Interaction of ac Josephson Currents with Surface Plasmons in Thin Superconducting Films

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The coupling of low-frequency surface plasmons to the ac Josephson currents in superposed thin dielectric and superconducting metal films is examined. The I-V characteristics for various superposed film systems are derived. The theory permits the magnitude of the parameters entering in the expression for the I-V characteristics, as well as their dependence on various quantities (temperature, mean free path, etc.), to be determined and compared with existing experiments. We have extended the theory of steplike structures in the I-V characteristics of a single Josephson junction to multiple films, and have found that more than one series of such steplike structures is possible. The theory provides a natural explanation for a three-film tunneling experiment of Giaever. Finally, the modifications of the dispersion relations for surface plasma oscillations in multiple-film systems, when admitting the presence of the ac Josephson effect, are derived.

I. INTRODUCTION

ELECTROMAGNETIC wave modes in superposed metal and dielectric films of various geometrical configurations have been examined by Economou[1] (whose paper will be referred to from now on as I). When the metals are superconducting and the thicknesses of the intervening dielectric films are sufficiently thin (~20 Å) so that supercurrents can tunnel, Josephson effects occur. The Josephson currents are coupled with the multiple-film em modes and various interesting results can be expected. Several aspects of this phenomenon provide the subject for the investigations of this paper.

Swihart[2] was the first to show theoretically the existence of resonant oscillations of electromagnetic waves in tunnel junctions. He showed that there can propagate slowed-down transverse electromagnetic waves with phase velocity

$$\bar{c}=c\left(\frac{d_i}{\epsilon_i[d_i+2\lambda_L \coth(d_m/\lambda_L)]}\right)^{1/2}$$

in a dielectric cavity between two superconducting metal films. Here c is the velocity of light, d_i is the thickness, and ϵ_i is the dielectric constant of the dielectric film, while d_m is the thickness and λ_L the London penetration depth of the metal films. Fiske,[3] and since then various other workers,[4-8] have observed experimentally the coupling of this junction mode to the ac Josephson currents. In the preceding paper, such considerations have been extended to multiple films which may be regarded as forming superposed junction structures. It was shown in I that in such geometries more than one such low-frequency electromagnetic mode is possible, and through these resonant modes the cavities associated with each junction are coupled together electromagnetically. This latter point is especially easy to visualize in the limiting situation when $d_m \ll \lambda_p$, where a surface plasma-oscillation type of description is valid.[9]

The present work discusses in a general way the excitation of resonant electromagnetic waves in such multiple-film superposed junctions by the alternating Josephson current. The influence of these resonances on the I-V characteristics of such systems are worked out as well as their consequences for some of the more interesting cases. The results for a single Josephson junction have essentially the same features as obtained before.[4,5] However, our approach enables now a quantitative calculation of the parameters of interest as a function of temperature, bias voltage, and mean free path. For example, we have some success in accounting quantitatively for the decrease of the ratio of the voltage at which a resonance peak occurs to the corresponding applied magnetic field.[4] The loss represented by a Q factor is given as a function of frequency and can be calculated for any temperature at which an experiment is performed. The variation of the positions of the current steps with temperature is discussed, with the result that the steps tend to shift their positions nonuniformly and

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[1] E. N. Economou, preceding paper, Phys. Rev. **181**, 539 (1969).

[2] J. C. Swihart, J. Appl. Phys. **32**, 461 (1961).

[3] M. D. Fiske, Rev. Mod. Phys. **36**, 221 (1964).

[4] R. E. Eck, D. J. Scalapino, and B. N. Taylor, Phys. Rev. Letters **13**, 15 (1964).

[5] R. E. Eck, D. J. Scalapino, and B. N. Taylor, in *Proceedings of the Ninth International Conference on Low-Temperature Physics*, edited by J. G. Daunt (Plenum Publishing Corp., New York, 1965), p. 415.

[6] I. M. Dmitrenko and I. K. Yanson, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **2**, 242 (1965) [English transl.: Soviet Phys.—JETP Letters **2**, 154 (1965)].

[7] C. B. Satterthwaite, M. G. Craford, R. N. Peacock, and R. P. Ries, in *Proceedings of the Ninth International Conference on Low-Temperature Physics*, edited by J. G. Daunt (Plenum Publishing Corp., New York, 1965), p. 443.

[8] D. C. Coon and M. D. Fiske, Phys. Rev. **138**, A744 (1965).

[9] E. N. Economou and K. L. Ngai, Phys. Rev. Letters **20**, 547 (1968); **20**, 701(E) (1968).

cease to be equally spaced as $T \to T_c$. The dependence on the electron mean free path is also given.

For the general multiple-film systems we have supplied a theoretical picture of what one expects when surface plasmon modes are excited by supercurrents. We have found, in general, that these electromagnetic modes cause an interaction of the barriers of the films system with each other, and thereby alter their operating characteristics. Moreover, for multiple films, several low-lying electromagnetic modes can be present, and consequently the same number of series of steps will appear in the I-V characteristics. Our results provide an alternative physical explanation of a Giaever's three-film experiment to detect the radiation of a Josephson junction by use of a continuous cavity to couple the two barriers tightly with the intention of eliminating the bad impedance mismatch between the junction and free space. Subsequent experiments[10] throw some support for this alternative explanation. In Sec. III, we examine the modifications of the dispersion relations of the low-frequency surface plasma oscillations (discussed in I) for multiple-film systems in which ac supercurrents can flow. This study is stimulated by the recent experimental observation[11] of the Josephson plasma resonance in superconducting barriers originally predicted by Josephson.[12,13] A general method is devised to treat similar resonances in multiple films, which, when specialized to the case of a barrier between semi-infinite superconductors, enables us to recapture the results of Josephson. The method provides a good description of the modes near $k \approx 0$ and explains why retardation effects are negligible by use of a criterion due to Ferrell.[14] Special features in the dispersion relations for these modes occur when the external metal films have thicknesses comparable or smaller than the penetration depth. Interaction of modes is sufficiently strong to cause a splitting to occur at the point of degeneracy. One of these modes of oscillation can still be described by the dispersion relation $\omega^2=\omega_0^2+\bar{c}^2k^2$ for $k=0$ up till $k \sim \omega_0/c$, where ω_0 is the Josephson plasma frequency, and therefore lies in the region $\omega > ck$ in the k-ω plane. Although this oscillation has a radiative character, yet the microwave radiation that can possibly be given off is negligible even if the metal films are thin enough. In general, the present results would be helpful for further studies of such resonances in multiple-film systems.

II. EXCITATION OF RESONANT ELECTROMAGNETIC MODES BY ac JOSEPHSON CURRENTS

When a dc voltage V_0 is maintained across a Josephson tunnel junction, alternating currents exist between the two superconductors separated by a barrier.[15] If, in addition, there is a uniform static magnetic field H_0 applied parallel to the barrier along the y axis, the alternating currents are given by

$$j_T=\mathrm{Re}\, j_1 e^{i(\omega t-kz)}, \tag{2.1}$$

where $\omega=2eV_0/\hbar$ and $k=2e(2\lambda_p+d_i)H_0/\hbar c$. The last relation between k and H_0 holds only under the assumption that the thicknesses of the metal films are much larger than the penetration depth. This relation is expected to be considerably modified in the opposite limit when d_m is small compared with λ_p. It was pointed out to us[16] that in this limit, some characteristic length like d_m would replace the role of λ_p in that relation. The quantitative result requires the solution of the microscopic problem in the presence of thin metallic films. This work has been carried out by Ivanchenko[17] and his final result is, in our notation,

$$k=\frac{2e}{\hbar c}\left[H_0\left(2\lambda_p \coth\frac{d_m}{\lambda_p}+d_i\right) - H_0' 2\lambda_p\left(\sinh\frac{d_m}{\lambda_p}\right)^{-1}\right], \tag{2.2}$$

where H_0 is the magnetic field inside the barrier and H_0' is the field on the outer surfaces of the superconductors. For $d_m \gg \lambda_p$, the above relation reduces to the one preceding it. The fields H_0 and H_0' are in general interdependent with their relation determinable by solving the magnetostatic problem presented by the configuration considered. In the limit $d_m/\lambda_p \to 0$, then the difference H_0-H_0' can be shown[18] to be of second order in d_m/λ_p and consequently in this thin-film limit the k-H relation reduces to the form

$$k=(2e/\hbar c)H_0(2d_m+d_i).$$

Such a current density wave propagating along the junction in the z direction can excite electromagnetic modes in the structure. When the phase velocity of this wave becomes equal to the electromagnetic-mode phase velocity, we may expect a resonance phenomenon to manifest itself in the tunneling characteristics of the superposed films. We therefore wish to find solutions of

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[10] I. Giaever, communication at the Advanced Study Institute and International Conference on Tunneling Phenomena in Solids, Risø, Denmark, 1967 (unpublished).

[11] A. J. Dahm, A. Denenstein, T. F. Finnegan, and D. N. Langenberg, Phys. Rev. Letters **20**, 859 (1968).

[12] B. D. Josephson, Advan. Phys. **14**, 419 (1965).

[13] B. D. Josephson, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Co., Amsterdam, 1966).

[14] R. A. Ferrell, Phys. Rev. **111**, 1214 (1958).

[15] B. D. Josephson, Phys. Letters **1**, 251 (1962).

[16] J. W. Wilkins (private communication).

[17] Yu. M. Ivanchenko, Zh. Eksperim. i Teor. Fiz. **51**, 337 (1966) [English transl.: Soviet Phys.—JETP **24**, 225 (1967)].

[18] We have calculated the field in the presence of a thin-film barrier by use of a perturbative expansion in powers of d_m/λ_p with the zero-order solution taken as the homogeneous external field. To lowest order, we find $H_0-H_0'=\frac{1}{2}(d_m/\lambda_p)^2H_0$.

the Maxwell equations for a geometry of alternating superposed superconducting metal and insulator films. As has been justified in I, the superconducting metal is described by the dielectric function

$$\epsilon_s(\omega)=1-(\omega_{ps}^2/\omega^2)(1+i\sigma_1/\sigma_2)\,, \tag{2.3}$$

and then $\lambda_p=c/\omega_{ps}$ is the actual penetration depth. The Maxwell equations are supplemented by the boundary conditions of continuity of the tangential fields at every boundary. The type of solution we seek corresponds to wave propagation along a direction parallel to the boundary surfaces separating the different metals, which we designate as the z axis. With the x axis normal to these surfaces, we further assume that there is no y dependence of any of the fields and that $H_x=H_z=E_y=0$ in all of the media. The solution for any component of the fields can thus be represented in the form $\mathcal{F}(x,z,t)=\mathrm{Re}F(x)e^{i(\omega t-kz)}$, with $\mathrm{Re}k>0$ and $\mathrm{Im}k<0$ so that the wave travels and is attenuated in the positive z-direction. The Maxwell equations determine the field amplitudes $F(x)$ through the ordinary differential equations

$$\begin{aligned} E_z(x)&=-(i/k)(dE_x/dx)\,,\\ H_y(x)&=(\omega\epsilon_s/ck)E_x\,,\\ d^2E_x(x)/d^2x&-K_m^2E_x=0 \end{aligned} \tag{2.4a}$$

inside the metal and

$$\begin{aligned} E_z&=-(i/k)(dE_x/dx)\,,\\ H_y&=(\epsilon_i\omega/ck)E_x-(4\pi i/ck)j_1\,,\\ d^2E_x/dx^2&-K_i^2[E_x+(i4\pi\omega/c^2K_i^2)j_1]=0 \end{aligned} \tag{2.4b}$$

in the dielectric region where a supercurrent of magnitude j_1 flows. Here

$$K_m^2=k^2-\omega^2\epsilon_s/c^2$$

and

$$K_i^2=k^2-\omega^2\epsilon_i/c^2\,.$$

Equations (2.4b) describe mathematically the excitation of the resonant modes by including the contribution to the fields due to the tunneling currents in the dielectric regions only. Assuming j_T has no x dependence, the electric field E_x can be expressed in the form

$$E_x=E(e^{K_ix}+e^{-K_ix})-i(4\pi\omega/c^2K_i^2)j_1.$$

With these modified solutions in the dielectric and the solutions of (3.8)–(3.11) in I for the metals, we can write the boundary conditions for the continuity of E_z and H_y. The number of such equations is the same as the number of unknown field amplitudes. Thus we have a system of linear equations which can be arranged to have the form that the left side of the equation is homogeneous in the unknown field amplitudes while the right side contain terms related to the tunneling currents. In this form it is obvious that we have a problem in which the electromagnetic modes of the system are driven by Josephson currents. Instead of going on to discuss the solution to the problem in this generality, it is perhaps profitable to consider some specific model systems.

A. Single Josephson Junction

We take as our first example a model junction consisting of two superconducting metal films of equal thickness d_m separated by a dielectric film of thickness d_i through which the supercurrent (2.1) flows. The quasiparticle tunneling current has no effect on the following considerations except that it contributes to losses and hence damping of the resonances. The magnitude of this damping can always be estimated[13] and hence we shall neglect quasiparticle currents from now on. A particular case of this when d_m is assumed to be infinite has been considered both experimentally and theoretically for the coupling of the ac Josephson currents to the junction modes.[5,6,19] Essentially the same results as theirs are obtained here by our method but now all the important parameters can be calculated explicitly using the model for the dielectric response of superconducting films as explained in I. Another reason for a repeated discussion of this by now well-understood case is to illustrate our general procedure, which will be applied to more complex structures afterwards.

Figure 6 of I represents the present problem if we take $d_1=d_2=d_m$. Eigensolutions for the fields are either symmetric or antisymmetric, and hence we can restrict our attention to the half-space $x>0$ and write E_x as

$$\begin{aligned} E_x&=E_1(e^{K_ix}\pm e^{-K_ix})/2-(i4\pi\omega/c^2K_i^2)j_1\\ &=E_2(e^{K_mx}+Be^{-K_mx})\\ &=E_3e^{-K_ix} \end{aligned} \tag{2.5}$$

inside the barrier, in the metal film, and outside the junction, respectively. Then via (2.2), E_z and H_y can be expressed in terms of E_x, and the boundary conditions to be satisfied by them yield the set

$$\begin{aligned} &-K_ie^{-K_i(d_m+\frac{1}{2}d_i)}E_3=K_mE_2(e^{K_m(d_m+\frac{1}{2}d_i)}-Be^{-K_m(d_m+\frac{1}{2}d_i)})\,,\\ &(k^2-K_i^2)e^{-K_i(d_m+\frac{1}{2}d_i)}E_3\\ &\qquad=(k^2-K_m^2)E_2(e^{K_m+\frac{1}{2}d_i)}+Be^{-K_m(d_m+\frac{1}{2}d_i)})\,,\\ &K_mE_2(e^{\frac{1}{2}K_md_i}-Be^{-\frac{1}{2}\mathbf{K}_md_i})=K_iE_1(e^{\frac{1}{2}\mathbf{K}d_i}\mp e^{-\frac{1}{2}K_id_i})/2\,,\\ &(k^2-K_m^2)E_2(e^{\frac{1}{2}K_md_i}+Be^{-\frac{1}{2}K_md_i})\\ &\qquad=(k^2-K_i^2)E_1(e^{\frac{1}{2}K_id_i}\pm e^{-\frac{1}{2}K_id_i})/2\\ &\qquad\qquad-(i4\pi\omega/c^2)(k^2/K_i^2)j_1\,, \end{aligned} \tag{2.6}$$

from which we can solve for E_1. The alternating voltage

[19] I. O. Kulik, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **2**, 134 (1965) [English transl.: Soviet Phys.—JETP Letters **2**, 84 (1965)].

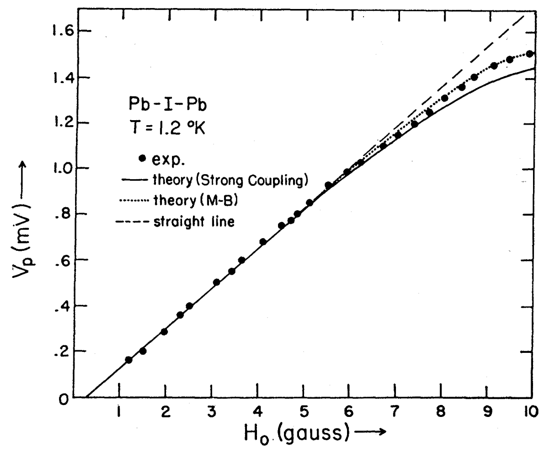

FIG. 1. The dots are experimental results (Ref. 4) for the positions of the observed resonant peak versus applied magnetic field. The curves give the calculated positions.

across the barrier is given by

$$v_0 \equiv E_x d_i = E_1 d_i - i(4\pi\omega/K_i c^2) j_1 d_i$$

$$= \frac{i4\pi d_i j_1}{\epsilon_i \omega K_i^2} \times \left(k^2 \cdot \frac{2C_+R - 2C_-R^2}{-C_-A_- + (C_+A_+ + C_+A_-)R - C_-A_\pm R^2} - \frac{\epsilon_i \omega^2}{c^2} \right), \quad (2.7)$$

where

$$A_\pm = (e^{\frac{1}{2}K_i d_i} \pm e^{-\frac{1}{2}K_i d_i}),$$

$$C_\pm = (e^{K_m d_m} \pm e^{-K_m d_m}),$$

and

$$R = \frac{K_m \epsilon_i}{K_i \epsilon_m}.$$

If we take R and $K_i d_i$ to be small, as is true in practice, we can simplify this to read

$$v_0 = \frac{i4\pi j_1 d_i}{\epsilon_i \omega K_i^2} \times \left(\frac{k^2(1-\epsilon_i \bar{c}^2/c^2)}{(1-k^2\bar{c}^2/\omega^2) + i(\epsilon_i \bar{c}^2/c^2)(k^2c^2/\epsilon_i\omega^2 - 1)\delta} - \frac{\epsilon_i \omega^2}{c^2} \right),$$

with

$$\bar{c} = c\left(\frac{d_i}{\epsilon_i[d_i + 2\lambda_p \coth(d_m/\lambda_p)]} \right)^{1/2}$$

and

$$\delta = [\sigma_1(\omega)/2\sigma_2(\omega)][1 + 2d_m/\lambda_p \sinh(2d_m/\lambda_p)]. \quad (2.8)$$

This can be reduced further to

$$v_0 = \frac{i4\pi d_i j_1}{\epsilon_i \omega} \left(\frac{1}{(1-\bar{c}^2k^2/\omega^2) + i/Q} \right), \quad (2.9)$$

where

$$Q = 2\frac{\sigma_2}{\sigma_1}\left(\frac{\omega}{\bar{c}k}\right)^2 \Big/ \left(1 - \frac{\epsilon_i \bar{c}^2}{c^2}\right)\left(1 + \frac{2d_m}{\lambda_p \sinh(2d_m/\lambda_p)}\right), \quad (2.10)$$

by making use of the fact that $\sigma_1/\sigma_2 \ll 1$ for the frequency range of our present interest. Here $\bar{c}$ is the phase velocity of a low-frequency branch of the dispersion relations for surface plasma oscillations in this geometry as given also by Eq. (3.29) of I. The ratio $\sigma_1/2\sigma_2$ is graphed in Fig. 11 of I, where it is seen to be a smooth function of its argument as long as $T \ll T_c$. The last equation indicates that resonance steps will occur whenever $k\bar{c} = \omega$ is satisfied and the resonance losses are given by the value of Q. Results identical in form to this have been obtained by Eck *et al.*[4,5] For their case the losses are represented by a Q factor which has not been given explicitly. From our expression for Q we can estimate it to be of the order of magnitude of 10^2. Strictly speaking, the discussion until now holds only for a junction of infinite extent in the z direction. If the actual length of the junction is L, then every function defined in this region can be expanded in the complete set of functions $\cos(n\pi z/L)$ and $\sin(n\pi z/L)$, $n = 0, 1, 2, \cdots$. The voltage normal modes of an open-circuited transmission line vary as $\cos(n\pi z/L)$. Expanding v_0 in these modes, we naturally obtain results similar in form to those in Ref. 5, that resonant jumps appear at equally spaced voltages $V_n = n(h\bar{c}\pi/2eL)$ which gradually merge into resonance-shaped peaks as the magnetic field is increased. The resonance-shaped peaks occur near the voltage at which $\omega = k\bar{c}$. Since k is proportional to H_0 and ω is proportional to V, the dependence of peak position V_p on applied magnetic field H_0 would be linear if $\bar{c}$ were frequency-independent. The experiments[4,5] verify such a linear relation except that a deviation from linearity is observed for higher voltages. The explanation of this deviation is the frequency dependence of $\bar{c}$, as is clear from Eqs. (2.8) and (2.3). An attempt to account for this quantitatively can be carried out by use of the frequency dependence of ω_{ps} or λ_p as given in I. The results of this calculation are represented in Fig. 1 by the dash-dotted line. However, the experiment was done with Pb-*I*-Pb junctions, and since Pb is a strong-coupling superconductor, the actual frequency dependence can only be obtained via the strong-coupling theory. For this reason we have taken the results of the calculation of the complex electrical conductivity of superconducting lead by Shaw and Swihart.[20] The theoretical curve that describes the relation between V_p and H_0 is shown in Fig. 1 as the solid line.

In addition to being frequency-dependent, $\bar{c}$ is also temperature-dependent. This would imply a temperature dependence of the position of the steps in the *I*-*V* characteristics which has also been experimentally ob-

[20] W. Shaw and J. C. Swihart, Phys. Rev. Letters **20**, 1000 (1968).

served.[6,7] The steps occur at voltages corresponding to the frequencies $\omega=(n\pi/L)\bar{c}$. Hence the temperature dependence of the steps is determined by a complicated equation of the form

$$\omega_n(T)=(n\pi/L)\bar{c}(T,\omega_n(T)). \tag{2.11}$$

The temperature and frequency dependence of $\bar{c}(\omega,T)$ comes from the corresponding dependence of the penetration depth $\lambda_p(\omega,T)$. Miller[21] has calculated $\lambda_p(\omega,T)$ assuming an infinite mean free path l. However, it is reasonable to assume that the results of these calculations can be used in our case (where the mean free path is short) since what we need is only the temperature and frequency dependence of the ratio $\lambda(T)/\lambda(0)$ and it is known[22] that the temperature dependence of this ratio is almost unaffected by the presence of a short mean free path at least in the static limit. Having thus the frequency and temperature dependence of $\bar{c}$, we can solve Eq. (2.11) graphically for each step. The resulting behavior for the steps expected theoretically for a Sn-I-Sn junction like that used by Dmitrenko *et al.*[6] is detailed in Fig. 2 (solid lines). Dmitrenko *et al.* studied experimentally the temperature dependence only of the first step; their results are also displayed in Fig. 2 (circles). The theoretical temperature variation of steps differ from one step to another when $T/T_c>0.6$ because of the frequency dependence of $\bar{c}$, Eq. (2.11). This means that the steps start to shift nonuniformly and consequently cease to occur at equally spaced voltages.

Another point worthy of comment is the following. It has previously been stated[6] that the temperature variation of the steps is given by the function $[1-(T/T_c)^4]^{1/4}$. We disagree with this statement, since it is not the *static* penetration depth which enters the expression for $\bar{c}$, as the derivation of the above expression requires, but the *frequency-dependent* ac penetration depth. That the temperature variation is not given by the function $[1-(T/T_c)^4]^{1/4}$ can be seen from the fact that $\lambda_p(\omega(T),T)$ does not tend to infinity even for a normal metal, i.e., when $T=T_c$, in contrast to the above function, which assumes that the penetration depth tends to infinity as $[1-(T/T_c)^4]^{-1/2}$ as $T\rightarrow T_c$.

Assuming that $T/T_c\ll 1$ and $\hbar\omega/kT_c\ll 1$, it can be shown by use of the calculations of Miller[22] that $\bar{c}$ decreases, but only slowly when l decreases.

In the above analysis we have assumed the current amplitude j_1 is uniform over the barrier and the junction size is small compared to the Josephson penetration depth λ_J,[12] so that the magnetic field associated with the dc current can be neglected. Under these assumptions and when the applied magnetic field H_0 is zero, the exciting supercurrent is uniform. According to our picture that structures in the I-V characteristics are due to the resonance excitation of surface plasmons, this would imply, at $H_0=0$, the absence of current jumps. However, in actual experiments[23] these structures persist. The explanation lies in the fact that the junction oxide layer is nonuniform and the current amplitude j_1 depends sensitively on the thickness of the barrier. This makes j_1 a function of the coordinates on the surface of the barrier and, when j_1 is expanded in the complete set $\cos(n\pi z/L)$ and $\sin(n\pi z/L)$, it becomes clear that resonant surface plasmon oscillations can be excited even when $H_0=0$ to produce the equally spaced current jumps.

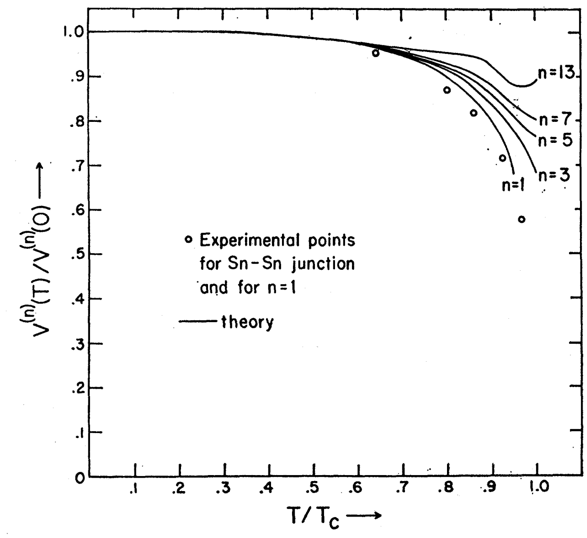

FIG. 2. Relative change in the position of the nth step versus relative temperature.

B. Superposed Tunnel Junctions

We proceed to the study of more complex superposed-film systems. Our interest in these systems was initiated by experiments performed by Giaever[24] in which two superposed junctions, one of which exhibits the ac Josephson effect and the other a conventional type with a continuous cavity linking them together, are used to detect the Josephson radiation. A voltage V_1 is applied across the Josephson junction which serves as a generator of fields of frequency $h\nu=2eV_1$. The conventional junction serves as a detector for these fields. Current steps are obtained in the I-V characteristic of the latter junction at voltages $V_2=(1/e)(2\Delta\pm n\hbar\nu)$ in the same manner as in the Dayem-Martin effect.[25] It seems natural to explain the result as a detection of the microwave fields in the Josephson junction by the conventional junction when the two cavities are coupled via a continuous link connecting them to eliminate the bad impedance mismatch between the junction and free space which is unavoidable when a conventional detec-

[21] P. B. Miller, Phys. Rev. 118, 928 (1960).
[22] P. B. Miller, Phys. Rev. 113, 1209 (1959).
[23] H. Fritzsche (private communication).
[24] I. Giaever, Phys. Rev. Letters 14, 904 (1965).
[25] A. H. Dayem and R. J. Martin, Phys. Rev. Letters 8, 246 (1962).

tor is used. However, other experiments[10] with this type of superposed junctions in which no continuous cavity linked them together still showed the same effect. On the other hand, experiments[10] were also performed with two junctions not superimposed but fabricated in such a way that they lie side by side with a continuous cavity connecting them so as to obtain a tighter electromagnetic coupling between the two junctions. Nevertheless, no effect was observed in such experiments. The only way to explain the behavior of the superposed junctions not connected by a continuous cavity is to assume that the fields from one junction penetrate directly into the other through the superconducting middle film; such an effect would be absent for the junctions which were side by side. We offer here a more precise picture for the understanding of these experimental findings by recognizing that these phenomena are just consequences of the excitation of those wave modes which couple the two barriers together electromagnetically. It is worthwhile to note that this new interpretation still implies that Giaever's experiments serve to detect the ac Josephson effect, though not with the help of a continuous cavity, but rather through the collective surface plasma oscillations that couple the two junction barriers.

We shall consider an idealized symmetrical system of two superposed junctions as shown in Fig. 8(a) of I. This choice has the merit of requiring algebra that can be easily handled, while the results nevertheless permit discussion of the most general situation. If we further assume either a symmetric or antisymmetric distribution of supercurrents of amplitude j_1 in the two barriers, the fields can be obtained in terms of symmetrical or antisymmetrical eigensolutions which can be represented by

$$\begin{aligned} E_x &= E_1 e^{-K_i x}, && \text{in } 1\\ &= E_2(e^{K_m x}+B_2 e^{-K_m x}), && \text{in } 2\\ &= E_3(e^{K_i x}+B_3 e^{-K_i x})-(i4\pi\omega/c^2K_i^2)j_1, && \text{in } 3\\ &= E_4(e^{K_m x}\pm e^{-K_m x}), && \text{in } 4. \end{aligned} \tag{2.12}$$

Six boundary conditions for the continuity of E_z and H_y lead to the system

$$\begin{aligned} &-K_iE_1e^{-K_i(d_m/2+d_m'+d_i)} = K_mE_2(e^{K_m(d_m/2+d_m'+d_i)}-B_2e^{-K_m(d_m/2+d_m'+d_i)}),\\ &(k^2-K_i^2)E_1e^{-K_i(d_m/2+d_m'+d_i)}=(k^2-K_m^2)E_2 \times(e^{K_m(d_m/2+d_m'+d_i)}-B_2e^{-K_m(d_m/2+d_m'+d_i)}),\\ &K_mE_2(e^{K_m(d_m/2+d_i)}-B_2e^{-K_m(d_m/2+d_i)}) = K_iE_3(e^{K_i(d_m/2+d_i)}-B_3e^{-K_i(d_m/2+d_i)}),\\ &(k^2-K_m^2)E_2(e^{K_m(d_m/2+d_i)}+B_2e^{-K_m(d_m/2+d_i)}) = K_iE_3(e^{K_i(d_m/2+d_i)}+B_3e^{-K_i(d_m/2+d_i)}) -(i4\pi\omega/c^2)(k^2/K_i^2)j_1,\\ &K_iE_3(e^{K_i(d_m/2)}-B_3e^{-K_i(d_m/2)}) = K_mE_4(e^{K_m(d_m/2)}\mp e^{-K_m(d_m/2)}),\\ &(k^2-K_i^2)E_3(e^{K_i(d_m/2)}+B_3e^{-K_i(d_m/2)}) -(i4\pi\omega/c^2)(k^2/K_i^2)j_1 =(k^2-K_m^2)E_4(e^{K_m(d_m/2)}\pm e^{-K_m(d_m/2)}). \end{aligned} \tag{2.13}$$

From this set of linear equations we can solve for E_3 and E_3B_3, and the alternating voltage developed across either barrier is

$$v_0=(E_3e^{K_md_m/2}+E_3B_3e^{-K_md_m/2})d_i-(i4\pi\omega/c^2K_i^2)d_ij_1 =\frac{i4\pi d_ij_1}{\epsilon_i\omega K_i^2}\left(\frac{k^2}{1-\xi_-\alpha_-\beta_\pm/W}-\frac{\epsilon_i\omega^2}{c^2}\right), \tag{2.14}$$

where

$$W=\xi_-\alpha_-\beta_\mp R^3-(\xi_+\alpha_+\beta_\mp+\xi_+\alpha_-\beta_\mp+2\xi_-\beta_\pm)R^2 +(\xi_-\alpha_+\beta_\mp+2\xi_+\beta_\pm)R,$$

with

$$\alpha_\pm=e^{K_id_i}\pm e^{-K_id_i},\quad \beta_\pm=e^{K_md_m/2}\pm e^{-K_md_m/2},\quad \xi_\pm=e^{K_md_m'}\pm e^{-K_md_m'}.$$

When restricted to the range of frequency of oscillations and the geometric sizes that occur in conventional Josephson tunnel junctions, the R^2 and R^3 terms in W can be neglected and v_0 becomes

$$v_0=\frac{i4\pi d_ij_1}{\epsilon_i\omega K_i^2}\times\left(\frac{k^2(1-\bar{c}_\pm^2/c_i^2)}{(1-k^2\bar{c}_\pm^2/\omega^2)+i(\bar{c}_\pm^2/c_i^2)(k^2c_i^2/\omega^2-1)\delta_\pm}-\frac{\omega^2}{c_i^2}\right),$$

with

$$\bar{c}_\pm=c_i\left(\frac{d_i}{d_i+\lambda_p\mathfrak{F}_\pm}\right)^{1/2},\quad c_i=\frac{c}{\sqrt{\epsilon_i}},\quad \mathfrak{F}_\pm=\coth(d_m'/\lambda_p)+\begin{pmatrix}\tanh(d_m/2\lambda_p)\\ \coth(d_m/2\lambda_p)\end{pmatrix}, \tag{2.15}$$

and

$$\delta_\pm=\frac{\sigma_1(\omega)}{2\sigma_2(\omega)}\left\{1+\left[d_m'\operatorname{csch}^2(d_m'/\lambda_p)/\lambda_p+\begin{pmatrix}\operatorname{sech}^2(d_m/2\lambda_p)\\ \operatorname{csch}^2(d_m/2\lambda_p)\end{pmatrix}d_m/2\lambda_p\right]\Big/\mathfrak{F}_\pm\right\}.$$

Here $\bar{c}_\pm$ are the phase velocities of two low-frequency surface plasma oscillations in this geometry. Further reduction yields the resonant form

$$v_0=\frac{i4\pi d_ij_1}{\epsilon_i\omega}\frac{1}{(1-\bar{c}_\pm^2k^2/\omega^2)+i/Q_\pm}, \tag{2.16}$$

where

$$Q_\pm(k,\omega)=\frac{(\omega/\bar{c}_\pm k)^2}{\delta_\pm(1-\bar{c}_\pm^2/c_i^2)}. \tag{2.17}$$

The plus sign holds when the excitation current distribution is symmetric and the minus when antisymmetric.

Following the procedure of Eck *et al.*,[17] we expand v_0 in the normal modes $\cos(n\pi z/L)$ of the junction. (L is the length of the junction in the z direction) and assume the junction to be of infinite extent in the y direction. We find

$$v_0(z,t)=\frac{4\pi d_i j_1}{\epsilon_i\omega}\sum_{n=0}^{\infty}\cos(n\pi z/L)\times\frac{a_n\cos(\omega t+\theta_n)+b_n\sin(\omega t+\theta_n)}{\{[1-(n\pi\bar{c}_\pm/\omega L)^2]^2+(1/Q_{\pm,n})^2]\}^{1/2}},\quad(2.18)$$

with

$$\theta_n=\tan^{-1}\frac{1/Q_n}{1-(n\pi\bar{c}_\pm/\omega L)^2},$$

$$a_n=\frac{2kL\sin(kL-n\pi)}{(kL)^2-(n\pi)^2}\begin{matrix}\times\frac{1}{2}, & n=0\\ \times 1, & n\neq 0\end{matrix}$$

$$b_n=\frac{2kL[1-\cos(kL-n\pi)]}{(kL)^2-(n\pi)^2}\begin{matrix}\times\frac{1}{2}, & n=0\\ \times 1, & n\neq 0\end{matrix}$$

and $Q_{\pm,n}=Q_\pm(n\pi/L,\omega)$ represents the losses in the nth mode.

This alternating voltage can in turn give rise to a dc current. To first order in v_0/V, the dc component of the tunneling current is

$$j_{\pm DC}=\int_0^L\frac{dz}{L}\int_0^T\frac{dt}{T}j_1\sin\left(\omega t-kz+\frac{2e}{\hbar}\int^t v_0(t')dt'\right)$$

$$=\frac{4\pi e d_i j_1^2}{\epsilon_i\hbar\omega}\times\sum_{n=0}^{\infty}\frac{1/Q_{\pm,n}}{[1-(n\pi\bar{c}_\pm/\omega L)^2]^2+(1/Q_{\pm,n})^2}F_n(k),\quad(2.19)$$

with

$$F_n(k)=\left(\frac{\sin(kL-n\pi)/2}{(kL-n\pi)/2}\right)^2\frac{1}{(1+n\pi/kL)^2}\begin{matrix}\times\frac{1}{2}, & n=0\\ \times 1, & n\neq 0.\end{matrix}$$

This equation exhibits resonant current jumps at two series of equally spaced voltages

$$V_{r,n}=n(\hbar\bar{c}_r\pi/2eL),\quad r=\pm,$$

(one series corresponds to the symmetric configuration, the other to the antisymmetric one) and the height of the nth current jump is modulated by the Fraunhofer term

$$\left(\frac{\sin(kL-n\pi)/2}{(kL-n\pi)/2}\right)^2.$$

Hence this type of behavior is analogous to that of a simple Josephson junction as considered in Ref. 5. The new feature worth our emphasis is the appearance of two distinct linear $\bar{c}$ modes—thus two series of resonant jumps.

When, in general, the two barriers are biased at different dc voltages V_1 and V_2 corresponding to supercurrents $j_1\sin(\omega_1 t-kz+\alpha_1)$ and $j_2\sin(\omega_2 t-kz+\alpha_2)$, respectively (here we do not require $j_1=j_2$), the alternating voltages v_1 and v_2 developed across the two barriers can be obtained as linear combinations of the symmetric and antisymmetric eigensolutions that correspond to supercurrent distributions

$$\begin{bmatrix}\frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\\ \frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\end{bmatrix}+\begin{bmatrix}\frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\\ -\frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\end{bmatrix}+\begin{bmatrix}\frac{1}{2}j_2\sin(\omega_2 t-kz+\alpha_2)\\ \frac{1}{2}j_2\sin(\omega_2 t-kz+\alpha_2)\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}j_2\sin(\omega_2 t-kz+\alpha_2)\\ \frac{1}{2}j_2\sin(\omega_2 t-kz+\alpha_2)\end{bmatrix}$$

written as a column vector, the top element of which is the current in the first junction and the bottom element the current in the second junction. These solutions have been obtained as in the form of Eq. (2.18) when expanded in the normal modes of the junctions. Calculating the dc component of the current density in the manner used before, we find for the first junction

$$j_1^{\mathrm{dc}}=\frac{2\pi e d_i j_1^2}{\epsilon_i\hbar\omega_1^2}\times\sum_{r=\pm}\sum_{n=0}^{\infty}\frac{1/Q_{r,n}}{[1-(n\pi\bar{c}_r/\omega_1 L)^2]^2+(1/Q_{r,n})^2}F_n(k),\quad(2.20)$$

and for the second

$$j_2^{\mathrm{dc}}=\frac{2\pi e d_i j_2^2}{\epsilon_i\hbar\omega_2^2}\times\sum_{r=\pm}\sum_{n=0}^{\infty}\frac{1/Q_{r,n}}{[1-(n\pi\bar{c}_r/\omega_2 L)^2]^2+(1/Q_{r,n})^2}F_n(k).\quad(2.21)$$

In the particular case when $\omega_1=\omega_2\equiv\omega$, an extra term proportional to j_1j_2 and given by

$$\frac{2\pi e d_i j_1 j_2}{\epsilon_i\hbar\omega^2}\sum_{r=\pm}\sum_{n=0}^{\infty}\frac{\mathrm{sgn}(r)/Q_{r,n}}{[1-(n\pi\bar{c}_r/\omega L)^2]^2+(1/Q_{r,n})^2}\times F_n(k)\cos(\alpha_1-\alpha_2),\quad(2.22)$$

where

$$\mathrm{sgn}(r)=1,\quad r=+$$
$$=-1,\quad r=-$$

has to be added to each of the two expressions [(2.20) and (2.21)] for the dc component of the current density. In this case each current j_1^{dc} and j_2^{dc} will have a double series of steps defined by

$$V_n=n(\hbar\bar{c}_+\pi/2eL)$$

and

$$V_{n'}=n'(\hbar\bar{c}_-\pi/2eL).$$

Of course, in order to have two distinguishable series of steps, $\bar{c}_+$ and $\bar{c}_-$ should be sufficiently different but this is equivalent to the requirement that d_m is not much larger than λ_p. Within the linear approximation of regarding $v/V\ll 1$, the supercurrents of frequency ω_2 present in the second junction, although giving rise to an ac voltage of the same frequency in the first junction, have no effect on the dc current of this junction and vice versa even if the two frequencies are so related that one is an integral multiple (other than one) of the other, for the reason that any such interference effect is higher-order in v/V, which is assumed to be small, and hence negligible. This is transparent when we write the expression for the supercurrent when the voltage across the barrier is the sum of the applied dc potential $V_1=\hbar\omega_1/2e$ and the oscillating potential $v\cos\omega_2 t$:

$$j=j_1\sin[\omega_1 t+(2ev/\hbar\omega_2)\sin\omega_2 t+\alpha]$$
$$=j_1\sum_{n=-\infty}^{\infty}\left[J_n\left(\frac{2ev}{\hbar\omega_2}\right)\sin[(n\omega_2+\omega_1)t+\alpha]\right],$$

where J_n is the Bessel function of order n. If $n\omega_2=\omega_1$ for a certain integer n, a dc component is obtained and is given by

$$j^{\mathrm{dc}}=(-1)^n j_1 J_n(nv/V_1)\sin\alpha.$$

If $v/V_1\ll 1$, this is of higher order in v/V_1 relative to the result when $n=1$. On the other hand, if the linear approximation breaks down and v/V cannot be assumed small, we would expect a non-negligible contribution to a dc current jump whenever $\omega_1=n\omega_2$ or the reverse is satisfied. Specifically in this case when one junction, say, the lower, is biased at the voltage $V_2=\hbar\omega_2/2e$ such that ω_2 corresponds to a resonant frequency of the system while the upper barrier is biased at $V_1=nV_2$ for any nonzero integer n, a sizable constant voltage step is expected in a manner completely analogous to the behavior observed by Shapiro.[26] In a similar situation but if $n=0$, i.e., $V_1=0$ and the upper junction is not externally biased, the resonant fields caused by the supercurrents in the lower junction at a resonant frequency would be expected to cause induced dc voltages across the upper junction analogous to the observation of Langenberg *et al.*[27]

We come back to discuss further Giaever's experiment for the detection of the ac Josephson effect in the light of the results just obtained for two superimposed barriers. We can simulate a generator and a detector junction superposed on each other as in the actual experimental situation of Giaever by taking our model junction of Fig. 8(a) of I and assuming that there is supercurrent only in the upper barrier, with the lower one regarded as if it were a conventional junction. We can approach this problem by again decomposing the zero-order ac supercurrents to symmetric and antisymmetric components as

$$\begin{bmatrix}\frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\\ \frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\end{bmatrix}+\begin{bmatrix}\frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\\ -\frac{1}{2}j_1\sin(\omega_1 t-kz+\alpha_1)\end{bmatrix}.$$

From (2.18), we have for the ac voltages the expressions

$$v_1(z,t)=\frac{2\pi d_i j_1}{\epsilon_i\omega}\sum_{r=\pm}\sum_{n=0}^{\infty}\cos\left(\frac{n\pi z}{L}\right)\times\frac{a_n\cos(\omega t+\theta_n)+b_n\sin(\omega t+\theta_n)}{[(1-n\pi\bar{c}_r/\omega L)^2+(1/Q_{r,n})^2]^{1/2}},\quad(2.23)$$

$$v_2(z,t)=\frac{2\pi d_i j_1}{\epsilon_i\omega}\sum_{r=\pm}\sum_{n=0}^{\infty}\operatorname{sgn}(r)\cos\left(\frac{n\pi z}{L}\right)\times\frac{a_n\cos(\omega t+\theta_n)+b_n\sin(\omega t+\theta_n)}{[(1-n\pi\bar{c}_r/\omega L)^2+(1/Q_{r,n})^2]^{1/2}}.\quad(2.24)$$

Note that for v_1 the symmetric and antisymmetric contributions add, whereas for v_2 it is the difference between these two contributions. If $d_m/\lambda_p\to\infty$, $\bar{c}_+\to\bar{c}_-$, and $Q_{+,n}\to Q_{-,n}$, then v_1 reduces to the usual result for a single barrier and v_2 becomes zero, as is expected since this limit the fields cannot penetrate through the middle metal film. In fact, for $d_m\gg\lambda_p$, $c_+-c_-\propto e^{-d_m/\lambda_p}$, $Q_+-Q_-\propto e^{-d_m/\lambda_p}$, and $v_2/v_1\propto e^{-d_m/\lambda_p}$. In the opposite case when $d_m\lesssim\lambda_p$, $\bar{c}_+$ and $\bar{c}_-$ are distinct, there is no overlapping of the (+) and (−) resonances, and consequently v_1 and v_2 are of the same order of magnitude. It is now clear that in view of our previous discussions when the generator is biased at a resonant step, a resonant electromagnetic mode of the multiple-film system is excited and the detector junction will have simultaneously *resonant* alternating fields across it, thus modifying its own *I-V* characteristic in essentially the same way as in the Dayem-Martin experiment[25] and explained by Tien and Gordon.[28] This would be a reasonable explanation for the experiments of Giaever and indicates clearly that even in the case when no effort was made to form a continuous cavity linking them together the experimental results would be unchanged. Under this explanation, however, two distinct series of step structures would be observable. This seems to be consistent with the observed *I-V* characteristic of the generator in that there are features which would be very difficult to explain with one series only. As a final remark, it can be said that the observations of Giaever verify the excitation of electromagnetic wave modes in the superposed-films system by ac Josephson currents.

We have assumed geometrical symmetry in the previous example to make the algebra manageable. The conclusions that we derived from there do not depend on this assumption, however. To show this indirectly we shall work out another multiple-film system with no

[26] S. Shapiro, Phys. Rev. Letters **11**, 80 (1963).

[27] D. N. Langenberg, D. J. Scalapino, B. N. Taylor, and R. E. Eck, Phys. Letters **20**, 563 (1966).

[28] P. K. Tien and J. P. Gordon, Phys. Rev. **122**, 647 (1963).

symmetry assumed as shown in Fig. 7 of I. The configuration can be easily realized in experiment and deserves a study of its behavior. We follow the procedure illustrated by the last two examples. Express the fields E_x in terms of the eigenfunctions in the separate regions and write the fields E_z and H_y in terms of E_x. The boundary conditions satisfied by E_z and H_y again lead to a determination of the unknown field amplitudes in terms of the driving currents $j=j_1\sin(\omega t-kz+\alpha)$ and $j'=j_1'\sin(\omega' t-kz+\alpha')$ in the two dielectric regions caused by the ac Josephson effect when external dc voltages $V=\hbar\omega/2e$ and $V'=\hbar\omega'/2e$ are biased across them. In I it was shown that the electromagnetic modes of the system in the frequency range of interest in tunnel junctions consist of two linear modes with phase velocities given by

$$\bar{c}_{1,2}=c_i/\{1+\lambda_p\mathfrak{F}_{1,2}\}^{1/2}, \tag{2.25}$$

where

$$\mathfrak{F}_{1,2}=\frac{(d_i+d_i')\pm[(d_i-d_i')^2\pm 4e^{-2d_m/\lambda_p}d_id_i']^{1/2}}{d_id_i'(1-e^{-2d_m/\lambda_p})}.$$

The alternating voltages induced across the two barriers are then given by

$$v=\frac{i4\pi d_i}{\epsilon_i\omega K_i^2}\left[j\left(k^2\frac{\mathfrak{H}(R)}{\mathfrak{L}(R)}-\frac{\epsilon_i\omega^2}{c^2}\right)+j'k^2\frac{\mathfrak{K}(R)}{\mathfrak{L}(R)}\right], \tag{2.26a}$$

with

$$\mathfrak{H}(R)=4R^2-2K_id_i'R,$$
$$\mathfrak{K}(R)=4e^{-K_md_m}K_id_i'R,$$
$$\mathfrak{L}(R)=4R^2-2K_i(d_i+d_i')R+K_i^2d_id_i'(1-e^{-2K_md_m}),$$

and

$$v'=\frac{i4\pi d_i'}{\epsilon_i\omega K_i^2}\left[j'\left(k^2\frac{\mathfrak{H}'(R)}{\mathfrak{L}'(R)}-\frac{\epsilon_i\omega^2}{c^2}\right)+j\frac{\mathfrak{K}'(R)}{\mathfrak{L}'(R)}\right]. \tag{2.26b}$$

$\mathfrak{H}'(R)$ and $\mathfrak{K}'(R)$ are, respectively, obtained from $\mathfrak{H}(R)$ and $\mathfrak{K}(R)$ by the interchange $d_i'\leftrightarrow d_i$, and note that $\mathfrak{L}'(R)\equiv\mathfrak{L}(R)$. By use of the usual approximations, (2.26a) and (2.26b) can be rewritten to exhibit their resonant forms as follows:

$$v=\frac{i4\pi d_ij}{\epsilon_i\omega K_i^2}\left\{\frac{k^2}{(1-e^{-2d_m/\lambda_p})d_id_i'}\right.$$
$$\left.\times\left[\frac{(2d_i'\lambda_p/c^2)(1-\bar{c}_1^2/c^2)-4\lambda_p^2\bar{c}_1^2/c^2}{(1/\bar{c}_1^2-1/\bar{c}_2^2)(1-k^2\bar{c}_1^2/\omega^2)}+1\leftrightarrow 2\right]-\frac{\epsilon_i\omega^2}{c^2}\right\}$$
$$+\frac{i4\pi d_ij'}{\epsilon_i\omega'}\frac{k^2}{K_i^2}\frac{e^{-d_m/\lambda_p}2d_i'\lambda_p/c^2}{(1-e^{-2d_m/\lambda_p})d_id_i'}$$
$$\times\left[\frac{(1-\bar{c}_1^2/c^2)}{(1/\bar{c}_2^2-1/\bar{c}_1^2)(1-k^2\bar{c}_1^2/\omega^2)}+1\leftrightarrow 2\right], \tag{2.27}$$

where the second term of each of the two square-bracketed expression is obtained from the first by an interchange $\bar{c}_1\leftrightarrow\bar{c}_2$. The corresponding expression for v' can be obtained from (2.27) by the interchange of $d_i\leftrightarrow d_i'$, $j\leftrightarrow j'$, and $\omega\leftrightarrow\omega'$. The Q factor can be derived also from (2.27) if we replace λ_p that occurs there explicitly or implicitly through $\bar{c}_1$ and $\bar{c}_2$ as in (2.25) by $\lambda_p[1-i\sigma_1(\omega)/\sigma_2(\omega)]$. The resulting expression for Q is a complicated expression which shall not be recorded here, but is again of the order of σ_1/σ_2. Under the special assumption that $d_i=d_i'$, $\omega=\omega'$, and $j=j'$, i.e., a symmetric excitation current distribution, (2.27) reduces to

$$v_2=\frac{i4\pi d_ij}{\epsilon_i\omega}\frac{1}{(1-\bar{c}_s^2k^2/\omega^2)+i/Q},$$

with

$$\bar{c}_s=c_i\frac{d_i}{d_i+\lambda_p(1+\tanh d_m/2\lambda_p)}$$

as it should. It is clear from these explicit expressions for v and v' that supercurrents can excite resonant electromagnetic modes of the system. Supercurrents inside one barrier can have a significant influence on the other only if $d_m\lesssim\lambda_p$ as it is clear from the appearance of the factor e^{-d_m/λ_p} in the second term of (2.27). Then, in that case, the expected result of this resonant excitation is the appearance of two distinct series of current jumps in the I-V characteristics of the barriers. Various other consequences that have been discussed before in connection with the previous geometry also apply to the present case.

One can go on to examine more and more complex geometries of superposed films. However, from the study in I, it seems clear that the essential physics of such systems has already been embodied by the few simple cases examined here. As the number of metal-dielectric interfaces increases, the number of eigenmodes increases correspondingly. Each of these modes corresponds to surface plasma oscillations of the system as a whole. When these inhomogeneous systems exhibit the ac Josephson effect, under certain conditions, some of these modes can be resonantly excited via the action of the supercurrents. These resonant excitations are observable in the I-V characteristics of the barriers as the appearance of several different series of equally spaced current jumps. The number of such series is the same as the number of the low-lying eigenmodes of the system.

III. MODIFICATION OF DISPERSION CURVES BY THE JOSEPHSON EFFECT

In I, electromagnetic wave modes in superposed normal or superconducting metal and dielectric films are described and their dispersion relations derived without the inclusion of any ac Josephson effect that may exist in such inhomogeneous systems when the metals are superconducting. Section II of the present

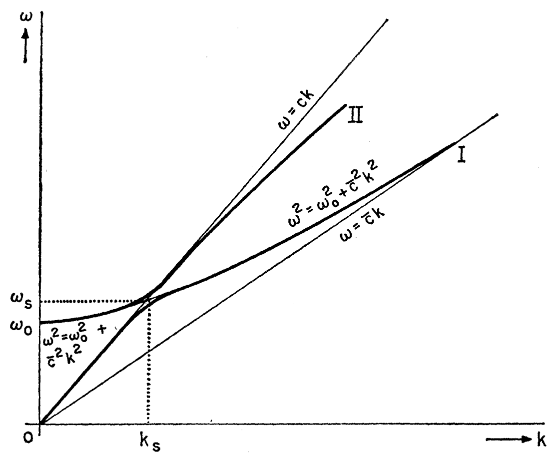

FIG. 3. Dispersion relations for the low-frequency modes as modified by the ac Josephson effect of the symmetric geometry of three films (metal-insulator-metal) between two semi-infinite insulators, $\omega_s\sim\omega_0(1+d_i/2\lambda_p)^{1/2}$, $k_s\sim\omega_s/c$.

work examines the excitation of these modes by the ac supercurrents when external dc voltages are applied. The treatment therein amounts to finding the response of the surface plasma oscillations to the exciting supercurrents in a perturbative manner. This approach is correct if a linear approximation is valid and when we are interested in such responses as the I-V characterisitics. There are situations, however, where this approach is invalid. One simple example is given by the case of a Josephson junction with no applied dc voltage when oscillations of the system either in the form of propagation of small-amplitude electromagnetic waves along the barrier or oscillations that resemble ordinary plasma oscillations commonly termed the Josephson plasma resonance are sought. Other examples are general superposed-film systems in the absence of applied dc voltages and oscillations of these types are examined. The study of such oscillations in multiple films will constitute Sec. III. We start with a discussion of the nature of these oscillations in a Josephson junction.

The existence of a plasma resonance in Josephson tunnel junctions was predicted theoretically by Josephson[12,13] and recently observed experimentally by Dahm *et al.*[11] The plasma resonant mode corresponds to a situation when the current and electric field are normal to the barrier, the magnetic field is zero, and there is a periodic exchange of energy between the electric field and the electrons (longitudinal oscillations). In addition to the plasma oscillations, a plane-wave solution $\sim e^{i(\omega t-k\cdot z)}$ which is physically equivalent to the propagation of electromagnetic waves along the barrier is also admitted. The dispersion formula is given by

$$\omega^2=\omega_0^2+\bar{c}^2k^2, \qquad (3.1)$$

where $\omega_0^2=8\pi ed_ij_1\cos\phi_0/\epsilon_i\hbar$ is the Josephson plasma frequency and $\bar{c}=c[d_i/\epsilon_i(2\lambda_p+d_i)]^{1/2}$ is the same as the phase velocity (3.19) derived in I for a barrier between two semi-infinite metals. In fact, (3.1) was derived for this model, and the appearance of $\bar{c}$, which is the propagation velocity of electromagnetic oscillations in the absence of supercurrents, is quite natural. The modification due to the supercurrents is clear from (3.1), which says that for $\omega<\omega_0$, k is imaginary, so that waves cannot be propagated, while for $\omega\gg\omega_0$ the propagation velocity tends to $\bar{c}$, its value in the absence of the Josephson effects. Before we examine these interesting modifications for the multiple films we studied in I, we shall sketch a procedure that will be applicable for the various configurations.

We assume that φ, the difference between the values of the phase of the superconducting order parameter of the two sides of the barrier, undergoes small oscillations $\delta\varphi$ of the plane-wave form $\sim e^{i(\omega t-kz)}$ about φ_0. From Josephson's equation, we have

$$i\omega\delta\varphi=(2e/\hbar)\delta v, \qquad (3.2)$$

which relates the oscillating part of the phase to the accompanying oscillating voltage δv across the barrier. The expression for the supercurrents can be linearized under the assumption that $\delta\varphi$ is small, and rewritten in terms of δv as

$$j_T=j_1\sin\varphi_0+(2ej_1/i\hbar\omega)\cos\varphi_0\delta v. \qquad (3.3)$$

The second term of the right-hand side is an oscillating current with time and spatial dependence of the form $e^{i(\omega t-kz)}$. As we have seen already in Sec. II, these supercurrents will cause electromagnetic fields in the multiple-film system and from these fields we can calculate the oscillating voltages across the separate Josephson junctions that are present in the system. For self-consistency these must be the same as the δv's that appear in (3.3). We are thus lead to an eigenequation which in general is of the matrix type from which the dispersion relations can be obtained.

A. Single Josephson Junction

As an orientation for use of this self-consistency method, we reexamine the case of a barrier between two semi-infinite superconductors. The voltage across the dielectric film of thickness d_i is $\delta v=E_id_i$, where E_i is the component of the electric field inside the barrier and perpendicular to the plane of the junction. With the method explained in Sec. II, E_i can be calculated and expressed in terms of the tunneling current j_T. Explicitly it is

$$\begin{aligned}\delta v=E_id_i&=\frac{i4\pi d_ij_T}{\epsilon_i\omega K_i^2}\left(\frac{k^2}{1-K_id_i/2R}-\frac{\epsilon_i\omega^2}{c^2}\right)\\&=\frac{i4\pi d_ij_T}{\epsilon_i\omega}\frac{1}{(1-\bar{c}^2k^2/\omega^2)+i/Q},\end{aligned} \qquad (3.4)$$

where $\bar{c}$ and Q can be obtained from Eqs. (2.8) and (2.10) in the limit $d_m/\lambda_p \rightarrow \infty$. For self-consistency, we also have to satisfy

$$j_T = (2ej_1/i\hbar\omega)\cos\varphi_0\delta v. \tag{3.5}$$

From (3.4) and (3.5) it follows that the dispersion relation is given in (3.1) with $\omega_0^2 = 2ej_1\cos\phi_0/\hbar C$, where C is the capacitance per unit area of the junction. At $k=0$, $\omega=\omega_0$ is the Josephson plasma frequency. The situation corresponding to such a plasma oscillation has been described by Josephson. Our present picture illustrates the fact that the magnetic fields associated with the junction modes excited by the supercurrents vanish as $k \rightarrow 0$. In particular, the magnetic field H_y, being first-order in k for small k, vanishes at $k=0$. We have seen in I that when $k \gg k_p$, the magnetic field and hence the retardation effects are also negligible. The physical explanation for why, in the regions of small k and large k, the retardation effects are negligible can be given via a criterion due to Ferrell[14] and elucidated in I. The criterion is to compare the ratio of the distance to the time over which charges are transferred with the propagation velocity of electromagnetic waves, $c_i = c/\epsilon_i^{1/2}$ and $c_m = c/\epsilon_m^{1/2}$ in the insulator and metal, respectively. If this ratio is much smaller than both c_i and c_m, retardation effects are negligible and the electrostatic theory is adequate for describing the phenomenon. For the region of large k, the distance is the wavelength $\sim 1/k$ and the time is $\sim 1/\omega$. Retardation effects are negligible if $\omega/k \ll c_i$ and c_m (region of surface plasmon). For the region of small k and ω close to ω_0, the appropriate distance is d_i since now the transfer of charges is due to the tunnel effect across the barrier. The criterion for negligible retardation effect is $d_i\omega \ll c_i$, c_m, which is obviously satisfied, for typically $d_i \sim 20$ Å, $\omega \sim 10^{11}$ sec^{-1}, $\lambda_p \sim 500$ Å, and $\bar{c} \sim 10^9$ cm/sec. As we move towards higher k, a considerable amount of charge will start to travel parallel to the surface and the characteristic distance involved in this process is $1/k$ as we have discussed before. When $k \sim \omega_0/\bar{c}$ this process is no longer negligible compared to tunneling across the barrier. Hence the characteristic distance changes gradually from being d_i to $1/k$ as we move from $k \simeq 0$ to $k \gg \omega_0/\bar{c}$. Now it is clear that we can say that oscillations near the point, $\omega=\omega_0$, $k \simeq 0$ are true plasma oscillations without retardation effects and of the same nature as plasma oscillation in the bulk of a metal. Their basic difference is that in the Josephson plasma oscillation only the charges that tunnel participate and not the bulk charge density. This also accounts for ω_0 being several orders of magnitude below ω_p.

So far we have examined the case of a barrier with semi-infinite metals. When the metal films are of finite thickness additional surfaces are present and give rise to additional electromagnetic wave modes as we can see from I. For convenience we take a symmetric junction where both metal films have the same thickness d_m. Adopting the self-consistency procedure illustrated previously, we find the equation

$$\omega^4 - B\omega^2 + \Gamma = 0, \tag{3.6}$$

where

$$B = \omega_0^2(1+\bar{t}(d_i/2\lambda_p)) + \bar{t}c^2K_i/\lambda_p + \bar{t}c^2K_i^2 d_i/2\lambda_p,$$
$$\Gamma = \omega_0^2 c^2 K_i(d_i/2\lambda_p + \bar{t})/\lambda_p + c^4 K_i^3 d_i/2\lambda_p^2,$$
$$t = \tanh d_m/\lambda_p, \quad \bar{t} = \coth d_m/\lambda_p,$$

whose solutions define the dispersion formula. Explicit solution of the equation for all regions of interest in the k-ω plane is difficult. This difficulty can be circumvented by solving for ω^2 in terms of K_i^2 to obtain

$$\omega^2 = \tfrac{1}{2}B \pm (B^2 - 4\Gamma)^{1/2}, \tag{3.7}$$

which will map out the dispersion curves by giving their intersections with the lines $K=$const. Graphically the qualitative behavior of the dispersion curves can be represented as in Fig. 3.

For small k, branch II can be described by

$$\omega^2 = \omega_0^2 + k^2\bar{c}^2$$

and I is close to the line $\omega = kc$. Near

$$k \cong \omega_0(1+d_i/2\lambda_p)^{1/2}/c$$

these two modes interact sufficiently strongly to split them apart into two branches which approach asymptotically the lines $\omega = kc$ and $\omega = k\bar{c}$ as shown in the figure. A minimum separation of the two branches along a direction parallel to the line $\omega = kc$ has been estimated to be $\propto e^{-d_m/\lambda_p}$. The cause for the complications in the dispersion formula for a junction with finite thickness d_m of metal films as compared with that when $d_m \rightarrow \infty$ can be explained by comparing Figs. 6 with 3 of I. The additional branch III in Fig. 6 can be described as associated with oscillation on the outer surfaces of the metal films when d_m is large and hence can be neglected if $d_m \rightarrow \infty$. On the other hand, if d_m is small, this description ceases to be valid and this mode interacts with the others. In the present discussion when the supercurrents are put in self-consistently, the branch $\omega = \bar{c}k$ becomes $\omega^2 = \omega_0^2 + \bar{c}^2k^2$, which would intersect branch III except for the interaction which lifts the degeneracy and splits them apart. The amount of splitting is measured by the minimum distance e^{-d_m/λ_p} given before, which shows that the splitting is dependent on the degree of interaction between the two modes.

It is worth noting from Fig. 3 that the branch starting from $\omega=\omega_0$ at $k=0$ remains above the $\omega = ck$ line throughout the range $0 \leqslant k \lesssim k_s$. In this range, $K_i = (k^2 - \omega^2/c^2)^{1/2}$ is imaginary and the corresponding modes are not confined to oscillations in the surface but have acquired the nature of propagation into the infinite medium in which the junction is imbedded. This possibility arises because the phase velocity of surface plasma oscillation exceeds the velocity of light.[14] Nevertheless, an order of magnitude estimation shows that the intensity of the expected microwave radiation is negligibly small to be detectable.

We are now ready to consider some multiple-film systems that have more than one dielectric barrier, and as a first example we take the configuration as illustrated by Fig. 6 of I. Assume the situation when the phase difference $\delta\varphi_1$ or $\delta\varphi_2$ across each of the two barriers oscillates about equilibrium value φ_0 according to the form $e^{i(\omega t-kz)}$ and allow $\delta\varphi_1$ to be either in phase or 180° out of phase with $\delta\varphi_2$. Thus, by this assumption the supercurrent distribution is either symmetric or antisymmetric and the solution of the problem can be easily obtained by invoking the result of Sec. II for the voltages induced across the barriers due to the excitation of electromagnetic modes by the supercurrents. The requirement for self-consistency leads us to the equation

$$1=\frac{\omega_0^2}{\omega^2}\frac{k^2}{K_i^2}\left(\frac{1}{1-\xi_-\alpha_-\beta_\pm/W}-\frac{\epsilon_i\omega^2}{c^2k^2}\right),\qquad(3.8)$$

which determines the dispersion relation for a simultaneous propagation of small-amplitude electromagnetic waves along the barriers. A general solution of this equation is rather difficult but unnecessary. For physical reasons we expect, for small k, some dispersion branches still described by the form $\omega^2=\omega_0^2+\bar{c}^2k^2$ for some $\bar{c}$. Indeed, substitution of this into (3.8) will satisfy the equation if $\bar{c}=c[d_i/\epsilon_i(d_i+\lambda F_\pm)]^{1/2}=c_\pm$, which are the phase velocities as defined in Eq. (2.13). The situation is then similar to the last case of a junction with metals of finite thickness. The branch $\omega^2=\omega_0^2+k^2c_+^2$ interacts with branch III of Fig. 8 in I and a splitting occurs which eventually makes the former and the latter approach the lines $\omega=ck$ and $\omega^2=\omega_0^2+\bar{c}_+^2k^2$, respectively. The same remarks apply to $\omega^2=\omega_0^2+\bar{c}_-^2k^2$ and branch IV of Fig. 8 in I. Since there is no demand for a detailed quantitative solution, we are satisfied with the picture given in Fig. 3, with branch II there now representing $\omega^2=\omega_0^2+\bar{c}_\pm^2k^2$ for small k.

The preceding results are derived through quite restrictive and nonrealistic assumptions about (a) geometric symmetry, (b) the identity of the tunneling amplitudes for the two barriers, and (c) the relation between the phases of the oscillations $\delta\varphi_1$ and $\delta\varphi_2$. We do not expect the physics of the problem to be modified drastically when the geometric symmetry is removed, except for a complication in the calculation. However, if we relax the last two assumptions, we would obtain instead a matrix equation for the solution of the dispersion formula. The treatment in this case corresponds to a decomposition of the current distribution in the two barriers into symmetrical and antisymmetrical components as has been discussed in Sec. II. The voltages induced across the barriers by these components of the current distribution can be immediately written by using the results of Sec. II and their sum gives the desired total voltage δv_1 and δv_2 across the two barriers. Self-consistency requirements as explained then lead to a 2×2 matrix equation

$$A\begin{pmatrix}\delta v_1\\ \delta v_2\end{pmatrix}=0$$

and the dispersion formula is given by equating the determinant of A to zero. The matrix elements of A can be easily derived but shall not be recorded because there is no interest for a general solution of this problem. Also, we are going to deal next with a similar case where the matrix equation is explicitly derived and discussed.

As a final example we discuss the two-barrier assembly as in Fig. 7 of I. The by-now-familiar procedure would yield via Eqs. (2.26a) and (2.26b) and a self-consistency argument the following matrix equation:

$$\begin{pmatrix}\delta v\\ \delta v'\end{pmatrix}=\frac{k^2}{\omega^2K_i^2}\begin{bmatrix}\omega_0^2\left(\dfrac{\mathcal{K}(R)}{\mathcal{L}(R)}-\dfrac{\epsilon_i\omega^2}{c^2k^2}\right) & \omega_0'^2\dfrac{d_i}{d_i'}\dfrac{\mathcal{K}(R)}{\mathcal{L}(R)}\\ \omega_0^2\dfrac{d_i'}{d_i}\dfrac{\mathcal{K}'(R)}{\mathcal{L}(R)} & \omega_0'^2\left(\dfrac{\mathcal{K}'(R)}{\mathcal{L}(R)}-\dfrac{\epsilon_i\omega^2}{c^2k^2}\right)\end{bmatrix}\begin{pmatrix}\delta v\\ \delta v'\end{pmatrix},\qquad(3.9)$$

where

$$\omega_0^2=8\pi ed_ij\cos\phi_0/\epsilon_i\hbar$$

and

$$\omega_0'^2=8\pi ed_i'j'\cos\phi_0'/\epsilon_i\hbar.$$

We rewrite this in the form

$$M\begin{pmatrix}\delta v\\ \delta v'\end{pmatrix}\equiv\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\delta v\\ \delta v'\end{pmatrix}=0.$$

Several facts can be deduced from this matrix equation by inspection. At $k=0$ we have

$$M_{11}=-\omega_0^2\omega^2-c^2K_i^2\omega^2,\quad M_{22}=-\omega_0'^2\omega^2-c^2K_i^2\omega^2,$$
$$M_{12}=M_{21}=0.$$

Nonzero solutions for

$$\begin{pmatrix}\delta v_1\\ \delta v_2\end{pmatrix}$$

are obtained for $\omega=\omega_0$ and also for $\omega=\omega_0'$. These correspond to the Josephson plasma frequency of each of the barriers itself and oscillations occur as if the rest of the multiple films have no effect on each individual barrier. This happens because at $k=0$ the matrix M is diagonal, which means that the oscillation in the two junctions are decoupled. Physically the situation corresponds to the appearance of two layers of equal and opposite charges at the two surfaces of each barrier. The field created by such a charge distribution is thus confined to inside the barriers only and decoupling of the two junctions is clear.

By making the usual approximations for the case when $\omega \ll \omega_p$, $k \ll k_p$, we can write the matrix elements as

$$M_{11}=\omega^2(\omega^2-\omega_0{}^2)-c_{11}k^2,$$
$$M_{12}=c_{12}k^2,$$
$$M_{22}=\omega^2(\omega^2-\omega_0'^2)-c_{22}k^2,$$
$$M_{21}=c_{21}k^2,$$

where

$$c_{11}=c^2\omega^2-c^2\omega_0{}^2\frac{2\lambda_p(2\lambda_p+d_i')}{4\lambda_p{}^2+2\lambda_p(d_i+d_i')+d_id_i'(1-e^{-2d_m/\lambda_p})},$$

$$c_{12}=-c^2\omega_0'^2\frac{2\lambda_p d_i e^{-d_m/\lambda_p}}{4\lambda_p{}^2+2\lambda_p(d_i+d_i')+d_id_i'(1-e^{-2d_m/\lambda_p})}.$$

c_{22} can be obtained from c_{11} and c_{21} from c_{12} by the interchanges $\omega_0 \leftrightarrow \omega_0'$ and $d_i \leftrightarrow d_i'$. Then the dispersion relation is given by

$$\begin{aligned}\omega^4(\omega^2-\omega_0{}^2)(\omega^2-\omega_0'^2)\\ -c_{11}\omega^2(\omega^2-\omega_0'^2)k^2-c_{22}\omega^2(\omega^2-\omega_0{}^2)k^2\\ +(c_{11}c_{22}-c_{12}c_{21})k^4=0.\end{aligned} \quad (3.10)$$

If $\omega_0 \neq \omega_0'$, in the neighborhood of the point $\omega=\omega_0$, $k=0$ (or $\omega=\omega_0'$, $k=0$), the last two terms of the above equation are of higher order and can be neglected. This means that not only at $k=0$ but also for small values of k the oscillations in the two barriers are independent. These independent modes are described by

$$\omega^2=\omega_0{}^2+\bar{c}^2k^2 \quad (3.11)$$

and

$$\omega^2=\omega_0'^2+\bar{c}'^2k^2, \quad (3.12)$$

where

$$\bar{c}^2=c^2\frac{d_i[2\lambda_p+d_i'(1-e^{-2d_m/\lambda_p})]}{4\lambda_p{}^2+2\lambda_p(d_i+d_i')+d_id_i'(1-e^{-2d_m/\lambda_p})} \quad (3.13a)$$

and

$$\bar{c}'^2=\bar{c}^2(d_i \leftrightarrow d_i'). \quad (3.13b)$$

We note that if either $d_i'=0$ or $d_m \to \infty$,

$$\bar{c}=c[d_i/(d_i+2\lambda_p)]^{1/2}$$

the same as the propagation velocity in a barrier with semi-infinite metals, as it should.

If $\omega_0=\omega_0'$, the nondiagonal terms cannot be neglected in this way and they cause a coupling of the oscillations in the two barriers. In this case the dispersion relations are given as

$$\omega^2=\omega_0{}^2+\bar{c}_{1,2}{}^2k^2, \quad (3.14)$$

where $\bar{c}_{1,2}$ are given as in (2.25).

We conclude this section by considering the modifications caused by the Josepson effect in the dispersion relations of plasma oscillations in the periodic structure of alternating metal and insulating films examined in I. Using the same techniques as before and assuming that the quantity ω_0 is the same for all barriers, we find that the dispersion relations for the low-frequency modes is changed from $\omega=\bar{c}(\alpha)k$ to

$$\omega^2=\omega_0{}^2+\bar{c}^2(\alpha)k^2 \quad (3.15)$$

for small k. Here $\bar{c}(\alpha)$ is given by

$$\bar{c}(\alpha)=c\left[d_i\Big/\left(d_i+2\lambda_p\frac{e^{2d_m/\lambda_p}+1-2\cos(\alpha d)e^{d_m/\lambda_p}}{e^{2\lambda_m/\lambda_p}-1}\right)\right]^{1/2}$$

(where $d=d_i+d_m$) and is the same as (3.52) of I.

IV. CONCLUSION

In this work we have thus given a thorough investigation of the interaction of the ac Josephson currents with surface plasmons in multiple-film systems and various consequences that can be observed experimentally. We have seen that the supercurrents can excite surface plasma oscillations whose characteristics have been well described in I. When specialized to a single Josephson junction, the theory leads to the occurrence of current jumps and resonant peaks in the I-V characteristics. It permits also the relevant parameters such as phase velocity of the oscillation and the loss at resonance to be calculated as a function of frequency, temperature, and mean free path. From the frequency dependence of the phase velocity, the decrease of the ratio of the voltage at which a resonance peak occurs to the corresponding applied magnetic field has been accounted for quantitatively. Meanwhile, its temperature dependence gives indication that when the temperature $T \gtrsim 0.6T_c$, the steps would start to shift nonuniformly with T and consequently cease to occur at equally spaced voltages. For superposed junctions, our results show that the collective nature of the surface plasma modes causes a coupling between the junctions. This provides an explanation to a three-film tunneling experiment of Giaever. Furthermore, this coupling will alter the operating characteristic of each individual junction and several series of equally spaced steplike structures appear in the I-V characteristic of each junction. It should be noted that in the study of the resonant excitation of surface plasma modes by supercurrents we have restricted ourselves to the situation when the linear approximation is justifiable. However, nonlinear effects which we have thus neglected can be important in certain circumstances. Nonlinear effects, if nonnegligible, would give rise to the appearance of harmonics and subharmonics, and would distort the line shape from Lorentzian to cusplike.[29] The recent experimental observation[11] of the Josephson plasma resonance has stimulated a good deal of interest in surface plasma oscillations when modified by the Josephson currents. We have studied such modified oscillations in multiple-film systems and their nature. The results of Sec. III would be useful for further experimental efforts in this area.

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Tunneling into Low-Carrier-Density Superconductors

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A theory of tunneling between a normal metal and a low-carrier-density superconductor at zero temperature is developed. Because of the small Fermi energy of low-carrier-density superconductors and the energy dependence of the tunneling matrix elements for these junctions, the conductance-versus-voltage curves are quite different from those for tunneling between a metal and a high-density (metallic) superconductor. The tunneling displays both the voltage-dependent conductance associated with tunneling between a metal and a degenerate semiconductor and the peaks in conductance arising from the large quasiparticle density of states at voltages slightly larger than the superconducting energy gap.

I. INTRODUCTION

Electron tunneling in metal-insulator-metal (M-I-M) junctions when one metal is in the superconducting state and the other in the normal state has been successfully used to determine the tunneling density of states and the superconducting energy gap as a function of energy, $\Delta(E)$, in superconducting metals.[1–4] The theory usually used to interpret these experiments assumes that the Fermi energy ϵ_F is much larger than any voltage applied across the junction, and that the tunneling probability is constant for all voltages of interest. While these approximations hold for M-I-M junctions, they are quite restrictive when applied to metal-insulator-superconducting degenerate semiconductor (M-I-SS) junctions. In this paper, we attempt to develop a theory of tunneling in M-I-SS junctions without making the above approximations.

At present, the low-carrier-density superconductors (superconducting semiconductors and semimetals) known to the author are GeTe,[5] $SrTiO_3$,[6] and SnTe.[7] Tunneling in Al-Al_2O_3-SnTe and Al-Al_2O_3-GeTe junctions has been observed when both Al and SnTe or Al and GeTe are in the normal state,[8] and in Al-Al_2O_3-GeTe junctions when Al is in the superconducting state and with GeTe in both the normal and superconducting states,[9] and $\Delta(E=\Delta)$ of GeTe has been measured in this way.

Although tunneling from In into $SrTiO_3$ has been observed at temperatures below the superconducting transition temperature of the $SrTiO_3$ specimen used (and also below the In transition temperature), the superconducting energy gap of $SrTiO_3$ was not observed.[10] The dc Josephson effect has been seen in tunneling from In points into $SrTiO_3$.[11]

Because low-carrier-density superconductors have Fermi energies of the order of magnitude of, or less than, the phonon frequencies important for superconductivity, it is expected[12] that $\Delta(E)$ will have a

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