Inhomogeneous Broadening of Ferromagnetic Resonance Lines

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A general theory of ferromagnetic resonance is developed assuming that the local, effective magnetic field is inhomogeneous. The scattering processes induced by the inhomogeneity may be classified as "primary" processes (which couple the uniform mode to nonuniform modes) and "secondary" processes (which couple nonuniform modes). In previous work only the primary processes were taken into account. The effect of secondary processes upon the susceptibility is calculated in the present paper. A perturbation series for the complex, effective resonant frequency of the uniform mode is derived. It is shown that the important terms of this infinite series can in part be generated by a self-consistency condition for the complex effective resonant frequency of spin waves. An approximate solution of this self-consistency condition is derived. Applied to polycrystals with cubic crystal structure, the theory predicts a linewidth of $\simeq 2.07H_a^2/4\pi M_0$ for spherical samples and $H_a \ll 4\pi M_0$ but $\simeq 0.87H_a$ for $H_a \gtrsim 4\pi M_0$. Here H_a is the anisotropy field and M_0 the saturation magnetization. The off-resonance absorption is characterized by the existence of a "strong absorption" region. When the intrinsic damping of the spin waves is assumed to approach zero the absorption goes to zero in the exterior of this region but to a finite value in its interior. If the Fourier spectrum of the inhomogeneity has significant components only at long wavelength and the inhomogeneity is weak, the strong absorption region coincides with the dc field interval in which the signal frequency is degenerate with resonant frequencies of long-wavelength spin waves. With increasing inhomogeneity, the width of the strong absorption region increases by approximately twice the width of the resonance line.

1. INTRODUCTION

IT is generally believed that the width of the ferromagnetic resonance line observed in polycrystalline ferro- and ferrimagnetic materials is largely due to the presence in the material of inhomogeneous, effective local magnetic fields.[1] Such effective magnetic fields can arise from crystalline anisotropy[2–4] (in conjunction with variations in orientation of different grains), from strain-induced anisotropy (in conjunction with local strains) and from inhomogeneity of the local saturation magnetization[5–7] (caused for instance by the presence of pores or inclusions of a secondary phase). Even in single crystals, the linewidth is frequently largely caused by surface roughness[6–8] which may be considered as a special type of inhomogeneity of the local saturation magnetization.

Two basically different approaches to the theory of inhomogeneous line broadening can be envisaged.[2] If the coupling between spins is very weak, the absorption profile (absorption versus dc magnetic field for a given frequency) is substantially determined by the volume fraction in which the magnitude of the additional field lies inside a given field interval. This "distribution" of the additional field has been calculated for simple cases, such as polycrystals of cubic[4] or hexagonal[9] materials assuming random orientation of the grains. The absorption profiles observed in materials, for which this theory is expected to apply, agree reasonably well with the theoretical predictions.[2,9,10]

Because of the strong coupling between spins produced by the dipolar interaction and the exchange interaction, the theory sketched in the preceding paragraph is valid only for rather extreme situations. In general, the coupling is not negligible and must be taken into account from the outset. Thus a collective approach to the problem (i.e., a formulation in terms of the normal modes of the system of coupled spins) is in general more appropriate.[11–13] For an infinite medium (or a rectangular test volume with periodic boundary conditions) the normal modes are plane waves, spin waves. If the material is perfectly homogeneous and the amplitudes of the waves are very small, they are exact normal modes. Each one can be excited separately. No energy is exchanged between them. If an inhomogeneous effective magnetic field is present in the material, however, the plane waves are no longer correct normal modes. If one wave is excited initially its energy is scattered to other waves, primarily those that have the same resonant frequency.

In ferromagnetic resonance experiments the rf magnetic field is substantially homogeneous over the volume of the sample. Thus it excites only the homogeneous (or uniform) mode of the spin-wave spectrum. An inhomogeneous additional magnetic field broadens

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the resonance line because it leads to scattering of energy into nonuniform modes. In general, the scattering processes may be classified into "primary" processes (which couple the uniform mode to nonuniform modes) and "secondary" processes (which couple nonuniform modes). In all previous theoretical work only the primary scattering has been taken into account. The objective of the present paper is to describe a method of taking the secondary scattering into account. It will be shown that the secondary scattering modifies the results of the collective theory of inhomogeneous line broadening quite significantly.

In most cases of practical interest the additional magnetic field varies relatively slowly with position and, therefore, has significant Fourier components only at low wave numbers. This implies that the uniform mode interacts substantially only with spin waves of low wavenumber, for which the exclange forces are of negligible importance. If only the primary scattering is taken into account, significant line broadening is expected only as long as the resonant frequency of the uniform mode lies inside the frequency band corresponding to low wave number spin waves. By the same token the losses in the wings of the absorption line are increased by the scattering only as long as the signal frequency lies inside this frequency band. At the high-frequency edge of the band the linewidth is expected to become very large, due to a singularity of the density of states into which energy can be scattered from the uniform model.

Experiments designed to test these rather dramatic predictions of the collective theory of inhomogeneous line broadening have generally shown that the effects are considerably weaker than expected theoretically.[14–19] Measurements of the linewidth as a function of frequency on spherical samples indeed showed a detectable peak in the vicinity of the point at which the uniform mode coincides with the upper edge of the spin wave band.[14–17] The height of the peak is much smaller than expected on the basis of the existing they, however. Measurements of the off-resonant absorption coefficient as a function of dc field strength have also confirmed that the loss diminishes considerably as the frequency moves out of the band corresponding to spin waves of long wavelength.[19] However, the transition is not as rapid as expected theoretically. Considerable losses are observed even when the resonant frequencies of all long wavelength spin waves are much larger or much smaller than the signal frequency.

These shortcomings of the existing theory of inhomogeneous line broadening are due to the neglect of secondary scattering. The principal effect of the secondary scattering is to increase the damping of the spin waves with which the uniform mode interacts. If the density of interacting modes is exceptionally large, the damping of these modes is also exceptionally large. Thus the peak expected in the linewidth versus frequency curve is considerably reduced. By the same token the secondary scattering increases the off-resonance absorption in the far wings of the resonance line.

The theory described in the present paper is primarily applicable to broadening due to crystalline and/or strain-induced anisotropy. The same principles apply in the case of broadening due to inhomogeneity of the saturation magnetization. The details are a little more involved in the latter case, and are, therefore, not considered in the present paper. In order to focus attention on the essential points certain approximations have been adopted, which can in principle be avoided without major difficulty. It has been assumed, for instance, that the effect of local (crystalline or strain-induced) anisotropy can be described by an inhomogenerous magnetic field $A(\mathbf{r})$ which is applied parallel to the dc field. Thus the tendency of the anisotropy to make the spin procession elliptical (rather than circular) has been neglected in this approach, as well as the effect of the local anisotropy upon the alignment of the local (static) magnetization vector with the dc field. The dipolar interaction also tends to make the spin precession elliptical. This ellipticity is also neglected for similicity.

A quantum-mechanical formulation has been chosen for convenience, even though the problem is basically a classical one and can be discussed consistently on the basis of the classical equations of motion. The canonical formulation described in the present paper has the advantage that the results can be readily applied to other physical systems, that are characterized by a similar Hamiltonian.

A general theory of the high-frequency susceptibility is developed in Sec. 2. The effect of the inhomogeneous magnetic field $A(\mathbf{r})$ upon the susceptibility can be expressed in terms of a complex effective resonant frequency $\Omega_{0\,\mathrm{eff}}$ which is given as an infinite series of ascending powers of the perturbation $A(\mathbf{r})$. The first correction term of the series depends only upon the primary scattering; the higher correction terms are due jointly to primary and secondary scattering. In Sec. 3, the case in which the inhomogeneous effective magnetic field $A(\mathbf{r})$ varies sinusoidally with position is investigated. It is shown that in this case the susceptibility can rigorously be expressed as a simple continued fraction. In Sec. 4, the general case [arbitrary variation of $A(\mathbf{r})$] is further considered. It is shown that the most significant terms of the infinite series, which represents $\Omega_{0\,\mathrm{eff}}$, can alternatively be generated by a self-consistency condition, which determines the effective complex resonant frequencies of the spin waves.

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An explicit approximate solution of the self-consistency condition is constructed in Sec. 5, assuming that the Fourier components $A(\mathbf{k})$ of the inhomogeneous magnetic field $A(\mathbf{r})$ are significantly large only for small wavenumbers $|\mathbf{k}|$, so that the effect of exchange forces becomes negligible. The results of detailed calculations concerning linewidth and off-resonance absorption are described in Secs. 6 and 7. The validity of the approximations upon which these results are based is reviewed in Sec. 8.

2. GENERAL THEORY OF SUSCEPTIBILITY

We begin by expressing the Hamiltonian which characterizes the system under consideration in terms of creation and annihilation operators $a^\dagger(\mathbf{r})$ and $a(\mathbf{r})$ which obey the commutation relations

$$a(\mathbf{r})a^\dagger(\mathbf{r}')-a^\dagger(\mathbf{r}')a(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r}'), \tag{1}$$

where $\delta(\mathbf{r})$ is the δ function. The components of the local magnetization vector $\mathbf{M}(\mathbf{r})$ can be expressed in terms of these operators as

$$\begin{aligned} M_+&=M_x+iM_y=(2\gamma\hbar M_0)^{1/2}a^\dagger[1-(\gamma\hbar/2M_0)a^\dagger a]^{1/2},\\ M_z&=M_0-\gamma\hbar a^\dagger a, \end{aligned} \tag{2}$$

where γ is the gyromagnetic ratio, $\hbar$ Planck's constant, and M_0 the saturation magnetization.[20,21,7] As long as the spins are substantially completely aligned (as we assume), the square root in Eq. (2) can be approximated by unity.

Consider now a small, rectangular test volume, which is representative of the sample as a whole. It is permissible to assume that all local properties are periodically repeated outside the test volume, and can hence be expanded in Fourier Series. The Fourier transforms

$$a(\mathbf{k})=V^{-1/2}\int d^3\mathbf{r}\, a(\mathbf{r})\exp(-i\mathbf{k}\cdot\mathbf{r}) \tag{3}$$

of the operator $a(\mathbf{r})$ obey the commutation relation

$$a(\mathbf{k})a^\dagger(\mathbf{k}')-a^\dagger(\mathbf{k}')a(\mathbf{k})=\delta_{kk'}, \tag{4}$$

where V is the periodicity volume and $\delta_{kk'}$ the Kronecker symbol.

The energy of the system under consideration comprises Zeeman, dipolar, and exchange contributions. The Zeeman energy [including the contribution due to the inhomogeneous field $A(\mathbf{r})$] is given by

$$\begin{aligned} \mathcal{H}_{\mathrm{Zee}}&=-\int(\mathbf{M}\cdot\mathbf{H}+M_zA)d^3\mathbf{r}\\ &=\mathrm{const}+\gamma\hbar H\sum_k a^\dagger(\mathbf{k})a(\mathbf{k})\\ &\quad+\gamma\hbar\sum A(\mathbf{k}'-\mathbf{k})a(\mathbf{k})a^\dagger(\mathbf{k}'), \end{aligned} \tag{5}$$

where

$$A(\mathbf{k})=V^{-1}\int d^3\mathbf{r}\, A(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{6}$$

is the Fourier transform of the inhomogeneous magnetic field. It may be assumed without loss of generality that $A(\mathbf{k}=0)=0$, because such a term (if nonzero) can always be included in the magnetic field H.

The dipolar energy is

$$\mathcal{H}_{\mathrm{dip}}=-\frac{1}{2}\int\mathbf{M}\cdot\mathbf{H}_{\mathrm{dip}}d^3\mathbf{r}, \tag{7}$$

where $\mathbf{H}_{\mathrm{dip}}$ is the magnetic field generated by the magnetization. In the quasistatic approximation, this part of the energy can be expressed as

$$\begin{aligned} \mathcal{H}_{\mathrm{dip}}&=\gamma\hbar 2\pi M_0\sum_k\sin^2\theta_k\{a^\dagger(\mathbf{k})a(\mathbf{k})\\ &\quad+\tfrac{1}{2}[e^{-2i\phi_k}a^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})+\mathrm{H.c.}]\}, \end{aligned} \tag{8}$$

where higher powers of the operators a and $a^\dagger$ have been neglected, and H.c. denotes the Hermitian conjugate of the expression preceding it, θ_k is the angle between the propagation vector $\mathbf{k}$ and the z axis, and ϕ_k is the angle between the projection of $\mathbf{k}$ upon the x-y plane and the x axis.

The exchange energy is

$$\mathcal{H}_{\mathrm{ex}}=\gamma\hbar D\sum k^2a^\dagger(\mathbf{k})a(\mathbf{k}), \tag{9}$$

where D is a phenomenological exchange constant ($\simeq 5\times10^{-9}$ Oe cm^2 for YIG).

We assume that a rotating driving field is applied:

$$\begin{aligned} h_x&=h_t\cos\omega t,\\ h_y&=h_t\sin\omega t. \end{aligned} \tag{10}$$

The contribution of this field to the energy is

$$\mathcal{H}_{\mathrm{driv}}=-hg_0h_t[a^\dagger(0)e^{-i\omega t}+\mathrm{H.c.}], \tag{11}$$

where

$$g_0=(\gamma M_0V/2\hbar)^{1/2}. \tag{12}$$

The second term in the dipolar energy [Eq. (8)] relates to the ellipticity of the spin precession. In principle, it can be transformed away by a canonical transformation.[20,21,7] Since this contribution to the energy does not add anything essential in the present context, we neglect it for simplicity at this point.

The complete Hamiltonian is, thus,

$$\begin{aligned} \mathcal{H}/\hbar&=\sum_k\omega_ka^\dagger(\mathbf{k})a(\mathbf{k})+\sum_{kk'}P_{kk'}a(\mathbf{k})a^\dagger(\mathbf{k}')\\ &\quad-g_0h_t[a^\dagger(0)e^{-i\omega t}+\mathrm{H.c.}], \end{aligned} \tag{13}$$

where

$$\omega_k=\gamma(H+2\pi M_0\sin^2\theta_k+Dk^2) \tag{14}$$

is the resonant frequency of the spin wave with wave number k and

$$P_{kk'}=\gamma A(\mathbf{k}'-\mathbf{k}). \tag{15}$$

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The equation of motion for $a^\dagger(\mathbf{k})$ is

$$\dot{a}^\dagger(\mathbf{k}) = -(i/\hbar)[a^\dagger(\mathbf{k}),\mathcal{H}] = (i/\hbar)\partial\mathcal{H}/\partial a(\mathbf{k}). \quad (16)$$

Hence, according to Eq. (13),

$$\dot{a}^\dagger(\mathbf{k}) = i[\omega_k a^\dagger(\mathbf{k}) + \sum_{k'} P_{kk'} a^\dagger(\mathbf{k}') - g_0 h_t \delta_{k0} e^{i\omega t}]. \quad (17)$$

The equations of motion in the form of Eq. (17) take no account of the damping of the spin waves. A plausible way of correcting this shortcoming would be to assign a positive imaginary part to the resonant frequencies ω_k. Such a damping term in the equations of motion would describe relaxation towards the state in which $a(k)$ vanishes. This is not entirely satisfactory, because actually relaxation proceeds towards the instantaneous equilibrium value of $a(k)$, which is consistent with the driving forces acting upon this mode. Such a damping can be described mathematically by multiplying the left hand side of Eq. (17) by $(1-i\epsilon_k)$, where ϵ_k is a small real parameter. It may in general be different for different modes. If the driving field is periodic [as we assume in Eq. (17)], this damping is equivalent to assigning an imaginary part to the resonant frequency, but the imaginary part of the resonant frequency is now proportional to the signal frequency ω. We define the complex resonant frequency of the spin wave with wave number k as

$$\Omega_k = \omega_k + i\eta_k = \omega_k + i\omega\epsilon_k. \quad (18)$$

Here $\eta_k = \omega\epsilon_k$ is the relaxation rate of mode k due to all processes not explicitly considered in the theory, the "intrinsic" relaxation rate.

In the formalism used in this paper a negative ω corresponds to a driving field having "negative" circular polarization [see Eq. (10)]. If η_k did not change sign with ω, one would obtain negative absorption for negative circular polarization.

The phenomenological description of losses in the equations of motion is to some extent ambiguous. Instead of modifying the left-hand side of Eq. (17) by $1-i\epsilon_k$, it would be equally as justifiable to modify it by $(1+i\epsilon_k)^{-1}$ or by $e^{-i\epsilon k}$. In the present context these differences are of no importance, because we are primarily interested in the limit in which the intrinsic damping is very small compared to the effective damping induced by the inhomogeneity.

The solution of the equations of motion (17) is obviously of the form

$$a^\dagger(\mathbf{k}) = b(\mathbf{k}) e^{i\omega t}, \quad (19)$$

where the $b(\mathbf{k})$ are determined by

$$\begin{aligned} &(\Omega_0 - \omega) b(0) + \sum_k P_{0k} b(\mathbf{k}) = g_0 h_t, \\ &(\Omega_k - \omega) b(\mathbf{k}) + \sum_{k'} P_{kk'} b(\mathbf{k}') = 0, \quad k \neq 0. \end{aligned} \quad (20)$$

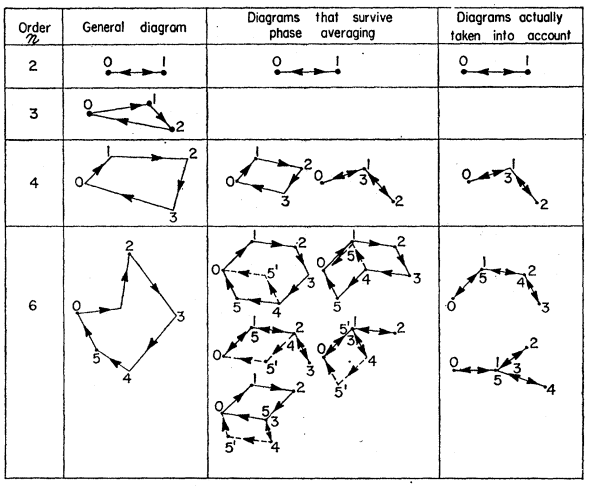

FIG. 1. Diagrams representing the lower-order corrections to the effective resonant frequency of the uniform mode. For $n=6$ each drawing in column 3 represents two different types of diagrams.

The circular susceptibility χ_+ is the ratio of the rotating magnetization to the rotating driving field, i.e.,

$$\chi_+ = (2\gamma\hbar M_0/V)^{1/2} b(0)/h_t. \quad (21)$$

To find the ratio $b(0)/h_t$ we solve the second line of Eq. (20) for $b(\mathbf{k})$, and insert the result repeatedly into the first line, collecting at each step of this iteration procedure the terms that are proportional to $b(0)$. The result is

$$(\Omega_{0\,\mathrm{eff}} - \omega) b(0) = g_0 h_t, \quad (22)$$

where

$$\begin{aligned} \Omega_{0\,\mathrm{eff}} = \Omega_0 &- \sum_{k_1 \neq 0} \frac{P_{0k_1} P_{k_1 0}}{\Omega_{k_1} - \omega} + \sum_{k_1, k_2 \neq 0} \frac{P_{0k_1} P_{k_1 k_2} P_{k_2 0}}{(\Omega_{k_1} - \omega)(\Omega_{k_2} - \omega)} \\ &- \sum_{k_1, k_2, k_3 \neq 0} \frac{P_{0k_1} P_{k_1 k_2} P_{k_2 k_3} P_{k_3 0}}{(\Omega_{k_1} - \omega)(\Omega_{k_2} - \omega)(\Omega_{k_3} - \omega)} + \cdots. \end{aligned} \quad (23)$$

None of the summation indices in Eq. (23) can be allowed to be zero, because the corresponding terms are separately taken into account. The circular susceptibility is now according to Eqs. (21), (22), and (12)

$$\chi_+ = \gamma M_0/(\Omega_{0\,\mathrm{eff}} - \omega). \quad (24)$$

The lowest correction term in Eq. (23) (i.e., the second term) is the contribution to the effective resonant frequency arising entirely from the primary scattering. The subsequent terms are due jointly to primary and secondary scattering.

It is useful to visualize the various contributions to the effective resonant frequency $\Omega_{0\,\mathrm{eff}}$ pictorially. If we picture the wave numbers $\mathbf{k}_1, \mathbf{k}_2, \cdots$, of the modes that interact with the uniform mode as points in a plane (actually they are, of course, distributed in three dimensions), then the nth order correction to $\Omega_{0\,\mathrm{eff}}$ is represented by an n-sided polygon which starts and ends at $\mathbf{k}=0$ and connects $n-1$ intermediate points

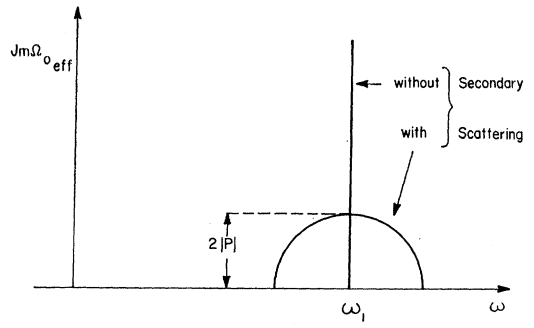

FIG. 2. Effective damping constant of the uniform mode in the presence of a sinusoidal inhomogeneity of long wavelength, considered as a function of signal frequency ω. The conventional theory, which neglects secondary scattering, predicts a sharp peak at ω_1, the resonant frequency of spin waves that interact with the uniform mode. The effect of secondary scattering is to broaden the sharp peak into a semicircular profile. The area under the sharp peak is the same as that under the semicircle.

$\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_{n-1}$. Some of the intermediate points may coincide, but none may coincide with the zero. The second-order diagram consists of a doubly traversed line. Some simple diagrams are shown in Fig. 1.

Before proceeding with an approximate evaluation of the complex resonant frequency $\Omega_{0\,\mathrm{eff}}$ according to Eq. (23) it is advantageous to consider a simple model, for which the problem can be solved rigorously.

3. RIGOROUS CALCULATION OF EFFECTIVE RESONANT FREQUENCY FOR A SIMPLE MODEL

Assume that the inhomogeneous magnetic field varies sinusoidally. In this case the scattering coefficients $P_{kk'}$ are nonzero only if $\mathbf{k}-\mathbf{k}'=\pm\mathbf{k}_0$, where $\mathbf{k}_0$ is the wave number of the inhomogeneity. Only those spin waves whose wave numbers are multiples of $\mathbf{k}_0$ are excited. For simplicity, let

$$P_{0k_0}=P, \qquad b(n\mathbf{k}_0)=b(n). \tag{25}$$

Equation (20) then becomes

$$(\Omega_0-\omega)b(0)+Pb(1)+P^*b(-1)=g_0h_t, \tag{26}$$

$$(\Omega_n-\omega)b(n)+Pb(n+1)+P^*b(n-1)=0, \quad n\neq 0. \tag{27}$$

For $n\geq 1$, Eq. (27) can alternatively be expressed as

$$(\tilde{\Omega}_n-\omega)b(n)+P^*b(n-1)=0, \tag{28}$$

where

$$\tilde{\Omega}_n=\Omega_n-\frac{|P|^2}{\tilde{\Omega}_{n+1}-\omega} \tag{29}$$

is an effective resonant frequency for mode n which is recursively defined by Eq. (29). The explicit solution of this equation is a continued fraction

$$\tilde{\Omega}_n=\Omega_n-\cfrac{|P|^2}{\Omega_{n+1}-\omega-|P|^2/[\Omega_{n+2}-\omega-|P|^2/(\Omega_{n+3}-\omega\cdots)]}. \tag{30}$$

To show that Eqs. (28) and (29) are equivalent with Eq. (27), insert (29) into (28) and then apply (28) with n replaced by $n+1$. This results in Eq. (27).

For $n\leq -1$, Eq. (27) can similarly be replaced by

$$(\tilde{\Omega}_n-\omega)b(n)+Pb(n+1)=0, \tag{31}$$

where

$$\tilde{\Omega}_n=\Omega_n-(|P|^2/\tilde{\Omega}_{n-1}-\omega). \tag{32}$$

Since $\Omega_{-n}=\Omega_n$, it follows that $\tilde{\Omega}_{-n}=\tilde{\Omega}_n$.

For $n=\pm 1$, Eqs. (28) and (31) can be solved for $b(\pm 1)$ in terms of $b(0)$. By substituting these solutions into Eq. (26), one finds the effective resonant frequency of the uniform mode as

$$\Omega_{0\,\mathrm{eff}}=\Omega_0-2[|P|^2/(\tilde{\Omega}_1-\omega)], \tag{33}$$

where $\tilde{\Omega}_1$ is the resonant frequency of mode 1 (or -1) defined by Eq. (29) or alternatively by Eq. (30).

The results can be simplified further when the unperturbed spin-wave frequencies $\Omega_n (n\neq 0)$ are all equal. This condition is approximately satisfied when k_0 is small (i.e., $Dk_0^2 \ll 4\pi M_0$). In this case, it follows from Eq. (29) that the effective spin-wave frequencies are also all equal and given by

$$\tilde{\Omega}_n-\omega=\tfrac{1}{2}(\Omega_1-\omega)+[\tfrac{1}{4}(\Omega_1-\omega)^2-|P|^2]^{1/2}. \tag{34}$$

The positive sign of the square root must be taken to assure that $\tilde{\Omega}_n$ reduces to Ω_n in the limit $|P|\rightarrow 0$. The effective resonant frequency of the uniform mode according to Eqs. (33) and (34) is now

$$\Omega_{0\,\mathrm{eff}}=\Omega_0-(\Omega_1-\omega)+[(\Omega_1-\omega)^2-4|P|^2]^{1/2}. \tag{35}$$

Consider now the effective damping constant of the uniform mode (i.e., $\mathrm{Im}\Omega_{0\,\mathrm{eff}}$) in the limit of vanishing intrinsic damping. According to Eq. (35),

$$\lim \mathrm{Im}\Omega_{0\,\mathrm{eff}} = [4|P|^2-(\omega_1-\omega)^2]^{1/2}, \quad \text{if} \quad |\omega_1-\omega|<2|P| \qquad =0, \quad \text{otherwise}. \tag{36}$$

Thus the effective damping constant considered as a a function of $\omega_1-\omega$ is represented by a semicircle as shown in Fig. 2.

This should be contrasted with the behavior that would be deduced for the same model (sinusoidal inhomogeneity of long wavelength) but neglecting the secondary scattering. In this approximation the effective spin-wave frequency $\tilde{\Omega}_1$ in Eq. (33) should be replaced by the unperturbed frequency Ω_1. Hence in

the limit of zero intrinsic damping in this case

$$\lim \mathrm{Im}\Omega_{0\,\mathrm{eff}} = 2\pi |P|^2 \delta(\omega_1 - \omega). \tag{37}$$

It is interesting to note that the frequency integral of the effective damping constant is the same in the two cases considered [Eqs. (36) and (37)].

The effect of the secondary scattering upon the effective damping constant of the uniform mode is thus to broaden the sharp peak expected on the basis of conventional theory by an amount comparable to the maximum damping of the uniform mode.

4. SELF-CONSISTENCY CONDITION FOR EFFECTIVE RESONANT FREQUENCY OF SPIN WAVES

A complete evaluation of all the sums in Eq. (23) does not appear feasible. It is, therefore, useful to ask, which of the many terms in the summations are likely to give strong contributions to the effective resonant frequency. To help answer this question we note that in actuality the inhomogeneous field $A(\mathbf{r})$ is not known in detail. Only statistical statements about certain averages can be made. It is, therefore, reasonable to average all quantities of interest over an ensemble of macroscopically equivalent samples with microscopically different local anisotropy fields $A(\mathbf{r})$. The ensemble may be characterized by the autocorrelation function

$$\Phi(\mathbf{r}) = V^{-1} \int d^3\mathbf{r}' A(\mathbf{r}+\mathbf{r}') A(\mathbf{r}') = \langle A(\mathbf{r}+\mathbf{r}') A(\mathbf{r}') \rangle. \tag{38}$$

Here the angular brackets denote the ensemble average. The autocorrelation function $\Phi(\mathbf{r})$ determines only the amplitudes of the Fourier coefficients $A(\mathbf{k})$ not their phases. The phases may, therefore, be assumed to be randomly distributed.

The physical quantity that is directly measured in most resonance experiments is the (complex) frequency shift of a resonant cavity due to the presence of the sample. When the sample is small, the frequency shift is proportional to an "effective" or "external" susceptibility χ_e, which is related to the "true" or "internal" susceptibility χ by

$$\chi_{+e}^{-1} = \chi_{+}^{-1} + 4\pi N_t. \tag{39}$$

Here it is assumed for simplicity that the microwave field is circularly polarized and that the sample is a spheroid magnetized along its axis. N_t is the transverse demagnetizing factor of the sample. The directly measured quantity χ_{+e}^{-1} is thus according to Eqs. (39) and (24) linearly related to $\Omega_{0\,\mathrm{eff}}$.

Since $P_{kk'} = \gamma A(\mathbf{k}'-\mathbf{k})$ [see Eq. (15)] and since $A(\mathbf{k})$ has a random phase, only those diagrams contribute to the ensemble average of $\Omega_{0\,\mathrm{eff}}$ that consist entirely of pairs of branches that are parallel and are traversed in opposite directions. The two branches may (but need not) fall on top of each other. Some examples of diagrams that contribute to the ensemble average of $\Omega_{0\,\mathrm{eff}}$ are shown in Fig. 1. All contributions of odd order in the perturbation vanish. The diagrams that contribute to the second-order correction ($n=2$ in Fig. 1) all survive the phase averaging. In fourth order ($n=4$), only those diagrams that either consist of two doubly traversed lines or have the shape of a parallelogram give a contribution.

Although the number of significant diagrams is considerably reduced by the random-phase postulate the complete evaluation of the perturbation series (23) to infinite order does not appear feasible in this approximation. Even the evaluation of the lowest significant correction term (proportional to $|P|^4$) is very difficult. A different approach has, therefore, been used for an approximate evaluation. The discussion of the rigorously solvable model in Sec. 3 suggests the use of a self-consistency condition for the effective resonant frequency of spin waves as an alternative method of taking account of the secondary scattering. Accordingly, we define the effective resonant frequency of the spin wave $\mathbf{k}$, in analogy with Eq. (32), by

$$\tilde{\Omega}_k = \Omega_k - \sum_{k' \neq 0} \frac{|P_{kk'}|^2}{\tilde{\Omega}_{k'} - \omega}. \tag{40}$$

Note that the denominator on the right-hand side contains $\tilde{\Omega}_{k'}$ rather than $\Omega_{k'}$. Equation (4) also defines (for $k=0$) an effective resonant frequency of the uniform mode. Repeated application of the self-consistency condition yields an infinite series for $\tilde{\Omega}_0$ in terms of Ω_k and the squares of the scattering coefficients $|P_{kk'}|^2$. This infinite series contains many, but not all, of the terms in $\Omega_{0\,\mathrm{eff}}$ that survive the phase averaging. The diagrams contributing to $\tilde{\Omega}_0$ have only doubly transversed branches and are therefore a more restrictive class than the diagrams contributing to $\langle \Omega_{0\,\mathrm{eff}} \rangle$. Some examples of diagrams that are taken into account in the self-consisting condition (40) are given in column 4 of Fig. 1.

It can be shown[22] that $\langle \Omega_{0\,\mathrm{eff}} \rangle$ [see Eq. (23)] reduces almost rigorously to $\tilde{\Omega}_0$ [Eq. (40) with $k=0$], if the phase of each scattering coefficient $P_{kk'}$ (as characterized by the two wave numbers k and k') is randomly distributed (except for the condition $P_{kk'} = P_{k'k}^*$). This random-phase assumption is more restrictive than the random-phase postulate applied to the Fourier coefficient $A(\mathbf{k})$ previously discussed. The latter postulate is physically justified, the former is very problematic.

The significance of the terms neglected in the transition from $\langle \Omega_{0\,\mathrm{eff}} \rangle$ to $\tilde{\Omega}_0$ is difficult to assess. $\tilde{\Omega}_0$ agrees with $\langle \Omega_{0\,\mathrm{eff}} \rangle$ in regard to the lower-order terms, which are probably the most important in most cases of

[22] E. Schlömann, Raytheon Technical Memo No. T-758, 1967 (unpublished).

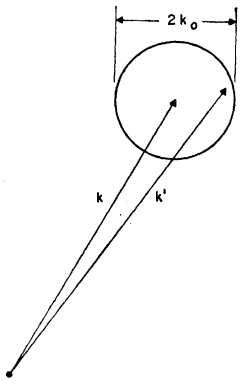

FIG. 3. The wave numbers $\mathbf{k}'$ of all spin waves that interact significantly with the mode $\mathbf{k}$ lie inside a sphere of radius k_0. The figure applies when $|\mathbf{k}| \gg k_0$.

interest. In addition $\tilde{\Omega}_0$ also contains selective terms of the perturbation series, summed to infinite order. It is, therefore, reasonable to expect that $\tilde{\Omega}_0$ represents a fairly good approximation of $\langle\Omega_{0\,\mathrm{eff}}\rangle$.

The self-consistency condition (40) is in principle an integral equation (provided that the summation over k' is replaced by an integration, which is permissible). Thus a general solution cannot readily be constructed. In the present case, and subject to certain simplifying assumptions, however, the integral equation can be reduced to a simple transcendental equation as discussed in the next section.

If the inhomogeneous magnetic field $A(\mathbf{r})$ varies sinusoidally (the model discussed in the preceding section), the wave-number vectors of all modes that couple to the uniform mode are parallel. Thus all diagrams have only multiply traversed branches in this case. It is, therefore, not surprising that the rigorous result obtained for this simple example is of the same form as the result (40), which is approximately valid under more general conditions.

5. APPROXIMATE SOLUTION OF THE SELF-CONSISTENCY CONDITION FOR COARSE-GRAIN INHOMOGENEITY

For the evaluation of the effective resonant frequency $\tilde{\Omega}_0$ the "spectrum" $|A(\mathbf{k})|^2$ of the inhomogeneous magnetic field $A(\mathbf{r})$ must be determined. We note first that the spectrum is generally related to the "autocorrelation" function $\Phi(\mathbf{r})$ given in Eq. (38) by

$$|A(\mathbf{k})|^2=\frac{1}{V}\int d^3\mathbf{r}\Phi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{41a}$$

The behavior of the autocorrelation function has previously been discussed on the basis of a simple one-dimensional model.[3] This model assumes in effect that all grains are equal in size and that the strengths of the inhomogeneous field in different grains are uncorrelated. On the basis of this assumption it is found that the correlation function has a maximum at the origin and that it decreases linearly with distance, becoming zero when the distance equals the grain diameter.

For the three-dimensional case it appears reasonable to assume that the correlation decreases exponentially with distance, i.e.,

$$\Phi(\mathbf{r})=\Phi(0)e^{-k_0 r}, \tag{41b}$$

where k_0 is of the order of the inverse grain size, and $\Phi(0)=\langle A^2\rangle$. Equation (41b) describes the correlation function in the interior of the periodicity volume. In the exterior of this volume the function is periodically repeated. The spectrum $|A(\mathbf{k})|^2$ corresponding to this correlation function is according to Eq. (41a)

$$|A(\mathbf{k})|^2=8\pi\Phi(0)/\{Vk_0^3[1+(k/k_0)^2]^2\}. \tag{41c}$$

Here the integration over r has been extended to infinity for the sake of simplicity. According to the definition (41a) the integration should actually be extended only over the periodicity volume. The error incurred in this step is very small, however, because the correlation function is very small for large r.

Equation (41c) shows that the spectrum is large when the wave number is smaller than the inverse grain size and rapidly approaches zero for wave numbers that are large compared to the inverse grain size.

It is unlikely that the autocorrelation actually realized in polycrystals is precisely of the exponential form assumed in Eqs. (41b) and (41c). The precise form of the k dependence of the spectrum is not actually used in the subsequent calculation, however. The feature that is actually used is that the spectrum $|A(\mathbf{k})|^2$ has significant components only below a certain cutoff wave number k_0. This property of the spectrum is likely to be independent of the detailed structure of the autocorrelation function.

Consider now the case that the cutoff wave number k_0 is sufficiently small, so that the effect of the exchange interaction upon the unperturbed resonant frequencies ω_k is negligible. This case is of practical interest because the grain size in polycrystals is typically of the order of several microns. The effect of the exchange interaction upon the resonant frequency of spin waves whose wavelength equals the grain size (10^{-4} cm) is equivalent to a magnetic field of approximately 20 Oe, i.e., much smaller than the saturation magnetization. Under those conditions it is permissible to disregard the effect of the exchange interaction by way of approximation. The unperturbed resonant frequencies ω_k of the spin waves that interact with the uniform mode then depend only upon the direction of the wave number $\mathbf{k}$, not upon its magnitude.

We discuss the solution of the self-consistency condition (40) for two limiting cases in which either $|\mathbf{k}|\gg k_0$ or $|\mathbf{k}|\ll k_0$. The wave numbers $\mathbf{k}'$ of the spinwaves that interact with the mode $\mathbf{k}$ are substantially confined to the interior of a sphere with radius k_0 and center at $\mathbf{k}$. If $|\mathbf{k}|\gg k_0$, the situation is qualitatively as shown in Fig. 3. The resonant frequencies of all modes $\mathbf{k}'$ that interact significantly with

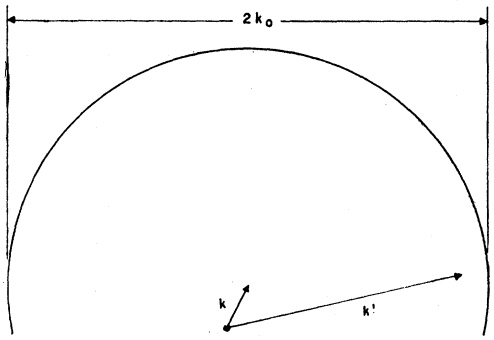

FIG. 4. The wave numbers $\mathbf{k}'$ of all spin waves that interact significantly with the mode $\mathbf{k}$ lie inside a sphere of radius k_0. The figure applies when $|\mathbf{k}|\ll k_0$.

the mode $\mathbf{k}$ are approximately equal. Since

$$\sum_{k'}|P_{kk'}|^2=\gamma^2\langle A^2\rangle, \tag{42}$$

the self-consistency condition (40), therefore, becomes approximately

$$\tilde{\Omega}_k=\Omega_k-\gamma^2\langle A^2\rangle/(\tilde{\Omega}_k-\omega) \tag{43a}$$

with the solution

$$\tilde{\Omega}_k-\omega=\tfrac{1}{2}(\Omega_k-\omega)+[\tfrac{1}{4}(\Omega_k-\omega)^2-\gamma^2\langle A^2\rangle]^{1/2}. \tag{43b}$$

In the limit of zero intrinsic loss ($\eta_k\rightarrow 0_+$) and at resonance the effective damping constant is according to Eq. (43b)

$$\tilde{\eta}_k=\mathrm{Im}(\tilde{\Omega}_k)=\gamma\langle A^2\rangle^{1/2}. \tag{44}$$

Note that the spinwaves that satisfy the condition $|\mathbf{k}|\gg k_0$ do not interact directly with the uniform mode.

If $|k|\ll k_0$ the spinwave $\mathbf{k}$ interacts equally effectively with waves propagating in all possible directions (see Fig. 4). We replace the summation in Eq. (40) by an integration. The first factor of the integrand $|P_{kk'}|^2$ depends only upon the magnitude of $\mathbf{k}-\mathbf{k}'$, i.e., for $|\mathbf{k}|\ll k_0$, substantially only upon the magnitude of $\mathbf{k}'$. The second factor $(\tilde{\Omega}_{k'}-\omega)^{-1}$ depends only upon the direction of $\mathbf{k}'$. The integration can, therefore, be carried out by first integrating $|P_{kk'}|^2$ over all $\mathbf{k}'$ (regardless of direction) and then averaging the second factor over all directions. Hence the self-consistency condition (40) becomes

$$\tilde{\Omega}_k=\Omega_k-\gamma^2\langle A^2\rangle\int_0^1\frac{d(\cos\theta_{k'})}{\tilde{\Omega}_{k'}-\omega}. \tag{45}$$

In the following we use this approximate self-consistency relation derived under the condition $|\mathbf{k}|\ll k_0$ to determine the effective resonant frequency of *all* spin waves that interact with the uniform mode. This procedure is only approximately correct, because not all of the spin waves that interact with the uniform mode satisfy the condition $|\mathbf{k}|\ll k_0$. The significance of this approximation is further discussed in Sec. 8.

For convenience we define a reduced, complex frequency shift

$$z=x+iy \tag{46}$$

by

$$\tilde{\Omega}_k=\Omega_k+\gamma 4\pi M_0 z. \tag{47}$$

It is readily shown that the self-consistency condition (45) leads to a transcendental equation for the frequency shift z:

$$z=\rho F(z,c), \tag{48}$$

where

$$\rho=\langle A^2\rangle/(4\pi M_0)^2, \tag{49}$$

$$F(z,c)=-2\int_0^1\frac{d\xi}{\xi_0^2-\xi^2}=-\frac{1}{\xi_0}\ln\frac{\xi_0+1}{\xi_0-1}, \tag{50}$$

$$\xi_0^2=c+2iy_0+2z, \tag{51}$$

$$y_0=\eta_k/(\gamma 4\pi M_0), \tag{52}$$

$$c=\cos^2\theta_0=(H+2\pi M_0-\omega/\gamma)/2\pi M_0. \tag{53}$$

θ_0 is the angle between the dc field and the propagation direction of those spinwaves that resonate under the given conditions of frequency and field strength.

If real and imaginary parts of the complex function $F(z,c)$ are defined as

$$F(z,c)=F_1(x,y,c)+iF_2(x,y,c), \tag{54}$$

one obtains from Eqs. (50) and (51),

$$F_1(x,y,c)=-\frac{1}{r}\left[-p\coth^{-1}\frac{r+1}{2p}+q\cot^{-1}\frac{r-1}{2q}\right], \tag{55}$$

$$F_2(x,y,c)=-\frac{1}{r}\left[q\coth^{-1}\frac{r+1}{2p}+p\cot^{-1}\frac{r-1}{2q}\right], \tag{56}$$

where

$$p=2^{-1/2}[r+c+2x]^{1/2}, \tag{57}$$

$$q=2^{-1/2}[r-(c+2x)]^{1/2}, \tag{58}$$

$$r=[(c+2x)^2+4(y_0+y)^2]^{1/2}=p^2+q^2. \tag{59}$$

The principal value of the $\cot^{-1}$ function, to be used in these expressions, is taken to be such that $0\leq\cot^{-1}x\leq\pi$. The complex self-consistency condition (48) is equivalent to a set of two coupled real, transcendental equations

$$\begin{aligned}x&=\rho F_1(x,y,c),\\ y&=\rho F_2(x,y,c).\end{aligned} \tag{60}$$

In the limit of zero damping (i.e., for $y_0+y\rightarrow 0_+$) the expressions for F_1 and F_2 simplify considerably. In this limit $q\rightarrow 0$ for $c+2x>0$ $p\rightarrow 0$ for $c+2x<0$.

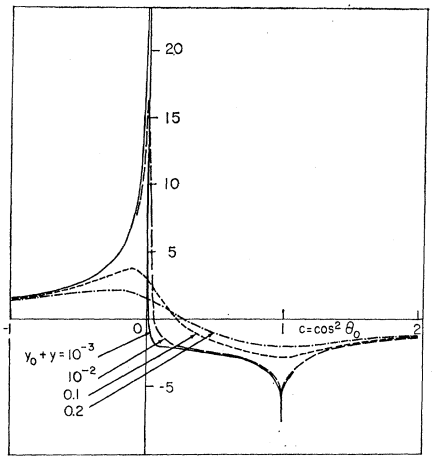

FIG. 5. Real part of the function $F(z,c)$ plotted versus $c=\cos^2\theta_0$ [see Eq. (53)] for various values of y_0+y.

Hence for $x=0$

$$F_1(0,0_+,c)=\frac{1}{c^{1/2}}\coth^{-1}\left(\frac{c+1}{2c^{1/2}}\right)=\frac{1}{c^{1/2}}\ln\left(\frac{c^{1/2}+1}{c^{1/2}-1}\right),$$

$$\text{for}\quad c>0$$

$$=\frac{1}{(-c)^{1/2}}\cot^{-1}\left(\frac{-c-1}{2(-c)^{1/2}}\right),\quad \text{for}\quad c<0 \qquad (61)$$

and

$$F_2(0,0_+,c)=\pi c^{-1/2},\quad \text{for}\quad 0<c<1$$
$$=0,\qquad \text{otherwise.} \qquad (62)$$

The values of F_1 and F_2 for $x\neq 0$ can generally be found from the values at $x=0$ by replacing c by $c+2x$.

In Figs. 5 and 6 the functions $F_1(x,y,c)$ and $F_2(x,y,c)$ are plotted versus c for $x=0$ and several assumed values of y_0+y. The function F_2 is always positive, as it should because it is proportional to the damping of the spinwaves. The function F_1 can be positive or negative.

The self-consistency condition (60) can be solved analytically both for $\rho\ll 1$ and for $\rho\gg 1$. It is clear that

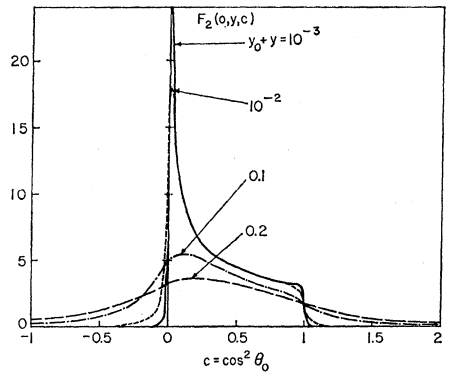

FIG. 6. Imaginary part of the function $F(z,c)$ plotted versus $c=\cos^2\theta_0$ [see Eq. (53)] for various values of y_0+y.

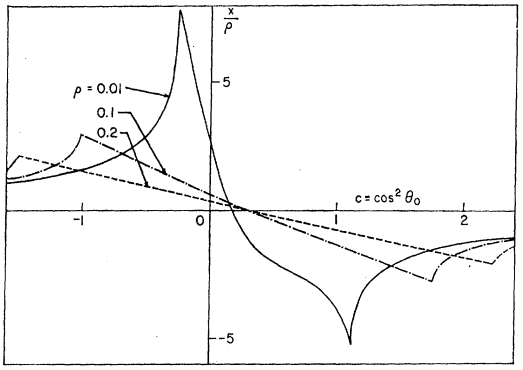

FIG. 7. Real part of the reduced frequency shift x divided by the inhomogeneity parameter ρ plotted versus $c=\cos^2\theta_0$. The curves were obtained by numerical solution of the self-consistency condition (60) assuming $y_0=0$.

for $\rho\to 0$ both x and y vanish, so that for $\rho\ll 1$,

$$x\simeq\rho F_1(0,0_+,c),$$
$$y\simeq\rho F_2(0,0_+,c). \qquad (63)$$

The behavior of the solution for $\rho\to\infty$ is discussed in Appendix A, where it is shown that for $\rho\gg 1$,

$$x\simeq\tfrac{1}{4}(\tfrac{1}{3}-c),$$
$$y\simeq\rho^{1/2}+O(\rho^{-1/2}). \qquad (64)$$

It should be noticed that x approaches a finite value, whereas y becomes very large, but only as $\rho^{1/2}$.

For intermediate values of ρ, the self-consistency condition (60) has been solved numerically on a digital computer. Typical results are shown in Figs. 7 and 8. Here x/ρ and y/ρ are plotted versus $c=\cos^2\theta_0$ for representative values of ρ, assuming $y_0=0$. It may be seen that the results of the present theory are qualitatively similar to the results of earlier calculations in which only the primary scattering was taken into account.[3] The effect of the secondary scattering is to "smear out" the effect of the primary scattering. Thus the peaks in the x/ρ and y/ρ versus c curves of Figs. 7 and 8 become less distinct as ρ increases. If only the primary scattering is taken into account, the corresponding curve becomes singular at the edges of the spin wave band, i.e., for $c=0$ and for $c=1$.

Subject to the approximations that have been adopted the complex shift of the resonant frequencies is the same for all modes, including the uniform mode. Thus y_0+y is, apart from a factor $\gamma 4\pi M_0$, equal to the imaginary part of $\tilde{\Omega}_0$. Its value at resonance determines the linewidth, its value away from resonance the off-resonance absorption.[19] These two consequences of the inhomogeneity-induced scattering are discussed in more detail in Secs. 6 and 7.

6. LINEWIDTH AND LINE SHIFT

The linewidth ΔH is usually defined as the width at half height of the absorption versus field curve, mea-

sured at constant frequency. For small samples of ellipsoidal shape the absorption is proportional to the imaginary part of an "effective" susceptibility. For a spheroidal sample magnetized along its axis the positive circular component of the effective susceptibility χ_{+e} is related to the "true" susceptibility χ_+ by [see Eq. (39)]

$$\chi_{+e}=(1+4\pi N_t\chi_+)^{-1}\chi_+, \tag{65}$$

where N_t is the transverse demagnetizing factor. By inserting Eq. (24) into Eq. (65) it may be seen that the effective susceptibility is given by

$$\chi_{+e}=\gamma M_0/(\Omega_{0\,\mathrm{eff}}+\gamma 4\pi M_0 N_t-\omega). \tag{66}$$

Thus the absorption coefficient $\mu_{+e}''=4\pi\chi_{+e}''$ is

$$\mu_{+e}''=(y+y_0)/((u-v+N_t+x)^2+(y+y_0)^2), \tag{67}$$

where

$$u=H/4\pi M_0, \quad v=\omega/\gamma 4\pi M_0. \tag{68}$$

As long as the complex frequency shift $z=x+iy$ is only weakly dependent upon the dc fields strength the linewidth is substantially determined by its imaginary part y taken at resonance:

$$\frac{\Delta H}{4\pi M_0}\simeq 2(y_0+y)_{\mathrm{res}}. \tag{69}$$

This equation is not rigorously correct, because at the half-maximum absorption points the complex frequency shift is in general different from that realized at resonance. A detailed study has shown, however, that this correction is quite small, even when the linewidth is of the order of the unperturbed spin-wave band ($2\pi M_0$). This comes about because the spin-wave band itself also spreads out by an amount that is even larger than the linewidth ΔH, as discussed in detail in Sec. 7.

Experimentally it is also possible to infer a "linewidth" from the magnitude of the absorption at resonance.[17] The theoretical expression (69) corresponds precisely to this definition of linewidth.

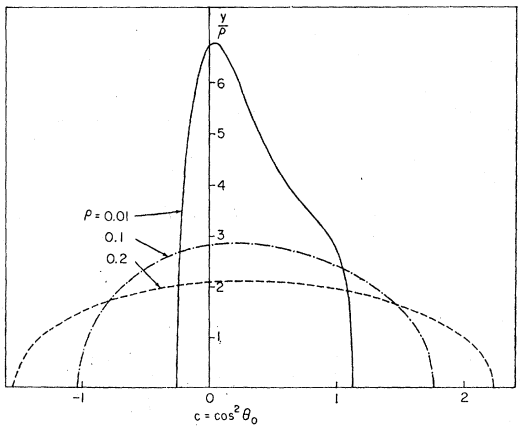

FIG. 8. Imaginary part of the reduced frequency shift y divided by the inhomogeneity parameter ρ plotted versus $c=\cos^2\theta_0$. The curves were obtained by numerical solution of the self-consistency condition (60) assuming $y_0=0$.

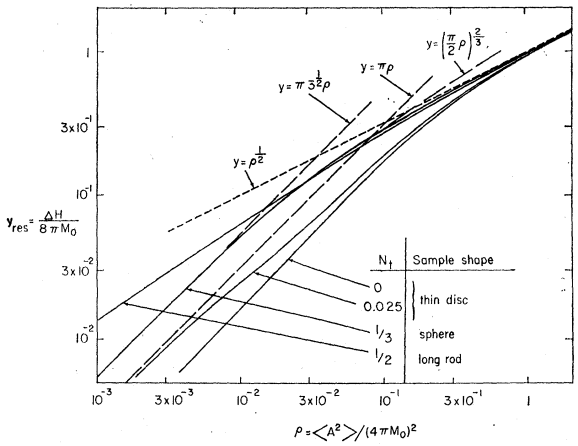

FIG. 9. Reduced linewidth y versus inhomogeneity parameter ρ for different sample shapes.

In order to calculate the linewidth the effective damping y has to be evaluated at resonance, i.e., when the first term in the denominator of Eq. (67) vanishes. According to Eq. (53) this is the case when

$$c+2x=1-2N_t. \tag{70}$$

The functions F_1 and F_2 of the self-consistency condition (60) depend upon c and x only through the combination $c+2x$ [see Eqs. (57)–(59)]. For finding y at resonance it is therefore sufficient to consider only the second line of Eq. (60), and solve it subject to the condition (70).

The results of numerical calculations of the reduced linewidth y_{res} for various sample shapes are shown in Fig. 9. For $\rho\ll 1$ and $N_t\neq\frac{1}{2}$ the asymptotic dependence of y_{res} is given by

$$y_{\mathrm{res}}\simeq\pi(1-2N_t)^{-1/2}\rho, \quad \rho\ll 1. \tag{71a}$$

For $N_t=\frac{1}{2}$ (the case of a long, axially magnetized cylinder)

$$y_{\mathrm{res}}\simeq(\pi\rho/2)^{2/3}, \quad \rho\ll 1. \tag{71b}$$

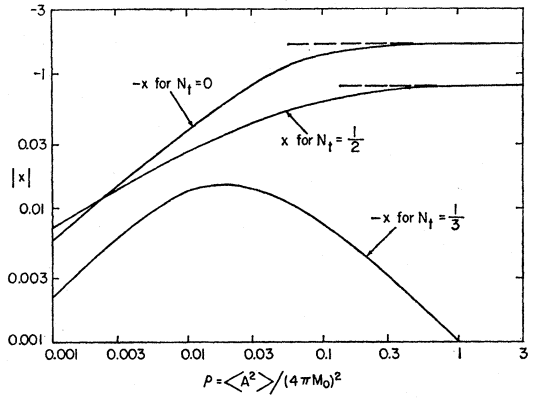

FIG. 10. Reduced line shift x versus inhomogeneity parameter ρ for different sample shapes.

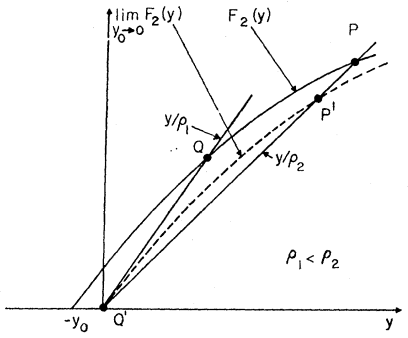

FIG. 11. Graphical method for solving the self-consistency condition.

For larger ρ, however,

$$y_{\text{res}} \simeq \rho^{1/2}, \quad \rho \gtrsim 1. \tag{71c}$$

The asymptotic formulas (71a)–(71c) are represented by the broken straight lines in Fig. 9.

It is interesting to note that for thin disks the predicted linewidth depends quite strongly upon the demagnetizing factor in the region of fairly small ρ (i.e., $\rho \simeq 10^{-2}$).

The line shift S of the absorption maximum is according to Eq. (67) given by

$$S/4\pi M_0 = -x, \tag{72}$$

where x is the solution of the first line of the self-consistency condition (60). Figure 10 shows a plot of the reduced line shift x versus ζ for spheres, thin disks, and long cylinders. For spheres and thin discs the solution x of the self-consistency condition is negative, for long cylinders it is positive. This may be anticipated from the behavior of the function $F_1(0,y,c)$ shown in Fig. 5. The predicted line shift is positive (shift towards higher fields) for spheres and thin discs and negative for long cylinders. In Fig. 10 $|x|$ is plotted versus ρ. As previously pointed out [Eqs. (63) and (64)] $|x|$ increases linearly with ρ for $\rho \ll 1$ and approaches a constant value for $\rho \gg 1$. In the case of spherical samples, the asymptotic limit of $|x|$ for $\rho \gg 1$ is zero. The asymptotic limiting values of $|x|$ for $\rho \gg 1$ are indicated by the broken lines in Fig. 10.

7. OFF-RESONANCE ABSORPTION

If the intrinsic damping is small, the primary scattering contributes to the off-resonance absorption only as long as the signal frequency lies inside the frequency band corresponding to spin waves of long wavelength. The previous theories of coarse-grain inhomogeneous line broadening,[1,3,12] in which secondary scattering was neglected, therefore, predict a sharp break in the absorption curve as the signal frequency goes through the edge of the spin wave band. It may be seen from Fig. 8 that with the inclusion of secondary scattering the theory predicts a much more gradual transition, even in the limit of zero intrinsic damping. As ρ increases, the region of substantial absorption spreads over a wider field interval. In the present section the width of this region of substantial absorption is discussed in detail.

Consider first the behavior of the effective damping constant y [the solution of Eq. (60)] in the limit of vanishingly small intrinsic damping y_0. Figure 11 demonstrates a graphical method for solving the self-consistency condition (60) in particular the second line of this equation. It can readily be shown that for small damping the function $F_2(x,y,c)$ increases approximately linearly with y_0+y provided that either $c+2x<0$ or $c+2x>1$. The function $F_2(x,y,c)=F_2(y)$ considered as a function of y for constant x, c is, therefore, qualitatively described by the full, curved line in Fig. 11. The solution of the self-consistency condition is given by the intersection of the curve $F_2(y)$ with the straight line y/ρ also shown in Fig. 11. Two values of ρ have been assumed giving rise to intersections at P and Q. Taking the limit of vanishingly small intrinsic damping ($y_0 \to 0$)

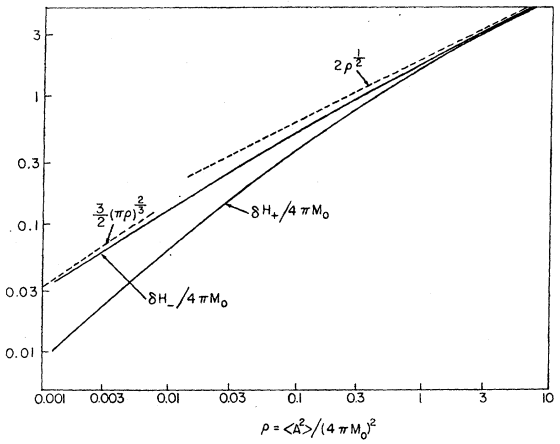

FIG. 12. Shift of the low- and high-field absorption edges (δH_- and δH_+) versus inhomogeneity parameter ρ.

corresponds to shifting the curve representing $F_2(y)$ to the right until it intersects the origin. The solution points P and Q then move to P' and Q', respectively. This demonstrates that for sufficiently large ρ the effective damping approaches a finite value in the limit $y_0 \to 0$, whereas it goes to zero for sufficiently small ρ. For given ρ it is, therefore, possible to define a "strong absorption region" in which y remains finite for $y_0 \to 0$. For $\rho \ll 1$ this region coincides with the spin wave band. It may be anticipated that with increasing ρ the width of this region will increase.

According to the geometrical construction shown in Fig. 11 the edges of the strong absorption region are determined by

$$\begin{aligned} x &= \rho F_1(x,0,c), \\ 1 &= \rho(\partial F_2/\partial y)(x,0,c). \end{aligned} \tag{73}$$

The solution of these equations for c as a function of ρ

is described in Appendix B. The results are summarized in Fig. 12, where the shift δH_- of the low-field absorption edge towards lower fields and the shift δH_+ of the high-field absorption edge toward higher fields are plotted versus ρ. For $\rho \ll 1$ the predicted behavior is

$$\begin{aligned} &\delta H_-/4\pi M_0 \simeq \tfrac{3}{2}(\pi\rho)^{2/3}, \\ &\delta H_+/4\pi M_0 \simeq \rho[1+\ln(2/\rho)], \end{aligned} \tag{74}$$

whereas for $\rho \gg 1$,

$$\delta H_-/4\pi M_0 \simeq \delta H_+/4\pi M_0 \simeq 2\rho^{1/2}. \tag{75}$$

It is interesting to note that for $\rho \gg 1$ δH_-, δH_+, and ΔH all become equal. Thus the strong absorption region is broadened approximately twice as much as the resonance line.

It is also interesting to note that the real and the imaginary parts of the frequency shift contribute approximately equally strongly to the broadening of the strong absorption region. It is therefore not permissible in the present context to disregard the real part of the frequency shift.

Finally, consider the absorption outside of the strong absorption region. In this case the damping is small, so that the general self-consistency condition (60) can be approximated by

$$x=\rho F_1(x,0,c), \quad y=\rho(y+y_0)\partial F_2/\partial y(x,0,c). \tag{76}$$

From the second line of this equation

$$y+y_0=y_0/(1-\rho(\partial F_2/\partial y)). \tag{77}$$

Consider now the absorption coefficient $\mu_{+\,\mathrm{eff}}''$ as given by Eq. (67). In the weak absorption regions the linewidth term $(y+y_0)^2$ is usually negligible compared to $(u-v+N_t+x)^2$ in the denominator of Eq. (67). For convenience we express the absorption coefficient as the product of two terms

$$\mu_{+\,\mathrm{eff}}''=[y_0/(u-v+N_t)^2]G(c,\rho). \tag{78}$$

The first factor is the absorption coefficient expected for a given intrinsic damping y_0 in the absence of scattering (i.e., for $\rho=0$). The second factor describes to what extent the absorption is increased by the scattering. According to Eqs. (67), (77), and (78) the enhancement factor is

$$G(c,\rho)=\{[1-\rho(\partial F_2/\partial y)](1+2x/(c-1+2N_t))^2\}^{-1}, \tag{79}$$

where u-v has been expressed in terms of c by means of Eq. (53) and x is to be determined as a function of ρ and c from the first line of Eq. (76). Equation (79) shows that the enhancement factor becomes infinite at the edges of the strong absorption region (since $\rho\partial F_2/\partial y \to 1$).

Explicit formulas for F_1 and $\partial F_2/\partial y$ are given in Appendix B. With the help of these formulas the enhancement factor has been calculated for various assumed values of ρ. The results are shown in Fig. 13.

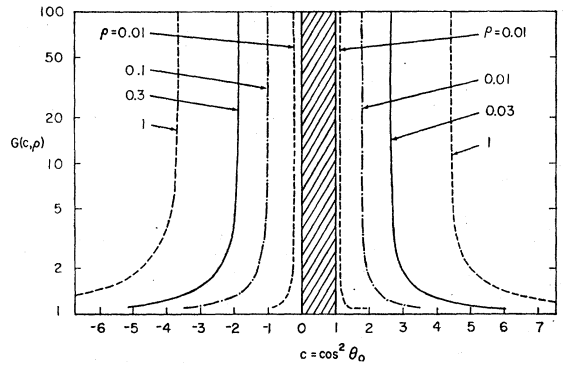

FIG. 13. Enhancement factor $G(c,\rho)$ which characterizes the absorption in the exterior of the strong absorption region plotted versus $c=\cos^2\theta_0$ for given ρ. The shaded area is the strong absorption region for $\rho \to 0$.

8. DISCUSSION

In order to compare the results of the theory described in the present paper with experimental data the inhomogeneity parameter ρ defined by Eq. (49) must be determined. For polycrystals of cubic materials, the contribution of crystalline anisotropy to the mean square of the inhomogeneous field is[3]

$$\langle A^2\rangle_{\mathrm{anis}}=(4/21)H_a^2, \tag{80}$$

where $H_a=2K_1/M_0$ and K_1 is the first-order anisotropy constant, provided that contributions of higher order are negligible. Thus, the inhomogeneity parameter ρ is in this case

$$\rho_{\mathrm{anis}}=(4/21)(H_a/4\pi M_0)^2. \tag{81}$$

Applying these results to the predictions concerning the linewidth, one finds that for spherical samples

$$\begin{aligned} \Delta H &\simeq 2.07H_a^2/4\pi M_0, \quad \text{for} \quad H_a \ll 4\pi M_0 \\ &\simeq 0.87H_a, \quad \text{for} \quad H_a \gtrsim 4\pi M_0. \end{aligned} \tag{82}$$

The transition between the two regimes occurs approximately when $\rho=0.05$, i.e., $H_a/4\pi M_0 \simeq \frac{1}{2}$. The result (82) for the weak-anisotropy case agrees exactly with previous calculations.[1,3] For the case of large anisotropy a rough estimate of the linewidth had previously lead to $\Delta H \simeq 0.5H_a$.[5] The new calculation is considered to be more reliable.

The theory described in the present paper is based upon three significant approximations. The first of these consists in neglecting those terms in the Hamiltonian that relate to the ellipticity of the spin precession. This approximation is well justified when $H \gg 2\pi M_0$, a condition which is reasonably well satisfied in many cases of practical interest. As a consequence of this approximation the resonance frequency of the uniform mode of long, axially magnetized cylinders coincides with the upper edge of the frequencyband corresponding to spin waves of long wavelength. In a more rigorous

calculation the resonant frequency of the uniform mode would fall outside of this band. The theory in its present form should, therefore, not be expected to give reliable predictions about the frequency dependence of the linewidth when the frequency is smaller than or comparable with $\gamma 4\pi M_0$. This applies particularly to the case of a weak inhomogeneity. For strong inhomogeneity the structure in the frequency and field dependence of the effective damping constant y is largely smeared out (see Fig. 8), so that the precise location of the uniform mode in relation to the spin-wave band becomes immaterial. A theory in which the ellipticity of the spin precession is correctly taken into account, but secondary scattering is neglected, has previously been described.[3]

The second significant approximation consists in replacing the infinite perturbation series which represents the effective resonant frequency [see Eq. (23)] by the self-consistency condition [see Eq. (40)]. In this transition all contributions of odd power in the perturbation have been neglected. In its application to cubic polycrystals the approximation has the consequence that the sign of the anisotropy constant K_1 becomes immaterial. Experimentally it is found that the maximum of the absorption line is shifted toward lower fields when $K_1<0$ and towards higher fields when $K_1>0$.[5,23] This agrees with theoretical predictions based upon the "independent grain" approach described briefly in the introduction. In the collective approach described in the present paper the effect is contained in the perturbation terms of odd order which were neglected by way of approximation.

The third significant approximation lies in replacing the self-consistence condition (40), which is an integral equation, by the much simpler condition (45) [or equivalently Eq. (48)]. The secondary scattering is actually stronger than taken into account in this approximation. A more rigorous calculation which avoids this approximation may, therefore, be expected to lead to a smaller linewidth ΔH and a larger broadening (δH_+ and δH_-) of the strong absorption region than the present calculation.

According to the theory described in the present paper the "effective" damping of the spin waves that interact with the uniform mode is of the same magnitude as the damping of the uniform mode itself. Here the effective spin wave damping is that experienced by any given mode if only this mode is driven by an rf field. It is in general different from the spin wave damping inferred from the instability threshold at high power levels.[24] A more detailed calculation on this subject has shown, in fact, that the scattering has only a minor effect upon the instability threshold provided that the wavenumber of the unstable modes is considerably larger than the cutoff wavenumber k_0 introduced in Sec. 5.

The linewidth of polycrystalline garnets containing calcium, vanadium and indium substitutions has recently been investigated by Van Hook *et al.*[25–27] In these materials the linewidth is primarily due to anisotropy broadening. The strength of the anisotropy field can be controlled by varying the temperature and the amount of indium substitution. The data show that the linewidth is proportional to $H_a^2/4\pi M$ for $H_a \ll 4\pi M_0$ but proportional to H_a for $H_a \gtrsim 4\pi M_0$. The linewidth measured in these materials agrees very well with the theory described in the present paper if allowance is made for some residual porosity broadening, which becomes important when the anisotropy is small. Reasonably good agreement with the theory is obtained over more than three decades of the anisotropy parameter ρ.

The off-resonance loss in porous polycrystalline YIG has been studied by Kohane and Schlömann.[19] The results demonstrate the existence of a well-defined strong absorption region, which is observed to be considerably broader than the spin-wave band at low wave numbers. The results agree qualitatively with the theory described in the present paper, but a quantitative comparison cannot be made, because the line broadening is predominantly caused by magnetic inhomogeneity in these materials.

Measurements of the off-resonance loss in materials with strong anisotropy broadening have been reported by Patton.[28] The effective linewidth, considered as a function of dc field strength, inferred from these measurements agrees very well with the predictions of the present theory.

Similar results have also been reported by Vrehen[29] and Vrehen *et al.*[30] These authors have independently developed a theory of inhomogeneous line broadening, in which the secondary scattering is taken into account in a different way.

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APPENDIX A: ASYMPTOTIC SOLUTION OF SELF-CONSISTENCY CONDITION (60) FOR LARGE ρ

For $\rho \gg 1$, the solution x, y of the self-consistency condition (60) can be represented by a power series in $\rho^{-1/2}$ of the form

$$\begin{aligned} x &= x_{-1}\rho^{1/2}+x_0+x_1\rho^{-1/2}+\cdots, \\ y &= y_{-1}\rho^{1/2}+y_0+y_1\rho^{-1/2}+\cdots. \end{aligned} \tag{A1}$$

The calculation shows that $x_{-1}=0$, $y_{-1}=1$, so that for $\rho \gg 1$, $y \gg 1$, and $y \gg x$. Anticipating that $x_{-1}=0$ and $y_{-1}=1$, one obtains from Eqs. (57)–(59),

$$\begin{aligned} r &\simeq 2\rho^{1/2}(1+\rho^{-1/2}y_0+\cdots), \\ p &\simeq \rho^{1/4}[1+\tfrac{1}{2}\rho^{-1/2}(y_0+x_0+\tfrac{1}{2}c)+\cdots], \\ q &\simeq \rho^{1/4}[1+\tfrac{1}{2}\rho^{-1/2}(y_0-x_0-\tfrac{1}{2}c)+\cdots]. \end{aligned} \tag{A2}$$

Hence

$$\begin{aligned} (r+1)/2p &= \rho^{1/4}[1+\tfrac{1}{2}\rho^{-1/2}(y_0-x_0+1-\tfrac{1}{2}c) \\ &\qquad +\cdots], \\ (r-1)/2q &= \rho^{1/4}[1+\tfrac{1}{2}\rho^{-1/2}(y_0+x_0-1+\tfrac{1}{2}c)+\cdots]. \end{aligned} \tag{A3}$$

Higher powers of $\rho^{-1/2}$ have been nelgected. Inserting Eqs. (A2) and (A3) into Eqs. (55) and (56), one obtains

$$\begin{aligned} F_1(x,y,c) &\simeq \rho^{-1}(\tfrac{1}{6}-x_0-\tfrac{1}{2}c)+O(\rho^{-3/2}), \\ F_2(x,y,c) &\simeq \rho^{-1/2}-\rho^{-1}y_0+O(\rho^{-3/2}). \end{aligned} \tag{A4}$$

Thus the solutions of the self-consistency condition (60) are, for $\rho \gg 1$, given by Eq. (A1) with

$$\begin{aligned} x_{-1} &= 0, \quad x_0 = \tfrac{1}{4}(\tfrac{1}{3}-\cos^2\theta_0), \\ y_{-1} &= 1, \quad y_0 = 0. \end{aligned} \tag{A5}$$

APPENDIX B: CALCULATION OF ABSORPTION EDGES

Two cases must be distinguished in which either $c+2x<0$ or $c+2x>1$.

Case 1. $c+2x<0$.

We define

$$\delta = -(c+2x) \tag{B1}$$

and obtain from Eqs. (57)–(59) for $y_0=0$, $y \to 0$,

$$r \simeq \delta+2y^2/\delta, \quad p \simeq y\delta^{-1/2}, \quad q \simeq \delta^{1/2}. \tag{B2}$$

Hence from Eqs. (55) and (56)

$$\begin{aligned} F_1(x,0,c) &= (1/\delta^{1/2})\ \cot^{-1}((\delta-1)/2\delta^{1/2}), \\ dF_2/dy(x,0,c) &= [2/\delta(\delta+1)] \\ &\qquad +1/\delta^{3/2}\cot^{-1}((\delta-1)/2\delta^{1/2}). \end{aligned} \tag{B3}$$

Inserting this into Eq. (73) the two variables δ and x can be determined for given ρ. The shift δH_- of the absorption edge is now according to Eq. (B1) given by

$$\delta H_-/4\pi M_0 = -\tfrac{1}{2}c = x+\tfrac{1}{2}\delta. \tag{B4}$$

The asymptotic formulas (74) and (75) are obtained by noting that for $\delta \ll 1$

$$F_1 \simeq \pi\delta^{-1/2}, \quad dF_2/dy \simeq \pi\delta^{-3/2}, \tag{B5}$$

whereas for $\delta \gg 1$,

$$F_1 \simeq 2\delta^{-1}, \quad dF_2/dy \simeq 4\delta^{-2}. \tag{B6}$$

Case 2. $c+2x>1$.

We define

$$\delta = c+2x-1 \tag{B7}$$

and obtain from Eqs. (57)–(59) for $y_0=0$, $y \to 0$,

$$\begin{aligned} r &\simeq 1+\delta+2y^2/(1+\delta), \\ p &\simeq (1+\delta)^{1/2}, \\ q &\simeq y(1+\delta)^{-1/2}. \end{aligned} \tag{B8}$$

Hence from Eqs. (55) and (56)

$$\begin{aligned} F_1(x,0,c) &= -(1+\delta)^{-1/2} \\ &\qquad \times \coth^{-1}[(1+\tfrac{1}{2}\delta)/(1+\delta)^{1/2}], \\ (dF_2/dy)(x,0,c) &= [2/\delta(1+\delta)]+(1+\delta)^{-3/2} \\ &\qquad \times \coth^{-1}[(1+\tfrac{1}{2}\delta)/(1+\delta)^{1/2}]. \end{aligned} \tag{B9}$$

These expressions are again inserted into Eq. (73) and x and δ are determined as functions of ρ. The shift δH_+ of the absorption edge is in this case

$$\delta H_+/4\pi M_0 = \tfrac{1}{2}(c-1) = -x+\tfrac{1}{2}\delta. \tag{B10}$$

The asymptotic formulas (74) and (75) follow from the observation that for $\delta \ll 1$

$$F_1 \simeq -\ln(4\delta^{-1}), \quad dF_2/dy \simeq 2\delta^{-1}, \tag{B11}$$

whereas for $\delta \gg 1$,

$$F_1 \simeq -2\delta^{-1}, \quad dF_2/dy \simeq 4\delta^{-2}. \tag{B12}$$