Theory of Pion-Nucleus Scattering Lengths*

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The impulse or single-scattering approximation for zero-energy pion-nucleus scattering gives results in disagreement with the experimental scattering lengths obtained from pionic atoms. We calculate multiple-scattering corrections in a simple nuclear model, and show that double scattering of the pion can account for a major portion of the disagreement. The largest contributions involve incoherent excitation of the nucleus and are not included in most optical models. These contributions include the effect of nuclear binding on the scattering.

I. INTRODUCTION

IN this paper, we apply the theory of multiple scattering to the problem of the scattering of zero-energy pions by light nuclei. This problem has previously been approached in several ways. Deser *et al.*[1] used the single-scattering (impulse) approximation to relate the pion-nucleus scattering length to the pion-nucleon scattering length. In Sec. III, we review the single-scattering approximation, and show that it does not lead to good agreement with experiment.

Ericson and Ericson[2] have computed a pion-nucleus optical potential for low-energy scattering, based on coherent multiple scattering of the pion in which the nucleus is not excited. Incoherent scattering effects were estimated by using a correlated Fermi-gas model. The Ericsons have also included an effect induced by nuclear absorption of pions, as suggested by Brueckner.[3] The size of this effect is not well known.

We shall demonstrate that multiple-scattering effects, not previously included, make a large contribution to the scattering length at low energies, because of the particular isospin structure of low-energy pion-nucleon scattering. The agreement with experiment is very much improved when these effects are included. The effects of interest are double scattering with excitation of the nucleus. This includes the correction to the impulse approximation due to nuclear binding. The theory and calculations of these effects are in Secs. IV and V.

The best data on low-energy pion-nucleus scattering come from measurements of level shifts in the pionic atoms. In Sec. II, we discuss the theory of the extraction of scattering lengths from these measurements.

II. RELATION OF SHIFTS TO SCATTERING LENGTHS

The work of Deser *et al.*[1] first made the connection between the S-level energy shift for a pion-atom and the zero-energy (non-Coulomb) scattering of a pion from the same nucleus. They found that

$$\Delta E = -(2\pi/\mu)\rho(0)a(Z, N), \tag{2.1}$$

where $\rho(0)$ is the probability density (unperturbed by scattering) for the atomic pion to be at the nucleus, that is, for a hydrogenlike atom, μ is the pion-nucleus reduced mass, and $a(Z, N)$ is the scattering length for a pion on the nucleus with Z protons and N neutrons after removing the effects of the Coulomb interaction. Equation (2.1) is approximate, requiring that the pion density $\rho(r)$ be essentially constant over the range of energy over which the pion-nucleus scattering amplitude varies significantly. Both conditions essentially require that the nuclear radius R, and therefore the pion-nucleus interaction range, be smaller than the average pion (Bohr) radius:

$$(Z\mu e^2)^{-1} \gg R. \tag{2.2}$$

This requirement is well obeyed for light nuclei ($Z<10$).

We have used (2.1) to produce scattering lengths $a(Z, N)$ from the measured level shifts of Jenkins *et al.*[4] and Harris *et al.*[5] for $6<A<23$. These are shown in Fig. 1, and are listed in column 5 of Table I.

Seki and Cromer[6] have attempted to calculate a more exact relation between ΔE and $a(Z, N)$, using a complex square-well potential for the pion-nucleus interaction. The pion-atomic scattering lengths are calculated in this potential with the Coulomb field from a finite charge distribution. The scattering lengths they obtained by this method using the level shifts of Jenkins *et al.*[4] are listed in Table I and shown in Fig. 1 (open triangles).

The radii of the square well were chosen in the work of Seki and Cromer to agree with the electron scattering radii. More recently, Seki[7] has used, instead, a simple A dependence for this radius, $R=1.3A^{1/3}$ fm, and has shown that the correction to (2.1) can be expressed

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[1] S. Deser, M. L. Goldberger, K. Baumann, and W. Thirring, Phys. Rev. **96**, 774 (1954).

[2] M. Ericson and T. E. O. Ericson, Ann. Phys. (N.Y.) **36**, 323 (1966).

[3] K. A. Brueckner, Phys. Rev. **98**, 769 (1955).

[4] D. A. Jenkins, R. Kunselman, M. K. Simmons, and T. Yamazaki, Phys. Rev. Letters **17**, 1 (1966).

[5] R. J. Harris, Jr., W. B. Shuler, M. Eckhouse, R. T. Siegel, and R. E. Welsh, Phys. Rev. Letters **20**, 505 (1968).

[6] R. Seki and A. H. Cromer, Phys. Rev. **156**, 93 (1967).

[7] R. Seki, Ph.D. thesis, Northeastern University, 1967 (unpublished).

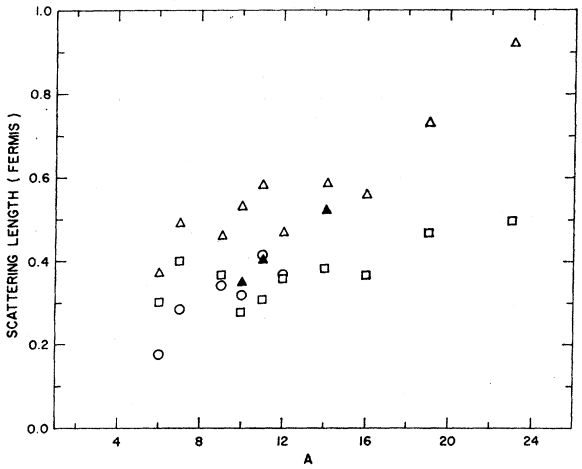

FIG. 1. Reduced experimental data from Table I. △, Seki and Cromer (Ref. 6) potential reduction using a varying radius for the potential from electron scattering. ▲, Seki (Ref. 7) potential reduction using an $A^{1/3}$ dependence for the radius of the potential and plotted only where it differs from the Seki and Cromer results. □, linear approximation according to Deser *et al.* [Eq. (2.1)]. All of these calculations used the data of Jenkins *et al.* (Ref. 4). ○, linear approximation using the data of Harris *et al.* (Ref. 5) for $A<12$.

approximately by a simple empirical relationship,

$$a_{\text{Seki}}=(1-0.02A)^{-1}a_{\text{lin}}, \tag{2.3}$$

where A is the number of nucleons and a_{lin} is given by (2.1). For the data of Jenkins *et al.*,[4] (2.3) gives essentially the same value of a as in the Seki-Cromer calculation, except for $A=10$, 11, and 14. We show these three points as dark triangles in Fig. 1.

We have selected the "best" values of the scattering lengths as follows: We use the data of Harris *et al.*[5] for $6<A<12$ and of Jenkins *et al.*[4] for $12<A$. We use (2.1) and (2.3) to obtain the scattering lengths, which are displayed, with bars for experimental uncertainties, in Fig. 5.

Note added in proof. The difference between (2.3) and the calculations of Seki-Cromer applied to the data of Jenkins *et al.* can be traced to our use of the Jenkins *et al.* Coulomb correction. Seki (private communcation) has not used the very large values of the charge radii quoted by Jenkins *et al.* for $A=10$, 11, and 14, and therefore there is no difference in the two approaches of Seki.

III. SINGLE-SCATTERING APPROXIMATION AND CORRECTIONS

The single-scattering (or impulse) approximation for $a(Z, N)$ was considered by Deser *et al.*[1] [Eq. (2.1)]. We shall restate it in terms of scattering matrices, using

$$a(Z, N)=-(\mu/2\pi)\langle 0 \mid T \mid 0\rangle, \tag{3.1}$$

where the angular brackets are understood to mean expectation value in the nuclear ground state and zero pion momentum state. The approximation can be put in terms of the pion–jth-nucleon scattering amplitudes $t(j)$; for $A=Z+N$,

$$T_{\text{ss}}=\sum_{j=1}^{A}t(j). \tag{3.2}$$

Note that there is no form factor since the momentum transfer is zero.

It is useful to write $t(j)$ in terms of an isoscalar and isovector component:

$$t(j)=[t_0+t_1\mathbf{i}\cdot\boldsymbol{\tau}(j)]\delta(\mathbf{k}_\pi+\mathbf{k}-\mathbf{k}_\pi'-\mathbf{k}_j'), \tag{3.3}$$

where we use $\mathbf{i}$ for the isospin operator of the pion, and $\boldsymbol{\tau}(j)$ is the usual (Pauli) isospin operator for the jth nucleon; $\mathbf{k}_\pi$ is the pion momentum before the scattering, while $\mathbf{k}_\pi'$ is the pion momentum after scattering, and similarly for the nucleon momenta $\mathbf{k}_j$ and $\mathbf{k}_j'$. The momentum δ function is just the expression of conservation of total momentum of the two interacting particles. For zero relative momentum between the pion and jth nucleon, the coefficients in (3.3) may be obtained from the pion-nucleon scattering lengths for isospin $\frac{1}{2}$ and $\frac{3}{2}$:

$$a_0=-(\mu/2\pi)t_0=\tfrac{1}{3}(a_{1/2}+2a_{3/2}), \tag{3.4a}$$

$$a_1=-(\mu/2\pi)t_1=-\tfrac{1}{3}(a_{1/2}-a_{3/2}). \tag{3.4b}$$

Then Eq. (3.2) may be written in terms of the total nuclear isospin operator

$$\begin{aligned}\langle 0 \mid T \mid 0\rangle&=At_0+\langle 2i\cdot I\rangle t_1\\&=At_0\pm(Z-N)t_1,\end{aligned} \tag{3.5}$$

where the $\pm$ are for $\pi^\pm$ scattering.

The most precise experimental determination of the scattering lengths come from measurements by Bierman[8] and by Fisher and Jenkins[9]:

$$a_{1/2}=0.182\pm0.006\mu^{-1}, \tag{3.6a}$$

$$a_{3/2}=-0.103\pm0.006\mu^{-1}, \tag{3.6b}$$

giving

$$a_0=-0.008\pm0.003\mu^{-1}, \tag{3.7a}$$

$$a_1=-0.095\pm0.003\mu^{-1}. \tag{3.7b}$$

The scattering lengths for several light nuclei, obtained by substituting these values into Eqs. (3.1) and (3.5), are listed in column 1 of Table II. One sees immediately that these results are too small when compared to the

[8] E. Bierman, Phys. Rev. **127**, 599 (1962).
[9] G. E. Fisher and E. W. Jenkins, Phys. Rev. **116**, 749 (1959).

TABLE I. Scattering lengths from experimental level shifts. All scattering-length terms should be multiplied by -1.

Nucleus	ΔE_{1s} (keV)	Seki-Cromer (fm)	Seki (fm)	Linear approx. (fm)
$_3Li^6$	0.60±0.20[a]	0.370±0.100	0.345±0.115	0.304±0.101
	0.35±0.06[b]		0.201±0.034	0.177±0.025
$_3Li^7$	0.80±0.20[a]	0.490±0.100	0.467±0.116	0.402±0.100
	0.57±0.06[b]		0.333±0.035	0.286±0.151
$_4Be^9$	1.75±0.20[a]	0.460±0.280	0.450±0.051	0.368±0.042
	1.63±0.08[b]		0.419±0.021	0.343±0.017
$_5B^{10}$	2.60±0.60[a]	0.530±0.030	0.349±0.079	0.279±0.064
	2.96±0.12[b]		0.397±0.014	0.318±0.011
$_5B^{11}$	2.90±0.70[a]	0.580±0.070	0.398±0.096	0.310±0.075
	3.85±0.12[b]		0.528±0.017	0.412±0.013
$_6C^{12}$	5.80±0.50[a]	0.470±0.040	0.471±0.471	0.358±0.031
	5.96±0.12[b]		0.484±0.009	0.368±0.007
$_7N^{14}$	9.80±1.10[a]	0.590±0.040	0.528±0.060	0.380±0.043
$_8O^{16}$	14.2±1.20[a]	0.560±0.020	0.542±0.046	0.368±0.031
$_9F^{19}$	25.8±1.10[a]	0.730±0.020	0.756±0.032	0.468±0.020
$_{11}Na^{23}$	49.8±1.40[a]	0.920±0.010	0.914±0.026	0.494±0.014

[a] Reference 4. [b] Reference 5.

experimental results in Table I, particularly for $Z=N$ nuclei, which are given by the isoscalar component alone. Therefore, we are led to consider possible corrections to the zero-energy first-order theory we have just presented.

As pointed out by Ericson and Ericson, although the pion is essentially at rest with respect to the nucleus in the pion-atom, the nucleons are moving with respect to the pion. One must correct Eq. (3.3) to include low, but nonzero, momentum effects which come from P-wave scattering. We write, for t_0,

$$t_0=-(2\pi/\mu)[a_0+b_0\mathbf{k}\cdot\mathbf{k}'+c_0\sigma(j)\cdot\mathbf{k}\times\mathbf{k}'], \quad (3.8)$$

with a similar equation for t_1, where $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final relative momenta of the pion and the jth nucleon. For a pion at rest with respect to the nucleus,

$$\mathbf{k}=\mathbf{k}'=[-\xi/(1+\xi)]\mathbf{P}(j), \quad (3.9)$$

where $\mathbf{P}(j)$ is the nucleon momentum and $\xi=\mu/M$ is the ratio of the pion and nucleon masses. The recoil effect is then (for spin-zero nuclei)

$$\Delta T_\pi=-(2\pi/\mu)[\xi/(1+\xi)]^2[b_0+b_1\mathbf{i}\cdot\boldsymbol{\tau}(j)]\langle P^2(j)\rangle. \quad (3.10)$$

Using the values for the coefficients

$$b_0=0.208\pm0.008\mu^{-3}, \quad (3.11a)$$

$$b_1=0.180\pm0.008\mu^{-3}, \quad (3.11b)$$

and an average $\langle P^2(j)\rangle_j$ calculated with harmonic-oscillator ground-state wave functions, we calculate the recoil effect. This correction is of the order of, but of opposite sign to, the single scattering. We list the recoil contributions in Table II.

Multiple Scattering

In general, at low energy, multiple scattering is a small correction to single scattering if $A(a/l)\ll1$, where A is the number of scatters, a is the scattering length, and l is the distance between scatters. In the case of $Z=N$ nuclei, for which only the isoscalar length contributes in first order, we have $a_0/l\sim10^{-2}$, with $l\sim\mu^{-1}$, so that this correction might appear to be small for light nuclei, e.g., $A<20$. However, as will be shown in the following sections, the double scattering in fact also contains a contribution from the isovector length, of order a_1^2/l, which is the same order of magnitude as the single scattering, $a_1^2/l\sim|a_0|$. Multiple scattering

TABLE II. Theoretical scattering lengths (fm). All scattering-length terms should be multiplied by -1.

Nucleus	T_{ss}	$\langle r^{-1}\rangle$ (fm^{-1})	ΔT_2	$T_{ss}+\Delta T_2+\Delta T_{ms}$ Total series	ΔT_b	Recoil	Sum
$_2He^4$	0.041±0.015	0.860	0.097±0.007	0.128±0.025	0.044±0.022	−0.031±0.001	0.141±0.048
$_3Li^6$	0.061±0.023	0.598	0.099±0.007	0.150±0.033	0.066±0.033	−0.050±0.005	0.166±0.070
$_3Li^7$	0.186±0.029	0.770	0.126±0.007	0.278±0.040	0.078±0.039	−0.059±0.005	0.297±0.085
$_4Be^9$	0.206±0.037	0.710	0.173±0.011	0.311±0.050	0.100±0.050	−0.069±0.003	0.342±0.103
$_5B^{10}$	0.100±0.038	0.672	0.181±0.015	0.246±0.055	0.110±0.055	−0.076±0.006	0.280±0.115
$_5B^{11}$	0.225±0.041	0.660	0.194±0.017	0.342±0.060	0.122±0.061	−0.092±0.003	0.372±0.124
$_6C^{12}$	0.120±0.045	0.616	0.197±0.018	0.276±0.065	0.132±0.066	−0.084±0.003	0.324±0.135
$_7N^{14}$	0.139±0.053	0.589	0.218±0.021	0.307±0.077	0.154±0.077	−0.096±0.004	0.365±0.157
$_8O^{16}$	0.159±0.060	0.542	0.227±0.023	0.333±0.085	0.176±0.088	−0.101±0.004	0.408±0.177
$_9F^{19}$	0.304±0.074	0.470	0.200±0.026	0.444±0.105	0.210±0.105	−0.115±0.007	0.539±0.217
$_{11}Na^{23}$	0.344±0.088	0.430	0.223±0.042	0.494±0.135	0.254±0.127	−0.127±0.010	0.621±0.272

therefore provides a major correction to single scattering. This contribution from the isovector part of the pion-nucleon scattering is not included in coherent (ground-state) multiple-scattering theories.

The evaluation of the double-scattering term is broken into two parts. The first part is the usual double-scattering effect and is evaluated in Sec. IV, while the second part is called the binding correction and is evaluated in Sec. V. The binding correction is due to the scattering amplitude for a pion on a single nucleon, when the nucleon is not free, but bound to a nucleus. This may be thought of as a double-scattering effect, in which the bound particle is excited by collision with the pion and then is deexcited by further scattering. The lowest-order contribution (in powers of the scattering length) will be of order a_1^2/l', where l' is some distance of the order of the nuclear size, and may be thought of as the distance the struck nucleon moves between two collisions with the pion. We calculate the binding correction with a simple model for the nuclear binding in Sec. V.

Absorption Effect

Brueckner pointed out that virtual absorption and reemission of pions by the target nucleus contributes to the elastic scattering amplitude. The contribution of absorption plus reemission by a single nucleon is presumably already included in the pion-nucleon amplitude $t(j)$ [Eq. (3.8)]. However, a virtual absorption involving two or more nucleons would contribute separately from the multiple-scattering terms already considered. The contribution of this correction has been crudely estimated to be of the same order as a_0, by Brueckner,[3] by Thouless,[10] and by Ericson and Ericson.[2] We do not treat this process in this paper.

[10] D. J. Thouless, Proc. Phys. Soc. (London) 69, 280 (1956).

IV. MULTIPLE SCATTERING

Now we need a brief review of the multiple-scattering formalism. We define T to be the scattering matrix for a pion scattering on an A-nucleon system (target nucleus) (this satisfies the usual integral equation of Lippmann and Schwinger[11] for scattering):

$$T=V+VGT, \tag{4.1}$$

with the many-body propagator

$$G=(E-H+i\eta)^{-1}, \tag{4.2}$$

$$E=\epsilon_\pi+E_{gs}, \qquad H=K_\pi+H_0. \tag{4.3}$$

K_π is the kinetic energy operator of the pion and H_0 is the Hamiltonian for the A-body system. ϵ_π is the energy of the pion in the center of mass of the A-nucleon system and E_{gs} is the binding energy of the ground state of the A-nucleon system. V is the potential interaction of the pion-A system and is taken in the usual manner to be the sum of the interaction potentials between the pion and each nucleon:

$$V=\sum_j v(j), \tag{4.4}$$

where $v(j)$ is the potential interaction between the pion and the jth nucleon. Taking the matrix element of Eq. (4.1) in any state of the target, gives

$$\langle A\,|\,T\,|\,A\rangle=\langle A\,|\,V\,|\,A\rangle + \sum_N \langle A\,|\,V\,|\,N\rangle\langle N\,|\,G\,|\,N\rangle\langle N\,|\,T\,|\,A\rangle, \tag{4.5}$$

where $|\,A\rangle$ is antisymmetric in all particles. The complete set of intermediate eigenfunction states $|\,N\rangle$ need not be restricted only to the antisymmetric states of

[11] See, e.g., M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley-Interscience, Inc., New York, 1964).

the A-body system. This follows from the fact that V is a symmetric sum in the particles and therefore does not change the symmetry of the initial state $|A\rangle$. Therefore, we choose the set $|N\rangle$ to include all symmetries and to form a complete set, so that

$$\sum_N |N\rangle\langle N| = 1, \tag{4.6}$$

although only the antisymmetric states actually contribute in (4.5). For example, if we consider a single-particle model for the target, then each $|N\rangle$ may be chosen as a product of single-particle wave functions.

The two-body scattering matrix $t(j)$ for the pion and the jth nucleon satisfies an integral equation similar to Eq. (4.1):

$$t(j) = v(j) + v(j)g(j)t(j), \tag{4.7}$$

where

$$g(j) = [E_0 - K_\pi - K(j) + i\eta]^{-1} \tag{4.8}$$

is the propagator of the pion scattering on a free nucleon and is a one-body operator. For the energy we have $E_0 = \epsilon_\pi + E_{K(j)}$, with $E_{K(j)}$ the initial kinetic energy of the nucleon.

An equation for T in terms of $t(j)$ is obtained formally by solving Eq. (4.7) for $v(j)$ and then, with the help of Eq. (4.4), substituting for V into Eq. (4.1). This relationship can be written

$$T = T_{ss} + \Delta T_2 + \Delta T_b + \Delta T_{ms}. \tag{4.9}$$

T_{ss} is the part of the scattering due to a single scattering, and has been seen earlier in Eq. (3.2) as

$$T_{ss} = \sum_{j=1}^{A} t(j). \tag{4.10}$$

ΔT_2 and ΔT_b are both to second order in t and so are double-scattering terms:

$$\Delta T_2 = \sum_{j\neq l}\sum t(j)Gt(l), \tag{4.11a}$$

$$\Delta T_b = \sum_j t(j)[G - g(j)]t(j). \tag{4.11b}$$

ΔT_2 is a conventional double-scattering term in the sense that it involves two nucleons, while ΔT_b involves two scatterings of the pion on the same nucleon. We call ΔT_b the binding correction, because it is induced by the potential which binds the nucleon to the rest of the target. Figure 2 illustrates ΔT_2, and Fig. 3 illustrates ΔT_b. This latter term is treated in Sec. V.

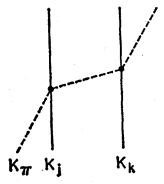

FIG. 2. Double-scattering term in the multiple-scattering series. The dashed line represents the pion line and the solid lines are the two nucleons.

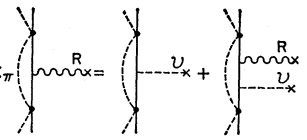

FIG. 3. Representation of the integral equation (5.5) for the binding-energy correction R, where $\mathcal{U}$ is the binding potential. The pion (dashed upgoing line) scatters twice on the same nucleon (solid upgoing line), while the nucleon interacts with the rest of the nuclear system through $\mathcal{U}$ any number of times between the two pion scatterings.

ΔT_{ms} is the remaining correction due to the pion scattering more than twice, or multiple scattering:

$$\Delta T_{ms} = \sum_{j\neq l\neq m}\sum\sum t(j)Gt(l)Gt(m) + \cdots. \tag{4.12}$$

It is treated approximately in the Appendix.

Let us look more closely at the second-order term ΔT_2. In general, in order to evaluate the terms to second order in $t(j)$, one expands in a complete set of intermediate states:

$$\Delta T_2 = \sum_{j\neq l}\sum_N \langle 0|t(j)|N\rangle\langle N|G|N\rangle\langle N|t(l)|0\rangle. \tag{4.13}$$

If it is assumed that we have a single-particle model, then $|N\rangle$ can be chosen to be product states of the single-particle wave functions. As shown earlier, only the antisymmetric part of $|N\rangle$ will contribute. Now $t(l)$ excites the lth particle, so that $|N\rangle$ would have the form, ordered by the particle number,

$$|N_l\rangle = |\alpha_1\rangle|\alpha_2\rangle\cdots|\alpha_j\rangle\cdots|n_l\rangle\cdots|\alpha_A\rangle, \tag{4.14}$$

where the α's are the same quantum numbers as in the ground-state wave functions. Similarly, $t(j)$ de-excites the jth particle so that the intermediate state that it comes from has quantum numbers:

$$|N_j\rangle = |\alpha_1\rangle|\alpha_2\rangle\cdots|n_j\rangle\cdots|\alpha_l\rangle\cdots|\alpha_A\rangle. \tag{4.15}$$

In order for $|N_l\rangle$ and $|N_j\rangle$ to be the same in the intermediate state, we must have $N_l = N_j$, but this violates the conditions on the sum in Eq. (4.13) that $j\neq l$, so we see that no excited states contribute to the matrix element (4.13). The only states that contribute are the states that have the same quantum numbers as the ground state. Because of this, the matrix element of the propagator is

$$\langle N|G|N\rangle = (\epsilon_\pi - K_\pi + E_{gs} - E_{gs} + i\eta)^{-1} = (\epsilon_\pi - K_\pi + i\eta)^{-1}, \tag{4.16}$$

so that $\langle N|G|N\rangle$ is independent of N. Using closure, we get

$$\Delta T_2 = \sum_{j\neq l}\sum \langle 0|\frac{t(j)t(l)}{\epsilon_\pi - K_\pi + i\eta}|0\rangle. \tag{4.17}$$

This can be written

$$\Delta T_2=\sum_{j\neq l}\sum d^3r_1\cdots d^3r_A\psi^*(1,\cdots,A)\times\langle k_\pi\mid M(\mathbf{r}_j;\mathbf{r}_l)\mid k_\pi'\rangle\psi(1,\cdots,A),\quad(4.18)$$

where

$$\langle k_\pi\mid M(\mathbf{r}_j;\mathbf{r}_l)\mid k_\pi'\rangle=-\frac{\mu}{2\pi}\exp[i(\mathbf{k}_\pi\cdot\mathbf{r}_j-\mathbf{k}_\pi'\cdot\mathbf{r}_l)]\times\frac{\exp(ik_0\mid\mathbf{r}_j-\mathbf{r}_l\mid)}{\mid\mathbf{r}_j-\mathbf{r}_l\mid}P_2(j,l),\quad(4.19)$$

$$P_2(j,l)=[t_0+t_1\mathbf{i}\cdot\boldsymbol{\tau}(j)][t_0+t_1\mathbf{i}\cdot\boldsymbol{\tau}(l)],\qquad k_0^2=(2\mu/\hbar^2)\epsilon_\pi.$$

Define $\mathbf{r}_{jl}=\mid\mathbf{r}_j-\mathbf{r}_l\mid$. Then in the limit k_π, k_π', and ϵ_π go to 0, we get

$$\Delta T_2=-\frac{\mu}{2\pi}\sum_{j\neq l}\sum\int d^3r_1\cdots d^3r_A\psi^*(1,\cdots,A)\times\frac{P_2(j,l)}{r_{jl}}\psi(1,\cdots,A).\quad(4.20)$$

If an average $\langle r^{-1}\rangle$ can be used for the matrix element of $(r_{jl})^{-1}$, we get the separation of coordinate space and isospin space. Then

$$\Delta T_2=-(\mu/2\pi)\langle r^{-1}\rangle\sum_{j\neq l}\sum\langle P_2(j;l)\rangle,\quad(4.21a)$$

$$\Delta T_2=-(\mu/2\pi)\langle r^{-1}\rangle\{A(A-1)t_0^2+2(A-1)t_0t_1(2\mathbf{i}\cdot\mathbf{I})+[(2\mathbf{i}\cdot\mathbf{I})^2-A\mathbf{i}^2]t_1^2\}.\quad(4.21b)$$

In this order (second) the average $\langle r^{-1}\rangle$ can be found exactly for each term. For instance, the t_0^2 term just takes the average of all of the pairs of particles, while the t_1^2 term takes the average only between the space-symmetric pairs of particles. In order to evaluate $\langle r^{-1}\rangle$, we choose the single-particle wave functions to be harmonic oscillators, since they can be separated into relative and center-of-mass coordinates using the Moshinsky tables.[12] $\langle r^{-1}\rangle$ should not change much for other choices of single-particle wave functions if the value of $\langle r^2\rangle$ is fitted for these wave functions. We have used electron scattering values of $\langle r^2\rangle$ for the different targets.[13] Because the t_1^2 term is the largest part of ΔT_2 for light nuclei, we calculate the space-symmetric average for $\langle r^{-1}\rangle$. The calculation was done exactly for He^4 and O^{16} and then the $\langle r^{-1}\rangle$ for nuclei between these two was interpolated. This procedure should be accurate enough for an estimate of $\langle r^{-1}\rangle$ since it is a very slowly varying function of nucleon number A. Values of $\langle r^{-1}\rangle$ and ΔT_2 for some light nuclei are listed in Table II. Clearly, in the case $Z=N$, the double scattering is more important than the single scattering. Most of the double-scattering contribution comes from the isovector term t_1^2 in (4.21b).

In the Appendix, it is shown that if one makes the assumption that an effective $\langle r^{-1}\rangle$ can be factored out for each G in (4.12), then the complete multiple-scattering series without binding can be summed in closed form. This summation shows that terms above second order in t are about 10% of the first two terms in the multiple-scattering series, and so are not important. However, these terms have been included in Table II in the partial sum (column 4) and total sum (column 7) for completeness.

V. BINDING EFFECT

We have called the term [in Eq. (4.11b)]

$$\Delta T_b=\sum_j t(j)[G-g(j)]t(j)\quad(5.1)$$

the binding correction. Estimates of Eq. (5.1) were made by Breit[14] and by Lippmann[15] for neutron scattering on the hydrogen molecule. These approaches suffer from the difficulty in calculating the difference of two approximately equal terms, one involving a sum over the excited states of the nucleus (in G), and the other the integral over the recoil states of the nucleon (in g). We shall not use this separation.

We treat the nucleus in a single-particle model. This implies that the nuclear Hamiltonian H_0, which appears in G [Eq. (4.2)], is of the form

$$H_0=\sum_j K(j)+\mathcal{U}(j).\quad(5.2)$$

Then it is clear that (5.1) separates into a sum of terms, each involving only one nucleon. We shall therefore consider each nucleon separately. Note that the Pauli principle does not enter, because we are treating the correction as a one-body problem. Consider the nucleon as bound to the rest of the nuclear system by a potential $\mathcal{U}$ (we drop the label j). Define an R such that

$$G-g=gRg;\quad(5.3)$$

then B, the matrix element in the ground state, is

$$B_j=\langle 0\mid t(j)[G-g(j)]t(j)\mid 0\rangle=\langle 0\mid tgRgt\mid 0\rangle,\quad(5.4)$$

and R satisfies the integral equation

$$R=\mathcal{U}+\mathcal{U}gR.\quad(5.5)$$

R is of the form of a reaction matrix for the scattering of the nucleon by the potential $\mathcal{U}$. Since we shall calculate R exactly in a model, we have avoided the cancellation problem discussed following Eq. (5.1). The integral equation (5.5) is shown diagramatically in Fig. 3, and, for comparison, Fig. 2 shows the diagram for the second-order scattering discussed in Sec. IV. Note that both diagrams are to second order in t, but

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[13] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).
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ΔT_b only deals with one nucleon while ΔT_2 involves two nucleons.

The binding correction problem has now been reduced to a modified three-body problem involving the pion, the nucleon, and a fixed heavy "particle" which gives the potential $\mathcal{U}$ and which represents the rest of the nucleus. This three-body problem is soluble for the case where the interactions are all separable.[16] In our problem, the pion-nucleon interaction is separable since we use a scattering length approximation. We now choose a separable form for the binding potential $\mathcal{U}$, following Yamaguchi[17]:

$$\langle k \mid \mathcal{U} \mid k' \rangle = \lambda u(k) u(k'), \tag{5.6}$$

with

$$u(k) = (q^2+k^2)^{-1}. \tag{5.7}$$

We solve explicitly for R of Eq. (5.3), where λ has been chosen to give a bound state $E=-\mid E_b \mid$, and a wave function, which is given below (in momentum-space representation) in Eq. (5.13). The expectation value (5.4) is taken in this bound state. We obtain nuclear matrix elements of R of the form

$$\langle k \mid R \mid k' \rangle = \gamma(P) u(k) u(k'), \tag{5.8}$$

where k is the momentum of the nucleon in the intermediate state of Eq. (5.4), and where

$$\gamma(P) = (\hbar^2 q/2M\pi^2)[(q+P)^{-2}-(q+p)^{-2}]^{-1}, \tag{5.9}$$

with

$$p^2 = (2M/\hbar^2) \mid E_b \mid, \tag{5.10a}$$

$$P^2 = \xi^{-1} k_\pi^2 + p^2. \tag{5.10b}$$

Note that R is diagonal in k_π; only the nucleon is scattered by $\mathcal{U}$. In this model, the sign of B is always negative, as can be seen from Eq. (5.9). This is the same sign as the first and second terms (T_{ss} and ΔT_2) in the multiple-scattering series, and hence adds to them.

We now reduce Eq. (5.4) to calculable form. We define the following unitless quantities:

$$\rho = p/q, \quad \eta = P/q, \quad x = k_\pi/q, \quad y = k/q, \tag{5.11}$$

where y is the normalized momentum of the intermediate nucleon and x is the normalized momentum of the intermediate pion. Then we have

$$\gamma(x) = -\frac{\hbar^2 \xi q^3}{2M\pi^2} \frac{(1+\rho)^2(1+\eta)^2}{x^2+2\xi(\eta-\rho)}. \tag{5.12}$$

In these formulas the mass M is the nucleon mass. The wave function for the potential we have used is

$$\langle y \mid \psi \rangle = N[(\rho^2+y^2)(1+y^2)]^{-1}, \tag{5.13}$$

with the normalization

$$N^2 = \pi^{-2} q^2 \rho (1+\rho)^3. \tag{5.14}$$

[16] See, e.g., J. Hetherington and L. Schick, Phys. Rev. **137**, B935 (1965); **139**, B1164 (1965).

[17] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

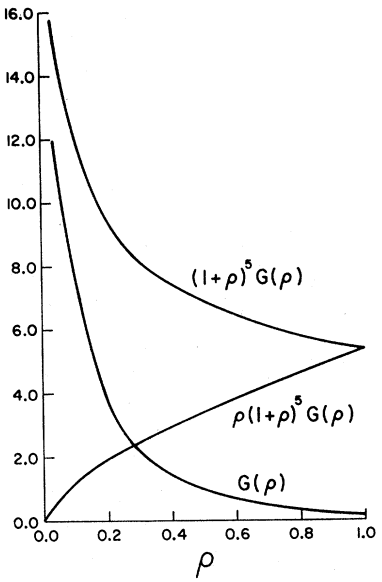

FIG. 4. Graph of integral equation (5.16), $D(\rho)$, and two convenient functional forms as a function of ρ [(Eq. 5.11)].

In the zero-energy limit, the value of B is

$$B_j = -(2/\pi^2) p (1+\rho)^5 D(\rho), \tag{5.15}$$

where

$$D(\rho) = \int_0^\infty \frac{F^2(\rho, x)\, dx}{(1-\eta)^2[x^2+2\xi(\eta-\rho)]}, \tag{5.16}$$

in which

$$F(\rho, x) = \int_0^\infty \frac{y dy}{(1+y^2)(\rho^2+y^2)} \times \ln\left(\frac{[1+(y-x)^2][\eta^2+(y+x)^2]}{[1+(y+x)^2][\eta^2+(y-x)^2]}\right). \tag{5.17}$$

Integrals (5.16) and (5.17) were computed numerically, and $D(\rho)$ is given in graphical form in Fig. 4. It is clear from Fig. 4 that if ρ is kept constant while p goes to 0 (q must also go to 0), then we have B also going to 0, so that in this zero-energy and zero-range limit there is no binding effect.

Two pieces of data are needed to fit the constants in the binding effect. Equation (5.10a) relates the binding energy directly to p. We shall choose ρ [or, equivalently, the range q in (5.11)] to give a fixed value of $\langle r^2 \rangle$, for fixed binding energy.

For example, for He^4 we choose the value $E_b = 20$ MeV, which gives $p = 0.982$ F^{-1}, and $\langle r^2 \rangle = 2.6$ fm^2, which gives $\rho = 0.31$. For these values, $B_j = -0.46$. This is the effect per particle without the isospin. To get the whole effect we sum over the individual particles. This gives

$$\Delta T_b = \sum_j B_j[t_0 + t_1 \mathbf{i} \cdot \boldsymbol{\tau}(j)]^2 = -0.0438\, \mu^{-1}. \tag{5.18}$$

For He^4 this is the same order of magnitude as T_{ss}.

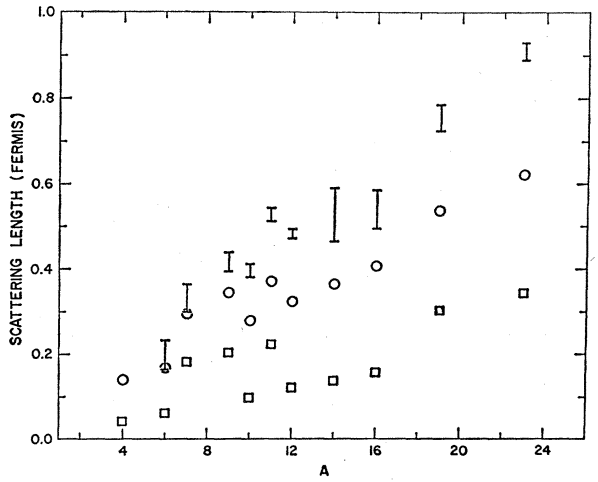

FIG. 5. Comparison of experimental and theoretical results. Bars represent Seki's smoothed radius function using Harris *et al.* (Ref. 5) data where available ($A\leq 12$) and Jenkins *et al.* (Ref. 4) for the rest of the energy shift data. □ is the single-scattering theory or impulse approximation [Eq. (3.5)]. ○ is the present theory with all of the corrections discussed in the text included. (See Table II.)

For all other light nuclei we choose $\rho=-0.46$, which is a good average value, for $E_b=8$ MeV and $\langle r^2\rangle=7$ fm^2. This value was not refitted for each nucleus, since (a) the average single-particle binding E_b is not accurately known and (b) this separable model was calculated for S-state particles. The summed result (ΔT_b) is shown in column 5 of Table II.

VI. RESULTS AND CONCLUSIONS

Our multiple-scattering calculations for nuclei $4\leq A\leq 23$ are summarized in Table II, where we show all the individual contributions to the scattering lengths. Column 1 gives the results of the single-scattering approximation (3.7), which is also displayed in Fig. 5 (open boxes). In Fig. 5, we also show (bars) the experimental scattering lengths as obtained by Seki (Table I, column 3) using the data of Harris *et al.*, where available, and data of Jenkins *et al.* for $A>12$. In Table II, column 7, and also in Fig. 5 (open circles), the total scattering length from our calculations is shown. From Table II, we see that the multiple-scattering corrections are considerably larger than the single-scattering contributions for the $T=0$ systems, and not a negligible contribution for the $T\neq 0$ cases. The total multiple-scattering lengths are much closer to those extracted from experiment than are the single-scattering approximations.

For the $T=0$ systems the binding effect alone is comparable to the single scattering, while double scattering (ΔT_2) is even larger. Both of these terms contribute largely because of the large isovector π-N scattering [see Eq. (3.8)]. It is interesting that both of these isovector contributions are "incoherent" in that they involve excitations of the target out of the ground state. They therefore depend linearly on the nucleon number A rather than as $A(A-1)$, as in the case for coherent double scattering (isoscalar).

It should be pointed out that the separation of the double-scattering and binding terms in (4.11) is somewhat arbitrary, although the sum of the two is well defined. For example, both the forms and numerical values of the two terms would change if we used only antisymmetric states in G, but the total contribution would not change. Our choice of product states leads to the closure result (4.17) for ΔT_2 (in a single-particle model), and to the reduction of ΔT_b to a single-nucleon problem. All excited states are allowed.

It is possible that the remaining disagreement with experiment is in part due to the omission of the absorption-reemission effect of Brueckner. This effect depends on correlations in the target, which we have not treated. A theoretical question which remains is to what extent we may use the scattering-length approximation (3.5) for π-N amplitude in the multiple-scattering series, for which nonzero intermediate pion energies appear.

ACKNOWLEDGMENT

We wish to thank Dr. R. Seki for his helpful discussions and communication of unpublished results.

APPENDIX

Equation (4.9) was written

$$T=T_{ss}+\Delta T_2+\Delta T_b+\Delta T_{ms}. \tag{A1}$$

In Sec. V, it was shown that ΔT_b was small but not negligible. Make the assumption that the particles are weakly bound so that ΔT_b and all terms like ΔT_b in ΔT_{ms} are negligible. Then Eq. (A1) becomes

$$T=\sum_j t(j)+\sum_{j\neq l}\sum t(j)Gt(l)+\sum_{j\neq l\neq m}\sum\sum t(j)Gt(l)Gt(m)\cdots. \tag{A2}$$

It was shown, following Eq. (4.13), that closure was good for the complete product set of states. For higher order in t this is not correct, but we shall assume it here for purposes of estimation. Because of the closure approximation we can factor out an effective $\langle r^{-1}\rangle$ from each term. This leaves Eq. (A1) in the form

$$T=\sum_j t(j)+\langle r^{-1}\rangle\sum_{j\neq l}\sum t(j)t(l)+\langle r\rangle^{-2}\sum_{j\neq l\neq m}\sum\sum t(j)t(l)t(m)\cdots=\sum_{n=1}^{\infty}P_n\langle r^{-1}\rangle^{n-1}, \tag{A3}$$

with

$$P_n=\sum_{j\neq l\neq}\sum\cdots\sum_{\neq p}t(j)t(l)\cdots t(p), \tag{A4}$$

where there are n t's in P_n. Strictly speaking $\langle r^{-1}\rangle$ is

not the same for each term; however, since the second-order term is the largest term in the multiple-scattering series, we will use the $\langle r^{-1}\rangle$ calculated for second order. If we finally assume that only the S-wave part of t contributes, then the series can be summed in closed form.

The first term in the series is just T_{ss} and is given

$$P_1=T_{ss}=\sum_j t(j)=At_0+(2i\cdot I)t_1, \qquad \text{(A5)}$$

where A is the number of nucleons, i is the isospin of the pion, and I is the isospin of the A-nucleon system. The double-scattering term is then

$$P_2-\sum_{J\neq l}\sum t(j)t(l)$$
$$=A(A-1)t_0^2+2(A-1)t_0t_1(2i\cdot I)+[(2i\cdot I)^2-Ai^2]t_1^2. \qquad \text{(A6)}$$

Note that the two-nucleon term P_2 for the case $I=0$ is

$$P_2{}^0=A[(A-1)t_0^2-i^2t_1^2]. \qquad \text{(A7)}$$

The t_1 part dominates P_2 for small nuclei; thus, we observe that P_2 is almost proportional to the number of nuclei for small nucleus.

For higher P_n, a recursive relationship can be found between P_n, P_{n-1}, and P_{n-2}. This relationship is

$$P_n=[(A-2)t_0+(2\mathbf{i}\cdot\mathbf{I})t_1]P_{n-1}+(A-1)(t_0^2-i^2t_1^2)P_{n-2}. \qquad \text{(A8)}$$

Using Eqs. (A3) and (A5)–(A7), the series for T can be summed in closed form. For the case $I=0$, the result of this summation is

$$T=A\,\frac{t_0+\langle r^{-1}\rangle(t_0^2-i^2t_1^2)}{1-(A-2)t_0\langle r^{-1}\rangle-(A-1)(t_0^2-i^2t_1^2)\langle r^{-1}\rangle^2}. \qquad \text{(A9)}$$

As an example, substitute t_0, t_1, A, and $\langle r^{-1}\rangle$ for He^4 into Eq. (A9). We find that the rest of the series $\sum_{n>2}P_n\langle r^{-1}\rangle^{n-1}$ is approximately 10% of $P_1+\langle r^{-1}\rangle P_2$ for this case and is of the opposite sign of these first two terms. Results for other light nuclei are shown in Table II.

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Gamma-Ray Transitions Involving Isobaric-Spin Mixed States in Be^8

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The cross sections for the reactions $Li^7(p,\gamma)Be^{8*}$ (16.63 and 16.90 MeV) $\rightarrow 2\alpha$ have been measured over the range of bombarding energies 0.441–2.45 MeV. The nonresonant portion of the cross section for the population of the 16.63-MeV level has been compared successfully with cross sections calculated on the assumption of an extranuclear direct-capture process. Resonant cross sections populating the 16.63- and 16.90-MeV levels of Be^8 were measured at incident energies of 0.441, 1.03, and 1.89 MeV, corresponding to levels at 17.64, 18.15, and 18.9 MeV in Be^8. Partial widths and reduced matrix elements were calculated for the resonant contributions to the cross sections. The experimental results are consistent with the hypothesis of nearly maximal isobaric-spin mixing in the $J^\pi=2^+$ levels of Be^8 (16.63 and 16.90 MeV). There is no evidence for transitions from the $J^\pi=3^+$ levels near 19 MeV to either of the 16-MeV states, but decay to these states was observed from a level (probably $J^\pi=2^-$) at 18.9 MeV.

I. INTRODUCTION

THE occurrence of isobaric-spin mixing in the $J^\pi=2^+$ Be^8 levels at 16.63 and 16.90 MeV has been firmly established by a review[1] of the experimental evidence. A vital experimental result which led to the proposal of isobaric-spin mixing in these levels[2] was the strong γ-ray transition[3–5] between the $J^\pi=1^+$ Be^8 level at 17.64 MeV (which is predominantly $T=1$) and 16.63-MeV level (which must contain appreciable $T=1$ strength analog to the Li^8 and B^8 ground states). This strong transition, later shown to be primarily an $M1$ transition,[6,7] cannot connect two states of the same

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