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One-Dimensional Ising Model with Random Exchange Energy

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The one-dimensional Ising model with random exchange energy is formulated in terms of a homogeneous integral equation. Assuming the input distribution has a narrow width proportional to N^{-1}, the integral equation is solved by perturbation in N^{-1}. The shift in the free energy of the system up to the order of N^{-2} is given. It is found that for a symmetrical distribution, the shift due to the randomness is second order in N^{-1} and *negative*, depending only upon the variance of the input distribution. The first-order shift for the asymmetric distribution comes entirely from the asymmetry. After a shift in energy is made to account for this asymmetry, the effective shift is identical to that for the symmetrical case. The shifts of all the thermodynamic properties of the system are also given. The randomness is found to *decrease* the magnetization for all temperature and applied field. However, shifts in magnetic susceptibility and specific heat are *oscillatory* in sign.

1. INTRODUCTION

RECENTLY the two-dimensional Ising model with limited amount of randomness in exchange energy was studied by McCoy and Wu.[1] The object of their study was to ascertain what the effect of random impurities would be on the nature of the phase transition. Because of the mathematical complexity associated with the phase transition, they were able to study this problem only for one particular narrow distribution of the exchange energy. They conjectured that the results are much more general than the particular example that they solved and stated their belief that the qualitative effect of randomness on the phase transition should be the same for a large class of narrow distributions of exchange energy.

It is the purpose of this paper to study the corresponding problem of randomness in the 1-dimensional Ising model in the presence of a magnetic field. Randomness is introduced by allowing each exchange energy to be an independent random variable E with a probability density function $P(E)$. Because there is no phase transition at nonzero temperature,[2,3] the mathematics is sufficiently simple to allow us to study the problem for a large class of narrow distributions. We demonstrate that for any symmetrical narrow distribution $P(E)$ the free energy is decreased by an amount proportional to the second moment of $P(E)$. We furthermore find for an asymmetrical $P(E)$ the only effect of the asymmetry is to replace the most probable energy by the mean energy. Once this additional first-order shift is made, the second-order term is the same as in the symmetrical case. This calculation therefore explicitly exhibits a system where the effect of randomness is determined by general properties of the distribution rather than its detailed form.

Along with the change in the free energy due to randomness in Eq. (4.18) and Fig. 3, we have also computed leading order shifts of several thermodynamic quantities: the magnetization M_1, the magnetic susceptibility χ_1, and the specific heat C_{v_1}. They are expressed in Eqs. (5.2), (5.4), (5.6), and Figs. 4–6. The shift of the magnetization is *negative* and vanishes at zero field. It approaches zero exponentially as $H\to\infty$, where H is the external magnetic field. The shift of magnetic susceptibility χ_1 is found to be negative at $H=0$. With increasing H is rises to a *positive* maximum and then decreases to zero, also exponentially. The shift of the specific heat C_{v_1} at any magnetic field vanishes both at zero temperature and infinite temperature. It oscillates in sign as temperature rises from zero to infinity. This shift due to randomness is seen to flatten the nonrandom specific heat curve.

We present the mathematical formulation of this problem in Sec. 2 which is based upon the application of theory of noncommuting random products by Furstenberg.[4] We closely follow the procedure of Ref. 1 and express the free energy per site in the presence of a constant magnetic field in terms of an average over an auxiliary variable x whose distribution function $\nu(x)$ is determined by a homogeneous integral equation. In Sec. 3, we discuss the properties of this integral equation. We assume that $P(E)$ has a narrow width proportional to N^{-1} and possesses well-defined asymptotic behavior. We then obtain a set of integral equations for respective orders in N^{-1} of $\nu(x)$ which are readily solved by Fourier transform. The shift in the free energy due to this randomness in energy is given in Sec. 4. It is related to the moments of $P(E)$ and $\nu(x)$. In Sec. 5, we give the corresponding shifts in the thermodynamic quantities as mentioned above.

2. GENERAL FORMULATION

We consider a one-dimensional array of $\mathfrak{N}$ Ising spins, which are labelled from 1 to $\mathfrak{N}$ along the chain from left to right. The Hamiltonian of the system is

$$\mathcal{H}=-\sum_{n=1}^{\mathfrak{N}-1} E(n)\sigma_{n+1}\sigma_n - H\sum_{n=1}^{\mathfrak{N}}\sigma_n\,, \tag{2.1}$$

[1] B. M. McCoy and T. T. Wu, Phys. Rev. **176**, 631 (1968).
[2] G. Rushbrooke and H. Ursell, Proc. Cambridge Phil. Soc. **44**, 263 (1948).
[3] L. Van Hove, Physica **16**, 137 (1950).
[4] H. Furstenberg, Ann. Math. Soc. Trans. **108**, 377 (1963).

where $\sigma_n = \pm 1$, $E(n)$ is the exchange energy between nth and $(n+1)$th sites, H is the magnetic field, and sites 1 and $\mathfrak{N}$ are not connected together.

Denote the partition function for $n(n<\mathfrak{N})$ spins counted from the left end of the linear chain from 1 to n by Z_n. Clearly, Z_n can be separated into two parts corresponding to the two states of the last (nth) spin.

$$Z_n = U_n + V_n ,$$

where U_n is the part of Z_n with $\sigma_n = +1$ and V_n is the part of Z_n with $\sigma_n = -1$.

Consider the addition of the $(n+1)$th spin to the chain. The partition function Z_{n+1} is built up from Z_n by the following transfer matrix which transforms the column vector:

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} \text{ into } \begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix}, \quad \begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = T_n \begin{pmatrix} U_n \\ V_n \end{pmatrix}, \tag{2.2}$$

where

$$T_n = (wz^*(n))^{-1/2} \begin{pmatrix} 1 & z^*(n) \\ wz^*(n) & w \end{pmatrix} \tag{2.3}$$

and

$$w = e^{-2\beta H}, \quad z^*(n) = e^{-2\beta E(n)}, \quad \beta = (kT)^{-1}, \tag{2.4}$$

and k is Boltzmann's constant.

Equation (2.2) represents a recursion relation for the two-dimensional vectors

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

with an initial condition for the column vector of

$$\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = w^{-1/2} \begin{pmatrix} 1 \\ w \end{pmatrix}.$$

The free energy per site of the Ising system $\mathfrak{F}$ in the thermodynamic limit is

$$\mathfrak{F} = -\beta^{-1} \lim_{\mathfrak{N}\to\infty} \mathfrak{N}^{-1} \ln(U_{\mathfrak{N}} + V_{\mathfrak{N}})$$
$$= -\beta^{-1} \lim_{\mathfrak{N}\to\infty} \mathfrak{N}^{-1} \sum_{n=1}^{\mathfrak{N}} \ln\left[\frac{U_{n+1} + V_{n+1}}{U_n + V_n}\right]. \tag{2.5}$$

We consider $E(n)$ to be random variables with their probability distribution $P(E)$. Then T_n is a random matrix acting upon a two-dimensional vector space.

We define

$$x_n = V_n / U_n , \tag{2.6}$$

which corresponds to the tangent of the angle which the random vector

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

makes with the V axis. From (2.2) we see that whenever T_n acts on

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix},$$

x_n is changed into x_{n+1} by

$$x_{n+1} = w(z^*(n) + x_n)/(1 + z^*(n) x_n) . \tag{2.7}$$

The matrices in (2.3) form a noncommuting set of random matrices. Furstenberg[4] has proved that as n becomes large the random variable x_n will approach a limiting stationary distribution $\nu(x)$ that is independent of the initial vector. This stationary distribution is characterized by the property that if we apply a random matrix (2.3) to it and average the resulting distribution over $P(E)$ that $\nu(x)$ will transform into itself. Therefore

$$\nu(x) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dE\, P(E) \nu(x')$$
$$\times \delta[x - w(z^* + x')/(1 + z^* x')]. \tag{2.8}$$

The existence and uniqueness of a solution to this equation is also proven by Furstenberg.[4]

With the stationary distribution $\nu(x)$, we can then replace the average over n in Eq. (2.5) by an average over the random variables E and x. Using (2.2), we have

$$\mathfrak{F} = -\beta^{-1} \lim_{\mathfrak{N}\to\infty} \mathfrak{N}^{-1} \sum_{n=1}^{\mathfrak{N}} \ln\{[wz^*(n)]^{-1/2}$$
$$\times [1 + z^*(n) x_n + wz^*(n) + wx_n]/(1 + x_n)\} \tag{2.9}$$

and we arrive at the final result that with probability 1

$$\mathfrak{F} = -\beta^{-1} \lim_{\mathfrak{N}\to\infty} \mathfrak{N}^{-1} \ln(U_{\mathfrak{N}} + V_{\mathfrak{N}})$$
$$= -\int EP(E) dE - H - \beta^{-1} \int dE \int dx$$
$$\times \nu(x) P(E) \ln[(1 + z^* x + wz^* + wx)/(1 + x)]. \tag{2.10}$$

It is expected that, since the free energy in (2.10) is essentially the average rate of growth of the quantity $(U_n + V_n)$, it must be equal to the average rate of growth of each of the component of the column vector, U_n and V_n. We denote

$$\mathfrak{F}^1 = -\beta^{-1} \lim_{\mathfrak{N}\to\infty} \mathfrak{N}^{-1} \sum_{n=1}^{\mathfrak{N}} \ln\left(\frac{U_{n+1}}{U_n}\right),$$
$$\mathfrak{F}^2 = -\beta^{-1} \lim_{\mathfrak{N}\to\infty} \mathfrak{N}^{\ 1} \sum_{n=1}^{\mathfrak{N}} \ln\left(\frac{V_{n+1}}{V_n}\right). \tag{2.11}$$

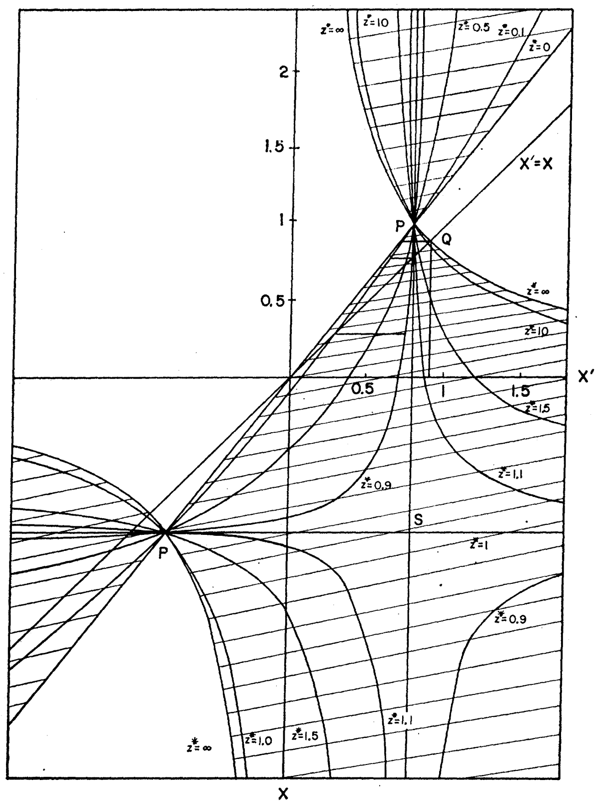

FIG. 1. Contours along which the function

$$P(-\tfrac{1}{2}\beta^{-1}\ln[(x-wx')/(w-xx')])$$

is constant. The kernel of the integral equation for $\nu(x)$ is different from zero only in the shaded region. The value of w is taken to be 0.8.

We shall prove that indeed $\mathfrak{F}=\mathfrak{F}^1=\mathfrak{F}^2$. Following the same line of argument, we find with probability 1

$$\mathfrak{F}^1=-\int EP(E)dE-H-\beta^{-1}\int dE\int dxP(E)\nu(x)\times\ln(1+z^*x), \quad (2.12)$$

$$\mathfrak{F}^2=-\int EP(E)dE-H-\beta^{-1}\int dE\int dxP(E)\nu(x)\times\ln(wx^{-1}(z^*+x)). \quad (2.13)$$

We call $d_1=\mathfrak{F}^1-\mathfrak{F}^2$, $d_2=\mathfrak{F}-\mathfrak{F}^1$; from (2.12) and (2.13) we have

$$d_1=-\beta^{-1}\int dE\int dxP(E)\nu(x)\ln(xw^{-1}(1+z^*x)/(z^*+x)),$$

$$=-\beta^{-1}\int dx\nu(x)\ln x+\beta^{-1}\int dE\int dxP(E)\nu(x)\times\ln[w(z^*+x)/(1+z^*x)]. \quad (2.14)$$

Now we use the relation

$$\ln[w(z^*+x)/(1+z^*x)]=\int d\zeta\,\delta(\zeta-w(z^*+x)/(1+z^*x))\ln\zeta, \quad (2.15)$$

$$d_1=-\beta^{-1}\int dx\nu(x)\ln x+\beta^{-1}\int dE\int dx\int d\zeta P(E)\nu(x)\times\delta(\zeta-w(z^*+x)/(1+z^*x))\ln\zeta. \quad (2.16)$$

Interchanging the order of integration, we have

$$d_1=-\beta^{-1}\int dx\nu(x)\ln x+\beta^{-1}\int d\zeta\ln\zeta\int dE\int dxP(E)\nu(x)\times\delta(\zeta-w(z^*+x)/(1+z^*x)). \quad (2.17)$$

As seen from the integral Eq. (2.8), d_1 is identically zero.

Similarly, using

$$\ln(1+w(z^*+x)/(1+z^*x))=\int d\zeta\,\delta(\zeta-w(z^*+x)/(1+z^*x))\ln(1+\zeta), \quad (2.18)$$

$$d_2=0.$$

We note that when the distribution function $P(E)$ is a δ function, $P(E)=\delta(E-E_0)$, then $\nu(x)$ will also be a δ function at that value of x which remains unchanged by the application of the transform of Eq. (2.7),

$$x_0=w(z_0^*+x_0)/(1+z_0^*x_0), \quad (2.19)$$

where

$$z_0^*=e^{-2\beta E_0}.$$

There are two solutions to this equation because the 2 by 2 matrix has two eigenvalues. Since the free energy is obtained through the selection of the larger eigenvalue, we must accordingly choose the correct one for x

$$x_0(E_0)=\tfrac{1}{2}[w-1+((w-1)^2+4z_0^{*2}w)^{1/2}]/z_0^*. \quad (2.20)$$

Inserting x_0 into (2.12), we find

$$\mathfrak{F}_0=-E_0-\beta^{-1}\ln(1+z_0^*x_0)-H,$$
$$=-E_0-\beta^{-1}\ln[\cosh\beta H+(\cosh_2\beta H-2e^{-2\beta E_0}\sinh 2\beta E_0)^{1/2}], \quad (2.21)$$

which is exactly the solution obtained by Ising in 1925.[5]

For the case where the magnetic field is zero ($w=1$), we have from (2.10) and (2.4)

$$\mathfrak{F}=-\beta^{-1}\int\ln(2\cosh\beta E)P(E)dE. \quad (2.22)$$

We observe that in this expression there is no dependence upon x and $\nu(x)$; the free energy is just an

[5] E. Ising, Z. Physik 31, 253 (1925).

average over $P(E)$. This result is to be expected because when $H=0$ it is elementary to show that for any set of $E(n)$ that

$$-\beta \ln Z_{\mathfrak{N}} = \sum_{n=1}^{\mathfrak{N}-1} \ln[2 \cosh\beta E(n)]. \qquad (2.23)$$

3. INTEGRAL EQUATIONS

In this section, we study the integral Eq. (2.8). Integrating over E, we obtain

$$\nu(x) = \int_{-\infty}^{\infty} \tfrac{1}{2}\beta^{-1} w(1-x'^2)(w-xx')^{-1}(x-wx')^{-1}$$
$$\times P(-\tfrac{1}{2}\beta^{-1} \ln[(x-wx')/(w-xx')])\nu(x')dx'. \qquad (3.1)$$

We readily see that the kernel in Eq. (3.1) will not become singular if the given distribution $P(E)$ vanishes rapidly at infinity such that

$$e^{E}P(E) \to 0 \quad \text{as} \quad E \to \infty. \qquad (3.2)$$

The integration limits formally written as $-\infty$ to $+\infty$ are actually determined by the domain of $P(E)$ and Eq. (2.11), that is,

$$\begin{aligned} &\text{if} \quad P(E)=0, \quad \text{unless} \quad E_1<E<E_2 \\ &\text{then} \quad \nu(x)=0, \quad \text{unless} \quad x_2=x<x_1 \end{aligned} \qquad (3.3)$$

where $x_1=x_0(E_1)$, $x_2=x_0(E_2)$. The detail is shown in Figs. 1 and 2.

The exact solution to the integral Eq. (3.1) is difficult, here we shall only explore some general characteristic of the solution for the class of narrow input distributions.

Assume that the input distribution $p(E)$ is of the following form:

$$P(E) = N\Delta^{-1}f(N\Delta^{-1}(E-E_0)) + N^{-1}h(E-E_0), \qquad (3.4)$$

where Δ is a unit for the width of the energy spread, N is a dimensionless scaling factor introduced to indicate the narrowness of the width, E_0 is the most probable energy where f is a maximum. Since $P(E)$ is a probability distribution function so clearly, we have

$$\int_{-\infty}^{\infty} f(y)dy = 1; \qquad (3.5)$$

for concreteness we further make the following assumptions:

(i) $f(y)$ is an entire function.

(ii) It also possesses finite moments of every order which behave asymtotically as

$$\int y^n f(y)dy = 0(n!/s^n) \quad \text{as } n \to \infty \qquad (3.6\text{a})$$

for some positive number s.

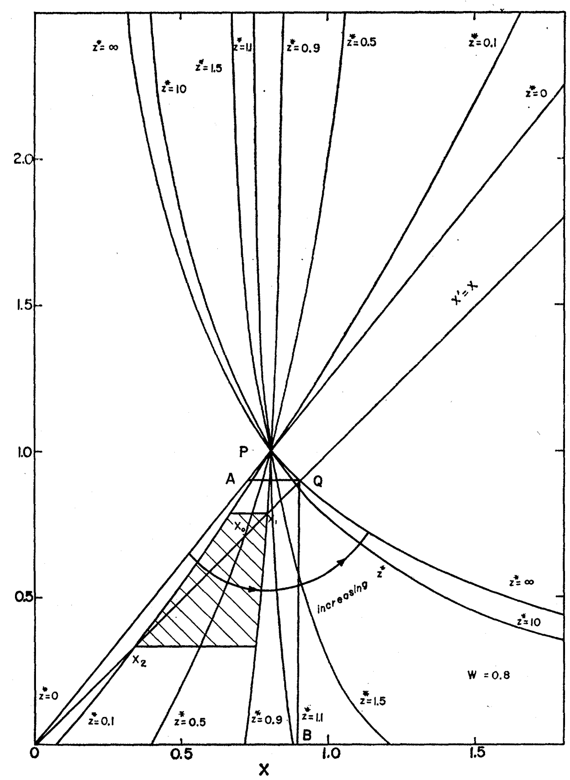

FIG. 2. Enlargement of Fig. 1. The shaded region is the only region in which $\nu(x)$ and $\nu(x')$ are different from zero.

(iii) $h(y)$ is regular at $y=0$.[6]

The second assumption directly implies[7] that:

(a) The condition (3.2) is met by this class of distribution functions; hence no singularity will occur in the kernel of the integral equations.

(b) The Fourier transform of $f(y)$, $F(k)$, where

$$F(k) = \int_{-\infty}^{\infty} e^{-iky} f(y)dy,$$

possesses derivatives of arbitrarily high order on an interval $-s<k<s$ and it has a Taylor series near the origin with a radius of convergence s, i.e.,

$$\max|(d^n/dk^n)F(k)| = 0(n!/s^n) \quad \text{for} \quad -s<k<s. \qquad (3.6\text{b})$$

(c) The moments of the distribution $f(y)$ are related to the derivatives of $F(k)$ at the origin by

$$my_n = (-i)^{-n}(d^n/dk^n)F(k)|_{k=0}. \qquad (3.7)$$

With these conditions $P(E)$ will approach δ function $\delta(E-E_0)$ as $N\to\infty$. Correspondingly, $\nu(x)$ will also be a function of $N(x-x_0)$, where $x_0 \equiv x_0(E_0)$. We further-

[6] These assumptions are not the weakest possible set and are made merely to simplify some of our notations. They are broad enough, however, to illustrate the physics of the situation.

[7] A. Winter, *The Fourier Transform of Probability Distributions* (Johns Hopkins University Press, Balitmore, Md., 1947).

more expand $\nu(x)$ as

$$\nu(x)=N\bar{g}_0(N(x-x_0))+\bar{g}_1(N(x-x_0))+N^{-1}\bar{g}_2(N(x-x_0))+\cdots, \quad (3.8)$$

where $\bar{g}_{01}$, $\bar{g}_1$, and $\bar{g}_2$ are all unknown functions to be determined from the integral equation with the normalization requirement

$$\int_{-\infty}^{\infty}\bar{g}_i(x)dx=\delta_{i0} \quad i=1,2,0\cdots. \quad (3.9)$$

We change the variables to

$$\xi=N(x-x_0),\quad \xi'=N(x'-x_0). \quad (3.10)$$

The argument in $P(E)$ in (3.1) can be expanded in terms of ξ and ξ' by substitution of (3.4) and (3.8) and an expansion in series of N^{-1}. Thus,

$$P(-\tfrac{1}{2}\beta^{-1}\ln[(x-wx')(w-xx')])=N\Delta^{-1}f(y), \quad (3.11)$$

where

$$y=-\tfrac{1}{2}(\beta\Delta x_0)^{-1}(1-w)^{-1}\{(1+z_0^*x_0)(\xi-A\xi')-\tfrac{1}{2}[Nx_0(1-w)]^{-1}((1-z_0^{*2}x_0^2)(\xi-A\xi')^2-4z_0^*x_0(\xi-A\xi')\xi'+2z^*{}_0x_0(w-1)A\xi'^2)+O(N^{-2})\} \quad (3.12)$$

and

$$A=(w-z_0^*x_0)/(1+z_0^*x_0). \quad (3.13)$$

We note also that $0<A<1$.

Define the new variables

$$\eta=-\tfrac{1}{2}(\beta\Delta)^{-1}\xi(1+z_0^*x_0)/(x_0(1-w)). \quad (3.14)$$

We then have

$$y=(\eta-A\eta')+\beta\Delta N^{-1}(1+z_0^*x_0)^{-2}[(\eta-A\eta')^2(1-z_0^{*2}x_0^2)-4z_0^*x_0(\eta-A\eta')\eta'+2z_0^*x_0(w-1)A\eta'^2]+O(N^{-2}).$$

From assumption (i) we can expand $f(y)$ into a Taylor series near the value $\eta-A\eta'$.

$$f(y)=f(\eta-A\eta')+\beta\Delta N^{-1}f'(\eta-A\eta')(1+z_0^*x_0)^{-2}\times[(1-z_0^{*2}x_0^2)(\eta-A\eta')^2-4z_0^*x_0(\eta-A\eta')\eta'+2z_0^*x_0(w-1)A\eta'^2]+O(N^{-2}). \quad (3.15)$$

Similarly, we expand

$$\tfrac{1}{2}\beta^{-1}w(1-x'^2)/[(w-xx')(x-wx')]=\tfrac{1}{2}(\beta x_0)^{-1}(1+z_0^*x_0)/(1-w)[1-2\beta\Delta N^{-1}\times(2z_0^*x_0\eta'(1+z_0^*x_0)^{-2}+(1+z_0^*x_0)^{-2}\times(z_0^{*2}x_0^2-1)(\eta-A\eta'))+O(N^{-2})]. \quad (3.16)$$

We further expand

$$\bar{\nu}(\eta)=g_0(\eta)+N^{-1}g_1(\eta)+O(N^{-2}), \quad (3.17)$$

where $\bar{\nu}(\eta)$ is related to $\nu(x)$ by $\nu(x)=\bar{\nu}(\eta)|d\eta/dx|$.

Thus, we have

$$g_0(\eta)+N^{-1}g_1(\eta)+O(N^{-2})=\int_{\eta_1}^{\eta_2}[1-2\beta\Delta N^{-1}(1+z_0^*x_0)^{-2}(2z_0^*x_0\eta'+(z_0^{*2}x_0^2-1)(\eta-A\eta'))+O(N^{-2})]$$
$$\times[f(\eta-A\eta')+\beta\Delta N^{-1}(1+z_0^*x_0)^{-2}f'(\eta-A\eta')((1-z_0^{*2}x_0^2)(\eta-A\eta')^2-4z_0^*x_0(\eta-A\eta')\eta'+2z_0^*x_0(w-1)A\eta'^2)]$$
$$\times[g_0(\eta')+N^{-1}g_1(\eta')+O(N^{-2})]d\eta', \quad (3.18)$$

where

$$\eta_i=-\tfrac{1}{2}N(\beta\Delta x_0)^{-1}(1+z_0^*x_0)(x_i-x_0)/(1-w), \qquad i=1,2. \quad (3.19)$$

For very large N, the integration limits can be approximated by $-\infty$ to ∞ since as a consequence of assumption (ii) the error introduced by this approximation is exponentially small in N. The contribution of $N^{-1}h(E-E_0)$ in (3.4) is only of N^{-2} order which can be neglected as we are only interested in the terms up to N^{-1} in (3.16).

Equating the two sides with respect to orders in N^{-1}, we obtain

$$g_0(\eta)=\int_{-\infty}^{\infty}f(\eta-A\eta')g_0(\eta')d\eta', \quad (3.20a)$$

$$g_1(\eta)=\int_{-\infty}^{\infty}f(\eta-A\eta')g_1(\eta')d\eta'-2\beta\Delta(1+z_0^*x_0)^{-2}\bigg\{\int_{-\infty}^{\infty}d\eta' g_0(\eta')f(\eta-A\eta')[2z_0^*x_0\eta'+(z_0^{*2}x_0^2-1)(\eta-A\eta')]$$
$$+\int_{-\infty}^{\infty}d\eta' g_0(\eta')f'(\eta-A\eta')[\tfrac{1}{2}(z_0^{*2}x_0^2-1)(\eta-A\eta')^2+2z_0^*x_0(\eta-A\eta')\eta'-z_0^*x_0A(w-1)\eta'^2]\bigg\}. \quad (3.20b)$$

We may readily solve (3.20a) by Fourier transform. Define

$$G_0(k)=\int_{-\infty}^{\infty} g_0(\eta)e^{-ik\eta}d\eta \tag{3.21a}$$

and

$$F(k)=\int_{-\infty}^{\infty} f(y)e^{-iky}dy. \tag{3.21b}$$

Note in particular that from (3.5) $F(0)=1$. Then (3.18a) yields

$$G_0(k)=G_0(Ak)F(k). \tag{3.22}$$

Since $A<1$, we may formally solve (3.22) to find

$$G_0(k)=G_0(0)\prod_{n=0}^{\infty} F(A^n k). \tag{3.23}$$

The infinite product converges if $F(k)$ is analytic at $k=0$. This follows from assumption (ii). Furthermore, from (3.9) we find $G(0)=1$, so that

$$G_0(k)=\prod_{n=0}^{\infty} F(A^n k). \tag{3.24}$$

We similarly Fourier transform (3.20b) to obtain

$$\begin{aligned}
G_1(k)&=\int_{-\infty}^{\infty} g_1(\eta)e^{-ik\eta}d\eta \\
&=\int_{-\infty}^{\infty} g_1(\eta')e^{-ikA\eta'}d\eta'\int_{-\infty}^{\infty} f(\eta-A\eta')e^{-ik(\eta-A\eta')}d\eta \\
&\quad -2\beta\Delta(1+z_0^*x_0)^{-2}\bigg\{\int_{-\infty}^{\infty} 2z_0^*x_0\eta' g_0(\eta')e^{-ikA\eta'}d\eta'\int_{-\infty}^{\infty} f(\eta-A\eta')e^{-ik(\eta-A\eta')}d\eta \\
&\quad +\int_{-\infty}^{\infty}(z_0^{*2}x_0^2-1)f(\eta-A\eta')(\eta-A\eta')e^{-ik(\eta-A\eta')}d\eta\int_{-\infty}^{\infty} g_0(\eta')e^{-ikA\eta'}d\eta'+\int_{-\infty}^{\infty} g_0(\eta')e^{-ikA\eta'}d\eta' \\
&\quad \times\int_{-\infty}^{\infty} \tfrac{1}{2}(z_0^{*2}x_0^2-1)(\eta-A\eta')^2 f'(\eta-A\eta')e^{-ik(\eta-A\eta')}d\eta+\int_{-\infty}^{\infty}\eta' g_0(\eta')e^{-ikA\eta'}d\eta' \\
&\quad \times\int_{-\infty}^{\infty} 2z_0^*x_0(\eta-A\eta')f'(\eta-A\eta')e^{-ik(\eta-A\eta')}d\eta-\int_{-\infty}^{\infty} z_0^*x_0A(w-1)\eta'^2 g_0(\eta')e^{-ikA\eta'}d\eta' \\
&\qquad\qquad \times\int_{-\infty}^{\infty} f'(\eta-A\eta')e^{-ik(\eta-A\eta')}d\eta\bigg\} \\
&=G_1(Ak)F(k)+\mathcal{L}(k),
\end{aligned} \tag{3.25}$$

where $\mathcal{L}(k)$ is the expression with the curly bracket.

With the assumptions for $f(y)$

$$\int_{-\infty}^{\infty} f'(\eta)e^{-ik\eta}d\eta=ikF(k), \tag{3.26a}$$

$$\int_{-\infty}^{\infty} \eta^n f(\eta)e^{-ik\eta}d\eta=(-i)^{-n}\frac{d}{dk}F(k). \tag{3.26b}$$

We then have

$$\begin{aligned}\mathcal{L}(k)=2i\beta\Delta k(1+z_0^*x_0)^{-2}[\tfrac{1}{2}(z_0^{*2}x_0^2-1)G_0(Ak)F''(k) \\ +2z_0^*x_0A^{-1}F'(k)G_0'(Ak)-A^{-1}z_0^*x_0(w-1) \\ \times G_0''(Ak)F(k)].\end{aligned} \tag{3.27}$$

With the requirement (3.9) that $G_1(0)=0$, Eq. (3.25) is readily solved as

$$G_1(k)=\sum_{n=0}^{\infty}\mathcal{L}(A^n k)\prod_{m=0}^{n-1}F(A^m k). \tag{3.28}$$

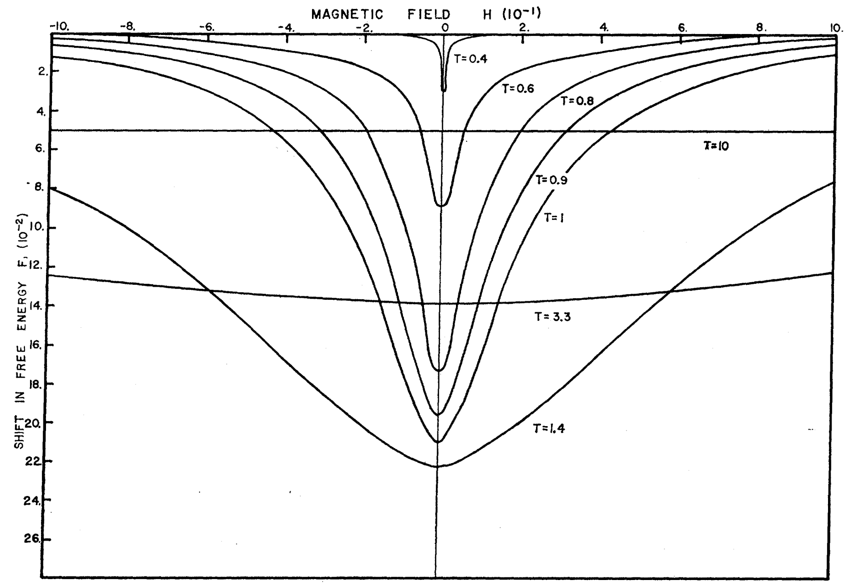

FIG. 3. Shift of the free energy $\mathfrak{F}_1$ as a function of the applied field H. $\mathfrak{F}_1$ has the unit of $E_0\sigma N^{-2}$, while H has the unit of H/E_0. Δ is taken to be equal to E_0. Temperature is used as a parameter in unit of E_0/k.

This infinite series is easily proved to be convergent using assumptions (i) and (ii).

With (3.24), (3.27), and (3.28), we have obtained the Fourier transform of an approximate solution to the integral Eq. (3.1).

4. FREE ENERGY

In Sec. 2, we showed that $\mathfrak{F}$ may be represented by the three equivalent expressions (2.10), (2.12), and (2.13). To compute the N^{-1} and N^{-2} correction to the free energy, it is most convenient to use (2.12) and expand the integrand about x_0 and E_0 to obtain

$$\mathfrak{F}=\mathfrak{F}_0+\mathfrak{F}_1\,, \tag{4.1}$$

where $\mathfrak{F}_0$ is the nonrandom part of the free energy, it is just the Ising expression as in (2.21).

$\mathfrak{F}_1$ represents the correction due to presence of randomness in the exchange energy

$$\begin{aligned}\mathfrak{F}_1=&-\Delta N^{-1}\int yf(y)dy-\beta^{-1}[2z_0{}^*x_0/(1+z_0{}^*x_0)]\\&\times\int\int d\eta dy\bar{\nu}(\eta)f(y)\{\beta\Delta N^{-1}[(w-1)(1+z_0{}^*x_0)^{-1}\eta-y]\\&+(\beta\Delta/N)^{-2}((1+z_0{}^*x_0)^{-1}y^2-z_0{}^*x_0(w-1)^2(1+z_0{}^*x_0)^{-3}\\&\times\eta^2-2(w-1)(1+z_0{}^*x_0)^{-2}\eta y)+O(N^{-3})\}\,.\end{aligned} \tag{4.2}$$

We denote the moments of the distributions $f(y)$, $g(\eta)$, etc. by the following convention:

$$my_n=\int y^nf(y)dy\,,$$
$$m\eta_{lm}=\int \eta^m g_l(\eta)d\eta\,. \tag{4.3}$$

The first index is the order of perturbation, the second index indicates the order of the moments.

In terms of this notation (4.2) becomes

$$\begin{aligned}\mathfrak{F}_1=&-\Delta N^{-1}my_1-\beta^{-1}[2z_0{}^*x_0/(1+z_0{}^*x_0)]\\&\times\{(\beta\Delta N^{-1})(-my_1+(w-1)(1+z_0{}^*x_0)^{-1}m\eta_{01})\\&+2z_0{}^*x_0(1+z_0{}^*x_0)^{-2}(\beta\Delta/N)^2(my_2-(w-1)^2\\&\times z_0{}^*x_0(1+z_0{}^*x_0)^{-2}m\eta_{02}-2(w-1)(1+z_0{}^*x_0)^{-1}\\&\times m\eta_{01}my_1)+(\beta\Delta N^{-2})(w-1)(1+z_0{}^*x_0)^{-1}m\eta_{11}\}\\&+O(N^{-3})\,.\end{aligned} \tag{4.4}$$

These moments of the output distributions, $m\eta_{01}$, $m\eta_{02}$, and $m\eta_{11}$ can be expressed explicitly in terms of the moments of the input distribution $f(y)$ by using Eqs. (3.24), (3.25), (3.28), and (3.7). We have

$$\begin{aligned}m\eta_{11}=&i\mathcal{L}'(0)/(1-A)=-2\beta\Delta(1+z_0{}^*x_0)^{-2}(1-A)^{-1}\\&\times[\tfrac{1}{2}(1-z_0{}^{*2}x_0{}^2)my_2-2z_0{}^*x_0my_1m\eta_{01}\\&+z_0{}^*x_0(w-1)Am\eta_{02}]\,,\end{aligned} \tag{4.5}$$

$$m\eta_{01}=my_1/(1-A)\,, \tag{4.6}$$

$$m\eta_{02}=my_1{}^2/(1-A)^2+\sigma/(1-A^2) \tag{4.7}$$

[where σ is the variance of $f(y)$],

$$\sigma=my_2-my_1{}^2\,. \tag{4.8}$$

With these relations (4.4) becomes

$$\begin{aligned}\mathfrak{F}_1=&-\Delta N^{-1}my_1+\beta^{-1}(2z_0{}^*x_0)(2z_0{}^*x_0+2z_0w)\\&\times(1+z_0{}^*x_0)^{-1}(2z_0{}^*x_0+1-w)^{-1}\\&\times\{\beta\Delta N^{-1}my_1-(\beta\Delta/N)^2(1+z_0{}^*x_0)^{-1}\\&\times[(1+(w-1)^2(2z_0{}^*w+1-w)^{-2}\\&+2z_0{}^{*2}x_0{}^2(w-1)(2z_0{}^*x_0+1-w)^{-2})my_1{}^2\\&+(2z_0{}^*x_0+1-w)^{-1}[(1-w)+z_0{}^*x_0(1+w)]\sigma]\}\\&+O(N^{-3})\,.\end{aligned} \tag{4.9}$$

From (4.9) we observe that for all symmetric distributions, i.e., $my_1=0$, we only have second-order term in $\mathfrak{F}_1$, which is negative and proportional to σ. In the case of an asymmetric $P(E)$ $\mathfrak{F}_1$ will have first-order term with opposite sign to my_1. It can be accounted for entirely by the discrepancy between the mean value and the most probable value of the exchange energy, where the mean-energy value is defined by the expectation value.

$$\bar{E}=\int EP(E)dE=\int EN\Delta^{-1}f(N\Delta^{-1}(E-E_0))dE$$

$$=\int (E_0+\Delta N^{-1}y)f(y)dy$$

$$=E_0+\Delta N^{-1}my_1. \tag{4.10}$$

This shift in energy gives correction to the free energy in the Ising expression of (2.21). Denote this correction term by $\mathfrak{F}_1'$.

$$\mathfrak{F}_1'=(\mathfrak{F}_0)_{E=\bar{E}}-(\mathfrak{F}_0)_{E=E_0}. \tag{4.11}$$

By using a Taylor series expansion and the relation

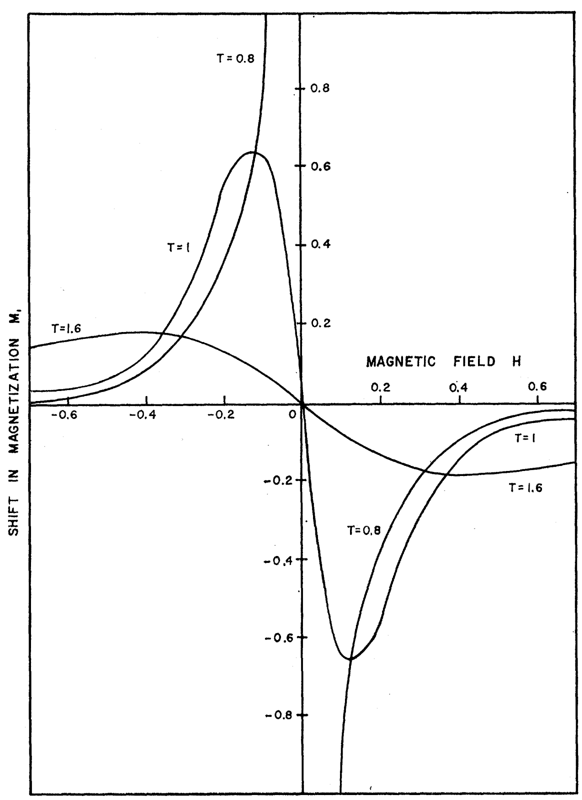

FIG. 4. Shift of the magnetization M_1 in unit of σN^{-2} versus the applied magnetic field in H/E_0 with temperature as parameter.

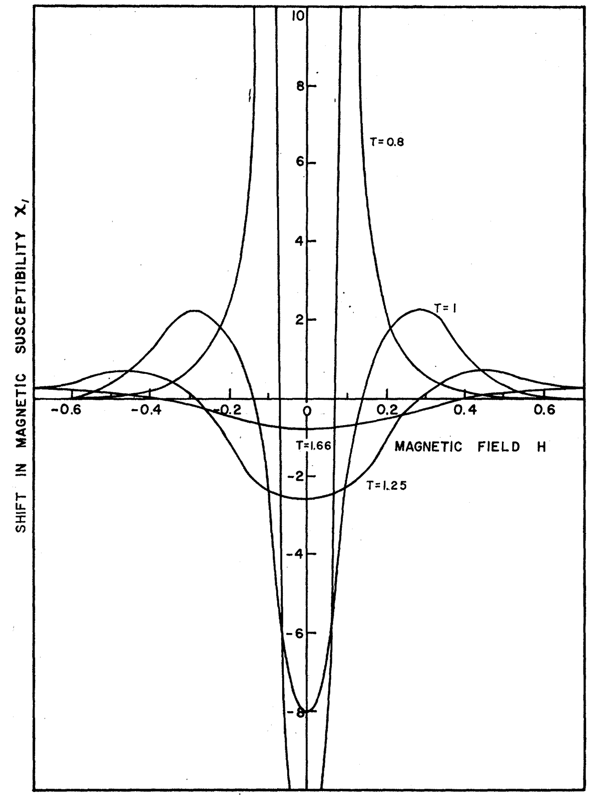

FIG. 5. Shift of the magnetic susceptibility χ_1 versus applied field. The unit for χ_1 is $\sigma/(N^2E_0)$.

(2.19) we have

$$\begin{aligned}\mathfrak{F}_1'=&-\Delta N^{-1}my_1-\beta^{-1}(2z_0{}^*x_0)(2z_0{}^*x_0+2-2w)\\&\times(1+z_0{}^*x_0)^{-1}(2z_0{}^*x_0+1-w)^{-1}\\&\times\{\beta\Delta N^{-1}my_1-(\beta\Delta/N)^2(1+z_0{}^*x_0)^{-1}\\&\times(1+(w-1)^2(2z_0{}^*x_0+1-w)^{-2}\\&+2z_0{}^{*2}x_0{}^2(w-1)(2z_0{}^*x+1-w)^{-2})my_1{}^2\}\\&+O(N^{-3}).\end{aligned} \tag{4.12}$$

The difference between (4.12) and (4.9) is then the effective correction which depends solely upon the variance of $P(E)$.

$$\begin{aligned}\mathfrak{F}_{1,\mathrm{eff}}=\mathfrak{F}_1-\mathfrak{F}_1'=&-\beta(\Delta/N)^2 2z_0{}^*x_0(1+z_0{}^*x_0)^{-2}\\&\times(2z_0{}^*x_0+1-w)^{-2}(2z_0{}^*x_0+2-2w)\\&\times(1-w+z_0{}^*x_0(1+w))\sigma.\end{aligned} \tag{4.13}$$

$\mathfrak{F}_{1,\mathrm{eff}}$, always negative and second-order, is identical to the correction by the symmetrical distribution in (4.5) when $my_1=0$.

Using the definition of x_0 in (2.19), we obtain

$$\mathfrak{F}_{1,\mathrm{eff}}=-\Delta N^{-2}(2\beta\Delta)2z_0{}^*x_0(1+x_0{}^2)(1+x_0{}^2+2z_0{}^*x_0)^{-2}\sigma. \tag{4.14}$$

As expected, this correction term to free energy is indeed symmetrical with respect to the magnetic field.

This can be seen more directly by using a further change of variable,

$$t=(2z_0^*x_0)/(1+x_0^2). \quad (4.15)$$

Following (2.19),

$$t=z_0^{*2}(z_0^{*2}+w'^2)^{-1}((z_0^{*2}+w'^2-z_0^{*2}w'^2)^{1/2}-w'^2), \quad (4.16)$$

where

$$w'=\tanh\beta H. \quad (4.17)$$

In terms of t we finally obtain

$$\mathfrak{F}_{1,\text{eff}}=-\Delta N^{-2}(2\beta\Delta)\sigma t/(1+t)^2. \quad (4.18)$$

The dependence of $\mathfrak{F}_{1,\text{eff}}$ upon the temperature and magnetic field is shown in Fig. 3. It can be seen that $\mathfrak{F}_{1,\text{eff}}$ vanishes both in the limit of infinite temperature and zero temperature. It is also seen that at low temperature, the correction due to the randomness is small and it is nonzero only in a very narrow low-field region.

We see for $|H|\gg|H_m|$, where $H_m=\beta^{-1}\sinh^{-1}z_0^*$, $\mathfrak{F}_{1,\text{eff}}$ will assume the following asymptotic expression,

$$\mathfrak{F}_{1,\text{eff}}=-4\beta\Delta^2N^{-2}wz_0^*\sigma+O(N^{-3}). \quad (4.19)$$

For low field $|H|\ll|H_m|$,

$$\mathfrak{F}_{1,\text{eff}}=-2\beta\Delta^2N^{-2}(z_0^*(1+z_0^*)^{-2} + \tfrac{1}{8}(z_0^*-1)[z_0^*(z_0^*+1)]^{-1}(w-1)^2). \quad (4.20)$$

It is obvious that the free energy behaves as a parabola near zero-field region, while at high field it approaches zero exponentially.

5. THERMODYNAMIC PROPERTIES

With the expression for the effective correction to free energy (4.18), we can obtain all the corresponding correction terms to thermodynamic quantities such as magnetization, magnetic susceptibility and specific heat by using the relations (2.4), (2.19), (4.16), and (4.17).

(i) *Magnetization*

$$M_1=\frac{\partial\mathfrak{F}_{1,\text{eff}}}{\partial H}=-\frac{\partial\mathfrak{F}_{1,\text{eff}}}{\partial t}\frac{\partial t}{\partial w'}\frac{\partial w'}{\partial H} \quad (5.1)$$

$$=(\Delta\sigma/N^2)(2\beta\Delta)\beta w'(1-w'^2)(w'^2+z_0^{*2}-z_0^{*2}w'^2)^{-2}z_0^{*4}(1-t)(z_0^{*2}+t)^{-1}. \quad (5.2)$$

This is plotted in Fig. 4. M_1 is strictly negative like $\mathfrak{F}_{1,\text{eff}}$, and vanishes at $H=0$.

(ii) *Magnetic Susceptibility*

$$\chi_1=(\partial M_1/\partial H)=(\partial w'/\partial H)[\partial M/\partial w'+(\partial M/\partial t)(\partial t/\partial w')] \quad (5.3)$$

$$=-2(\beta\Delta/N)^2\sigma z_0^{*4}\beta(1-w'^2)(z_0^{*2}+(1-z_0^{*2})w'^2)^{-2}[(1-t)(z_0^{*2}-3w'^2+w'^4-z_0^{*2}w'^4) \times(z_0^{*2}+t)^{-1}(z_0^{*2}+(1-z_0^{*2})w'^2)^{-1}+w'^2(1-w'^2)(1+z_0^{*2})((z_0^{*2}+w'^2)t+w'^2)^{-1}]. \quad (5.4)$$

This susceptibility is plotted in Fig. 5. It starts from an initial negative value which is

$$\chi_{1,0}=-(\beta\Delta/N)^2 2\beta\sigma e^{2\beta E_0}\tanh\beta E_0 \quad (5.5)$$

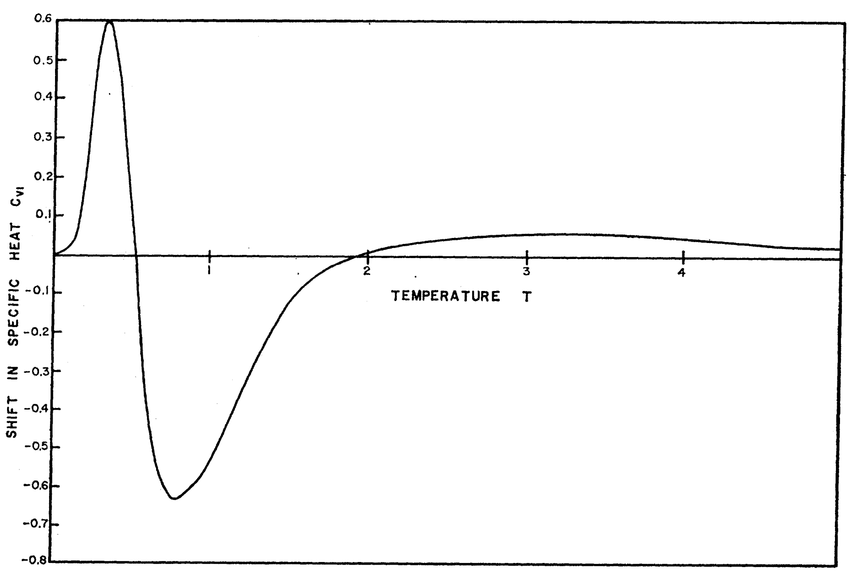

FIG. 6. Shift of the specific heat for $H=0$ in unit of $k\sigma N^{-2}$ versus temperature in E_0/k.

and then rises to a *positive* maximum and eventually decreases to zero exponentially at high field. The magnitude of this maximum is largest at zero temperature and decreases to zero exponentially at $T \to \infty$.

(iii) Specific Heat

$$C_{v,1}=-T\frac{\partial^2 \mathfrak{F}_{1,\mathrm{eff}}}{\partial T^2}=-k\beta\frac{\partial}{\partial\beta}\left(\beta^2\frac{\partial \mathfrak{F}_{1,\mathrm{eff}}}{\partial\beta}\right)$$
$$=2k(\beta\Delta/N^2)\sigma(1+t)^{-2}[2t-4\beta(1-t)B+\beta^2B^2(t-3)+\beta^2(t-1)(\partial B/\partial\beta)], \quad (5.6)$$

where

$$B=-(1+t)^{-1}(\partial t/\partial\beta)=(z_0^{*2}+t)^{-1}(w'^2+z_0^{*2}-z_0^{*2}w'^2)^{-1}[2E_0(w'^2t^2+z_0^{*4}(1-w'^2))+Hw'(1-w'^2)(z_0^{*2}+l)^2] \quad (5.7)$$

and

$$\frac{\partial B}{\partial\beta}=\frac{\partial B}{\partial t}\left(\frac{\partial t}{\partial w'}\frac{\partial w'}{\partial\beta}+\frac{\partial t}{\partial z_0^*}\frac{\partial z_0^*}{\partial\beta}\right)+\frac{\partial B}{\partial w'}\frac{\partial w'}{\partial\beta}+\frac{\partial B}{\partial z_0^*}\frac{\partial z_0^*}{\partial\beta}$$
$$=(z_0^{*2}+t)^{-2}((z_0^{*2}+w'^2)t+z_0^{*2}w'^2)^{-1}(w'^2+z_0^{*2}-z_0^{*2}w'^2)^{-1}[4E_0^2z_0^{*2}t^2(w'^2t^2+z_0^{*4}(\tfrac{1}{2}-w'^2))$$
$$-H^2w'^2(1-w'^2)(z_0^{*2}+t)^4-4E_0Hw'^3t(1-w'^2)(z_0^{*2}+t)^3]-4z_0^{*2}E_0(z_0^{*2}+t)^{-2}(w'^2+z_0^{*2}(1-w'^2)^{-2})$$
$$\times[2E_0(z_0^{*4}t(1-w'^2)^2+w'^2(1-w'^2)(z_0^{*4}+2z_0^{*2}t-2z_0^{*2}t^2-t^3)-w'^4t^2)+Hw'(1-w'^2)(z_0^{*2}+t)^2$$
$$\times(2w'^2+(2z_0^{*2}+t)(1-w'^2))]+(1-w'^2)H(w'^2+z_0^{*2}(1-w'^2))^{-2}[4E_0w'z_0^{*2}(z_0^{*2}+t)^{-1}$$
$$\times((t^2-z_0^{*2})-(1-w'^2)(\tfrac{1}{2}-z_0^{*2})^2)+(z_0^{*2}+t)H(z_0^{*2}(1-w'^2)^2-w'^2(1-w'^2)-2w'^4)]. \quad (5.8)$$

Similar to the magnetic susceptibility, this shift in specific heat does not have a constant sign. It vanishes both at zero temperature and infinite temperature. For $H=0$, it is reduced to the following:

$$C_{v1}=k(\beta\Delta/N)^2\sigma\,\mathrm{sech}^2\beta E_0(1-\beta^2E_0^2+3\beta^2E_0^2\tanh^2\beta E_0-4\beta E_0\tanh\beta E_0) \quad (5.9)$$

and is shown in Fig. 6. The magnitude of the oscillations depends upon the applied field. It is largest for zero field and approaches zero exponentially rapidly as $H \to \infty$.

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