dependent parameters is able to reproduce the differential cross sections and polarizations of elastically-scattered protons and neutrons from 1p-shell nuclei. The set of parameters has characteristics similar to the sets of parameters that fit elastic scattering data for heavier nuclei. The numerical systematics of the model differ somewhat between light and heavy nuclei. The Thomas form of the spin-orbit potential has a peculiar behavior for light nuclei which is compounded by the fact that at low energies the calculations are particularly sensitive to its strength. Over a large energy range, the radius parameter must be energy-dependent; and this dependence cannot be compensated by an increased energy dependence in the real potential. These differences, while small, seem to be significant.

It should be pointed out that the parameters of the present analysis are not necessarily the best set of parameters since they were not determined by a rigorous parameter search. The analysis does indicate that such an analysis would be meaningful. While the data used in the present analysis cover a wide range of energies, the measurements were not spaced at regular intervals over this energy range. When a more complete set of data becomes available, a more rigorous analysis can be undertaken.

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Three-Particle Channels in Nuclear-Reaction Theory*

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A new approach to the treatment of three-body channels in nuclear-reaction theory is proposed. The method is based on the R-matrix formalism. Instead of introducing three-particle final states as a new class of channels, it is suggested that they be described in terms of incoherent contributions from the various two-body channels having scattering-state residual-nucleus wave functions instead of the customary bound-state ones. The method is (a) illustrated with a simple one-dimensional three-body system, (b) applied to a general three-body system, and finally (c) used to set up a distorted-wave Born-approximation analysis of the general three-body system.

I. INTRODUCTION

FOR the most part, theoretical treatments of scattering and reactions have been restricted to the regime of two-body channels. While some efforts have been made to find the appropriate three-body channel generalizations,[1] useful methods of general applicability have not been forthcoming.

In this paper, we outline a new approach to the description of three-body channels which appears to be at once practical and completely rigorous. We propose to describe three-particle final states in terms of incoherent contributions from two-body channels for which the internal motion of one of the residual nuclei is a scattering state rather than a bound state. Thus, we do not find it necessary to introduce into the R-matrix formalism[2] a new class of three-particle channels to supplement the usual two-particle ones.

A preview sketch of our method is presented in Sec. II. In Sec. III, we demonstrate the method on a simple one-dimensional three-body system. A general three-particle system is treated in Sec. IV. In Sec. V, we show how our analysis of the three-particle scattering problem can provide the basis for a distorted-wave Born-approximation (DWBA) calculation.

II. PREVIEW OF METHOD

The basis of our analysis is the conventional R-matrix theory scheme for defining channels. The $(3N-3)$-dimensional relative-motion configuration space of a given N nucleon system is separated into an "inside region" and an "asymptotic region" by a large closed $(3N-4)$-dimensional hypersurface, the boundary hypersurface, centered at the center of mass. For the purposes of this analysis, this surface will be taken to be arbitrarily large. When the energy of the system is sufficiently small, the wave function will be found to be negligible everywhere on this boundary surface except at certain small patches. Each such patch corresponds to a partition of the N nucleons into two widely separated clusters. Take the boundary hypersurface to be

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[1] E. Gerjuoy, Ann. Phys. (N.Y.) **5**, 58 (1958); M. Danos and W. Greiner, Z. Physik **202**, 125 (1967); F. S. Levin, Ann. Phys. (N.Y.) **46**, 41 (1968).

[2] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958); L. Garside and W. Tobocman, Phys. Rev. **173**, 1047 (1968).

a polyhedron with a face centered on each such patch. Each face of this polyhedron will then be called the channel entrance for a particular class of channels.

Thus, a partition of the N nucleons into two clusters defines a class of channels and an associated channel entrance on the boundary hypersurface. Individual channels belonging to a particular class are each associated with a "total channel wave function" defined on the corresponding channel entrance. The total channel wave function is an eigenfunction of the total angular momentum resulting from vector coupling together the internal-motion wave functions for particular bound states of each of the two clusters of a given class with a spherical harmonic for the angular part of their relative motion.

When the energy is small, the asymptotic behavior of the system is adequately described in terms of just a few channels in each class. Since each of the two clusters of nucleons is in a bound state, the total channel wave function will vanish exponentially in approaching the outer limits of the channel entrance. This is consistent with the patch where the wave function is non-negligible being small in comparison to the size of the channel entrance. As the energy of the system is increased, these patches will grow in size as it becomes necessary to include additional channels corresponding to more highly excited states of the individual nucleon clusters.

As the energy of the system approaches the threshold for three-body reactions, the description of the asymptotic behavior of the wave function outlined above seems to become inadequate. The patches of non-negligible wave function expand to fill their respective channel entrances as it becomes possible for one of the two nucleon clusters to be in an unbound state. Previous attempts to describe this situation[1] have sought to supplement the finite set of two-body channels with a continuum of three-body channels. This approach has difficulties arising from the lack of orthogonality of the two-body and three-body channel wave functions.

The method we suggest is based on the idea that the framework set up to describe two-body channels can continue to provide a valid characterization of the asymptotic wave function even for energies above the three-body reaction threshold. For each class of two-body channels, we require that the total channel wave functions form a complete orthogonal set normalized over the channel entrance. The total channel wave functions belonging to a given set will thus fulfill homogeneous boundary conditions at the outer edge of their channel entrance. In this way, a complete orthonormal set of wave functions is created which can provide a true representation of the system wave function over the entire boundary polyhedron.

Being defined over a finite (but arbitrarily large) region, the channel wave functions belonging to a given class will form a denumerable set. This set will have a low-energy sparse-in-energy part corresponding to each of the two nucleon clusters being in bound states. It will also have a high-energy dense-in-energy part corresponding to at least one of the two nucleon clusters being in an unbound state. We distinguish between these two types of total channel wave functions by calling the former "bound-state channels" and the latter "scattering-state channels."

In this paper, we show that it is possible to give a natural interpretation of the collision matrix elements associated with scattering-state channels. We find that each open scattering-state channel makes an incoherent (with respect to the contributions of other scattering-state channels) contribution to a three-body reaction. We also show that the calculation of the collision matrix elements for scattering-state channels does not present any special problems.

III. ONE-DIMENSIONAL THREE-BODY SYSTEM

To illustrate our method, we apply it to a very simple three-body system. Suppose we have two particles, N and P, of equal mass and having but one degree of freedom. Let these particles interact with each other and with an infinite mass scattering center by means of short-range potentials. Then in appropriate units ($\hbar^2/2m=1$) the Hamiltonian will be given by

$$H=-(\partial^2/\partial r_N^2)-(\partial^2/\partial r_P^2)+v_N(r_N)+v_P(r_P)+2v_{NP}(|r_N-r_P|). \quad (1)$$

Both v_N and v_P become infinite at the origin. All the potentials vanish beyond certain finite ranges. Let K^2 be the energy of the system. The Shrödinger equation will be

$$\{K^2+(\partial^2/\partial r_N^2)+(\partial^2/\partial r_P^2)-v_N(r_N)-v_P(r_P)-2v_{NP}(|r_N-r_P|)\}\psi_\Gamma(r_N,r_P)=0 \quad (2a)$$

or, equivalently,

$$\{K^2+\tfrac{1}{2}(\partial^2/\partial R^2)+2(\partial^2/\partial r^2)-v_N(R+\tfrac{1}{2}r)-v_P(R-\tfrac{1}{2}r)-2v_{NP}(|r|)\}\chi_\Gamma(R,r)=0, \quad (2b)$$

where

$$R=\tfrac{1}{2}(r_N+r_P),\qquad r=r_N-r_P. \quad (2c)$$

The two-dimensional configuration space of the sysetm is plotted in Fig. 1. The cross-hatched regions contain those points of configuration space where v_N, v_P, and v_{NP} are not all zero. A boundary, separating configuration space into an asymptotic region and an inside region, is formed by the three line segments $r_P=a_P$, $r_N=a_N$, and $R=a_D$. It is understood that

$$b_P=2a_D-a_N \quad (3a)$$

is positive and much greater than the range of v_P.

It is also understood that

$$b_N = 2a_D - a_P \tag{3b}$$

is positive and much greater than the range of v_N. Finally, it is understood that

$$\rho_N = 2(a_N - a_D), \tag{3c}$$

$$\rho_P = 2(a_P - a_D) \tag{3d}$$

are both positive and much greater than the range of v_{NP}. The three segments of this boundary will be called the neutron, proton, and deuteron channel entrances, corresponding to the three classes of two-body channels available to the system.

For each channel entrance, we introduce a complete set of channel wave functions. For the neutron channel entrance, the channel wave functions will be the solutions of the equation

$$(q_{P\alpha}^2 + (d^2/dr_P^2) - v_P)\phi_{P\alpha}(r_P) = 0,$$
$$0 \leq r_P \leq b_P, \qquad \alpha = 1, 2, 3, \cdots. \tag{4a}$$

For the proton channel entrance the channel wave functions will be the solutions of

$$(q_{N\beta}^2 + (d^2/dr_N^2) - v_N)\phi_{N\beta}(r_N) = 0,$$
$$0 \leq r_N \leq b_N, \qquad \beta = 1, 2, 3, \cdots. \tag{4b}$$

And finally, for the deuteron channel entrance the channel wave functions will be

$$(q_{D\gamma}^2 + (d^2/dr^2) - v_{NP})\phi_{D\gamma}(r) = 0,$$
$$-\rho_N \leq r \leq \rho_P, \qquad \gamma = 1, 2, 3, \cdots. \tag{4c}$$

Being orthogonal and defined over finite regions, the channel wave functions have discrete spectra of energy eigenvalues q^2. Each spectrum will be sparse at its low-energy end and dense at its high-energy portion, with a sharp demarcation between the two parts. We will refer to the sparse low-energy spectrum as the bound-state spectrum and to the dense higher-energy spectrum as the scattering-state spectrum.

We regard each channel wave function as being identified with a distinct channel having an associated radial wave function u.

$$u_{P\alpha,\Gamma}(r_N) = \int_0^{b_P} dr_P\, \phi_{P\alpha}(r_P)^* \psi_\Gamma(r_N, r_P), \tag{5a}$$

$$u_{N\beta,\Gamma}(r_P) = \int_0^{b_N} dr_N\, \phi_{N\beta}(r_N)^* \psi_\Gamma(r_N, r_P), \tag{5b}$$

$$u_{D\gamma,\Gamma}(R) = \int_{-\rho_N}^{\rho_P} dr\, \phi_{D\gamma}(r)^* \chi_\Gamma(R, r). \tag{5c}$$

The radial wave functions will reflect free-particle behavior at the channel entrances. The specification of this behavior constitutes the specification of the asymptotic boundary conditions fulfilled by the system wave function ψ_Γ (or χ_Γ).

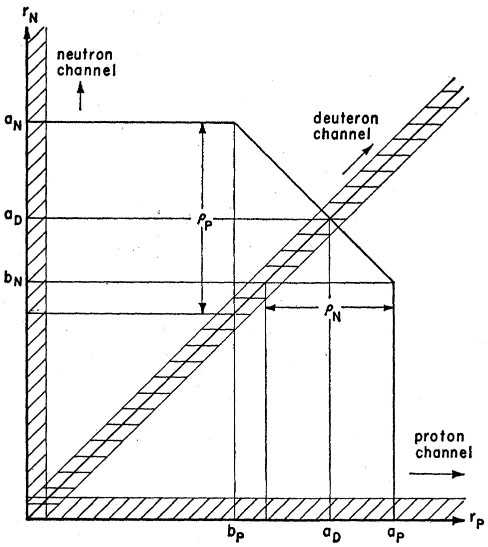

FIG. 1. Configuration-space diagram for a system consisting of two particles, N and P, each having one degree of freedom.

For the sake of definiteness we will consider the case of ground-state deuterons incident on the target. The asymptotic boundary conditions for this case are specified by

$$u_{P\alpha,D1}(a_N) = -(\hbar/2p_{P\alpha})^{1/2} \exp(ip_{P\alpha}a_N)\, U_{P\alpha,D1}, \tag{6a}$$

$$u_{N\beta,D1}(a_P) = -(\hbar/2p_{N\beta})^{1/2} \exp(ip_{N\beta}a_P)\, U_{N\beta,D1}, \tag{6b}$$

$$u_{D\gamma,D1}(a_D) = (\hbar/p_{D\gamma})^{1/2}\{\exp(-ip_{D\gamma}a_D)\,\delta_{\gamma,1} - \exp(ip_{D\gamma}a_D)\, U_{D\gamma,D1}\}, \tag{6c}$$

where the wave numbers p are defined by

$$p_{P\alpha}^2 + q_{P\alpha}^2 = K^2, \tag{7a}$$

$$p_{N\beta}^2 + q_{N\beta}^2 = K^2, \tag{7b}$$

$$\tfrac{1}{2}p_{D\gamma}^2 + 2q_{D\gamma}^2 = K^2. \tag{7c}$$

The parameters that specify the asymptotic boundary conditions are seen to be the elements of the collision matrix U. This matrix is required to be symmetric and "unitary" in the sense described below. It must be possible to express the transition probabilities for all physical processes in terms of the collision matrix.

Strictly speaking, the collision matrix is infinite-dimensional since there are infinitely many channel wave functions. However, there will only be a finite number of open channels, i.e., channels for which $p^2 > 0$. The conservation of flux requirement leads to the "unitarity condition"

$$\delta_{\Gamma,\Sigma} = \sum_{\Lambda} U_{\Lambda,\Gamma} U_{\Lambda,\Sigma}^*, \tag{8}$$

where U is regarded as a finite matrix through the exclusion of all closed channels.

If K^2 is sufficiently small, then all the open channels will be contained in the bound-state part of the channel-wave-function spectrum. This situation corresponds to having only two-body channels open. We are interested in trying to understand how to interpret the collision matrix for higher values of K^2 such that a range of scattering-state channels is included in the set of open channels.

The quantity $|U_{P\alpha,D1}|^2$ is to be interpreted as the outward flux of neutrons associated with protons captured into the $P\alpha$ state when there is unit incident flux in channel $D1$. Thus, the cross section for the (D, N) reaction to the ground state of the residual nucleus is proportional to $|U_{P1,D1}|^2$. *If $\phi_{P\alpha}$ is a scattering state, then $|U_{P\alpha,D1}|^2$ must simply be the flux of neutrons associated with protons that subsequently emerge with energy $q_{P\alpha}^2$.*

Suppose we set up an experiment with an incident unit flux of ground-state deuterons. What will be the emergent flux of neutrons of energy k^2 in the energy interval $\Delta E=2k\Delta k$? One might say that the flux is

$$F_P\Delta E=|U_{P\alpha,D1}|^2\rho_{P\alpha}\Delta E, \tag{9a}$$

$$q_{P\alpha}^2=K^2-k^2>0, \tag{9b}$$

where $\rho_{P\alpha}$ is the density of states in the $\phi_{P\alpha}$ spectrum at energy $q_{P\alpha}^2=K^2-k^2$. $F_P\Delta E$ is the flux of neutrons of energy k^2 which are followed by protons of energy K^2-k^2. But there are additional contributions to be included when $K^2-k^2>0$. We must include

$$F_N\Delta E=|U_{N\beta,D1}|^2\rho_{N\beta}\Delta E, \tag{10a}$$

$$q_{N\beta}^2=k^2, \tag{10b}$$

which is the flux of protons of energy K^2-k^2 which are followed by neutrons of energy k^2. $\rho_{N\beta}$ is the density of states in the $\phi_{N\beta}$ spectrum at energy $q_{N\beta}^2=k^2$. Finally, there is the contribution from the deuteron channel.

$$F_D\Delta E=|U_{D\gamma,D1}|^2\rho_{D\gamma}\Delta E, \tag{11a}$$

$$q_D^2=\tfrac{1}{4}\{k-(K^2-k^2)^{1/2}\}^2 =\tfrac{1}{4}K^2-\tfrac{1}{2}k(K^2-k^2)^{1/2}. \tag{11b}$$

This must be interpreted as the flux of neutrons of energy k^2 that emerge in the company of protons of energy K^2-k^2. $\rho_{D\gamma}$ is the density of states in the $\phi_{D\gamma}$ spectrum at energy $q_{D\gamma}^2=\frac{1}{4}K^2-\frac{1}{2}k(K^2-k^2)^{1/2}$.

The total emerging flux of neutrons is seen to be

$$F\Delta E=(F_N+F_P+F_D)\Delta E =\{|U_{P\alpha,D1}|^2\rho_{P\alpha}+|U_{N\beta,D1}|^2\rho_{N\beta} +|U_{D\gamma,D1}|^2\rho_{D\gamma}\}\Delta E, \tag{12}$$

when these neutrons are associated with unbound protons. We can summarize the description of three-particle channels provided by the formalism in the following manner. *The asymptotic behavior of the scattering wave function is described by a collision matrix that is formulated in terms of two-body channels only. Each class of two-body channels begins to make a contribution to the three-particle reaction when the energy becomes great enough so that the scattering-state two-body channels begin to open up. Each class of two-body channels contributes incoherently to the three-particle flux.*

The density of states ρ depends on the two-body interaction v, and it also depends on the extent of the channel entrance. Consider $\phi_{N\beta}$. At the far channel entrance boundary, $\phi_{N\beta}$ will have the form

$$\phi_{N\beta}(b_N)\sim\sin(q_{N\beta}b_N+\delta_{N\beta}). \tag{13}$$

The precise values of the energy $q_{N\beta}^2$ and the phase shift $\delta_{N\beta}$ will depend on the homogeneous boundary conditions we choose to impose on the $\phi_{N\beta}$ at b_N. In any case, the change of phase of the argument of the sine function in Eq. (13) must be π when β changes by one.

$$(q_{N,\beta+1}-q_{N,\beta})b_N+\delta_{N,\beta+1}-\delta_{N,\beta}=\pi. \tag{14}$$

We can take b_N very large so that the level spacing $\delta E=q_{N,\beta+1}^2-q_{N,\beta}^2$ is very small. Then,

$$\pi=\{(\partial\delta_N/\partial q_N^2)+(b_N/2q_N)\}\delta E. \tag{15}$$

It follows that the level density is

$$\rho_N=1/\delta E=\pi^{-1}\{(\partial\delta_N/\partial q_N^2)+(b_N/2q_N)\}. \tag{16}$$

The presence of the energy derivative of the two-body phase shift constitutes an effect of final-state interactions.[3]

The collision matrix elements $U_{\Lambda,\Gamma}$ can be calculated numerically from expressions provided by R-matrix theory.[2] In certain cases, values for the $U_{\Lambda,\Gamma}$ provided by DWBA calculations may be adequate.

IV. GENERAL CASE OF THREE-BODY SYSTEM

We consider a system consisting of three distinguishable particles which we will label N (neutron), P (proton), and A (α particle). Labels will also be used to identify pairs of particles: $N+P=D$ (deuteron), $N+A=H$(He$-$5), and $P+A=L$(Li$-$5). Let M_B be the mass of particle B and let M_{BC} be the reduced mass of particles B and C. Let $V_{BC}=(\hbar^2/2M_{BC})v_{BC}$ be the (finite-range) potential energy of interaction between particles B and C. We suppose the relative-motion Hamiltonian for our system to be

$$H=T+V, \tag{17a}$$

$$V=V_{NP}+V_{NA}+V_{PA} =(\hbar^2/2)\{M_{NP}^{-1}v_{NP}+M_{NA}^{-1}v_{NA}+M_{PA}^{-1}v_{PA}\}, \tag{17b}$$

$$T=-(\hbar^2/2)\{M_{DA}^{-1}\nabla_{DA}^2+M_{NP}^{-1}\nabla_{NP}^2\} =-(\hbar^2/2)\{M_{PH}^{-1}\nabla_{PH}^2+M_{NA}^{-1}\nabla_{NA}^2\} =-(\hbar^2/2)\{M_{NL}^{-1}\nabla_{NL}^2+M_{PA}^{-1}\nabla_{PA}^2\}. \tag{17c}$$

Here ∇_{BC}^2 is the Laplacian with respect to the coordi-

[3] K. M. Watson, Phys. Rev. **88**, 1163 (1952); G. C. Phillips, T. A. Griffy, and L. C. Biedenharn, Nucl. Phys. **21**, 327 (1960).

nate $\mathbf{r}_{BC}=\mathbf{r}_B-\mathbf{r}_C$. The Schrödinger equation for the system is

$$(E-H)\Psi_\Gamma=0. \tag{18}$$

We introduce a five-dimensional hypersurface in our six-dimensional configuration space separating the asymptotic region from the inside region. This hypersurface will be made up of the three segments which will serve as channel entrances for the three classes of two-body channels. These hypersurface segments are

$$r_{PH}=a_P, \quad \text{(proton ch ent)} \tag{19a}$$

$$r_{NL}=a_N, \quad \text{(neutron ch ent)} \tag{19b}$$

$$r_{DA}=a_D \quad \text{(deuteron ch ent)}. \tag{19c}$$

In each channel entrance, we construct a complete set of two-body eigenstates

$$(q_{L\alpha}^2+\nabla_{PA}^2-v_{PA})\chi_{L\alpha}(\mathbf{r}_{PA})=0, \quad \text{(neutron ch ent)} \tag{20a}$$

$$(q_{H\beta}^2+\nabla_{NA}^2-v_{NA})\chi_{H\beta}(\mathbf{r}_{NA})=0, \quad \text{(proton ch ent)} \tag{20b}$$

$$(q_{D\gamma}^2+\nabla_{NP}^2-v_{NP})\chi_{D\gamma}(\mathbf{r}_{NP})=0 \quad \text{(deuteron ch ent)}. \tag{20c}$$

These functions will fulfill homogeneous boundary conditions at the boundaries of their respective channel entrances. The angular momentum of the two-body eigenstate is then coupled to the spin of the third particle and to the relative orbital angular momentum to form the total channel wave function.

$$\phi_{L\alpha Sl}{}^{JM}=\sum(Slsm\mid JM)(j_{L\alpha}j_N\mu_{L\alpha}\mu_N\mid Ss)\times i^lY_l{}^m(\hat{r}_{NL})\chi_{L\alpha}{}^{j_{L\alpha}\mu_{L\alpha}}\chi_N{}^{j_N\mu_N}, \tag{21a}$$

$$\phi_{H\beta Sl}{}^{JM}=\sum(Slsm\mid JM)(j_{H\beta}j_P\mu_{H\beta}\mu_P\mid Ss)\times i^lY_l{}^m(\hat{r}_{PH})\chi_{H\beta}{}^{j_{H\beta}\mu_{H\beta}}\chi_P{}^{j_P\mu_P}, \tag{21b}$$

$$\phi_{D\gamma Sl}{}^{JM}=\sum(Slsm\mid JM)(j_{D\gamma}j_A\mu_{D\gamma}\mu_A\mid Ss)\times i^lY_l{}^m(\hat{r}_{DA})\chi_{D\gamma}{}^{j_{D\gamma}\mu_{D\gamma}}\chi_A{}^{j_A\mu_A}. \tag{21c}$$

With the help of the total channel wave functions we can define the radial wave functions. To make the notation more compact we will omit the angular momentum indices when they are not needed. Thus, the radial wave functions are defined by

$$u_{L\alpha,\Gamma}(a_{NL})=\langle\phi_{L\alpha}\delta(r_{NL}-a_{NL})\mid\Psi_\Gamma\rangle, \tag{22a}$$

$$u_{H\beta,\Gamma}(a_{PH})=\langle\phi_{H\beta}\delta(r_{PH}-a_{PH})\mid\Psi_\Gamma\rangle, \tag{22b}$$

$$u_{D\gamma,\Gamma}(a_{DA})=\langle\phi_{D\gamma}\delta(r_{DA}-a_{DA})\mid\Psi_\Gamma\rangle. \tag{22c}$$

The configuration-space integrations in the matrix elements that appear in Eq. (22) are understood to be confined to the appropriate channel entrance region of the hypersurface separating the inside and asymptotic regions.

The asymptotic boundary conditions are given by

$$u_{\Lambda,\Gamma}(a_\Lambda)=\zeta_\Lambda{}^{(-)}(a_\Lambda)\delta_{\Lambda,\Gamma}-\zeta_\Lambda{}^{(+)}(a_\Lambda)U_{\Lambda,\Gamma}, \tag{23}$$

where the $\zeta_\Lambda{}^{(-)}(\zeta_\Lambda{}^{(+)})$ are the incoming (outgoing) unit current radial wave functions which have the following forms in the asymptotic region:

$$\zeta_{L\alpha}{}^{(\pm)}(a_{NL})\to(M_{NL}/\hbar k_{L\alpha})^{1/2}i^{\mp l}\exp(\pm ip_{L\alpha}a_{NL}), \tag{24a}$$

$$\zeta_{H\beta}{}^{(\pm)}(a_{PH})\to(M_{PH}/\hbar k_{H\beta})^{1/2}i^{\mp l}\exp(\pm ip_{H\beta}a_{PH}), \tag{24b}$$

$$\zeta_{D\gamma}{}^{(\pm)}(a_{DA})\to(M_{DA}/\hbar k_{D\gamma})^{1/2}i^{\mp l}\exp(\pm ip_{D\gamma}a_{DA}), \tag{24c}$$

and U is the collision matrix. The wave numbers $\{p_\Lambda\}$ are defined by

$$E=(\hbar^2/2)\{M_{NL}{}^{-1}p_{L\alpha}{}^2+M_{PA}{}^{-1}q_{L\alpha}{}^2\}=E_{NL}+E_{PA} \tag{25a}$$

$$=(\hbar^2/2)\{M_{PH}{}^{-1}p_{H\beta}{}^2+M_{NA}{}^{-1}q_{H\beta}{}^2\}=E_{PH}+E_{NA} \tag{25b}$$

$$=(\hbar^2/2)\{M_{DA}{}^{-1}p_{D\gamma}{}^2+M_{NP}{}^{-1}q_{D\gamma}{}^2\}=E_{DA}+E_{NP}, \tag{25c}$$

where E is the total energy of the system.

As before, we interpret $|U_{L\alpha,\Gamma}|^2$ as the flux of emergent neutrons associated with $P+A$ complexes left in the state $L\alpha$ when there is unit flux incident in channel Γ. If the incident beam is a plane wave in channel $H1$ (protons incident on ground-state He$-$5's), then the flux of neutrons emerging in channel $L\alpha$ will be

$$F_{L\alpha}=\sum_{SsM}(2j_P+1)^{-1}(2j_{H1}+1)^{-1}\Big|\sum_{lm}U_{L\alpha S'l',H1Sl}{}^{JM}\times(Slsm\mid JM)Y_l{}^m(\hat{p}_{H1})^*\Big|^2(4\pi^2\hbar/M_{PH}p_{H1}), \tag{26}$$

where $\hat{p}_{H1}$ is a unit vector pointing in the same direction as the wave number $\mathbf{p}_{H1}$ for the relative motion of the incident particle and the target. This expression includes an average over the various possible initial spin states and a sum over final states.

$F_{L\alpha}$ is the flux of neutron+Li pairs which emerge with orbital angular momentum $l'\hbar$ coupled together to form a state of angular momentum $J\hbar$. This quantity is not as useful as the flux of neutron+Li pairs emerging per unit solid angle with relative linear momentum $\mathbf{p}_{L\alpha}\hbar$. To construct such a quantity we need to find the transition rate to a final state which is the appropriate coherent sum of states of the same energy but different values of J, M, l', and S'. The required weighting factor for such a sum is just

$$\sum_{m'}Y_{l'}{}^{m'}(\hat{p}_{L\alpha})(S'l's'm'\mid JM).$$

Thus, the flux of neutrons per unit solid angle emerging in the direction $\hat{p}_{L\alpha}$ is given by

$$\tilde{F}_{L\alpha,H1}(\hat{p}_{L\alpha}\Lambda\hat{p}_{H1})$$

$$=\sum_{S's'Ss}(2j_P+1)^{-1}(2j_{H1}+1)^{-1}\Big|\sum_{lml'm'}Y_{l'}^{m'}(\hat{p}_{L\alpha})$$

$$\times(S'l's'm'\,|\,JM)U_{L\alpha S'l',H1Sl}{}^{JM}$$

$$\times(Slsm\,|\,JM)Y_l{}^m(\hat{p}_{H1})^*\Big|^2(4\pi^2\hbar/M_{PH}p_{H1})$$

$$=(4\pi^2\hbar/M_{PH}p_{H1})F_{L\alpha,H1}(\hat{p}_{L\alpha}\Lambda\hat{p}_{H1}),\qquad(27)$$

so that the differential cross section for the (P,N) reaction is just

$$d\sigma_{L\alpha,H1}(\hat{p}_L\Lambda\hat{p}_{H1})/d\Omega_{NL}=(M_{PH}/\hbar p_{H1})\tilde{F}_{L\alpha,H1}(\hat{p}_{L\alpha}\Lambda\hat{p}_{H1})$$

$$=(4\pi^2/p_{H1}{}^2)F_{L\alpha,H1}(\hat{p}_{L\alpha}\Lambda\hat{p}_{H1}).\qquad(28)$$

The (P,N) differential cross section given above is appropriate for the situation where the state of the residual nucleus $\chi_{L\alpha}$ belongs to the bound-state part of the spectrum. If $\chi_{L\alpha}$ belongs to the continuum, then we must recognize that there will be many states that are nearby in energy and cannot be distinguished by counters with finite resolution in energy. Thus, the sum over final states in F must have a factor

$$\rho_{L\alpha}dE_{NL}=\rho_{L\alpha}d(\hbar^2p_{L\alpha}{}^2/2M_{NL})$$

$$=\pi^{-1}\{(\partial\delta_L/\partial E_{PA})+b_LM_{PA}/\hbar^2q_{L\alpha}\}$$

$$\times d(\hbar^2p_{L\alpha}{}^2/2M_{NL})\qquad(29)$$

included to account for this. The quantity $\rho_{L\alpha}$ is the density of states at energy $\hbar^2q_{L\alpha}{}^2/2M_{PA}$ in the $\chi_{L\alpha}$ energy spectrum. In addition, each energy level of the states $\chi_{L\alpha}$ will be highly degenerate. Each state $\chi_{L\alpha}{}^{j_{L\alpha}\mu_{L\alpha}}$ has the form

$$\chi_{L\alpha}{}^{j_{L\alpha}\mu_{L\alpha}}=\chi_{L\alpha\Sigma\lambda}{}^{j_{L\alpha}\mu_{L\alpha}}$$

$$=\sum(\Sigma\lambda\sigma\nu\,|\,j_{L\alpha}\mu_{L\alpha})i^\lambda Y_\lambda{}^\nu(\hat{r}_{PA})f_{\lambda\Sigma}(r_{PR})r_{PA}{}^{-1}$$

$$\times(j_Pj_A\mu_P\mu_A\,|\,\Sigma\sigma)\chi_P{}^{j_P\mu_P}\chi_A{}^{j_P\mu_A}\qquad(30)$$

for a sufficiently simple choice of V_{PH}. There will be many values of $j_{L\alpha}$, Σ, and λ which have very nearly the same energy.

For our final state, we form a superposition of such states $\chi_{L\alpha\Sigma\lambda}{}^{j_{L\alpha}\mu_{L\alpha}}$ which will correspond to constraining the $P+A$ relative motion to take place in a given direction. Thus, the differential cross section for the $H(P,NP)A$ reaction via the neutron channel is

$$d\sigma_{L\alpha;H1}(\hat{q}_{L\alpha}\Lambda\hat{p}_{L\alpha},\hat{p}_{L\alpha}\Lambda\hat{p}_{H1})/d\Omega_{q_{L\alpha}}d\Omega_{p_{L\alpha}}dE_{NL}$$

$$=(4\pi^2/p_{H1}{}^2)F_{L\alpha,H1}(\hat{q}_{L\alpha}\Lambda\hat{p}_{L\alpha},\hat{p}_{L\alpha}\Lambda\hat{p}_{H1})\rho_{L\alpha},\qquad(31a)$$

$$F_{L\alpha,H1}(\hat{q}_{L\alpha}\Lambda\hat{p}_{L\alpha},\hat{p}_{L\alpha}\Lambda\hat{p}_{H1})$$

$$=\sum_{Ss\Sigma\sigma\mu_N}(2j_P+1)^{-1}(2j_{H1}+1)^{-1}\Big|\sum_{\substack{lml'm'\lambda\nu\\S's'JMj_L\mu_L}}Y_\lambda{}^\nu(\hat{q}_{L\alpha})$$

$$\times(\Sigma\lambda\sigma\nu\,|\,j_L\mu_L)(j_Lj_N\mu_L\mu_N\,|\,S's')$$

$$\times Y_{l'}{}^{m'}(\hat{p}_{L\alpha})(S'l's'm'\,|\,JM)U_{L\alpha\Sigma\lambda S'l',H1Sl}{}^{JM}$$

$$\times Y_l{}^m(\hat{p}_{H1})^*(Slsm\,|\,JM)\Big|^2.\qquad(31b)$$

This differential cross section is not yet in the form we need for comparison with experiment. The solid angle $d\Omega_{q_{L\alpha}}$ refers to a counter in the center-of-mass frame of the $P+A=L$ system rather than to the center-of-mass frame for the three-body system. In the three-body center-of-mass frame,

$$\mathbf{k}_N=\mathbf{p}_{L\alpha}\qquad(32a)$$

is the wave number of the emerging neutrons discussed above. The associated protons emerge with a three-body center-of-mass-frame wave number of

$$\mathbf{k}_P=\mathbf{q}_{L\alpha}-(M_P/M_L)\mathbf{p}_{L\alpha}.\qquad(32b)$$

To transform the cross section into a form which refers to solid angles with respect to the directions of $\mathbf{k}_N$ and $\mathbf{k}_P$ instead of those of $\mathbf{p}_{L\alpha}$ and $\mathbf{q}_{L\alpha}$, we need only multiply the cross section by a factor w.

$$d\sigma_{La,H1}(\hat{k}_P\Lambda\hat{p}_{H1},\hat{k}_N\Lambda\hat{p}_{H1})/d\Omega_Pd\Omega_NdE_N$$

$$=(4\pi^2/p_{H1}{}^2)F_{La,H1}(\hat{q}_{L\alpha}\Lambda\hat{p}_{L\alpha},\hat{p}_{L\alpha}\Lambda\hat{p}_{H1})\rho_{L\alpha}w_L,\qquad(33a)$$

$$E_N=\hbar^2k_N{}^2/2M_N.\qquad(33b)$$

The explicit expression for the weight factor w_L is given in the Appendix.

Equation (33) is just the neutron channel contribution to the $H(P,PN)A$ cross section. There will be contributions from the proton channel and deuteron channel as well. These three contributions are simply additive; there is no interference since there is no overlap of total channel wave functions.

The proton channel contribution to the $H(P,PN)A$ differential cross section is

$$d\sigma_{Hb,H1}(\hat{k}_P\Lambda\hat{p}_{H1},\hat{k}_N\Lambda\hat{p}_{H1})/d\Omega_Pd\Omega_NdE_N$$

$$=(4\pi^2/p_{H1}{}^2)F_{Hb,H1}(\hat{q}_{H\beta}\Lambda\hat{p}_{H1},\hat{p}_{H\beta}\Lambda\hat{p}_{H1})\rho_{H\beta}w_H,\qquad(34a)$$

$$F_{Hb,H1}(\hat{q}_{H\beta}\Lambda\hat{p}_{H1},\hat{p}_{H\beta}\Lambda\hat{p}_{H1})$$

$$=\sum_{Ss\Sigma\sigma\mu_P}(2j_P+1)^{-1}(2j_{H1}+1)^{-1}\Big|\sum_{\substack{lml'm\lambda\nu\\S's'JMj_H\mu_H}}Y_\lambda{}^\nu(\hat{q}_{H\beta})$$

$$\times(\Sigma\lambda\sigma\nu\,|\,j_H\mu_H)(j_Hj_P\mu_H\mu_P\,|\,S's')$$

$$\times Y_{l'}{}^{m'}(\hat{p}_{H\beta})(S'l's'm'\,|\,JM)U_{Hb\Sigma\lambda S'l',H1Sl}{}^{JM}$$

$$\times Y_l{}^m(\hat{p}_{H1})^*(Slsm\,|\,JM)\Big|^2.\qquad(34b)$$

The expression for w_H is found in the Appendix. The wave numbers are defined by

$$\mathbf{p}_{H\beta}=\mathbf{k}_P, \quad (34c)$$

$$\mathbf{q}_{H\beta}=\mathbf{k}_N+M_N M_H^{-1}\mathbf{k}_P. \quad (34d)$$

The deuteron channel contribution to the $H(P, PN)A$ differential cross section is

$$d\sigma_{Dc,H1}(\hat{k}_P\Lambda\hat{p}_{H1}, \hat{k}_N\Lambda\hat{p}_{H1})/d\Omega_P d\Omega_N dE_N$$

$$=(4\pi^2/p_{H1}^2)F_{Dc,H1}(\hat{q}_{D\gamma}\Lambda\hat{p}_{H1}, \hat{p}_{D\gamma}\Lambda\hat{p}_{H1})\rho_{D\gamma}w_D, \quad (35a)$$

$$F_{Dc,H1}(\hat{q}_{D\gamma}\Lambda\hat{p}_{H1}, \hat{p}_{D\gamma}\Lambda\hat{p}_{H1})$$

$$=\sum_{Ss\Sigma\sigma\mu_A}(2j_P+1)^{-1}(2j_{H1}+1)^{-1}\Big|\sum_{\substack{lml'm'\lambda\nu\\ S's'j_D\mu_D JM}}Y_\lambda{}^\nu(\hat{q}_{D\gamma})$$

$$\times(\Sigma\lambda\sigma\nu\,|\,j_D\mu_D)(j_D j_A\mu_D\mu_A\,|\,S's')$$

$$\times Y_{l'}{}^{m'}(\hat{p}_{D\gamma})(S'l's'm'\,|\,JM)U_{Dc\Sigma\lambda S'l',H1Sl}{}^{JM}$$

$$\times Y_l{}^m(\hat{p}_{H1})^*(Slsm\,|\,JM)\Big|^2. \quad (35b)$$

The expression for w_D is found in the Appendix. The wave numbers are defined by

$$\mathbf{p}_{D\gamma}=\mathbf{k}_N+\mathbf{k}_P, \quad (35c)$$

$$\mathbf{q}_{D\gamma}=M_P M_D^{-1}\mathbf{k}_N-M_N M_D^{-1}\mathbf{k}_P. \quad (35d)$$

The complete expression for the $H(P, PN)A$ differential cross section is thus

$$d\sigma_{NP,H1}/d\Omega_N d\Omega_P dE_N$$

$$=(d\sigma_{La,H1}+d\sigma_{Hb,H1}+d\sigma_{Dc,H1})/d\Omega_N d\Omega_P dE_N. \quad (36)$$

V. COLLISION MATRIX IN DWBA

We have seen how the cross section for scattering into three-body final states can be written in terms of collision matrix elements for transitions between two-body channels. These elements of the collision matrix differ from the usual sort in that they refer to channel states which are scattering states instead of bound states. What we have called scattering-state channel wave functions are really discrete spectrum wave functions which satisfy homogeneous boundary conditions at the boundary of the channel entrance. By scattering-state functions we mean that these channel wave functions have oscillatory behavior near the boundary of the channel entrance. The bound-state channel wave functions are those which decay exponentially as they approach the boundary of the channel entrance.

Let us use the representation of the collision matrix provided by the extended R-matrix formalism.[4]

$$U_{\Lambda,\Gamma}=\exp(i\delta_\Lambda)\{(1+i\mathcal{K}^{(+)})^{-1}(1-i\mathcal{K}^{(-)})\}_{\Lambda,\Gamma}\exp(i\delta_\Gamma), \quad (37a)$$

$$\mathcal{K}_{\Lambda,\Gamma}{}^{(\pm)}=\langle\phi_\Lambda r_\Lambda^{-1}x_\Lambda(r_\Lambda)\,|\,X_{\Lambda,\Gamma}\,|\,\phi_\Gamma r_\Gamma^{-1}x_\Gamma(r_\Gamma)\rangle(1\mp is_\Gamma), \quad (37b)$$

$$X_{\Lambda,\Gamma}=V_\Lambda+V_\Lambda(E-H)^{-1}V_\Gamma. \quad (37c)$$

[4] L. Garside and W. Tobocman, Ann. Phys. (N.Y.) 53, 115 (1969).

Here ϕ_Λ is the channel wave function for channel $\Lambda=L\alpha$. Let $\mathcal{H}_L$ be the Hamiltonian of which ϕ_Λ is eigenfunction. Let $\mathcal{T}_L$ be the kinetic energy operator for the relative radial motion of the two nuclides that constitute channel L. Let $\mathcal{V}_{L\alpha}(\mathbf{r}_{L\alpha})$ be an optical potential for the relative motion in channel $L\alpha$. Then,

$$H_{L\alpha}=\mathcal{H}_L+\mathcal{T}_L+\mathcal{V}_{L\alpha} \quad (38a)$$

is the channel Hamiltonian, and

$$V_{L\alpha}=H-H_{L\alpha} \quad (38b)$$

is the interaction in channel $L\alpha$. The radial wave function x_Λ is the regular solution of

$$(E-H_\Lambda)\phi_\Lambda r_\Lambda^{-1}x_\Lambda(r_\Lambda)=0. \quad (39)$$

The optical-model phase shift δ_Γ characterizes the asymptotic behavior of x_Γ

$$x_\Gamma(r)\rightarrow(2M_\Gamma/\hbar^2 p_\Gamma)^{1/2}\sin(p_\Gamma r-l_\Gamma\pi/2+\delta_\Gamma). \quad (40)$$

Finally, the parameters s_Γ characterize the asymptotic behavior of the R-matrix Green's-function operator $R=(E-H)^{-1}$.

$$a_\Gamma(d/da_\Gamma)\ln\langle\phi_\Lambda\delta(a_\Lambda-r_\Lambda)\,|\,(E-H)^{-1}\,|\,\phi_\Gamma\delta(a_\Gamma-r_\Gamma)\rangle$$

$$=\frac{-p_\Gamma a_\Gamma[1+s_\Gamma\cot(p_\Gamma a_\Gamma-\frac{1}{2}l_\Gamma\pi+\delta_\Gamma)]}{[\cot(p_\Gamma a_\Gamma-\frac{1}{2}l_\Gamma\pi+\delta_\Gamma)-s_\Gamma]}, \quad (41)$$

a_Γ is the channel radius of channel Γ.

This exact representation for the collision matrix permits us to see very explicitly how the nature of the channel wave functions ϕ_Λ will affect the collision matrix. Clearly, it is simply a question of using scattering-state wave functions in place of bound-state wave functions in the evaluation of certain matrix elements.

To pursue the matter further, let us make the first-order approximation

$$X_{\Lambda,\Gamma}\approx V_\Gamma, \quad (42)$$

which might be accurate at sufficiently high energy. If we choose $s_\Gamma=-i$, then R becomes a scattering Green's-function operator, and the first-order approximation is essentially the DWBA. On the other hand, if we choose $s_\Gamma=0$, then $\mathcal{K}^{(\pm)}=\mathcal{K}$ becomes the K matrix and unitarity is preserved even in the first-order approximation.

To simplify discussion we assume that all two-body interactions are channel spin and orbital angular momentum scalars. We also set $s_\Gamma=-i$. Now let us examine the various elements of the K matrix. Since $s_\Gamma=-i$, $\mathcal{K}_{\Lambda,\Gamma}{}^{(+)}=0$. For the others,

$$\mathcal{K}_{L\alpha,H1}{}^{(-)}=2\langle\Phi_{L\alpha}\,|\,V_{NP}+V_{NA}-\mathcal{V}_{NL}\,|\,\Phi_{H1}\rangle, \quad (43a)$$

$$\mathcal{K}_{H\beta,H1}{}^{(-)}=2\langle\Phi_{H\beta}\,|\,V_{NP}+V_{PA}-\mathcal{V}_{PH}\,|\,\Phi_{H1}\rangle, \quad (43b)$$

$$\mathcal{K}_{D\gamma,H1}{}^{(-)}=2\langle\Phi_{D\gamma}\,|\,V_{NA}+V_{PA}-\mathcal{V}_{DA}\,|\,\Phi_{H1}\rangle, \quad (43c)$$

where

$$\Phi_\Lambda=\phi_\Lambda r_\Lambda^{-1}x_\Lambda(r_\Lambda). \quad (43d)$$

Using the post-prior equivalence, we can replace Eq.

(43c) by

$$\mathcal{K}_{D\gamma,H1}^{(-)}=2\langle\Phi_{D\gamma}\mid V_{NP}+V_{PA}-\mathcal{V}_{PH}\mid\Phi_{H1}\rangle. \qquad (43c')$$

Now we make the assumption that

$$V_{NA}-\mathcal{V}_{NL}\approx 0, \qquad (44a)$$

$$V_{PA}-\mathcal{V}_{PH}\approx 0, \qquad (44b)$$

which would be justified if $M_A \gg M_N$, M_P. Next we approximate V_{NP} by a zero-range potential,

$$V_{NP}=V_0R^3\delta(\mathbf{r}_N-\mathbf{r}_P). \qquad (45)$$

On the basis of these assumptions and Eqs. (21), (30), and (43) we find

$$\mathcal{K}_{L\alpha\Sigma'\lambda'S'l',H1\Sigma\lambda S}^{(-)JM}=2W_LM_L, \qquad (46a)$$

$$W_L=\sum_{K,k,m,\nu,\cdots}(S'l's'm'\mid JM)(j_Lj_N\mu_L\mu_N\mid S's')(\Sigma'\lambda'\sigma'\nu'\mid j_{L\alpha}\mu_{L\alpha})(j_Pj_A\mu_P\mu_A\mid\Sigma'\sigma')(Slsm\mid JM)$$
$$\times(j_Nj_P\mu_H\mu_P\mid Ss)(\Sigma\lambda\sigma\nu\mid j_{H1}\mu_{H1})(j_Nj_A\mu_N\mu_A\mid\Sigma\sigma)(l'\lambda'm'\nu'\mid Kk)(l\lambda m\nu\mid Kk)$$
$$\times(l'\lambda'00\mid K0)(l\lambda00\mid K0)\left[\frac{(2l'+1)(2\lambda'+1)(2l+1)(2\lambda+1)}{(4\pi)^2(2K+1)^2}\right]^{1/2}$$
$$=\sum_{K,f,g,h}\begin{Bmatrix}j_N & j_{L\alpha} & S'\\ J & l' & f\end{Bmatrix}\begin{Bmatrix}\Sigma' & j_{L\alpha} & \lambda'\\ K & l' & f\end{Bmatrix}\begin{Bmatrix}j_P & j_{H1} & S\\ J & l & g\end{Bmatrix}\begin{Bmatrix}\Sigma & j_{H1} & \lambda\\ K & l & g\end{Bmatrix}\begin{Bmatrix}h & j_P & \Sigma\\ K & g & J\end{Bmatrix}\begin{Bmatrix}h & j_N & \Sigma'\\ K & f & J\end{Bmatrix}\begin{Bmatrix}j_P & \Sigma' & j_A\\ \Sigma & j_N & h\end{Bmatrix}$$
$$\times\left[\frac{(2j_{L\alpha}+1)(2j_{H1}+1)(2h+1)(2l+1)(2l'+1)}{(4\pi)^2(2j_A+1)(2J+1)^2}\right]^{1/2}$$
$$\times(l'\lambda'00\mid K0)(l\lambda00\mid K0)(-1)^{\Sigma+\Sigma'+S+S'-j_N-j_P-2j_{L\alpha}-2j_{H1}-l-l'}, \qquad (46b)$$

$$M_L=R^3V_0\int_0^\infty dr\,r^2X_{l'S'}{}^{La}\left(\frac{M_A}{M_L}r\right)F_{\lambda'\Sigma'}{}^{La}(r)X_{lS}{}^{H1}\left(\frac{M_A}{M_H}r\right)F_{\lambda\Sigma}{}^{H1}(r), \qquad (46c)$$

where

$$X_\Lambda(r)=x_\Lambda(r)/r, \qquad (47a)$$

$$F_\Lambda(r)=f_\Lambda(r)/r, \qquad (47b)$$

and

$$W(abde;cf)=\begin{Bmatrix}a & b & c\\ d & e & f\end{Bmatrix}\Big/[(2c+1)(2f+1)]^{1/2} \qquad (47c)$$

is the Racah W coefficient.[5]

$$\mathcal{K}_{Hb\Sigma'\lambda'S'l',H1\Sigma\lambda Sl}^{(-)JM}=2W_HM_H, \qquad (48a)$$

$$W_H=\sum_{K,k,m,\nu,\cdots}(S'l's'm'\mid JM)(j_{H\beta}j_P\mu_{H\beta}\mu_P\mid S's')(\Sigma'\lambda'\sigma'\nu'\mid j_{H\beta}\mu_{H\beta})(j_Nj_A\mu_N\mu_A\mid\Sigma'\sigma')(Slsm\mid JM)$$
$$\times(j_{H1}j_P\mu_{H1}\mu_P\mid Ss)(\Sigma\lambda\sigma\nu\mid j_{H1}\mu_{H1})(j_Nj_{\mu_N}\mu_A\mid\Sigma\sigma)(l'\lambda'm'\nu'\mid Kk)(l\lambda m\nu\mid Kk)$$
$$\times(l'\lambda'00\mid K0)(l\lambda00\mid K0)\left[\frac{(2l'+1)(2l+1)(2\lambda'+1)(2\lambda+1)}{(4\pi)^2(2K+1)^2}\right]^{1/2}$$
$$=\sum_{K,f}\begin{Bmatrix}j_P & j_{H\beta} & S'\\ J & l' & f\end{Bmatrix}\begin{Bmatrix}\Sigma & j_{H\beta} & \lambda'\\ K & l' & f\end{Bmatrix}\begin{Bmatrix}j_P & j_{H1} & S\\ J & l & f\end{Bmatrix}\begin{Bmatrix}\Sigma & j_{H1} & \lambda\\ K & l & f\end{Bmatrix}\delta_{\Sigma,\Sigma'}\left[\frac{(2j_{H\beta}+1)(2j_{H1}+1)(2l'+1)(2l+1)}{(4\pi)^2(2f+1)^2}\right]^{1/2}$$
$$\times(l'\lambda'00\mid K0)(l\lambda00\mid K0)(-1)^{2\Sigma+S'+S-2j_P-2j_{H\beta}-2j_{H1}-l-l'}, \qquad (48b)$$

$$M_H=R^3V_0\int_0^\infty dr\,r^2X_{l's'}{}^{H\beta}\left(\frac{M_A}{M_H}r\right)F_{\lambda'\Sigma'}{}^{H\beta}(r)X_{ls}{}^{H1}\left(\frac{M_A}{M_H}r\right)F_{\lambda\Sigma}{}^{H1}(r), \qquad (48c)$$

$$\mathcal{K}_{D_c\Sigma'\lambda'S'l',H1\Sigma\lambda l}^{(-)JM}=2W_DM_D, \qquad (49a)$$

[5] L. C. Biedenharn, J. M. Blatt, and M. E. Rose, Rev. Mod. Phys. **24**, 249 (1952).

$$W_D = \sum_{K,k,m,\nu,\cdots} (S'l's'm' \mid JM)(j_{D\gamma} j_A \mu_{D\gamma} \mu_A \mid S's')(\Sigma'\lambda'\sigma'\nu' \mid j_{D\gamma}\nu_{D\gamma})(j_N j_P \mu_N \mu_P \mid \Sigma'\sigma')(Slsm \mid JM)$$

$$\times (j_{H1} j_P \mu_{H1} \mu_P \mid Ss)(\Sigma\lambda\sigma\nu \mid j_{H1}\mu_{H1})(j_N j_A \mu_N \mu_A \mid \Sigma\sigma)(l'\lambda'm'\nu' \mid Kk)(l\lambda m\nu \mid Kk)$$

$$\times (l'\lambda'00 \mid K0)(l\lambda 00 \mid K0)\left[\frac{(2l'+1)(2l+1)(2\lambda'+1)(2\lambda+1)}{(4\pi)^2(2K+1)^2}\right]^{1/2}$$

$$= \sum_{K,f,g,h} \begin{Bmatrix} j_A & j_{D\gamma} & S' \\ J & l' & f \end{Bmatrix} \begin{Bmatrix} \Sigma' & j_{D\gamma} & \lambda' \\ K & l' & f \end{Bmatrix} \begin{Bmatrix} j_P & j_{H1} & S \\ J & l & g \end{Bmatrix} \begin{Bmatrix} \Sigma & j_{H1} & \lambda \\ K & l & g \end{Bmatrix} \begin{Bmatrix} h & j_A & \Sigma' \\ K & f & J \end{Bmatrix} \begin{Bmatrix} h & j_P & \Sigma \\ K & g & J \end{Bmatrix} \begin{Bmatrix} j_P & \Sigma' & j_N \\ \Sigma & j_A & h \end{Bmatrix}$$

$$\times \left[\frac{(2j_{D\gamma}+1)(2j_{H1}+1)(2l'+1)(2l+1)(2h+1)}{(4\pi)^2(2J+1)^2(2j_N+1)}\right]^{1/2}$$

$$\times (l'\lambda'00 \mid K0)(l\lambda 00 \mid K0)(-1)^{S+S'+2h-2j_{D\gamma}-2j_{H1}-l-l'}, \quad (49b)$$

$$M_H = R^3 V_0 F_{\lambda'\Sigma'}{}^{D\gamma}(0) \int_0^\infty dr\, r^2 X_{l'S'}{}^{D\gamma}(r) F_{\lambda\Sigma}{}^{H1}(r) X_{lS}{}^{H1}\left(\frac{M_A}{M_H} r\right). \quad (49c)$$

$X^{L\alpha}$ is the radial wave function for the relative motion of N and the nuclide L. $F^{L\alpha}$ is the radial wave function for the relative motion of P and A in the nuclide $L\alpha$. The normalization of the X's is fixed by Eqs. (40) and (47a). F^{H1} is a bound-state wave function. Its presence in the integrand justifies our treating the integral over the inside region as an integral over all configuration space. The other F's are scattering wave functions in the sense already discussed. Their normalization is fixed by the requirement

$$1 = \int_0^{b_\Lambda} dr\, r^2 F_\Lambda(r)^2. \quad (50)$$

This requirement and the fact that b_Λ is very large tells us that the scattering-state F's have the following asymptotic behavior:

$$F_\Lambda(r) \to (2/b_\Lambda)^{1/2} r^{-1} \sin(q_\Lambda r - \lambda\pi/2 - \delta_\Lambda). \quad (51)$$

In the expression for the cross section, the square of the matrix element in which the scattering state F_Λ appears will be multiplied by the density of channel-wave-function states factor

$$\rho_\Lambda = \pi^{-1}\{(\partial\delta_\Lambda/\partial E_\Lambda) + (b_\Lambda M_\Lambda/\hbar^2 q_\Lambda)\}, \quad (52a)$$

$$E_\Lambda = \hbar^2 q_\Lambda^2 / 2M_\Lambda. \quad (52b)$$

In the limit as b_Λ becomes very large, the contribution of $\partial\delta_\Lambda/\partial E_\Lambda$ becomes negligible. Since all the b_Λ's will be chosen to be very large, we can set

$$\rho_\Lambda = \pi^{-1}, \quad (53a)$$

$$F_\Lambda(r) \to (2M_\Lambda/\hbar^2 q_\Lambda)^{1/2} r^{-1} \sin(q_\Lambda r - \lambda\pi/2 + \delta_\Lambda). \quad (53b)$$

Thus, all explicit dependence on the $\{b_\Lambda\}$, which serve to measure the extent of the channel entrances, disappears.

We see that the final-state-interaction effects do not arise from the $\partial\delta/\partial E$ term in ρ. Instead, final-state-interaction effects enter the formalism by way of the energy dependence of the scattering-state F_Λ's.

Since we have assumed that V_{NP} is zero range, the appropriate expression for $F_{\lambda'\Sigma'}{}^{D\gamma}(0)$ which appears in Eq. (49c) is

$$F_{\lambda'\Sigma'}{}^{D\gamma}(0) = (2M_{NP}/\hbar^2 q_{D\gamma})^{1/2} R^{-1} (\sin\delta_{\Sigma'}{}^{D\gamma})\, \delta_{\lambda',0}. \quad (54)$$

From the example just considered, it is seen that for the $H(P, NP)A$ reaction, the use of scattering-state channel wave functions causes no special difficulty. In particular, the radii $\{b_\Lambda\}$ which measure the extents of the channel entrances and the channel radii $\{a_\Lambda\}$ do not appear in the final expressions.

In the case of the $A(D, NP)A$ reaction, we encounter the familiar difficulty that plagues Born-approximation treatments of stripping to unbound states. All the radial wave functions that appear in the radial integrals M_L, M_H, and M_D are scattering wave functions. The absence of a convergence factor in the integrand causes some ambiguity in the value of the integral. However, these integrals can be interpreted if we follow the approach of Huby and Mines.[6]

APPENDIX

Suppose we have two pairs of vectors related by a linear transformation:

$$\mathbf{p} = a\mathbf{k}_N + b\mathbf{k}_P, \quad (A1a)$$

$$\mathbf{q} = c\mathbf{k}_N + d\mathbf{k}_P. \quad (A1b)$$

We seek a weight factor w which will serve to relate the differentials

$$2\alpha p\, dp\, d\Omega_p\, d\Omega_q = w\hbar^2 M_N^{-1} k_N\, dk_N\, d\Omega_N\, d\Omega_P, \quad (A2)$$

where Ω_q is the solid angle related to the orientation of the vector $\mathbf{q}$ and Ω_N is similarly related to $\mathbf{k}_N$. To

[6] R. Huby and J. R. Mines, Rev. Mod. Phys. **37**, 406 (1965).

do this, start by considering the element

$$dv = d\Omega_p p^2 dp d\Omega_q \int dq\, q^2 \delta(\alpha p^2+\beta q^2-E)\, \frac{2\beta}{qp}\frac{\alpha 2M_N}{\hbar^2}$$

$$= p dp d\Omega_p d\Omega_q \alpha 2M_N \hbar^{-2}. \quad \text{(A3)}$$

By requiring the transformation to be unimodular,

$$ad-bc=\pm 1, \quad \text{(A4)}$$

we force the Jacobian of the transformation to be unity so that

$$dv = d\Omega_N k_N^2 dk_N d\Omega_P \int dk_P k_P^2 \delta(\alpha p^2+\beta q^2-E)\, \frac{2\beta}{qp}\frac{\alpha 2M_N}{\hbar^2}. \quad \text{(A5)}$$

Comparison of Eqs. (2), (3), and (5) shows the weight factor to be

$$w = \int dk_P \frac{4\alpha\beta M_N}{\hbar^2}\frac{k_P^2 k_N}{qk}\,\delta(\alpha p^2+\beta q^2-E) \quad \text{(A6a)}$$

$$= (k_N k_P^2/qp2Py)(2\alpha\beta M_N/\hbar^2),$$

where

$$k_P = P - y^{-1} z k_N \cos\theta_{NP}, \quad \text{(A6b)}$$

$$\cos\theta_{NP} = (\mathbf{k}_N\cdot\mathbf{k}_P)/k_N k_P, \quad \text{(A6c)}$$

$$P = \{y^{-1}E - y^{-1}x k_N^2 + y^{-2}z^2 k_N^2 \cos^2\theta_{NP}\}^{1/2}, \quad \text{(A6d)}$$

$$y = \alpha b^2 + \beta d^2, \quad \text{(A6e)}$$

$$x = \alpha a^2 + \beta c^2, \quad \text{(A6f)}$$

$$z = \alpha ab + \beta cd, \quad \text{(A6g)}$$

$$p = \{a^2 k_N^2 + b^2 k_P^2 + 2ab k_N k_P \cos\theta_{NP}\}^{1/2}, \quad \text{(A6h)}$$

$$q = \{c^2 k_N^2 + d^2 k_P^2 + 2cd k_N k_P \cos\theta_{NP}\}^{1/2}. \quad \text{(A6i)}$$

To apply this result to specific cases, consider first the neutron channel where it is found that

$$\mathbf{p}_L = \mathbf{k}_N, \quad \text{(A7a)}$$

$$\mathbf{q}_L = \mathbf{k}_P + M_P M_L^{-1}\mathbf{k}_N, \quad \text{(A7b)}$$

$$\alpha p^2 + \beta q^2 = (\hbar^2/2)(M_{PA}^{-1} q_L^2 + M_{NL}^{-1} p_L^2). \quad \text{(A7c)}$$

Comparison with Eqs. (1) and (3) gives

$$a=1, \quad b=0, \quad c=M_P M_L^{-1}, \quad d=1, \quad \text{(A8a)}$$

$$\alpha = \hbar^2/2M_{NL}, \quad \beta = \hbar^2/2M_{PA}. \quad \text{(A8b)}$$

Substitution into Eq. (6) gives

$$w = \left(\frac{\hbar^2}{2M_{PA}}\right)^{1/2} \frac{(P - M_P M_L^{-1} k_N \cos\theta_{NP})^2}{P(E - M_N M_{NL}^{-1} E_N)^{1/2}} \frac{M_N}{M_{NL}} = w_L, \quad \text{(A9a)}$$

$$P = (\hbar^2/2M_{PA})^{-1/2}$$

$$\times (E - M_N M_{NL}^{-1} E_N - M_P M_N M_A^{-1} M_L^{-1} E_N \sin^2\theta_{NP})^{1/2}, \quad \text{(A9b)}$$

$$E_N = \hbar^2 k_N^2 / 2M_N. \quad \text{(A9c)}$$

Next consider the proton channel.

$$\mathbf{p}_H = \mathbf{k}_P, \quad \text{(A10a)}$$

$$\mathbf{q}_H = \mathbf{k}_N + M_N M_H^{-1}\mathbf{k}_P, \quad \text{(A10b)}$$

$$\alpha p^2 + \beta q^2 = (\hbar^2/2)(M_{NA}^{-1} q_H^2 + M_{PH}^{-1} p_H^2). \quad \text{(A10c)}$$

Comparison with Eq. (1) and (3) gives

$$a=0, \quad b=1, \quad c=1, \quad d=M_N M_H^{-1}, \quad \text{(A11a)}$$

$$\alpha = \hbar^2/2M_{PH}, \quad \beta = \hbar^2/2M_{NA}. \quad \text{(A11b)}$$

Substitution into Eq. (6) gives

$$w = k_N k_P \lambda^2 M_A M_H / 2q_H P M_N M_{PH} = w_H, \quad \text{(A12a)}$$

$$\lambda^2 = M_N M_H M_P / M_A (M_N M_P + M_A M_P + M_A M_H), \quad \text{(A12b)}$$

$$P = (2M_A^2 \lambda^2 / \hbar^2 M_N)^{1/2}$$

$$\times (E - M_H M_A^{-1} E_N + \lambda^2 E_N \cos^2\theta_{NP})^{1/2}, \quad \text{(A12c)}$$

$$k_P = P - M_A M_N^{-1} \lambda^2 k_N \cos\theta_{NP}, \quad \text{(A12d)}$$

$$q_H = (k_N^2 + M_N^2 M_H^{-2} k_P^2 + 2M_N M_H^{-1} k_N k_P \cos\theta_{NP})^{1/2}. \quad \text{(A12e)}$$

Finally, consider the deuteron channel

$$\mathbf{p}_D = \mathbf{k}_N + \mathbf{k}_P, \quad \text{(A13a)}$$

$$\mathbf{q}_D = M_P M_D^{-1}\mathbf{k}_N - M_N M_D^{-1}\mathbf{k}_P, \quad \text{(A13b)}$$

$$\alpha p^2 + \beta q^2 = (\hbar^2/2)(M_{NP}^{-1} q_D^2 + M_{DA}^{-1} p_D^2). \quad \text{(A13c)}$$

Comparison with Eq. (1) and (3) gives

$$a=1, \quad b=1, \quad c=M_P M_D^{-1}, \quad d=M_N M_D^{-1}, \quad \text{(A14a)}$$

$$\alpha = \hbar^2/2M_{DA}, \quad \beta = \hbar^2/2M_{NP}. \quad \text{(A14b)}$$

Substitution into Eq. (6) gives

$$w = M_D k_N k_P^2 / 2M_{NP} q_D p_D P = w_D, \quad \text{(A15a)}$$

$$P = (2M_D/\hbar^2\kappa)^{1/2}$$

$$\times (E - M_H M_A^{-1} E_N + M_D M_N M_A^{-2} \kappa^{-1} E_N \cos^2\theta_{NP})^{1/2}, \quad \text{(A15b)}$$

$$\kappa = 1 + M_D M_A^{-1} + M_N M_P^{-1}, \quad \text{(A15c)}$$

$$k_P = P - M_D M_A^{-1} \kappa^{-1} k_N \cos\theta_{NP}, \quad \text{(A15d)}$$

$$p_D = \{k_N^2 + k_P^2 + 2k_N k_P \cos\theta_{NP}\}^{1/2}, \quad \text{(A15e)}$$

$$q_D = \{M_P^2 k_N^2 + M_N^2 k_N^2 - 2M_N M_P k_N k_P \cos\theta_{NP}\}^{1/2} M_D^{-1}. \quad \text{(A15f)}$$