_0}^{t} kR(t)^{-1} dt'\right)\right]. \qquad (74)$$

Hence, ϕ can be written as

$$\phi = \frac{1}{[2\pi R(t)]^{3/2}} \sum_{\mathbf{k}} \frac{1}{[2\omega(k,t)]^{1/2}} \times \left\{ A_{\mathbf{k}} \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \int_{t_0}^{t} \omega(k,t') dt'\right)\right] + \text{H.c.}\right\}, \qquad (75)$$

where $\omega(k,t) = k/R(t)$, and the $A_{\mathbf{k}}$ are time-independent annihilation operators. This has the same form as Eq. (6), with $W(k,t) = \omega(k,t)$, and $a_{\mathbf{k}}(t) = A_{\mathbf{k}}$. Clearly Eq. (32) and the minimization postulate are satisfied, so that the $A_{\mathbf{k}}$ are the physical creation and annihilation operators, during the expansion, for particles satisfying Eq. (70) with the metric of (1). Since the $A_{\mathbf{k}}$ are constant, there is no creation of such particles during the expansion. We prefer to take Eq. (2), rather than Eq. (70), as the equation governing the physical massless spin-0 field because Eq. (2) is the simplest covariant generalization of the Klein-Gordon equation. Therefore, the conclusions drawn from Eq. (70) are merely illustrative of what can also be done with massless felds of higher spin.

For nonzero spin, the simplest covariant generalizations of the free-field equations are all conformally invariant.[23] Therefore, by means of an argument of the same type as used in connection with Eq. (70), we can show that there is no creation of massless particles of nonzero spin during the expansion. In the two-component spinor formalism,[24] the covariant equations governing the massless field of nonzero spin s can be written, following Penrose,[23] as

$$\nabla^{\nu_1 \dot{\sigma}} \xi_{\nu_1 \nu_2 \cdots \nu_{2s}} = 0. \qquad (76)$$

Under the conformal transformation (69), the equation can be written

$$\tilde{\nabla}^{\nu_1 \dot{\sigma}} \tilde{\xi}_{\nu_1 \nu_2 \cdots \nu_{2s}} = 0, \qquad (77)$$

with

$$\tilde{\xi}_{\nu_1 \nu_2 \cdots \nu_{2s}} = \Omega^{s+1} \xi_{\nu_1 \nu_1 \cdots \nu_{2s}}. \qquad (78)$$

As we did in the spin-0 case, we conformally transform the metric of (1) using $\Omega = R(t)$. Then Eq. (77) is the equation for a free field in special relativity, and it can easily be shown, using (78), that there are independent solutions of (76) having the form

$$\xi_{\nu_1 \cdots \nu_{2s}}(\mathbf{x},t) \propto R(t)^{-(s+1)} \times \exp\left[\pm i\left(\mathbf{p}\cdot\mathbf{x} - \int_{t_0}^{t} p_0 R(t')^{-1} dt'\right)\right] \chi_{\nu_1 \cdots \nu_{2s}}. \qquad (79)$$

During the expansion, ξ can be written in terms of the independent solutions of the form (79) with constant creation and annihilation operators, as ϕ was written in (75). Clearly, the positive- and negative-frequency parts of ξ will not be mixed by a statically bounded expansion, so that there will be no creation of massless particles with nonzero spin as the result of such an expansion. Without going into the details of quantization during the expansion of the universe, we can conclude that no creation of observable massless particles of nonzero spin occurs during the expansion because no creation of such particles takes place as the result of *any* statically bounded expansion. As stated earlier, the present result, in conjunction with the result obtained in Sec. F, implies that for the Friedmann universe with Euclidean 3-space, and filled with massless particles in equilibrium, there is no creation of massless particles of any spin.

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[24] L. Infeld and B. L. van der Waerden, Sitzber. Deut. Akad. Wiss. Berlin Kl. Math.-Naturw. 380 (1933); W. Bade and H. Jehle, Rev. Mod. Phys. **25**, 714 (1953).