Surface Plasmons in Thin Films

E. N. Economou*†

The James Franck Institute and Department of Physics, The University of Chicago, Chicago, Illinois 60637

(Received 15 January 1969)

The dispersion relations for surface plasma oscillations in normal metals are investigated for single- and multiple-film systems taking retardation effects into account. The simple dielectric function $\epsilon(\omega)=1-\omega_p^2/\omega^2$ is found to be adequate for the high-frequency region in which oscillations remain undamped. Two types of possible modes of oscillation are found. One type corresponds to dispersion relations which behave linearly for not-so-high frequency, with a phase velocity always smaller than the velocity of light in the dielectric, but at least ten times larger than the Fermi velocity, while the other type consists of high-frequency modes ($\omega\sim\omega_p$). The role of these oscillations in the problem of transition radiation is reexamined. In the case of a thin metal film, a new interpretation is proposed for the peak observed in the transition radiation spectrum. Finally, the work is extended to superconducting metals where, in the frequency range $\hbar\omega<2\Delta$ (2Δ is the superconducting energy gap), we have justified the use of a dielectric function of the same functional form as given above but with ω_p^2 replaced by an almost frequency-independent quantity ω_{ps}^2, where $\omega_{ps}=c/\lambda_{ps}$ and λ_{ps} is the actual penetration depth. In this frequency range, the oscillations are essentially undamped and play an important role in the electromagnetic properties of the multiple-film systems, and particularly when the systems exhibit the ac Josephson effect.

I. INTRODUCTION

A WELL-KNOWN property of an electron gas is its capacity to undergo collective motions, i.e., plasma oscillations. Much theoretical work has been done on the study of these oscillations, particularly on dispersion relations.[1] Surface effects are usually neglected in most treatments. However, there are cases in which the surface effects are quite important; the inelastic scattering of electrons by thin foils and the problem of transition radiation are two such examples.

In general, the presence of surfaces introduces new modes of plasma oscillations in addition to the bulk one with different properties and particularly with different dispersion relations. These modes can be excited by incident electrons[2,3] or photons[4,5] and can be detected experimentally.[6,7] An interesting feature of these modes is their strong dependence on the properties of the surfaces in such a way that they can be used as a tool for investigating surfaces and thin films.[8] These surface oscillations play also an important role in determining the electrical characteristics of superconducting tunneling junctions, especially when the Josephson effect is present.[9–11] The low-lying modes of surface plasma oscillations may also provide an additional mechanism for the creation of an attractive electron-electron interaction in multiple-film superconductors in a way analogous to the phonon-induced attractive interaction in an ordinary superconductor,[12] and, more generally, low-lying surface plasmons may show, through a residual interaction with electrons, effects analogous with those caused by the electron-phonon interaction.

Considerable theoretical and experimental work has been done in the study of these oscillations for the geometry of one surface separating a metal and a dielectric. The properties of these modes[13–15] and their excitation by incident electrons[15,16] and photons[17] have been studied. Ritchie[3] and Ferrell[2] have studied the surface plasma oscillations (SPO) in the case of a thin metal film using electrostatic theory instead of the complete set of Maxwell equations. It is known, however, that the retardation effects introduced through the Maxwell equations have a profound influence on the mode of single metal-insulator interface.[2] The purpose of this paper is to study the properties of SPO in geometries of experimental interest by taking into account retardation effects, i.e., by using the complete set of Maxwell equations. A simple dielectric function of the form $\epsilon(\omega)=1-\omega_p^2/\omega^2$ is used, where ω_p is the bulk plasma frequency. The limits of validity of this simple approximation are discussed. The result for a single metal-insulator interface is recaptured and new results

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* William Rainey Harper Fellow, University of Chicago, 1967–68.

† Submitted in partial fulfillment of the requirements for the Ph.D. degree at the University of Chicago.

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for more complex geometries are obtained. Some of these results are in disagreement with previous estimations made by Ferrell about the importance of retardation effects, thus requiring fundamental changes in the interpretation of the observed peak in the transition radiation emitted when a thin metal foil is bombarded by an electron beam.

The case when the metal films are superconducting requires special consideration. In normal metals the low-lying oscillations ($\omega<\omega_t$, where ω_t is no smaller than 10^{12} $\sec^{-1}$ in the cases of interest to us) are absent because of the very large attenuation due to bulk and surface collisions. However, for superconductors and for frequencies such that $\hbar\omega<2\Delta$ (2Δ is the superconducting energy gap), we expect real oscillations. It is shown in this paper via the theory of Mattis and Bardeen[18] and the numerical calculations of Miller[19] that a local dielectric function of the form $\epsilon_s(\omega)=1-\omega_{ps}^2/\omega^2$ can still be used for $\hbar\omega<2\Delta$, under certain assumptions satisfied in the cases of interest. The almost frequency-independent parameter ω_{ps}, which replaces here the plasma frequency ω_p, is temperature-dependent and is related with the actual penetration depth λ_{ps} via the simple equation $\lambda_{ps}=c/\omega_{ps}$, where c is the velocity of light. Using this formalism, the results of Swihart[9] for tunneling geometries were recaptured (with the difference that in our formalism the actual penetration depth replaces the London penetration depth appearing in Swihart's results) and new results for more complex geometries of experimental interest were derived. As is well known,[10,11] the presence of SPO[20] has important effects on the operating characteristics of the Josephson tunneling junctions. This subject is considered in the following paper.[21]

Discussion of plasma oscillations can be based on the Maxwell equations together with a constitutive equation relating the electric current $\mathbf{j}$ to the electric field $\mathbf{E}$. Assuming, as usual, that the magnetic permeability μ is equal to unity and that time enters through a factor of the form $e^{i\omega t}$, we want solutions with a nonzero $\mathbf{E}_l$ for the system of equations

$$\mathbf{E}=\mathbf{E}_t+\mathbf{E}_l\,, \tag{1.1}$$

$$\nabla\times\mathbf{E}_l=0\,, \tag{1.2}$$

$$\nabla\cdot\mathbf{E}_l=(4\pi i/\omega)\nabla\cdot\mathbf{j}\,, \tag{1.3}$$

$$\nabla\times\mathbf{E}_t=-(i\omega/c)\mathbf{H}\,, \tag{1.4}$$

$$\nabla\cdot\mathbf{E}_t=0\,, \tag{1.5}$$

$$\nabla\times\mathbf{H}=(i\omega/c)\mathbf{E}+(4\pi/c)\mathbf{j}\,, \tag{1.6}$$

$$\nabla\cdot\mathbf{H}=0\,, \tag{1.7}$$

supplemented by a linear functional relation of the form

$$\mathbf{j}=\hat{\sigma}_l\mathbf{E}_l+\hat{\sigma}_t\mathbf{E}_t\,, \tag{1.8}$$

where $\hat{\sigma}_l$ and $\hat{\sigma}_t$ are conductivity tensors, integral operators in general, and the subscripts l and t indicate the longitudinal and the transverse parts, respectively. If it were justifiable to assume $\nabla\cdot\hat{\sigma}_t\mathbf{E}_t=0$ everywhere, with $\mathbf{E}_t$ not identically zero, then the system of equations for the longitudinal electric field would decouple from that of the transverse fields, thus implying that purely transverse solutions can be found. If, on the other hand, the solution is such that $(i\omega/c)\mathbf{E}_l+(4\pi/c)\hat{\sigma}_l\mathbf{E}_l=0$ everywhere with $\mathbf{E}_l$ not identically zero, then the system of equations for the transverse electric and magnetic fields would decouple from that of the longitudinal field, permitting so solutions of purely longitudinal nature. Now, if the system considered is just one material of infinite extension and if $\hat{\sigma}_t$ were simply a proportionality constant, Eqs. (1.2) and (1.3) of the above set could be rewritten as follows by taking Fourier transforms:

$$\mathbf{k}\times\mathbf{E}_l(\mathbf{k},\omega)=0\,, \tag{1.9}$$

$$\epsilon_l(\mathbf{k},\omega)\mathbf{k}\cdot\mathbf{E}_l(\mathbf{k},\omega)=0\,, \tag{1.10}$$

where

$$\epsilon_l(\mathbf{k},\omega)=1-(4\pi i/\omega)\sigma_l(\mathbf{k},\omega) \tag{1.11}$$

is the dielectric function and

$$\mathbf{j}_l(\mathbf{k},\omega)=\sigma_l(\mathbf{k},\omega)\mathbf{E}_l(\mathbf{k},\omega)\,. \tag{1.12}$$

In order to have nonzero $\mathbf{E}_l$ solutions we must have

$$\epsilon_l(\mathbf{k},\omega)=0\,, \tag{1.13}$$

the condition which gives the bulk plasma dispersion relation, and which also decouples the transverse fields from the longitudinal. So, in the case of one material with $\hat{\sigma}_t$ just a proportionality constant, all the possible plasma oscillations are of purely longitudinal nature in the sense that the nonzero $\mathbf{E}_l$ field *does not create* transverse fields. On the othe hand, if different materials (with $\hat{\sigma}_t$ just a proportionality constant) are present, there are two groups of plasma oscillations. One consists of the bulk oscillations and the other of surface oscillations. The first group is characterized by the fact that each material sustains independent oscillations (not plane waves in general) where a surface charge on the boundary is so distributed as to compensate *outside* the material the field created by the space charge inside. These bulk oscillations have the two characteristics of the oscillations in a material of infinite extent: They are longitudinal, and their frequency is determined by the relation $\epsilon_l(\mathbf{q},\omega)=0$, where $\mathbf{q}$ is the set of eigenvalues labeling the solution. The only role that the surfaces have in these oscillations is to determine the eigensolutions through the boundary conditions. In the geometries of alternating metal and insulating films, which are the subject of this paper, one set of bulk solutions

[18] D. C. Mattis and J. Bardeen, Phys. Rev. 111, 412 (1958).
[19] P. B. Miller, Phys. Rev. 118, 928 (1960).
[20] These oscillations are referred as "cavity modes" in the literature of tunneling junctions.
[21] K. L. Ngai, following paper, Phys. Rev. 182, 555 (1969).

can be found quite easily: Each metal film sustains normal oscillations of the form $E_x=E_0e^{i(kx-\omega t)}$ (the x axis is chosen normal to the interfaces) with space charge $\rho_V=(ik/4\pi)E_0e^{i(kx-\omega t)}$ and surface charges at the two boundaries $x=a$ and $x=a+d$ equal to $\rho_{S\,x=a}=(E_0/4\pi)e^{ika-\omega t)}$ and $\rho_{S\,x=a+d}=-(E_0/4\pi)e^{i(ka+kd-\omega t)}$, where ω and k satisfy the relation $\epsilon_l(\omega,k)=0$; it can be seen by inspection that the total charge per unit length of the film is exactly zero and consequently the field created by such a charge distribution is exclusively confined inside the metal film, as it should, in order to have independent oscillations. The surface oscillations, to which we shall confine ourselves exclusively in this paper, are of different nature. First, $\hat{\epsilon}_l$ is not identically zero and consequently there is no space charge inside the metals. Second, the nonzero $\mathbf{E}_l$ *does* create transverse fields which in turn *do* modify the $\mathbf{E}_l$, since the condition $\nabla\cdot\hat{\sigma}_t\mathbf{E}_t=0$ is no longer satisfied at the surfaces separating any two different materials. [$\nabla\cdot\hat{\sigma}_t\mathbf{E}_t \propto \delta(f)$, where $f=0$ describes the surface.] A physical criterion for an estimation of the relative importance of the retardation effects introduced via the coupling with the transverse fields is obtained by comparing the ratio of the distance over which electrical charges are transferred to the time required for such a transport with the velocities of propagation of electromagnetic waves, $c_i=c\epsilon_i^{-1/2}$ and $c_m=c|\epsilon_m|^{-1/2}$ in the insulator and metal, respectively. If this ratio is much smaller than both c_i and c_m, retardation effects are negligible and the electrostatic theory is adequate for describing the phenomenon. The electrostatic theory gives also good results for over-all quantities, such as dispersion relations, etc., in the case where only one of the above inequalities is satisfied, if a relatively small amount of the total field energy is stored in the material which violates the corresponding inequality.

In Sec. II, a discussion of the current-field relation for normal metals is given. It is shown that a local relation with $\sigma=\sigma_0/(1+i\omega\tau)$, where σ_0 is the static conductivity and τ is the relaxation time, is adequate for describing the surface plasma dispersion relation in the interesting region where it is essentially real. Using this form for the conductivity, the dispersion relations for various geometrical configurations are derived in Sec. III. These results are discussed in conjunction with the published works that exist in the literature. Finally, in Sec. IV the work has been extended to the case when superconducting materials are present and essentially undamped oscillations are shown to occur at low frequencies. One of the modes we find, the strip-line mode most interesting for conventional superconductive tunnel junctions, has already been studied theoretically by Swihart.[9]

II. CURRENT-FIELD RELATION

In general, the current-field relation can be found from the Boltzmann equation which determines the distribution function as a functional of the field. The linearized Boltzmann equation can be written

$$\frac{\partial f_1}{\partial x}+\frac{f_1}{\tau v_x}(1+i\omega\tau-ikv_z\tau)=\frac{|e|}{mv_x}\mathbf{E}\cdot\nabla_v f_0, \qquad (2.1)$$

where $f_0(\mathbf{v})$ is the equilibrium distribution function, $\mathbf{v}$ is the electron velocity, and $f_1(\mathbf{v},x)e^{i(\omega t-kz)}$ is the first-order correction due to the electric field. The current is given by the relation

$$j_\alpha=-2|e|m^3h^{-3}\int v_\alpha f_1 d^3v. \qquad (2.2)$$

If the terms $\partial f_1/\partial x$ and $(ikv_z/v_x)f_1$ are negligible compared with $f_1(1+i\omega\tau)/\tau v_x$, the solution of Eq. (2.1) is

$$f_1=[|e|\tau/m(1+i\omega\tau)]\mathbf{E}\cdot\nabla_v f_0, \qquad (2.3)$$

which in turn gives that

$$\mathbf{j}=[\sigma_0/(1+i\omega\tau)]\mathbf{E}, \qquad (2.4)$$

with

$$\sigma_0=\omega_p^2\tau/4\pi\ . \qquad (2.5)$$

The last two equations describe what we call the local approximation to the current-field relation.

As has been discussed by other authors,[22,23] a criterion for the applicability of the local approximation is that the quantity $l/|1+i\omega\tau|$ should be much smaller than any other characteristic length of the problem. Here l is the mean free path and $l/|1+i\omega\tau|$ gives a dynamical or effective mean free path and is related to the distance over which an electron can move without suffering any collisions. For low frequencies ($\omega\tau\ll1$) this distance is l, but for high frequencies ($\omega\tau\gg1$) it is $l/\omega\tau\sim v_F/\omega$, the distance transversed by an electron in one period (v_F is the Fermi velocity). The characteristic lengths of our present problem of SPO in thin films are (a) the thicknesses of the metal films d, (b) the wavelength of the oscillation λ, and (c) the quantity

$$\left|\frac{c(1+i\omega\tau)^{1/2}}{(2\pi\omega\sigma_0)^{1/2}(1+i)}\right|=\lambda_p\left|1+\frac{1}{\omega^2\tau^2}\right|^{1/4},$$

which measures the frequency-dependent classical penetration depth, where $\lambda_p=c/\omega_p$ is the London penetration depth.[24] Thus, the local approximation holds when

$$l/\lambda_p\ll\omega\tau(1+1/\omega^2\tau^2)^{3/4}, \qquad (2.6)$$

$$l/d\ll(1+\omega^2\tau^2)^{1/2}, \qquad (2.7)$$

and

$$l/\lambda\ll(1+\omega^2\tau^2)^{1/2} \qquad (2.8)$$

[22] G. E. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

[23] R. Englman and E. H. Sondheimer, Proc. Phys. Soc. (London) **B69**, 449 (1956).

[24] In the literature the quantity $2\pi c/\omega_p$, the plasma wavelength, is also denoted by the symbol λ_p.

are simultaneously satisfied. For our present interest in real oscillations of the system, we shall restrict our consideration to frequency ranges where these oscillations are essentially undamped. Scattering of electrons in the bulk, described by τ or by l, and scattering from boundary surfaces contribute to the damping. From physical considerations and the results of previous calculations,[22] the bulk scattering does not cause appreciable attenuation if

$$\omega\tau \gg 1. \tag{2.9}$$

On the other hand, the contribution to damping from surface scattering is also negligible if

$$l/d \ll (1+\omega^2\tau^2)^{1/2} \tag{2.7}$$

(a diffuse boundary scattering condition is assumed to be a good approximation). The conditions (2.9) and (2.7) for real oscillations can be written as

$$\omega\tau \gg \max(1, l/d). \tag{2.10}$$

Under (2.10) the local approximation is valid provided

$$\omega\tau \gg \max(l/\lambda,\ l/\lambda_p). \tag{2.11}$$

Therefore, if $\max(1,l/d) \gtrsim \max(l/\lambda, l/\lambda_p)$, all real oscillations can be found via the local approximation. On the other hand, if $\max(l/\lambda, l/\lambda_p) \gg \max(1,l/d)$, there would be a range of frequency in which the local approximation is invalid and yet real oscillations can still be found. This latter case occurs when

$$\min(\lambda,\lambda_p) \ll \min(l,d). \tag{2.12}$$

However, in many practical investigations of plasma oscillations in metal films, conditions such that $l \approx 10^{-6}$ cm, $d \approx 10^{-5}$ cm, and $\lambda_p \approx 3\times 10^{-6}$ cm occur. This implies that (2.12) is not in general satisfied and the local approximation is quite adequate for our present study. For (2.12) to be satisfied, l would have to approach d, with $d \gtrsim 10^{-5}$ cm. In other words, the films would have to be pure and perfect, and the experiments conducted at low temperatures. There is some reason to believe that Au films may perhaps be good enough.[25] If, under these special circumstances, (2.12) is satisfied, then there exist situations in which either

$$\omega\tau \lesssim l/\lambda, \tag{2.13a}$$

which is equivalent to

$$\omega/k \lesssim v_F \tag{2.13b}$$

or

$$\omega\tau \lesssim l/\lambda_p. \tag{2.14}$$

In the first case even the Boltzmann equation is not adequate for investigating the current-field relation. A microscopic quantum theory is necessary, and the current-field relation for the transverse part of the electric field is in general different from that of the longitudinal part.[26] In the second case when $\omega\tau \lesssim l/\lambda_p$, we are in the regime of the anomalous skin effect[22] and the solution of an involved integrodifferential equation that results from combining the Maxwell equations with the Boltzmann equation is required; this might be the case for the Au films cited above.

Since the conditions for these complications can be realized only with difficulty in applications of our present physical interest, the local approximation is adequate and the following formula for the dielectric function $\epsilon_m(\omega)$ in the metal,

$$\epsilon_m(\omega) = 1 - \frac{\omega_p^2}{\omega^2}\frac{1}{1-i/\omega\tau}, \tag{2.15}$$

is satisfactory.

In deriving this formula, the effects of the periodic crystal potential and the core polarization have been neglected. It should be stated, however, that for certain metals, e.g., Al, Mg, Be, K, Na, etc.,[8] this is a good approximation for the frequency range of interest. For other metals, e.g., Ag, the interband transitions or the core polarization change considerably the simple formula (2.15). The core polarization replaces the unity in the right-hand side of (2.15) by ϵ_0, which can be considered as a constant larger than unity, and the interband transitions contribute significantly to both the imaginary and the real part of ϵ_m, changing the functional dependence of the $\mathrm{Re}(\epsilon_m)$ on ω and increasing appreciably the $\mathrm{Im}(\epsilon_m)$. In this case we should expect changes in the dispersion relations derived by using the simple formula (2.15), but more significantly the oscillations should be attenuated sometimes so strongly that they should not appear at all.

A final remark should be added about the dielectric function of the insulating materials. In what follows it has been assumed that ϵ_i is a real constant which for simplicity has been taken equal to unity in most cases. This assumption is satisfactory for a lot of dielectric layers on metal surfaces.[27] However, it may very well happen in actual cases that the dielectric medium is strongly absorptive at frequencies below ω_p. If that is the case, then the dielectric constant ϵ_i acquires an imaginary part which is strongly dependent on frequency and which adds to the attenuation of the oscillations. When $\mathrm{Im}(\epsilon_i)$ approaches or exceeds $\mathrm{Re}(\epsilon_i)$, then the damping can be so large that one can no longer speak of a collective oscillation. Thus, the results derived in this paper hold in the case where ω_p is smaller than the absorption edge in the insulator or if $\mathrm{Im}(\epsilon_i)$ is very small. Since we have restricted ourselves to cases of essentially real oscillations, the imaginary parts of the dielectric functions have no appreciable influence on the dispersion relations; they only determine the

[25] H. E. Bennett and J. M. Bennett, in *Optical Properties and Electronic Structure of Metals and Alloys*, edited by F. Abeles (North-Holland Publishing Co., Amsterdam, 1966), p. 175.

[26] A. R. Melnyk and M. J. Harrison, Phys. Rev. Letters **21**, 85 (1968).

[27] E. A. Stern and R. A. Ferrell, Phys. Rev. **120**, 130 (1960).

attenuation of these oscillations. Hence in what follows the $\mathrm{Im}(\epsilon_m)$ and $\mathrm{Im}(\epsilon_i)$ are neglected. Also, for the sake of simplicity the contribution of the core polarization to ϵ_m is neglected and it is assumed that $\epsilon_i = 1$ in almost all cases. These restrictions are not by any means necessary; they are introduced in order to simplify the subsequent calculations in Sec. III.

III. LOCAL THEORY FOR MULTIPLE-FILM SYSTEM

We wish to find solutions which satisfy Maxwell equations with a local current-field relation as follows:

$$\nabla \cdot \mathbf{D} = 0, \tag{3.1}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{3.2}$$

$$\nabla \times \mathbf{E} = -(1/c)\partial \mathbf{H}/\partial t, \tag{3.3}$$

$$\nabla \times \mathbf{H} = (1/c)\partial \mathbf{D}/\partial t, \tag{3.4}$$

$$\mathbf{D} = \epsilon(\omega)\mathbf{E}, \tag{3.5}$$

for a geometry of alternating superposed flat metal and insulator films, where

$$\epsilon(\omega) = 1 - \omega_p^2/\omega^2 \tag{3.6}$$

for the metal films and is constant for the dielectrics. Maxwell equations are supplemented by the boundary conditions of continuity of the tangential fields at every boundary. The type of solution we seek corresponds to wave propagation along a direction parallel to the boundary surfaces separating the different materials, which we designate as the z axis.[9] With the x axis normal to these surfaces, we further assume that there is no y dependence of any of the fields and that $H_x = H_z = E_y = 0$ in all of the media. This last assumption means that we restrict ourselves to the so-called "electric (or TM) waves" and we neglect the other group of possible solutions, the "magnetic (or TE) waves," which are of no interest to us, since they are purely transverse waves (in the sense that $\nabla \cdot \mathbf{E} = 0$). The solution for any component of the fields can thus be represented in the form

$$\mathfrak{F}(x,z,t) = \mathrm{Re} F(x) e^{i(\omega t - kz)}, \tag{3.7}$$

with $\mathrm{Re}k > 0$ and $\mathrm{Im}k < 0$, so that the wave travels and is attenuated in the positive z direction. The Maxwell equations (3.1)–(3.6) determine the field amplitudes $F(x)$ inside each material through ordinary differential equations as follows:

$$E_z(x) = -(i/k)(dE_x/dx), \tag{3.8}$$

$$H_y(x) = (\omega \epsilon_{i,m}/ck) E_x, \tag{3.9}$$

$$d^2E_x/dx^2 - K_{i,m}^2 E_x = 0, \tag{3.10}$$

where

$$K_{i,m} = (k^2 - \omega^2 \epsilon_{i,m}/c^2)^{1/2}, \quad \mathrm{Re}K_{i,m} > 0. \tag{3.11}$$

The permeability of each medium is assumed to be that of vacuum. It may be noted here that a general solution

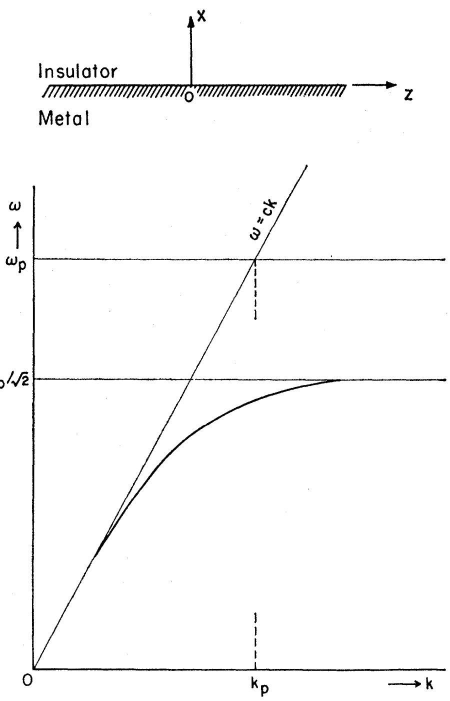

FIG. 1. Geometry and the dispersion relation for SPO of an insulator-metal interface ($\epsilon_i = 1$, $\epsilon_m = 1 - \omega_p^2/\omega^2$). The analytical expression for the curve shown schematically here is (3.15).

to (3.10) is a linear combination of the two independent solutions $e^{K_{i,m}x}$ and $e^{-K_{i,m}x}$. Several multiple-film systems will now be discussed separately.

A. Single Metal-Dielectric Interface

We assume that the $x = 0$ plane separates a semi-infinite metal ($x < 0$) from a semi-infinite dielectric ($x > 0$) (Fig. 1). The solution for E_x that remains finite at infinity is

$$E_x = A_I e^{K_m x}, \quad x < 0$$
$$= A_{II} e^{-K_i x}, \quad x > 0. \tag{3.12}$$

The corresponding E_z and H_y can be calculated via Eqs. (3.8) and (3.9); the continuity of these fields across the boundary gives the dispersion relation implicitly as

$$R \equiv -\frac{(K/\epsilon)_{\mathrm{metal}}}{(K/\epsilon)_{\mathrm{insulator}}} = 1. \tag{3.13}$$

From this equation we get the dispersion relation

$$k^2 = \frac{k_p^2 + \omega^2/\epsilon_i c^2 - 2k_p^2/\epsilon_i + k_p^2 \omega_p^2/\epsilon_i \omega^2 - \omega^2/c^2}{\omega_p^4/\epsilon_i^2 \omega^4 + 1/\epsilon_i^2 - 1 - 2\omega_p^2/\omega^2 \epsilon_i^2}, \tag{3.14}$$

where $k_p = \omega_p/c = 1/\lambda_p$. If the dielectric constant for the insulator ϵ_i is unity, this solution can be recast in the form

$$\omega = \omega_p [1 + 1/2q^2 + (1 + 1/4q^4)^{1/2}]^{-1/2}, \tag{3.15}$$

where $q=k/k_p$. The dispersion formula (3.15) has previously been obtained by Sommerfeld[13] and by Stern[14] and has recently been verified experimentally.[15] For $q \gg 1$, ω tends to $\omega_p/\sqrt{2}$, the surface plasma frequency resulting from electrostatic theory for the geometry considered. For $q \ll 1$, ω approaches ck, the dispersion relation of a wave propagating in vacuum. It is obvious that retardation effects are important for $q<1$ and that they do not play any role for $q \gg 1$. To understand that we have to use the criteria stated in the Introduction, namely, that when both

$$\omega/k \ll c_i \quad \text{and} \quad \omega/k \ll c_m$$

are satisfied, retardation effects are negligible. Using the assumptions $\epsilon_i=1$ and $\epsilon_m=1-\omega_p{}^2/\omega^2$, we can divide the ω-k plane into regions in which none, one, or both of the above inequalities are satisfied (Fig. 2) as follows:

regions I, I′: $\omega/k>c$ and $\omega/k>c|\epsilon_m|^{-1/2}$;
region II: $\omega/k<c$ and $\omega/k>c|\epsilon_m|^{-1/2}$;
region III: $\omega/k>c$ and $\omega/k<c|\epsilon_m|^{-1/2}$;
region IV: $\omega/k<c$ and $\omega/k<c|\epsilon_m|^{-1/2}$.

In regions I and I′ the retardation effects are always important. In region II and far from the curve $\omega=ck$ the retardation effects can be unimportant for over-all quantities, such as phase velocity, etc., if the percentage of the field energy stored in the metal films is small, which means that the metal films should be thin. Similarly, in region III and far from the curves $\omega=ck|\epsilon_m|^{-1/2}$ the retardation effects can be unimportant for over-all quantities if the field energy stored in the insulators is small. Finally, well inside region IV the retardation effects are always unimportant. This general discussion not only explains the behavior of the dispersion relation under consideration but can also be used for

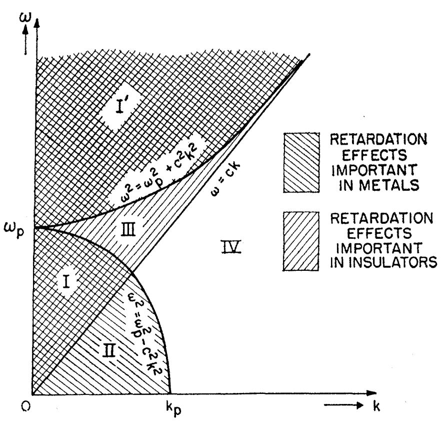

Fig. 2. Division of the ω-k plane according to the importance of the retardation effects (RE) under the assumptions $\epsilon_i=1$, $\epsilon_m=1-\omega_p{}^2/\omega^2$. RE are always important inside regions I, I′, and near their boundaries. RE are negligible in region IV. Inside region II (III) retardation effects can be unimportant for over-all quantities only when the metal (insulating) films are very thin.

estimation of the contribution of the retardation effects in all the modes, which are examined next.

It should be recalled at this point that what has been said holds for ω larger than a certain threshold frequency ω_t so that the assumption $\epsilon_m=1-\omega_p{}^2/\omega^2$ is a reasonable one. However, the fact that the dispersion relation tends to $\omega=ck$ for small k is insensitive to the detailed form of ϵ_m and even to its reality. This is a general property of all the modes with dispersion curves near the $\omega=ck$ curve. The fact that they lie near the $\omega=ck$ curve depends only on the assumption that $|\epsilon_m| \gg 1$, the condition for a planelike propagation of the "principal wave" in the theory of waveguides.[28] In any case, since for $\omega \to ck$ all the fields tend to zero for the TM waves, the portions of the dispersion relations near the $\omega=ck$ curve are of no interest to us.

It is worthwhile to note here that the same dispersion relation (3.15) can be obtained from the theory of transition radiation.[29] Transition radiation is caused by the change in the electromagnetic fields surrounding a charged particle as it is in transit from one medium to another with a different dielectric constant. If a photon is emitted, conservation of the momentum component parallel to the surface requires that

$$k=\omega \sin\theta/c, \tag{3.16}$$

where θ is measured from the x axis (normal to the surface boundary). If we examine the expression for the transition radiation per unit solid angle per unit frequency range, $dW/d\Omega d\omega$ [see, e.g., Garibian,[29] formula (21)] in conjunction with (3.16), it is not difficult to show that $dW/d\Omega d\omega$ blows up whenever (3.15) is satisfied. More explicitly, the factor $\epsilon_m \cos\theta + (\epsilon_m - \sin^2\theta)^{1/2}$ that occurs in the denominator of the expression $dW/d\Omega d\omega$ is proportional to $R-1$. Since $R-1$ vanishes only for $\omega<ck$ (see Fig. 1) while a real θ corresponds to the region $\omega>ck$, the quantity $dW/d\Omega d\omega$ never blows up for physical values of θ.

B. Insulating Film between Two Semi-Infinite Metals

Following the same procedure as in the last case, the dispersion relation appropriate to this geometry is given implicitly by

$$(1-R)/(1+R)=\pm e^{-K_i d_i}. \tag{3.17}$$

Although this cannot be solved explicitly for ω as a function of k, it allows for approximate solutions when the ω-k plane is divided into different regions. The result is best given graphically via Fig. 3.

Branches I and II are adequately described by the longitudinal electrostatic theory[2] when $k \gg k_p$, as can be

[28] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Co., New York, 1941), pp. 527–528.
[29] G. M. Garibian, Zh. Eksperim. i Teor. Fiz. **33**, 1403 (1958) [English transl.: Soviet Phys.—JETP **6**, 1079 (1958)].

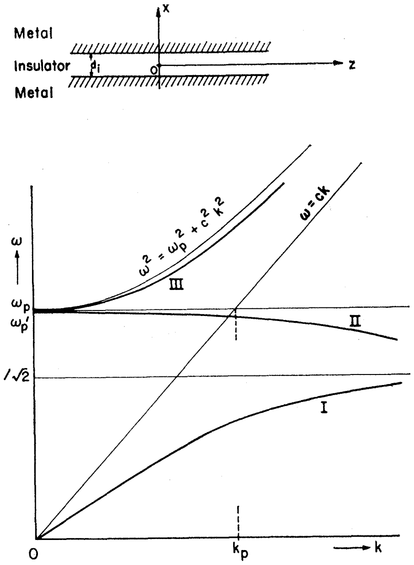

FIG. 3. Geometry and the dispersion relations for SPO of an insulating film between two semi-infinite metals ($\epsilon_i=1$, $\epsilon_m=1-\omega_p^2/\omega^2$). The analytical expressions for $k\gg k_p$ or $k\ll k_p$, $k_p=\omega_p/c$, for the curves shown schematically here are (3.18)–(3.21). ω_p' is given by (3.22). For normal metals only the higher part ($\omega>\omega_t$, where $\omega_t\approx 10^{12}$–10^{14} sec^{-1}) of curve I exists. For superconducting metals, besides the higher part, the lower part ($\hbar\omega<2\Delta$) of curve I is present, but with ω_p replaced by ω_{ps}.

seen by inspection of Fig. 2, and are given by

$$\omega=(\omega_p/\sqrt{2})(1\pm e^{-kd_i})^{1/2}, \qquad (3.18)$$

if $\epsilon_i=1$. The low-frequency $k<k_p$ part of I has the form

$$\omega=ck[d_i/(d_i+2\lambda_p)]^{1/2}. \qquad (3.19)$$

This expression for the junction rf mode has been derived in connection with the I-V characteristic of Josephson tunnel junction.[9,10] It is appropriate to recall here once again that for normal metals and for $\omega<\omega_t$ all the SPO (except those for which $\omega\approx ck$) disappear because of the strong attenuation.

If $k_pd_i\ll 1$, as is usually the case, branch III is given by

$$k^2=\omega^2/c^2+2/d_i^2\epsilon_m^2-(2/d_i^2\epsilon_m^2)\times(1+k_p^2d_i^2\epsilon_m^2)^{1/2}, \qquad (3.20)$$

and it is below but close to the curve $\omega^2=\omega_p^2+c^2k^2$. However, both $\omega=ck$ and $\omega^2=\omega_p^2+c^2k^2$ are trivial solutions for our problem with $E_x=E_z=H_y=0$. In order to prove this statement we have to use Eqs. (3.8)–(3.10), the boundary conditions, and the condition that the fields should be zero at infinity. Consequently, branch III corresponds to field solutions which are small and hence of no physical significance. Under the same assumption $k_pd_i\ll 1$, the $k\lesssim k_p$ portion of II is described by

$$k^2=\frac{4(1-\omega^2/\omega_p^2)}{d_i^2}\left(\frac{1-\omega^2/\omega_p^2}{\omega^4/\omega_p^4}-\tfrac{1}{4}d_i^2k_p^2\right), \qquad (3.21)$$

and the frequency at $k=0$ is

$$\omega_p'=\omega_p(1-\tfrac{1}{8}d_i^2k_p^2). \qquad (3.22)$$

Now, neglecting the retardation effects is equivalent mathematically to put $c=\infty$. If we follow the change in Fig. 3 as $c\to\infty$, we can convince ourselves that the dispersion relation reduces to the well-known expressions (3.18).[3,12] Branches I and III correspond to the antisymmetric solution and branch II to the symmetric one. Comparison of Figs. 2 and 3 shows that the retardation effects are important for branches I and III and for $k\lesssim k_p$, while branch II can be derived from electrostatic theory down to very small k if $d_ik_p\ll 1$. This last statement can be verified from Eq. (3.21). If d_i approaches zero, branches I and III disappear while II still exists with a dispersion relation which tends to the bulk one, in agreement with what should be expected from physical considerations.

C. Metal Film in Vacuum

The equation from which the dispersion relations are determined is

$$(1-R)/(1+R)=\pm e^{-K_md_m}. \qquad (3.23a)$$

Solutions to this equation are shown in Fig. 4. Approximate solutions can be obtained for separate regions. In particular, for $k<k_p$

$$k^2\simeq\omega^2/c^2+\omega^4/c^2\omega_p^2[\tanh(\tfrac{1}{2}k_pd_m)]^2, \qquad (3.23b)$$

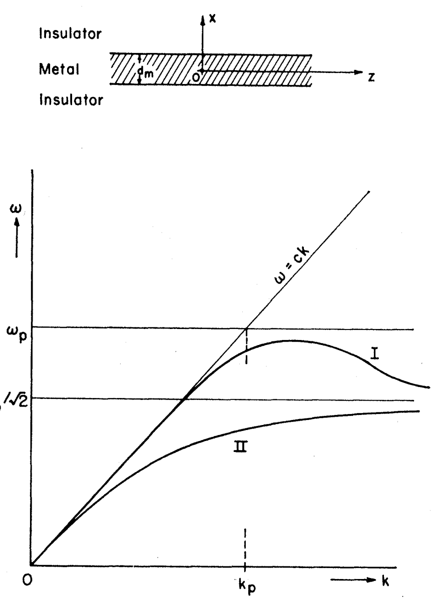

FIG. 4. Geometry and the dispersion relations for SPO of a metal film between two semi-infinite insulators ($\epsilon_m=1-\omega_p^2/\omega^2$, $\epsilon_i=1$). The analytical expressions for $k\gg k_p$ or $k\ll k_p$ for the curves shown schematically here are (3.23b) and (3.18).

for branch II (symmetric oscillation), and where tanh is replaced by coth for branch I (antisymmetric oscillation). For $k \gg k_p$ both I and II approach $\omega_p/\sqrt{2}$ in the manner described by (3.18).

If the metal film thickness d_m is such that $k_p d_m \ll 1$, these branches are quite separate, whereas when $k_p d_m \gg 1$, they are nearly degenerate because oscillations in the two surfaces now become decoupled.

It is worthwhile to note that these results are in disagreement with the estimations made by Ferrell[2] that the high branch should remain almost unaffected by the presence of retardation effects. The opposite behavior is exhibited by the previous results. Instead of starting from ω_p at $k=0$, it starts from zero and increases as k increases but remains below the $\omega = ck$ line. The fact that the retardation effects have a considerable influence on both modes for $k < k_p$ is in agreement with what was said previously, since both Ferrell's curves lie inside region I of Fig. 2. Another independent check of Eq. (3.23a) can be made by reference to the literature of transition radiation.[30–32] As is explained in Ref. 16, the SPO dispersion relations correspond to poles in the expressions for the transition radiation if we make the substitution $\sin\theta = ck/\omega$. It can be seen[30–32] quite easily that the expression for the transition radiation in the present geometry contains in the denominator a term proportional to $(1+R)^2 e^{-K_m d_m} - (1-R)^2 e^{K_m d_m}$ which vanishes on the dispersion curves given by (3.23a). It should be noted that Ferrell's arguments about the adequacy of the electrostatic theory for the high-lying mode and for $k \approx 0$ are correct but are not applicable to the SPO, since what he describes is the bulk solution (cf. the discussion of this point in the Introduction), which is indeed unaffected by retardation effects.

The dispersion relations given by (3.23a) can be checked experimentally by either electron-loss experiments or radiation measurement. The former[8] can measure the high-k part of the curves, while the latter can go to considerably lower k by following a technique similar to that applied by Teng and Stern.[15]

It was mentioned before that the dispersion relations of the SPO correspond to poles in the expressions for the transition radiation if we take into account the relation $\sin\theta = ck/\omega$. It should be added, however, that whenever the dispersion curves lie in the region $\omega/k < c$, the poles correspond to complex values of θ. In other words, for real values of θ we are far away from the poles. Consequently, they do not make any appreciable contribution to the transition radiation in accordance with the physically obvious fact that nonradiative ($\omega/k < c$) SPO cannot contribute to the emitted radiation. The relation of SPO with the problem of transition radiation has an interesting history. When the electromagnetic radiation arising from the passage of fast electrons through thin films was investigated[33,34] and a peak in the spectral distribution was found, it was assumed[33,34] that the observed radiation was emitted by SPO according to Ferrell's theory.[2] Silin and Fetisov[31] insisted that the observed radiation was nothing else than the transition radiation predicted by Ginzburg and Frank[35] as early as 1946. Indeed, they showed that the experimental results can be explained quite well using the formula for the transition radiation emitted by a thin film.[30] Stern[36] argued that Ferrell's theory and transition radiation theory are two different ways of considering the same phenomenon. Ferrell's method, he admitted, "only calculates the peak," but "shows the physical mechanism causing it, namely, the contribution of radiative SPO."

Since, however, our analysis shows that there are no radiative SPO in the present geometry, Stern's explanation[36] of the physical mechanism causing the peak cannot be correct. This can be seen also from the fact that as $\sin\theta$ decreases, the peak becomes more pronounced in spite of the fact that we are moving away from the poles corresponding to the SPO. Recall also that Ferrell's result of ω_p for the SPO frequency is incorrect [cf. Eq. (3.23b)], so that the observed coincidence of the peak frequency of the transition radiation and ω_p is not evidence for radiative SPO.

The denominator of the expression for the transition radiation near the peak (for which $\epsilon_m = 0$) has the form

$$\left(\frac{d_m}{\lambda_p/\sin\theta}\,\tfrac{1}{2}\tan\theta\right)^2 + \epsilon_m^2 .$$

The above formula suggests the following physical interpretation of the peak in question: At $\epsilon_m = 0$ the displacement current cancels out the convective current, so that no magnetic field is set up and no wave propagation inside the metal is possible. The incoming field is totally reflected, and consequently a maximum appears in the transition radiation. In order to obtain a sharp maximum the quantity $[d_m/(\lambda_p/\sin\theta)]\frac{1}{2}\tan\theta$ should be much smaller than unity, which physically means that the penetration depth of the field $1/K_m \approx \lambda_p/\sin\theta$ should be much larger than the thickness of the metal film. Thus, according to this picture, the peak in the transition radiation is due to a switch from conditions of total reflection at $\omega = \omega_p$, the *bulk* plasma frequency, to conditions of large transmission at the neighboring points. In order to support this explanation we consider the related problem of calculating the reflection coefficient for a plane electromagnetic wave

[30] V. E. Pafomov, Zh. Eksperim. i Teor. Fiz. **33**, 1074 (1958) [English transl.: Soviet Phys.—JETP **6**, 829 (1958)].
[31] V. E. Silin and E. P. Fetisov, Phys. Rev. Letters **7**, 374 (1961).
[32] H. Boersch and G. Sauerbrey, in *Optical Properties and Electronic Structure of Metals and Alloys*, edited by F. Abeles (North-Holland Publishing Co., Amsterdam, 1966), p. 386.
[33] W. Steinmann, Phys. Rev. Letters **5**, 470 (1960).
[34] R. W. Brown, P. Wessel, and E. P. Trounson, Phys. Rev. Letters **5**, 472 (1960).
[35] V. L. Ginzburg and I. M. Frank, J. Expt. Theoret. Phys. (USSR) **16**, 15 (1946).
[36] E. A. Stern, Phys. Rev. Letters **8**, 7 (1962).

incident on a metal foil of dielectric function ϵ_m and thickness d_m. The result is[37]

$$|r|^2=\left|\frac{r_{12}(e^{2\psi_m}-1)}{e^{2\psi_m}-r_{12}^2}\right|^2,$$

where $|r|^2$ is the reflection coefficient,

$$\psi_m=(\omega d_m/c)(\sin^2\theta-\epsilon_m)^{1/2},$$

$$r_{12}=\frac{\epsilon_m\cos\theta-i(\sin^2\theta-\epsilon_m)^{1/2}}{\epsilon_m\cos\theta+i(\sin^2\theta-\epsilon_m)^{1/2}},$$

and θ is the angle of incidence. From the above expressions it can be seen that $|r|^2$ has a maximum at $\epsilon_m=0$ and that near the maximum ($|\epsilon_m|\ll\sin^2\theta$) we can write

$$|r|^2=\left(\frac{d_m}{\lambda_p/\sin\theta}\,\tfrac{1}{2}\tan\theta\right)^2\Big/\left[\left(\frac{d_m}{\lambda_p/\sin\theta}\,\tfrac{1}{2}\tan\theta\right)^2+\epsilon_m^2\right],$$

in complete agreement with what was expected from the proposed physical picture.

Since the denominators in the expressions for the transition radiation and the reflection coefficient are exactly the same, the SPO should appear as poles of the reflection coefficient as well. However, in the present geometry and for real values of θ we are far away from the poles, as explained before, and no contribution is expected from them. It should be noted that in the geometries to be considered later, radiative SPO occur. In this case $|r|^2\to\infty$ as $\omega\to f(\sin\theta)$, where $\omega=f(ck/\omega)$ is the dispersion relation. The physical explanation of $|r|^2\to\infty$ has to do with the fact that the incoming wave excites SPO of infinite amplitude, since the resonance condition is satisfied and no attenuation is included. The violation of energy conservation in the present case ($|r|^2>1$) is caused by the omission of all kinds of losses, both of radiative and nonradiative nature. In the geometry shown, e.g., in Fig. 6, radiative SPO exist. Also, an abrupt change from $|r|^2=1$ for $\epsilon_m=0$ to large values of transmission at the neighboring points is always possible if the metal films are thin compared with the penetration depth. Thus, in this geometry two peaks will occur, one at $\omega=f(\sin\theta)$ due to radiative SPO and another at $\omega=\omega_p$ of the same nature as that appearing in the case of one film. Before we conclude this discussion, it is worthwhile to recall that nonradiative SPO can play a role in radiation measurements if there exists an effective mechanism for their transformation into three-dimensional waves, e.g., scattering by irregularities of the surface, nonlinear effects, etc.[16,17,38,39]

[37] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, Inc., Oxford, 1960), pp. 278–279.
[38] P. E. Fedders, Phys. Rev. **165**, 580 (1968).
[39] J. Bösenberg, Phys. Letters **26A**, 74 (1967).

D. Swihart's Geometry

This geometrical configuration has been considered by Swihart[9] in connection with the problem of the propagation of an electromagnetic wave in a superconducting transmission line with such a configuration. The SPO dispersion relations are now given by either

$$R=1 \qquad (3.24a)$$

or

$$[(1-R)/(1+R)]^2=e^{-2K_id_i}+e^{-2K_md_m}-e^{-2K_id_i-2K_md_m}. \qquad (3.24b)$$

Note that (3.24a) is identical to (3.13), which determines the dispersion formula for a single metal-dielectric interface. Hence the oscillation in the external metal-insulator surface is decoupled from the oscillation in the internal surfaces. The complete dispersion curves are given in Fig. 5.

Analytical expressions for these branches can be given. Thus, for branch I, we have

$$\omega=ck\left(\frac{d_i}{d_i+\lambda_p+\lambda_p\coth(d_m/\lambda_p)}\right)^{1/2}, \qquad (3.25)$$

for $k\ll k_p$, which is just the formula first given by Swihart.[9] Swihart confined himself to the low-frequency linear region of branch I only. Branch II is just (3.15) again and corresponds to oscillations on the external interface. Branch III starts from $\omega=\omega_p$ at $k=0$ and it

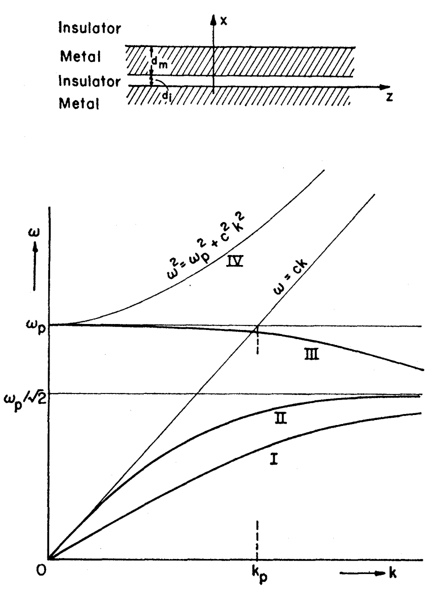

FIG. 5. Swihart's geometry and the dispersion relations of the corresponding SPO ($\epsilon_i=1$, $\epsilon_m=1-\omega_p^2/\omega^2$). The analytical expressions for $k\gg k_p$ or $k\ll k_p$ for the curves shown schematically here are (3.25) and (3.26). Curve II is the same as in Fig. 1. For normal metals only the higher part ($\omega>\omega_t$, where $\omega_t\approx10^{12}$–10^{14} sec^{-1}) of curve I exists. For superconducting metals, besides the higher part, the lower part ($\hbar\omega<2\Delta$) of curve I is present, but with ω_p replaced by ω_{ps}.

decreases according to the form

$$k^2=\omega^2/c^2-k_p^2+(1/d_id_m)[(\omega_p^2-\omega^2)/\omega^2]. \quad (3.26)$$

We have drawn branch III in Fig. 5 with the assumption that $d_id_mk_p^2\ll 1$, which is justified in most practical applications. Then the point where the intersection of branch III with the line $\omega=ck$ occurs is given by $[k=k_p(1-d_id_mk_p^2),\ \omega=\omega_p(1-d_id_mk_p^2)]$. As is always the case whenever we have $k^2\gg k_p^2+\omega^2/c^2$, retardation effects are negligible and an electrostatic theory is already sufficient. Branch IV behaves for small k as $\omega^2=\omega_p^2+c^2k^2$ for any finite d_m and corresponds to just the trivial solution when all fields are zero. In general, $k_p^2d_id_m\ll 1$ is satisfied and branch IV follows this null curve. On the other hand, if $k_p^2d_id_m\gg 1$ were to hold, branch IV would follow this null curve up to $1-\omega_p^2/\omega^2\sim(k_p^2d_id_m)^{-1}$ and then start bending slightly away from it, giving rise to very weak high-frequency oscillations.

E. Two Metal Films of Different Thicknesses

For the configuration depicted in Fig. 6 the dispersion relation is implicitly given by the equation

$$[(1-R)/(1+R)]^4-A_2[(1-R)/(1+R)]^2+e^{-2K_md_1-2K_md_2}=0, \quad (3.27)$$

where

$$A_2=e^{-2K_md_1}+e^{-2K_md_2}+e^{-2K_id_i}(1-e^{-2K_md_1})\times(1-e^{-2K_md_2}), \quad (3.28)$$

and explicitly represented in Fig. 6.

Branch V has essentially the same behavior as branch IV in Fig. 8 and shall not be discussed any further. For $k\ll k_p$, analytical expressions can be found for:

branch I:

$$\omega=ck\{d_i/[d_i+\lambda_p(\coth k_pd_1+\coth k_pd_2)]\}^{1/2}, \quad (3.29)$$

which has been given before by Swihart;

branches II, III:

$$k^2=\frac{\omega^2}{c^2}\left(1+\frac{\omega^2}{\omega_p^2}\{\tanh[k_p\tfrac{1}{2}(d_1+d_2)]\}^{\mp 2}\right); \quad (2.30)$$

branch IV:

$$k^2=\frac{\omega^2}{c^2}-k_p^2+\frac{d_1+d_2}{d_id_1d_2}\frac{\omega_p^2-\omega^2}{\omega^2}. \quad (3.31)$$

For $k\gg k_p$, all these four branches can then be well described by the electrostatic theory as is by now clear. As a check, formulas (3.27)–(3.31) do reduce to the corresponding results for the previous cases A–D if some of the lengths d_1, d_2, and d_i go to infinity such that the present geometry goes to the previous cases. If the thickness of the metal films is sufficiently small so that $d_m/\lambda_p\ll 1$, then $\coth(k_pd_1)$ and $\coth(k_pd_2)$ can be

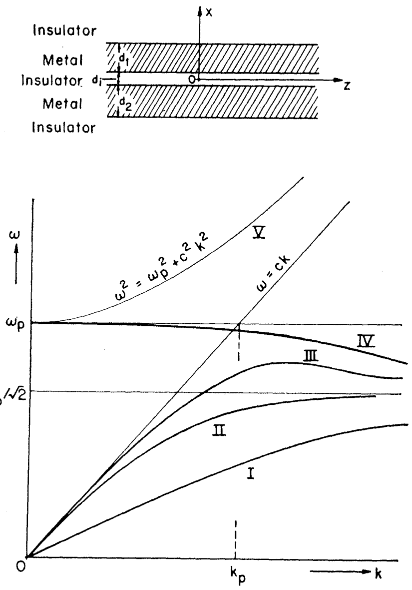

FIG. 6. Geometry and the dispersion relations for SPO of three films (metal-insulator-metal) between two semi-infinite insulators ($\epsilon_i=1$, $\epsilon_m=1-\omega_p^2/\omega^2$). The analytical expressions for $k\gg k_p$ or $k\ll k_p$ for the curves shown schematically here are (3.29)–(3.31). For normal metals only the higher part ($\omega>\omega_t$, where $\omega_t\approx 10^{12}$–10^{14} sec^{-1}) of curve I exists. For superconducting metals, besides the higher part, the lower part ($\hbar\omega<2\Delta$) of curve I is present, but with ω_p replaced by ω_{ps}.

replaced by λ_p/d_1 and λ_p/d_2, and from (3.29) the phase velocity of branch I becomes equal to

$$\omega_p[d_id_1d_2/(d_1+d_2)]^{1/2}. \quad (3.32)$$

We have assumed in deriving (3.32) that $d_id_1d_2/(d_1+d_2)\ll\lambda_p^2$ and recall that $c/\lambda_p=\omega_p$. It is important to note here that (3.32) can alternatively be obtained directly via electrostatic theory. This is natural according to what has been said before, since the curve lies well inside region II and the metal films, owing to their thinness, store a small portion of the total field energy. So, purely Coulomb interactions in the present geometry can lead to collective excitations with a linear dispersion relation of the form $\omega\propto k$ in analogy with zero sound, the high-frequency ($\omega\tau\gg 1$) collective excitation normally appearing only when short-range interactions are present. What makes this analogy possible is the fact that the long-range tail of the Coulomb forces associated with charges on any one surface is eliminated on the average by opposite charges on the other surfaces. Already in the problem of one metal (or dielectric) film, charges on one surface eliminate partially the tail of Coulomb forces due to equal (or opposite) charges on the other surface. This partial elimination causes a great relaxation in the frequency of oscillation, so that $\omega\propto k^{1/2}$ for small k.[2,3] But at least four surfaces are needed in order to have a complete elimination of the Coulomb tails and a linear dispersion relation characteristic of short-range interaction forces.

For the problem of transition radiation some quite new and important features emerge that are present in cases D and E. Branches III of Fig. 5 and IV of Fig. 6 have a portion with phase velocity larger than the velocity of light. Consequently, the corresponding modes are not surface waves anymore but acquire a radiative nature and can be detected directly. The reflection coefficients for plane waves incident on the considered geometrical configurations blow up for real values of θ, satisfying the relation $\omega = f(\sin\theta)$. Here $\omega = f(ck/\omega)$ gives the dispersion relation for branches III of Fig. 5 or IV of Fig. 6. Values of $|r|^2$ larger than 1 are due to the fact that the SPO are assumed to remain undamped in our formalism in spite of the energy losses due to collisions inside the materials and to radiation of electromagnetic energy. In order to have energy conservation, some external source should maintain the SPO undamped, supplying the missing energy. Otherwise the attenuation of SPO should be taken into account, in which case $|r|^2$ is always smaller than 1. These radiative SPO can be excited, of course, by incoming electrons, thus adding an extra term to the observed radiation. As is well known,[16,29] the fields in the radiation zone are given by expressions of the form $\int_0^\infty \varphi(k)dk$ and are calculated by the method of steepest descent, where the saddle point is $k = (\omega/c)\sin\theta$. The SPO appear as poles of the function $\varphi(k)$. If their phase velocity is smaller than c, they do not make any contribution, since the location of the pole is outside the region defined by the original and the transformed path.[29] But if $\omega(k)/k \gtrsim c$, the pole is located inside this region and very close to the saddle point. Consequently we should expect an extra term coming from the residue at the pole, in the same way as Čerenkov radiation adds an extra term, under certain conditions, in the problem considered by Garibian[29] and by Pafomov.[30]

The next two geometrical configurations to be considered are motivated by the various experiments performed on coupled superconducting junctions, notably by Giaever.[40] Furthermore, the presence of new branches in the last two cases, D and E, which correspond to radiative SPO, promotes interest in studying multiple films. In particular, case G below may be important in the problem of transition radiation. It should be noted, however, that no mode appears in these more complex geometries with qualitatively different features from those discussed above.

F. Two Dielectric Films

Two dielectric barriers separated by a metal film and bounded on the other sides by semi-infinite metals is the geometry to be considered here (Fig. 7). The dispersion relations are determined by

$$[(1-R)/(1+R)]^4 - A_2'[(1-R)/(1+R)]^2 + e^{-2K_id_1}e^{-2K_id_2} = 0, \quad (3.33)$$

[40] I. Giaever, Phys. Rev. Letters 14, 904 (1965).

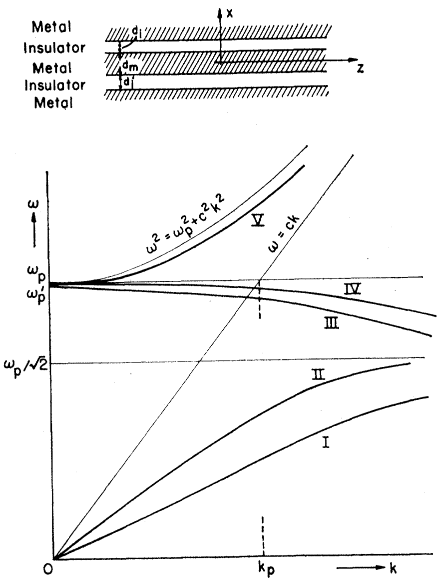

FIG. 7. Geometry and the dispersion relations for SPO of three films (insulator-metal-insulator) between two semi-infinite metals ($\epsilon_i = 1$, $\epsilon_m = 1-\omega_p^2/\omega^2$). The analytical expressions for $k \gg k_p$ or $k \ll k_p$ for the curves shown schematically here are (3.35)–(3.40). ω_p' is given by (3.39a). For normal metals only the higher parts ($\omega > \omega_t$, where $\omega_t \approx 10^{12}$–10^{14} sec^{-1}) of curves I and II exist. For superconducting metals, besides the higher parts, the lower parts ($\hbar\omega < 2\Delta$) of curves I and II are present, but with ω_p replaced by ω_{ps}.

with

$$A_2' = e^{-2K_id_1} + e^{-2K_id_2} + e^{-2K_md_m}(1-e^{-2K_id_1}) \times (1-e^{-2K_id_2}). \quad (3.34)$$

Graphically they are shown in Fig. 7.

For $k \ll k_p$ branches I and II are given by

$$\omega = c[(d_1+d_2 \mp \gamma^{1/2})/(d_1+d_2+4\lambda_p \mp \gamma^{1/2})]^{1/2}k, \quad (3.35)$$

where

$$\gamma = d_1^2 + d_2^2 + 2(2e^{-2k_pd_m}-1)d_1d_2. \quad (3.36)$$

In the special case when $d_1 = d_2 = d_i$ they correspond to symmetric and antisymmetric solutions and (3.35) reduces to the form

$$\omega = c\left(\frac{d_i(1\pm e^{-k_pd_m})}{d_i(1\pm e^{-k_pd_m})+2\lambda_p}\right)^{1/2} k. \quad (3.37)$$

In the limit when $k_pd_m \gg 1$, $\gamma \to (d_1-d_2)^2$, and (3.35) becomes

$$\omega = c[d_{1,2}/(d_{1,2}+2\lambda_p)]^{1/2}k, \quad (3.38)$$

as we expect, since when the thickness of the central metal film becomes very large, the two oscillations are decoupled. Branch III starts at

$$\omega_p' = \omega_p[1-\tfrac{1}{8}(d_1+d_2)^2k_p^2], \quad (3.39a)$$

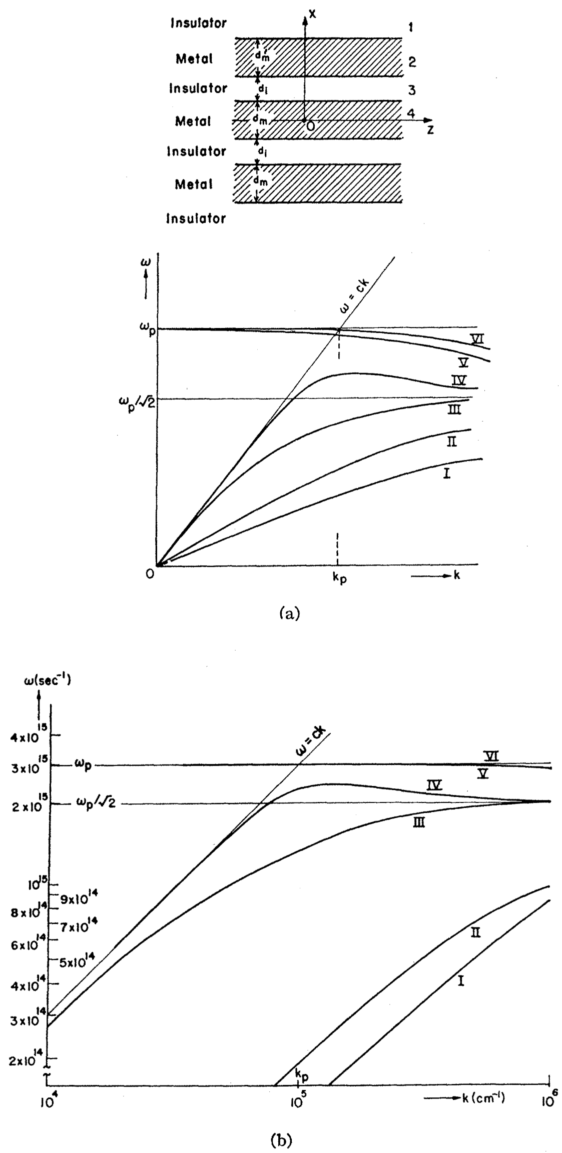

FIG. 8. (a) Geometry and the dispersion relations for SPO of five films (metal-insulator-metal-insulator-metal) between two semi-infinite insulators ($\epsilon_i=1$, $\epsilon_m=1-\omega_p^2/\omega^2$). The analytical expressions for $k\gg k_p$ or $k\ll k_p$ for the curves shown schematically here are (3.43)–(3.47). For normal metals only the higher parts ($\omega>\omega_t$, where $\omega_t\approx 10^{12}$–10^{14} sec^{-1}) of curves I and II exist. For superconducting metals, besides the higher parts, the lower parts ($\hbar\omega<2\Delta$) of curves I and II are present, but with ω_p replaced by ω_{ps}. (b) The high-frequency parts of these dispersion relations calculated numerically for $\lambda_p=1000$ Å, $d_i=20$ Å, $d_m=200$ Å, and $d_m'=200$ Å. Branches V and IV are almost degenerate.

when $k=0$ and goes like

$$k^2=\frac{4(1-\omega^2/\omega_p^2)}{(d_1+d_2)^2}\left(\frac{1-\omega^2/\omega_p^2}{\omega^4/\omega_p^4}-\tfrac{1}{4}(d_1+d_2)^2k_p^2\right), \quad (3.39b)$$

for $k\lesssim k_p$. This oscillation is exactly the same as the case B of a dielectric film of thickness d_1+d_2 between two semi-infinite metals, ignoring the presence of the middle metal film. For small k branch IV is described by

$$k^2=\omega^2/c^2-k_p^2+[(d_1+d_2)/d_md_1d_2](\omega_p^2-\omega^2)/\omega^2. \quad (3.40)$$

When $d_2\gg d_1$, this equation reduces to (3.26) of case D. This implies that branch IV corresponds to an oscillation which couples the two junctions, and, when one of the thicknesses of the dielectric films becomes large, the coupling is broken and the oscillation is confined to one of them. This physical description is correct if $k_pd_m<1$, but if $k_pd_m\gg 1$, as we go along branch IV in the direction of increasing k, this ceases to be valid when $\epsilon_md_m^2k_p^2\gtrsim 1$, for then the oscillations in the two junctions become independent and can be separately described by the formulas of case D. Again for $k\gg k_p$ branches I–IV approach $\omega_p/\sqrt{2}$ in accordance with the electrostatic theory. Also, branch V is given by a formula similar to (3.20) of case B.

G. Three Metal Films

Closely related to the last case is the configuration shown in Fig. 8. The dispersion relation is given by

$$\chi^3\pm B_2\chi^2-B_1\chi\mp B_0=0, \quad (3.41)$$

where

$$B_2=e^{-K_md_m}+e^{-2K_id_i-K_m(d_m+2d_m')}-e^{-K_md_m-2K_id_i},$$
$$B_1=e^{-2K_md_m'}+e^{-2K_id_i}-e^{-2K_md_m'-2K_id_i}, \quad (3.42)$$
$$B_0=e^{-K_m(d_m+2d_m')},$$
$$\chi=(1-R)/(1+R).$$

This equation as well as all the previous equations that determine the dispersion relation reduce to their analogs in the electrostatic theory when retardation effects are ignored (by letting $c\to\infty$), since then $\chi\to(\epsilon_m+1)/(\epsilon_m-1)$. The dispersion relations resulting from (3.41) are calculated numerically and are shown in Fig. 8(b). When $k\ll k_p$, analytical expressions can be written for:

branches I, II:

$$\omega=ck\left(\frac{d_i}{d_i+\lambda_p/B_{I,II}}\right)^{1/2}, \quad (3.43)$$

where

$$B_I=\left|\frac{e^{-k_pd_m''}(e^{-k_pd_m}-e^{-2k_pd_m'}-e^{-k_pd_m''})+2e^{-k_pd_m''}+e^{-2k_pd_m'}-e^{-k_pd_m}-1}{2(1-e^{-2k_pd_m''})}\right|, \quad (3.44a)$$

$$B_{II}=\left|\frac{e^{-k_pd_m''}(e^{-k_pd_m''}-e^{-2k_pd_m'}-e^{-k_pd_m})+2e^{-k_pd_m''}-e^{-2k_pd_m'}-e^{-k_pd_m}+1}{2(1-e^{-2k_pd_m''})}\right|, \quad (3.44b)$$

and $d_m''=d_m+2d_m'$;

branches III, IV:

$$k^2=(\omega^2/c^2)\{1+\omega^2/\omega_p^2[\tanh(\tfrac{1}{2}k_pd_m'')]^{\pm 2}\}; \quad (3.45)$$

branches V, VI:

$$k^2=\omega^2/c^2-k_p^2+(1/d_m'd_i)[(\omega_p^2-\omega^2)/\omega^2], \quad (3.46)$$

$$k^2=\omega^2/c^2-k_p^2+(d_m''/d_id_md_m')[(\omega_p^2-\omega^2)/\omega^2]. \quad (3.47)$$

The branches given by (3.43) with (3.44b), (3.47), and (3.45) (upper case) correspond to symmetric solutions, while (3.43) with (3.44a), (3.46), and (3.45) (lower case) correspond to antisymmetric solutions [see Fig. 8(b)].

We conclude this section by a discussion of a periodic structure of alternating metal and insulating films of thickness d_m and d_i, respectively. Periodicity implies that the eigensolutions obey the Floquet-Bloch theorem; namely,

$$\varphi_a(x+d)=e^{iad}\varphi_a(x),$$

with $d=d_i+d_m$ being the period. The secular equation is

$$(1/R)^2-A_p(1/R)+1=0, \quad (3.48)$$

where

$$A_p=\frac{2(e^{2K_id_i}+1)(e^{2K_md_m}+1)-8e^{K_md_m+K_id_i}\cos ad}{(e^{2K_id_i}-1)(e^{2K_md_m}-1)}. \quad (3.49)$$

For $k\gg k_p$ the solutions tend to the corresponding solutions for the electrostatic problem and eventually approach $\omega_p/\sqrt{2}$ when $k\to\infty$ .When either d_i or d_m becomes infinite, we retrieve the dispersion relations of the one-film geometry. The dispersion relations are pictorially represented in Fig. 9.

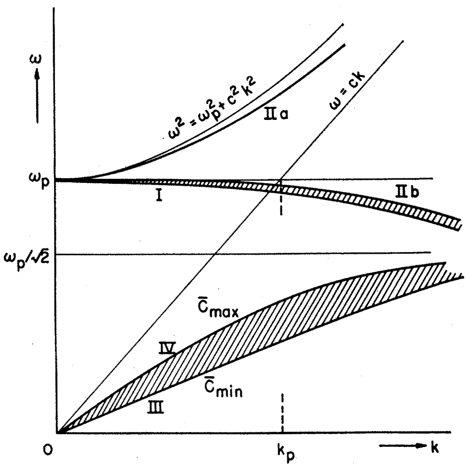

FIG. 9. Dispersion relations (shaded areas) for SPO in the periodic geometry of alternating metal and insulating films ($\epsilon_i=1$, $\epsilon_m=1-\omega_p^2/\omega^2$). The analytical expressions for $k\gg k_p$ or $k\ll k_p$ for the regions shown schematically here are (3.50)–(3.55). For normal metals only the higher parts ($\omega>\omega_t$, where $\omega_t\approx10^{12}$–10^{14} sec^{-1}) of the low-lying curves exist. For superconducting metals, besides the higher parts, the lower parts ($\hbar\omega<2\Delta$) of the low-lying curves are present, but with ω_p replaced by ω_{ps}.

Solutions are found in the shaded regions. For $k<k_p$ curves III and IV can be taken as straight lines with phase velocities

$$c_{\min}=c\{d_i/[d_i+2\lambda_p\coth(\tfrac{1}{2}k_pd_m)]\}^{1/2} \quad (3.50)$$

and

$$c_{\max}=c\{d_i/[d_i+2\lambda_p\tanh(\tfrac{1}{2}k_pd_m)]\}^{1/2}, \quad (3.51)$$

respectively; any intermediate solution has phase velocity

$$\bar{c}=c\left[d_i\Big/\left(d_i+2\lambda_p\frac{\exp(2k_pd_m)+1-2\cos(ad)\exp(k_pd_m)}{\exp(2k_pd_m)-1}\right)\right]^{1/2}. \quad (3.52)$$

The upper region within which solutions lie is bounded by I, a portion of IIa, and IIb. If $k_p^2d_id_m\ll1$, curve I is given by

$$k^2=(\omega_p^2-\omega^2)(4/d_id_m\omega^2-1/c^2), \quad (3.53)$$

while IIa is given by

$$k^2=\left(\frac{\omega^2}{c^2}-k_p^2\right)\left(1+\frac{\omega^2-\omega_p^2}{\omega^2(1+d_m/d_i)}\right). \quad (3.54)$$

Solutions near I are given by

$$k^2=(\omega_p^2-\omega)[2(1-\cos ad)/d_id_m\omega^2-1/c^2]. \quad (3.55)$$

Other details of various solutions can be obtained by numerical computation only. For $k\gg k_p$, the electrostatic theory is sufficient and the secular equation is

$$\epsilon_m^2+a_p\epsilon_m+1=0, \quad (3.56)$$

where a_p is obtained from (3.49) by putting $K_i=K_m=k$. Again as $k\to\infty$ the eigenfrequencies tend to $\omega_p/\sqrt{2}$.

IV. SUPERCONDUCTING METAL FILMS

In the previous sections surface plasmons in thin normal-metal films have been considered. As has been explained in the Introduction, owing to the finite mean free path or to the surface scattering, the damping of these modes becomes quite large when ω is lower than a given value, which in actual cases is usually no smaller than 10^{12} sec^{-1}. Consequently, for normal metals the lower parts of the dispersion relations derived in Sec. III are absent. When we consider superconductors instead of normal metals, the higher part of the curves remains unmodified, since for $\hbar\omega\gg2\Delta$ (2Δ is the energy gap) the superconductor behaves as a normal metal. In the lower part corresponding to $\hbar\omega<2\Delta$ some modifications appear, which are not so significant for the dispersion relation, i.e., $\mathrm{Re}\,\omega(k)$, but are quite important for the damping, i.e., $\mathrm{Im}\,\omega(k)$. This is due to the fact that for $\hbar\omega<2\Delta$ and low temperatures the number of quasiparticle excitations which contribute to the bulk and surface scattering is very small.

What we want in order to find the modifications arising from the presence of superconducting material instead of normal metal is the knowledge of the frequency-dependent conductivity $\hat{\sigma}(\omega)$ [or the dielectric operator $\hat{\epsilon}(\omega)\equiv 1-(4\pi i/\omega)\hat{\sigma}(\omega)$] for the superconducting films. It is reasonable to assume that, due to the polycrystalline and impure nature of most thin films, the case of short mean free path $l\ll\xi_0$ ($\xi_0=\hbar v_F/\pi\Delta$ is the coherence length, and v_F is the Fermi velocity) is realized. Mattis and Bardeen[18] have formulated the problem of the surface impedance for superconductors with arbitrary values of the mean free path. Mattis and Bardeen gave the current-field relation as

$$\mathbf{j}(\mathbf{r},t)=\sum_{\omega}\frac{e^2N(0)v_F}{2\pi^2\hbar c}\times\int\frac{\mathbf{R}[\mathbf{R}\cdot\mathbf{A}_\omega(\mathbf{r}')]I(\omega,R,T)e^{-R/l}d^3r'}{R^4},\quad(4.1)$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, $I(\omega,R,T)$ is a complicated function whose range with respect to R is ξ_0, and the vector potential is given by $A_\omega(\mathbf{r}')=(1/i\omega)\mathbf{E}_\omega(\mathbf{r}')$. An enormous simplification in the calculation of the dispersion relation for SPO is obtained when the current-field relation is local, i.e., $\hat{\sigma}(\omega)$ is just a number in ordinary space. Then, as can be seen from Eqs. (3.1) and (3.5), the field satisfies the simple relation

$$\nabla\cdot\mathbf{E}=0,\quad(4.2)$$

except at the surfaces and when ω becomes equal to the bulk plasma frequency. We shall be concerned with much lower frequencies. In this case we are completely justified to use Mattis and Bardeen's formulation since they have assumed in their calculations that $\nabla\cdot\mathbf{A}=0$, which is now a natural consequence of Eq. (4.2). For a local relation to hold, the range of the kernel in the integral relation between current and field should be much smaller than both the characteristic length over which the field changes (penetration depth λ_{ps}) and the thickness of the metal film. We see from Eq. (4.1) that the range of this kernel is determined by either ξ_0 or the mean free path l, whichever is smaller. So the condition for the applicability of the local theory is

$$\min(l,\xi_0)\ll\min(\lambda_{ps},d_m).\quad(4.3)$$

In actual experimental situations $l<\xi_0$ holds and the condition becomes

$$l\ll\min(\lambda_{ps},d_m).\quad(4.4)$$

Typical values for the quantity $\min(\lambda_{ps},d_m)$ are 500–2000 Å, while it is not unreaslistic to assume $l\approx 100$ Å in thin films, so that Eq. (4.4) is satisfied. Miller[19] has shown that even when (4.4) is not satisfied the results of a local approximation do not differ appreciably from the results of a nonlocal theory. It seems, therefore, that we are justified in using the local approximation under which Eq. (4.1) can be written

$$\mathbf{j}(\omega)=[\sigma_1(\omega)-i\sigma_2(\omega)]\mathbf{E}(\omega),\quad(4.5)$$

where the scalar functions $\sigma_1(\omega)$ and $\sigma_2(\omega)$ can be obtained from Eq. (4.1). Since $l\ll\xi_0$, λ_{ps}, we put $R=0$ in $I(\omega,R,T)$ and reduce Eq. (4.1) to

$$\mathbf{j}(\mathbf{r},\omega)=\mathbf{A}(\mathbf{r},\omega)I(\omega,0,T)\frac{e^2N(0)v_F}{2\pi^2\hbar c}\int\frac{\mathbf{R}\mathbf{R}e^{-R/l}d^3r'}{R^4}.\quad(4.6)$$

Comparing this with the corresponding equation for the normal metal[18] that holds in the limit $\omega\to 0$

$$\mathbf{j}(\mathbf{r},\omega)=-i\pi\hbar\omega\mathbf{A}(\mathbf{r},\omega)\frac{e^2N(0)v_F}{2\pi^2\hbar c}\int\frac{\mathbf{R}\mathbf{R}e^{-R/l}d^3r'}{R^4},\quad(4.7)$$

we obtain

$$\frac{\sigma_1(\omega)-i\sigma_2(\omega)}{\sigma_0}=\frac{I(\omega,0,T)}{-i\pi\hbar\omega},\quad(4.8)$$

where σ_0 is the static normal-state conductivity given by

$$\sigma_0=\omega_p{}^2l/4\pi v_F.\quad(4.9)$$

Formula (4.8) can also be obtained under the assumption that $\xi_0\gg\lambda_{ps}$, an assumption which is usually referred in the literature[18,19] as the extreme anomalous limit. In this case the complex conductivity ratio is defined simply as the ratio of the currents in the superconducting and normal states. The evaluation of $I(\omega,0,T)$ for $T\neq 0$ requires numerical integrations which have been performed by Miller[19]; the results are tabulated as functions of $\Delta(T)/kT$ and $\hbar\omega/kT$. Using the complex conductivity defined above, we can calculate the corresponding dielectric function as

$$\epsilon_s(\omega)\equiv 1-\frac{4\pi i}{\omega}\sigma(\omega)=1-\frac{4\pi\sigma_2(\omega)\cdot\omega}{\omega^2}\left(1+i\frac{\sigma_1(\omega)}{\sigma_2(\omega)}\right).\quad(4.10)$$

By use of values tabulated by Miller we can see (either from Fig. 10 or Table I) that the quantity $\sigma_2(\omega)\cdot\omega$ is almost independent of frequency except for $\hbar\omega$ close to 2Δ. Defining ω_{ps} by

$$\omega_{ps}{}^2=4\pi\sigma_2(\omega)\cdot\omega,\quad \hbar\omega<2\Delta,\quad(4.11)$$

we obtain

$$\epsilon_s(\omega)=1-(\omega_{ps}{}^2/\omega^2)(1+i\sigma_1/\sigma_2),\quad(4.12)$$

which has exactly the same form as in a normal metal with the role of ω_p played by an effective ω_{ps} which is of the order of magnitude 5×10^{15} sec^{-1} and is a function of temperature as well. The imaginary part σ_1/σ_2 is small (Fig. 11 or Table I) for low temperatures or ω

TABLE I. The quantities $\sigma_2\omega/\sigma_0$ and $\sigma_1/2\sigma_2$ as functions of frequency and temperature for Pb and Sn.

T/T_c Pb	T/T_c Sn	$\hbar\omega/\Delta(T)$ Pb and Sn	$\frac{\sigma_2\omega}{\sigma_0}$ (10^{12} sec^{-1}) Pb	$\frac{\sigma_2\omega}{\sigma_0}$ (10^{12} sec^{-1}) Sn	$10^3\frac{\sigma_1}{2\sigma_2}$ Pb and Sn	T/T_c Pb	T/T_c Sn	$\hbar\omega/\Delta(T)$ Pb and Sn	$\frac{\sigma_2\omega}{\sigma_0}$ (10^{12} sec^{-1}) Pb	$\frac{\sigma_2\omega}{\sigma_0}$ (10^{12} sec^{-1}) Sn	$10^3\frac{\sigma_1}{2\sigma_2}$ Pb and Sn
0	0	0	6.00	2.74	0	0.40	0.36	0.057	5.90	2.73	1.37
0	0	0.2	6.00	2.74	0	0.40	0.36	0.086	5.90	2.73	1.68
0	0	0.4	6.00	2.74	0	0.40	0.36	0.114	5.90	2.73	1.91
0	0	0.6	6.00	2.74	0	0.40	0.36	0.200	5.90	2.73	2.30
0	0	0.8	5.85	2.65	0	0.40	0.36	0.286	5.90	2.73	2.43
0	0	1.0	5.76	2.60	0	0.40	0.36	0.430	5.90	2.73	2.50
0	0	1.2	5.70	2.56	0	0.40	0.36	0.570	5.83	2.70	2.43
0	0	1.4	5.20	2.30	0	0.40	0.36	0.860	5.67	2.63	2.30
0	0	1.6	4.75	2.12	0	0.40	0.36	1.14	5.40	2.51	2.24
						0.40	0.36	2.00	3.80	1.75	3.02
0.56	0.51	0.059	5.25	2.46	6.00	0.685	0.630	0.064	4.42	2.10	12.5
0.56	0.51	0.086	5.25	2.46	7.60	0.685	0.630	0.094	4.42	2.10	16.2
0.56	0.51	0.119	5.25	2.46	8.90	0.685	0.630	0.128	4.42	2.10	19.2
0.56	0.51	0.208	5.25	2.46	11.4	0.685	0.630	0.223	4.51	2.14	25.3
0.56	0.51	0.297	5.25	2.46	17.7	0.685	0.630	0.319	4.52	2.15	29.4
0.56	0.51	0.445	5.22	2.44	18.9	0.685	0.630	0.480	4.57	2.18	33.6
0.56	0.51	0.595	5.22	2.44	14.0	0.685	0.630	0.640	4.60	2.19	34.8
0.56	0.51	0.890	5.15	2.41	14.3	0.685	0.630	0.960	4.50	2.14	36.2
0.56	0.51	1.19	4.94	2.31	14.3	0.685	0.630	1.28	4.33	2.06	37.1
0.56	0.51	2.08	3.24	1.53	53	0.685	0.630	2.28	2.66	1.27	153
0.820	0.770	0.075	2.99	1.45	26	0.920	0.880	0.100	1.73	0.85	49.2
0.820	0.770	0.112	3.08	1.49	34	0.920	0.880	0.153	1.69	0.83	68.0
0.820	0.770	0.150	3.12	1.52	40	0.920	0.880	0.204	1.72	0.85	83.0
0.820	0.770	0.263	3.18	1.55	55	0.920	0.880	0.356	1.80	0.89	119
0.820	0.770	0.375	3.25	1.58	65	0.920	0.880	0.510	1.86	0.92	138
0.820	0.770	0.562	3.33	1.62	76	0.920	0.880	0.764	2.02	1.0	158
0.820	0.770	0.750	3.37	1.64	82	0.920	0.880	1.00	2.01	0.99	180
0.820	0.770	1.12	3.37	1.64	90	0.920	0.880	1.53	2.01	0.99	208
0.820	0.770	1.50	3.21	1.52	96	0.920	0.880	2.04	1.75	0.86	255
0.820	0.770	2.63	1.77	0.858	500						
0.960	0.950	0.150	0.704	0.360	108	0.990	0.985	0.250	0.262	0.135	243
0.960	0.950	0.226	0.734	0.375	144	0.990	0.985	0.380	0.280	0.144	324
0.960	0.950	0.300	0.760	0.388	174	0.990	0.985	0.510	0.303	0.156	395
0.960	0.950	0.526	0.822	0.420	244	0.990	0.985	0.890	0.322	0.166	600
0.960	0.950	0.751	0.874	0.447	260	0.990	0.985	1.27	0.350	0.180	670
0.960	0.950	1.13	0.939	0.480	352	0.990	0.985	1.91	0.360	0.185	880
0.960	0.950	1.50	0.968	0.495	402	0.990	0.985	2.51	0.322	0.166	1340
0.960	0.950	2.26	0.816	0.417	603						

not so close to 2Δ and in a first approximation can be neglected. This means that all the results obtained in Sec. III are valid for superconductors for $\hbar\omega<2\Delta$ if we replace ω_p by ω_{ps} and $\lambda_p\equiv c/\omega_p$ (London penetration depth) by $\lambda_{ps}=c/\omega_{ps}$ (actual penetration depth). The losses other than those due to radiation can be calculated from the quantity σ_1/σ_2. Mattis and Bardeen have used the BCS weak-coupling theory for their calculations.

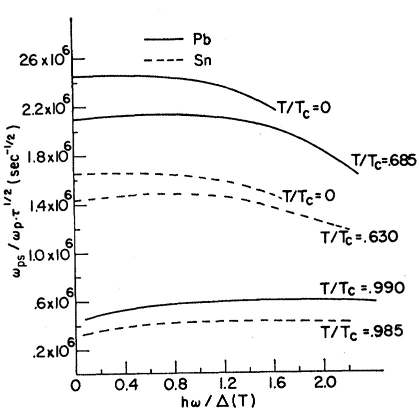

FIG. 10. Effective bulk plasmon frequency ω_{ps} as a function of frequency and temperature for superconducting Pb and Sn. The calculations are based on the Mattis-Bardeen theory (Ref. 18).

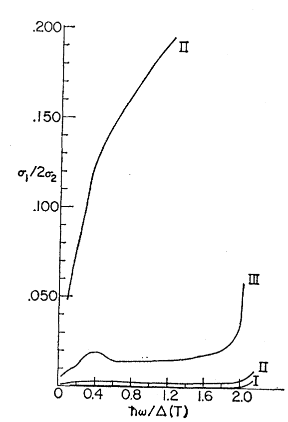

FIG. 11. The quantity $\sigma_1/2\sigma_2$ as a function of frequency for Pb and Sn at various temperatures: (I) $T/T_c=0$ (Pb and Sn); (II) $T/T_c=0.40$ (Pb) or $T/T_c=0.36$ (Sn); (III) $T/T_c=0.56$ (Pb) or $T/T_c=0.51$ (Sn); (IV) $T/T_c=0.92$ (Pb) or $T/T_c=0.88$ (Sn).

However, the strong-coupling theory[41] is definitely more appropriate for some superconducting materials, especially lead as brought out by tunneling experiments, in which the phonon spectrum was imaged in the I-V characteristics of the junction.[42] Further, the experimental observation[43] of an anomalously steep electromagnetic absorption edge in superconducting lead and the anomalously large transmission through thin films at the gap frequency require also the strong-coupling theory for their explanation. Theoretical calculations by the strong-coupling theory of the complex electrical conductivity at $T=0$ of superconducting lead have been performed by Nam[44] and more recently by Shaw and Swihart.[45] The calculations are in good agreement with experiment[42] and with each other. In Ref. 45 the authors have numerically integrated the expression for $\sigma_1(\omega)/\sigma_0$ in terms of the complex energy-gap function $\Delta(\omega)$ at $T=0$ which is obtained by solving the BCS complex gap equation from the strong-coupling theory. They expressed $\sigma_1(\omega)/\sigma_0$ as $\sigma_1(\omega)/\sigma_0=A_g\delta(\omega)+\sigma_1'(\omega)/\sigma_0$ and found σ_2 from the Kramers-Kronig relation

$$\frac{\sigma_2}{\sigma_0}=\frac{2}{\pi}\frac{A_g}{\omega}-\frac{2\omega}{\pi}\int_0^\infty \frac{\sigma_1'(\omega)/\sigma_0}{\omega_1^2-\omega^2}d\omega_1. \tag{4.13}$$

They observed that for $\hbar\omega<2\Delta$, the difference between their σ_2/σ_0 and the Mattis-Bardeen result is due mainly (99%) to a change in A_g and negligibly (1%) to a change in the second term of (4.13). From this information we can deduce that in this frequency range, the constancy of the product $\sigma_2(\omega)\cdot\omega$ is retained at $T=0$ for lead in the strong-coupling theory and the local approximation (4.12) is valid with ω_{ps}^2 and σ_1/σ_2 now given via the results of Nam or of Shaw and Swihart, which differ from those of Mattis and Bardeen by about 30%.

What this discussion indicates is that the figures given in Table I may be revised if the calculations were based on a more realistic theory, which consequently will change slightly the parameters entering into our expressions; the results we have reached, however, will remain essentially unchanged.

V. CONCLUSION

In summary, we can say that the various modes of SPO in multiple-film systems can be classified into two main groups. One group contains those modes whose dispersion curves start from zero frequency at $k=0$, increase as k increases, but remain below the line $\omega=ck$. The other group starts at $k=0$ from $\omega=\omega_p$ or a value slightly less than ω_p and remains close to the line $\omega=\omega_p$. For very large k, all the dispersion curves of both groups converge asymptotically to the classical surface plasmon frequency $\omega_p/\sqrt{2}$. In addition to these two groups, some uninteresting modes may appear with dispersion curves that lie just below the curve $\omega^2=\omega_p^2+c^2k^2$, which corresponds to the trivial solution of zero fields.

For normal metals this description is valid only for high enough frequencies so that oscillation damping is negligible. On the other hand, for superconducting metals, the picture is valid not only for the high-frequency region but also for low frequencies such that $\hbar\omega<2\Delta$.

We have seen that the SPO are not responsible for the observed peak in the spectral resolution of the transition radiation emitted from a thin foil, as has been generally accepted in the literature. A new physical picture was offered for the interpretation of this peak. At the same time it has been shown that in multiple-film structures radiative SPO exist, which should have observable effects in the radiation properties of these structures. In particular, there seems to be a possibility of obtaining intense radiation as the number of the films increases.

The modifications of the dispersion relations due to non-negligible retardation effects as we have discussed may be observed experimentally in energy loss experiments or in radiation measurements. Finally, the important role that the low-frequency modes can play in determining the electromagnetic properties of multiple superconducting film structures with the presence of the ac Josephson effect is examined in detail by Ngai in the following paper.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor Morrel H. Cohen for suggesting this problem and for stimulating discussions throughout the work, and to K. L. Ngai for detailed assistance and valuable discussions on the subject of this paper. This research was supported in part by the National Aeronautics and Space Administration and benefited from general support of Materials Science at the University of Chicago by the Advanced Research Projects Agency.

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