Noise in the ac Josephson Effect*

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A Langevin treatment of noise in the ac Josephson effect is given at finite temperatures, including the quasiparticle tunneling currents. Using quasilinear methods, the stability and small oscillations of the system are investigated. It is found that the system is unstable for phase fluctuations and that these phase fluctuations lead to a Lorentzian line for the radiation emitted from a Josephson junction. The power spectrum of voltage fluctuations of the junction and the statistics of the emitted radiation are investigated.

1. INTRODUCTION

IN a previous Letter,[1] a Langevin treatment of noise in the ac Josephson[2] effect was given. In this paper we extend the previous treatment to finite temperatures and include the quasiparticle tunneling currents. As before, the Langevin equations are investigated using quasilinear techniques. This method enables us to get most of the important results in a simple and direct manner. The Langevin equations have the advantage that we deal directly with the equations of motion of variables or operators representing physical quantities. The more involved Fokker-Planck techniques applied to this problem will be discussed elsewhere.

Using quasilinear techniques, the stability of the oscillator for small deviations from steady state is investigated. The oscillator is found to be stable under all displacements from steady state except one. This instability is a phase instability and can be simply understood as follows. The differential equations describing the oscillator do not depend explicitly on the time, so that if we have a solution which begins at t_0, we can construct from it other solutions beginning at different times. It costs no energy to pass from one solution to another, and fluctuations in phase which take us from one solution to another are not suppressed. The phase fluctuations are analogous to the Brownian motion of a free particle. This instability is very important for the linewidth of the radiation emitted from a Josephson junction. The effect of noise in the system is to broaden the δ-function spectrum of the oscillator into a finite Lorentzian line. Classical autonomous oscillators of this kind have been discussed by Lax.[3]

The linewidth of the radiation emitted from a Josephson junction has been studied experimentally by Parker, Taylor, and Langenberg[4] and by Parker, Dahm, and Denenstein[5] in connection with their measurements of e/h. Owing to the fundamental nature of these measurements, it is important to understand the origin of the linewidth of the radiation. In this paper it is found that the residual linewidth at low temperatures is due to photon shot noise associated with the dissipation of electromagnetic energy in the cavity formed by the two superconductors. At finite temperatures, quasiparticle tunneling currents give rise to voltage fluctuations across the junction which have an important effect on the linewidth. The quasiparticle noise has previously been considered by Scalapino.[6] Voltage fluctuations due to Johnson noise in the external circuit can also contribute to the linewidth but under most experimental conditions (large external resistance) this effect is small.[7]

In Sec. 2, we consider the modes of the superconducting cavity in the absence of tunneling currents. In Sec. 3 a simple description of the interaction of the tunneling pairs and the electromagnetic field in the cavity is developed. In I it was found that, apart from a small frequency pulling effect,[8] the only relevant variable needed to describe the superconductors was the phase difference between the two superconductors forming the junction. This is in agreement with Josephson's[2] original description. This enables us to largely eliminate the properties of the superconductors from the problem. In Sec. 4 we obtain the Langevin equations describing the oscillator. The remaining sections are devoted to an analysis of the Langevin equations using quasilinear techniques.

The ac Josephson effect has many features in common with the laser. The Langevin techniques used here are most closely similar to those used by Lax[9] and by Haken[10] in connection with lasers. In certain respects, the tunneling pairs are analogous to the active atoms injected into a laser. However, an important difference arises because all the pairs in a superconductor are phase-coherent and not independent like the atoms in a laser. It is for this reason that frequency pulling effects

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[7] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksperim. i Teor. Fiz. **53**, 2159 (1967) [English transl.: Soviet Phys.—JETP **26**, 1219 (1968)].

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[9] M. Lax, Phys. Rev. **145**, 110 (1966).

[10] H. Haken, Z. Physik **190**, 327 (1966).

are very small. For the small electromagnetic fields considered in this paper we need only consider the phase of the pairs and neglect any changes in the amplitudes of the pair wave functions. The other important difference with the laser is that the Josephson frequency $2eV/\hbar$ (where V is the voltage across the junction) fluctuates as the pairs tunnel. This frequency is analogous to the atomic frequency in the laser problem. It is this frequency modulation which leads to the relatively large linewidth (10^3–10^4 cps) of the Josephson radiation. The origin of the linewidth in the ac Josephson effect and the laser are essentially the same, i.e., the shot noise associated with dissipation of radiation in the cavity.

2. CAVITY MODES

In this section we consider the normal modes of the cavity formed by the two superconductors and the oxide layer separating them in the absence of any tunneling current. These modes have been studied in detail by Swihart.[11] We suppose that the junction lies in the xy plane and has dimensions $L_x \simeq L_y \simeq 10^{-1}$ cm. The thickness of the oxide layer is $l \simeq 10^{-7}$ cm. To describe the modes we use a vector potential $\mathbf{A}$ (with $\text{div}A=0$) satisfying

$$\left(\nabla^2 - \frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}. \tag{2.1}$$

In the oxide layer, $J=0$, and the appropriate solution of (2.1) for a standing wave along x is

$$\begin{aligned} A_x &= A_{x1}\sinh K_1 z \sin kx\, e^{-i\Omega t}, \\ A_z &= A_{z1}\cosh K_1 z \cos kx\, e^{-i\Omega t}, \end{aligned} \tag{2.2}$$

where

$$K_1^2 = k^2 - \epsilon\Omega^2/c^2, \quad A_{x1}/A_{z1} = -K_1/k, \tag{2.3}$$

and ϵ is the dielectric constant of the oxide. In spite of the fact that the ends of the cavity are open, there is a poor impedance match with the outside, and to a good approximation we can take the boundary conditions at the open ends $x=0$, L_x to be $H=0$. This determines $k=n\pi/L_x$, where n is an integer.

In the superconductor from London's equation $J=-(c/4\pi\lambda^2)A$, where λ is the penetration depth and the appropriate solution of (2.1) is

$$\begin{aligned} A_x &= A_{x2}e^{-K_2|z|}\sin kx\, e^{-i\Omega t}, \\ A_z &= A_{z2}e^{-K_2|z|}\cos kx\, e^{-i\Omega t}, \end{aligned} \tag{2.4}$$

where

$$K_2^2 = 1/\lambda^2 + k^2 - \Omega^2/c^2, \quad A_{x2}/A_{z2} = K_2/k. \tag{2.5}$$

Matching the tangential fields at the superconductor oxide interface leads to the dispersion relation at long wavelengths

$$\epsilon\Omega^2/c^2 = lk^2/(l+2\lambda). \tag{2.6}$$

We note from (2.2) that $A_x/A_z \simeq kl \simeq 10^{-5}$, so that for practical purposes we need only retain A_z. Therefore, at long wavelengths compared to l, the modes of the cavity are predominantly transverse with the electric field E_z perpendicular to the superconductors. The components of the fields in the superconductors may be neglected, since they are small—of the same order as A_x.

In order to quantize the electromagnetic field, we introduce the normal modes given in (2.2) and (2.4). We quantize the field in the gauge $\text{div}\mathbf{A}=0$ so that the scalar potential leading to the voltage across the junction is classical and calculated from Coulomb's law. Thus the charge Q on the one superconductor is related to the voltage by

$$Q = CV,$$

where C is the capacitance of the junction. After a standard calculation[12] we obtain the result required here

$$A_z(x,z) = \sum_n \left(\frac{2\pi\hbar c^2}{\epsilon\Omega_n}\right)^{1/2} U_n(x,z)(b_n + b_n^\dagger), \quad |z| < \tfrac{1}{2}l \tag{2.7}$$

where

$$U_n(x,z) = (2/L_xL_yl)^{1/2}\cosh K_n z \cos k_n x.$$

The normal modes are labeled by the index n, and $b_n^\dagger$ and b_n are operators creating and destroying quanta in the mode n and obey Bose commutation relations:

$$[b_n, b_{n'}^\dagger] = \delta_{nn'}. \tag{2.8}$$

The Hamiltonian of the radiation field is now

$$H_{\text{rad}} = \sum_n \hbar\Omega_n(b_n^\dagger b_n + \tfrac{1}{2}). \tag{2.9}$$

For typical junction dimensions $L_x = 10^{-1}$ cm, these modes are well separated in frequency. In what follows we will only consider the one mode closest to resonance with the Josephson frequency.

3. TUNNELING HAMILTONIAN

In this section we consider the interaction between the tunneling electrons and the electromagnetic field in the junction. Tunneling is most conveniently described by means of a Hamiltonian[13,14]:

$$H_T = \sum_{kq\sigma}(T_{kq}C_{k\sigma}^\dagger C_{q\sigma} + \text{c.c.}), \tag{3.1}$$

where $C_{k\sigma}^\dagger$, etc., is a creation operator for an electron in state (k,σ). We use the subscripts k and q to denote electron states in the left- and right-hand metals, respectively. The matrix element T_{kq} is given by

$$T_{kq} = \frac{\hbar^2}{2m}\int dx\,dy\left(\chi_q\frac{\partial}{\partial z}\phi_k^* - \phi_k^*\frac{\partial}{\partial z}\chi_q - \frac{2ei}{\hbar c}\phi_k^*\chi_q A_z\right), \tag{3.2}$$

[11] J. Swihart, J. Appl. Phys. **32**, 471 (1961).

[12] W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954).

[13] J. Bardeen, Phys. Rev. Letters **6**, 57 (1961).

[14] R. E. Prange, Phys. Rev. **131**, 1083 (1963).

when ϕ_k and χ_q are the states of the electrons and the integration is over the area of the junction. A term in the electromagnetic field has been included in (3.2). To determine the states ϕ_k and χ_q in the presence of the electromagnetic field in the junction we use a semiclassical approximation:

$$\phi_k=\phi_k^{(0)}\exp\left(\frac{-ie}{\hbar c}\int_{-\infty}^{z}A_z dz\right),\qquad \chi_q=\chi_q^{(0)}\exp\left(\frac{ie}{\hbar c}\int_{z}^{\infty}A_z dz\right),\tag{3.3}$$

where $\phi_k^{(0)}$ and $\chi_q^{(0)}$ are the states when there is no field in the junction. Substituting (3.3) in (3.2), we find

$$T_{kq}=\frac{\hbar^2}{2m}\int dx\,dy\left(\chi_q^{(0)}\frac{\partial}{\partial z}\phi_k^{(0)*}-\phi_k^{(0)*}\frac{\partial}{\partial z}\chi_q^{(0)}\right)\times\exp\left(\frac{ie}{\hbar c}\int_{-\infty}^{\infty}A_z dz\right).\tag{3.4}$$

A similar result was obtained in a different way by Ivanchenko.[15]

We replace the exponential in (3.4) by

$$\exp\left(\frac{ie}{\hbar c}\int_{-\infty}^{\infty}A_z dz\right)=1+\frac{iel}{\hbar c}A_z(x,z=0).\tag{3.5}$$

By retaining terms linear in A_z we only consider tunneling processes in which a single photon is absorbed or emitted and neglect multiple-photon processes. The expansion in (3.5) is adequate under most experimental conditions when the second term in (3.5) is about $\frac{1}{10}$ of the first. To get this estimate we have used the steady-state value of A_z obtained in Sec. 5. If the junction is very strongly coupled and the density of radiation in the cavity is large, it will become necessary to consider further terms in (3.5).

It is convenient to eliminate the properties of the superconductor as far as possible. Thus from (3.1), (3.4), and (3.5), following Ambegaokar and Baratoff,[16] the tunneling current up to terms linear in A_z is[17]

$$J_T=j_1\sin\theta+(2e/\hbar c)lA_z(x=0,z=0)j_1\times\cos\theta+J_n(V),\tag{3.6}$$

when θ is the phase difference of the order parameters in the two superconductors. The amplitude j_1 is given in Ref. 16 and for identical superconductors

$$j_1=(\pi\Delta_g/2R_0e)\tanh(\Delta_g/2kT),\tag{3.7}$$

[15] Yu. M. Ivanchenko, Zh. Eksperim. i Teor. Fiz. **52**, 1320 (1967) [English transl.: Soviet Phys.—JETP **25**, 878 (1967)].

[16] V. Ambegaokar and A. Baratoff, Phys. Rev. Letters **10**, 486 (1963); **11**, 104 (1963).

[17] In (3.6) a small correlation between the quasiparticle and pair currents is neglected.

when Δ_g is the energy gap and R_0 is the normal-state resistance of the junction. The quasiparticle current is given by the last term of (3.6) and approximately for identical superconductors[18] (with $eV<\Delta_g$)

$$J_n(V)=\frac{V}{R_0\cosh^2(\Delta_g/2kT)}\times\left(\frac{\Delta_g}{4kT}\ln\frac{\Delta_g}{eV}+\frac{2\cosh^2(\Delta_g/2kT)}{e^{\Delta_g/kT}+1}\right).\tag{3.8}$$

The logarithmic term arises from the large density of states in the superconductors. We have neglected the interaction of the field and the quasiparticles.

The second term of (3.6) arises from pair tunneling with the emission or absorption of a photon. We will only retain the energy-conserving parts of this term. This is equivalent to the rotating-wave approximation of atomic physics. Then from (2.7)

$$J_T=j_1\sin\theta+2e\sum_n T_n(b_ne^{i\theta}+b_n^\dagger e^{-i\theta})+J_n(V),\tag{3.9}$$

where

$$T_n=j_1(\pi l/\hbar\epsilon\Omega_nL_xL_y)^{1/2}.\tag{3.10}$$

Equation (3.6) arises from a coupling energy of the two superconductors

$$H_c=-(\hbar j_1/2e)[\cos\theta-(2e/\hbar c)lA_z\sin\theta].\tag{3.11}$$

Making the same rotating-wave approximation as in (3.9), we get

$$H_c=-\frac{\hbar j_1}{2e}\cos\theta+i\hbar\sum_n T_n(b_n^\dagger e^{-i\theta}-b_ne^{i\theta}),\tag{3.12}$$

where the second term describes the interaction of the tunneling pair and the electromagnetic field in the cavity and is the important term in the ac Josephson effect.

4. LANGEVIN EQUATIONS

In this section we obtain the Langevin equations describing the junction. As shown by Josephson,[2] the phase difference θ is related to the voltage V across the junction by

$$d\theta/dt=2eV/\hbar.\tag{4.1}$$

It was found in I when we considered in a detailed manner the addition and removal of pairs from the superconductors that there could be additional small terms in (4.1). These terms described the loss of phase memory of the superconductors and were only important for the small frequency pulling effect found in I. This effect will be neglected here.

[18] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksperim. i Teor. Fiz. **51**, 1535 (1966) [English transl.: Soviet Phys.—JETP **24**, 1035 (1967)].

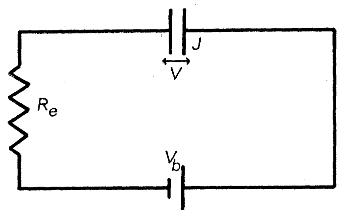

FIG. 1. Simple circuit for the ac Josephson effect. The junction is denoted by J and the voltage across the junction is V. R_e is the external resistance at temperature T_e, and V_b is the voltage of the battery.

To construct the second Langevin equation we imagine the junction connected into the simple circuit of Fig. 1. By considering the currents entering and leaving the left-hand superconductor we obtain the equation describing the time variation of the charge Q in the superconductor

$$dQ/dt=(V_b-V)/R_e-2eT(be^{i\theta}+b^\dagger e^{-i\theta})-J_n(V)+F(t). \quad (4.2)$$

The first term is due to the current entering the superconductor from the external circuit. The second and third terms are the negative of the tunneling current (3.9). We have omitted the first term of (3.9), which is rapidly varying when there is a voltage across the junction. We have only included the one mode of the electromagnetic field which has a frequency closest to $2eV/\hbar$. The noise source $F(t)$ in (4.2) has zero average value and a correlation function

$$\langle F(t_1)F(t_2)\rangle=2D_Q\delta(t_1-t_2). \quad (4.3)$$

There are two sources of noise in the circuit of Fig. 1. The external resistance R_e which is at room temperature T_e will contribute Johnson noise. The quasiparticle currents in the junction at temperature T will also contribute noise. This noise has been considered by Scalapino.[6] No noise is introduced by the coupling between the pairs and the field in (4.2). These considerations lead to the diffusion constant[19]

$$2D_Q=2kT_e/R_e+eJ_n(V)\coth(eV/2kT). \quad (4.4)$$

The first term is due to the Johnson noise in R_e and the second term is due to the quasiparticle currents in the junction. For $eV<kT$, which is usually the case, the quasiparticle noise reduces to Johnson noise in an effective resistance V/J_n at temperature T. At low temperatures when $eV\gg kT$, the quasiparticle noise has the form of shot noise and can be much larger than Johnson noise. This arises because the total voltage drop across the junction takes place in the oxide layer, i.e., effectively one mean free path.

[19] The model used in I leads to a diffusion constant of the shot-noise form in place of (4.4); i.e., $2eJ$ rather than $2kT/R_e$, where J is the dc current. Equation (4.4) is the correct result and leads to a much smaller noise component from the external circuit. The linewidth at $T=0$°K found in I was too large by a factor of 2 for this reason.

Finally, we obtain the equations obeyed by the radiation field operator b and $b^\dagger$. From (2.9) and (3.12) we find

$$\begin{aligned} db/dt&=-i[b,H_{\rm rad}+H_c]-\tfrac{1}{2}\gamma b+f(t)\\ &=(-i\Omega-\tfrac{1}{2}\gamma)b+Te^{-i\theta}+f(t), \end{aligned} \quad (4.5)$$

where Ω is the cavity mode frequency and γ is the cavity bandwidth. This is included phenomenologically and arises from the dissipation of radiation in the cavity. This could take place in a variety of ways. The second term in (4.5) comes from the interaction (3.12) with the pair tunneling current. The properties of the noise sources $f(t)$ in (4.5) have been given by Lax[9] and are

$$\begin{aligned} \langle f(t_1)f^\dagger(t_2)\rangle&=\gamma(\bar{n}+1)\delta(t_1-t_2),\\ \langle f^\dagger(t_1)f(t_2)\rangle&=\gamma\bar{n}\delta(t_1-t_2), \end{aligned} \quad (4.6)$$

where $\bar{n}=(e^{\hbar\Omega/kT}-1)^{-1}$ and is the number of blackbody photons at frequency Ω in the cavity at temperature T. In the absence of any tunneling current the cavity will have an incoherent blackbody distribution. This is ensured by the properties (4.6) of the noise sources. The noise sources also preserve the commutation relation (2.8) of the b operators.

The remainder of this paper is concerned with the solution of the Langevin equations (4.1), (4.2), and (4.5) using quasilinear methods.[20]

5. STEADY STATE

To obtain the operating point of the oscillator we neglect the noise sources in (4.2) and (4.5). We denote steady-state values by the subscript 0. The dc voltage across the junction is V_0 and from (4.1)

$$\theta_0=(2eV_0/\hbar)t=\omega_0 t. \quad (5.1)$$

From (4.5) we have $b=b_0e^{-i\omega_0 t}$, where

$$b_0=T/[i(\Omega-\omega_0)+\tfrac{1}{2}\gamma]. \quad (5.2)$$

From the second term of (4.2) the pair tunneling current is

$$J_s=2eT^2\gamma/[(\Omega-\omega_0)^2+\tfrac{1}{4}\gamma^2]. \quad (5.3)$$

The total dc current in the circuit is $J_T=J_s+J_n(V_0)$, where $J_n(V_0)$ is the quasiparticle tunneling current. The current voltage characteristic in (5.3) is a Lorentzian with a width determined by the cavity bandwidth γ. Equation (5.3) can also be written

$$J_sV_0=2eV_0\gamma|b_0|^2, \quad (5.4)$$

which is an expression of energy conservation. The left-hand side is the work done by the supercurrent and the right-hand side is the rate of dissipation of electromagnetic energy in the cavity.

[20] It was pointed out to me by M. Lax (private communication) that this procedure treats the noise sources in (4.6) in a symmetrical manner. Any errors introduced by this procedure are generally unimportant because under most experimental conditions $\bar{n}\gg 1$ and the order of noise sources is immaterial.

We define a dynamic resistance R_s for the pair current by

$$\frac{1}{R_s}=\left(\frac{\partial J_s}{\partial V}\right)_{V_0}=\frac{8e^2T^2\gamma\Delta}{\hbar(\Delta^2+\frac{1}{4}\gamma^2)^2}, \tag{5.5}$$

where $\Delta=\Omega-\omega_0$ is the detuning of the oscillator from the cavity frequency Ω. This resistance is positive if the detuning is positive.

We also define a dynamic resistance of the junction to quasiparticle currents by

$$1/R_n=[\partial J_n(V)/\partial V]_{V_0}, \tag{5.6}$$

where J_n is given by (3.8). The total dynamic resistance of the circuit in Fig. 1 is denoted by R, where

$$1/R=1/R_e+1/R_n+1/R_s. \tag{5.7}$$

It is convenient to define the resistance of the circuit to normal currents by

$$1/R_N=1/R_e+1/R_n. \tag{5.8}$$

6. SMALL OSCILLATIONS

We now linearize Eqs. (4.1), (4.2), and (4.5) around the steady operating point found in Sec. 5 by setting

$$\theta=\theta_0+\theta_1,\quad V=V_0+V_1,\quad b=b_0e^{-i\omega_0t}e^{u-i\phi}. \tag{6.1}$$

From (4.1)

$$d\theta_1/dt-2eV_1/\hbar=0. \tag{6.2}$$

From (4.5), (6.1), and (5.2)

$$d(u-i\phi)/dt=(i\Delta+\tfrac{1}{2}\gamma)(e^{-u+i(\phi-\theta_1)}-1)+(f/b_0)e^{i\omega_0t}$$
$$\simeq(i\Delta+\tfrac{1}{2}\gamma)[-u+i(\phi-\theta_1)]+(f/b_0)e^{i\omega_0t}.$$

Equating real and imaginary parts to zero gives

$$d\phi/dt+\tfrac{1}{2}\gamma(\phi-\theta_1)-\Delta u=-\mathrm{Im}[(f/b_0)e^{i\omega_0t}], \tag{6.3}$$

$$du/dt+\tfrac{1}{2}\gamma u+\Delta(\phi-\theta_1)=\mathrm{Re}[(f/b_0)e^{i\omega_0t}]. \tag{6.4}$$

In (4.2) we assume that the junction behaves like a capacitance, so that the charge Q is related to the voltage across the junction by $Q=CV$. The capacity C is approximately $L_xL_y/4\pi l\simeq10^4$ cm. Then substituting (6.1) in (4.2) and using (5.5), we get

$$dV_1/dt+V_1/R_NC+\alpha[\gamma u+2\Delta(\theta_1-\phi)]=F/C, \tag{6.5}$$

where

$$\alpha=\hbar(\Delta^2+\tfrac{1}{4}\gamma^2)/4e\Delta R_sC\gamma.$$

The four equations determining the modes of small oscillation of the system about the steady state are given by (6.2)–(6.5). These modes are driven by the noise sources in the system which appear on the right-hand sides. One special feature of these equations is that only the combination $\phi-\theta_1$ appears. Although ϕ and θ_1 separately may not be small, the difference $\phi-\theta_1$ can be small and is an appropriate variable in which to linearize. The occurrence of this difference is connected with the fact that the phase difference θ and the phase ϕ of the electromagnetic field are not determined separately but only their difference is fixed.

To determine the small modes of oscillation, we neglect the noise sources and assume that all the variables ϕ, θ_1, u, and V_1 vary like $e^{-\lambda t}$. The matrix of coefficients $A(\lambda)$ in (6.3), (6.2), (6.4), and (6.5) taken in that order is

$$A(\lambda)=\begin{bmatrix}\lambda-\frac{1}{2}\gamma & \frac{1}{2}\gamma & \Delta & 0\\ 0 & \lambda & 0 & 2e/\hbar\\ -\Delta & \Delta & \lambda-\frac{1}{2}\gamma & 0\\ 2\Delta\alpha & -2\Delta\alpha & -\alpha\gamma & \lambda-1/R_NC\end{bmatrix}. \tag{6.6}$$

The eigenvalues are the roots of the determinant of $A(\lambda)$, which is

$$\lambda F(\lambda)=0, \tag{6.7}$$

where

$$F(\lambda)=\lambda^3-\lambda^2\left(\gamma+\frac{1}{R_NC}\right)+\lambda\left[(\tfrac{1}{4}\gamma^2+\Delta^2)\left(1+\frac{1}{\gamma R_sC}\right)+\frac{\gamma}{R_NC}\right]-(\Delta^2+\tfrac{1}{4}\gamma^2)\left(\frac{1}{R_sC}+\frac{1}{R_NC}\right). \tag{6.8}$$

From (6.7) we see that the oscillator has one zero-frequency mode of oscillation. This is connected with the fact that it does not cost any energy to change the phases θ and ϕ simultaneously by the same amount. The eigenvector corresponding to the root $\lambda=0$ is

$$u=V_1=0,\quad \phi=\theta_1.$$

The other three eigenvalues are determined by $F(\lambda)=0$. In general, this equation must be solved by numerical methods. Provided the detuning Δ is positive and hence R_s is positive, the roots of (6.8) have positive real parts, indicating that the system is stable. It is possible for the last term in (6.8) to vanish when the detuning Δ, and hence R_s, is negative. The system will become unstable at this point for amplitude fluctuations. Physically we would expect the system to jump to some other mode of oscillation as we approach this point of instability.

7. LINEWIDTH OF THE RADIATION

In this section we study the phase fluctuations in the system which lead to the linewidth. We denote the cofactor of the matrix $A(\lambda)$ in (6.6) by $B_{ij}(\lambda)$. Then from (6.1)–(6.5), after eliminating all the variables except ϕ, we find that the equation satisfied by ϕ is

$$F\left(-\frac{d}{dt}\right)\frac{d\phi}{dt}=B_{11}\left(-\frac{d}{dt}\right)\mathrm{Im}[(f/b_0)e^{i\omega_0t}]-B_{13}\left(-\frac{d}{dt}\right)\mathrm{Re}[(f/b_0)e^{i\omega_0t}]-B_{14}\left(-\frac{d}{dt}\right)\frac{F}{C}. \tag{7.1}$$

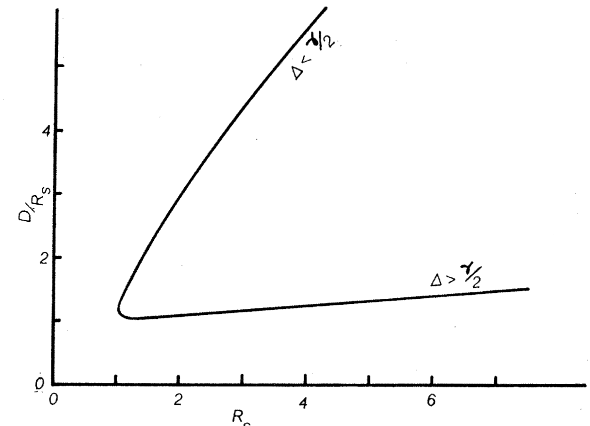

FIG. 2. Plot of D_0/R_s versus R_s at low temperatures [see Eqs. (7.6) and (5.5)]. Each quantity has been normalized to its minimum value. The magnitude of the detuning Δ is indicated on each branch.

In $F(\lambda)$ and $B(\lambda)$, λ is replaced by $-d/dt$. For purposes of investigating the narrow linewidth D of the radiation we can assume in (7.1) that the operation d/dt is of order D. D is very much smaller than the inverse relaxation times of the system γ, $(RC)^{-1}$, so that in F and B in (7.1) we need only retain the terms independent of d/dt. This leads to

$$\frac{d\phi}{dt}=\frac{R}{R_s}\,\mathrm{Im}[(f/b_0)e^{i\omega_0 t}]+\frac{R}{R_s}\frac{\Delta^2-\frac{1}{4}\gamma^2}{\Delta\gamma}\times\mathrm{Re}[(f/b_0)e^{i\omega_0 t}]+(2e/\hbar)RF. \quad (7.2)$$

This equation was given in I for the special case $R=R_s$. According to (7.2), the phase ϕ carries out a random walk. Using the properties (4.3) and (4.6) of the noise sources, we find the diffusion constant

$$D=\langle[\phi(t)-\phi(0)]^2\rangle/t$$
$$=\frac{4e^2}{\hbar^2}R^2\left(2eJ_s(2\bar{n}+1)+eJ_n(V_0)\coth\frac{eV_0}{2kT}+\frac{2kT_e}{R_e}\right). \quad (7.3)$$

The radiation line is Lorentzian and D is the full width at half-maximum in rad/sec. The first term in (7.3) is due to the cavity noise source f in (7.2) and the remaining terms come from the quasiparticle currents in the junction and the Johnson noise in the external resistance R_e. The effects of quasiparticle currents on the linewidth have been previously considered by Scalapino.[6]

Under most experimental conditions, $eV \ll kT$ and the external resistance is large, so that the last term in (7.3) can be neglected. Then[21]

$$D=(8e^3/\hbar^2)(kT/V_0)J_TR^2, \quad (7.4)$$

where J_T is the total dc current through the junction $(J_T=J_s+J_n)$ and R is total dynamic resistance $(dJ_T/dV)^{-1}$. We have assumed in (7.4) that the temperature of the blackbody distribution in the cavity and the temperature determining the quasiparticle current fluctuations are the same.

At sufficiently low temperatures when the quasiparticle currents in the junction can be neglected, we obtain the residual linewidth[22]

$$D_0=(8e^3/\hbar^2)J_sR_s^2\coth(eV_0/kT). \quad (7.5)$$

It is much simpler to consider D_0/R_s, which from (5.3) and (5.5) is

$$\frac{D_0}{R_s}=\frac{2e^2}{\hbar}\frac{\Delta^2+\frac{1}{4}\gamma^2}{\Delta}\coth\frac{eV_0}{kT}. \quad (7.6)$$

This is independent of the properties of the superconductors and only involves the cavity parameters. If these are independent of temperature, then (7.6) has a simple temperature dependence determined by the last factor. D_0/R_s has a minimum value when the detuning $\Delta=\frac{1}{2}\gamma$:

$$\left(\frac{D_0}{R_s}\right)_{\min}=\frac{2e^2\gamma}{\hbar}\coth\frac{eV_0}{kT}. \quad (7.7)$$

The dynamic resistance R_s is a minimum when $\Delta=\gamma/2\sqrt{3}$. A plot of D_0/R_s against R_s is given in Fig. 2. For $\Delta \gg \frac{1}{2}\gamma$ when R_s is large, D_0/R_s varies as $R_s^{1/3}$, and where $\Delta \ll \frac{1}{2}\gamma$ and R_s is again large, D_0/R_s varies linearly with R_s.

We now examine the temperature dependence of D/R. At low temperatures when we neglect quasiparticle currents, this is given by (7.6). As we approach T_c according to (3.7), (3.10), and (5.3), J_s and R_s^{-1} behave like $(\Delta_g \tanh\Delta_g/2kT)^2$ and become small. The linewidth is then entirely determined by the quasiparticle contribution, which becomes large close to T_c. Thus taking the limit as $T\rightarrow T_c$ in (7.4), which amounts to replacing J by V_0/R_0 and R by R_0, where R_0 is the normal-state resistance of the junction,

$$(D/R_0)_{T_c}=(8e^2/\hbar^2)kT. \quad (7.8)$$

The temperature dependence of the linewidth is shown in Fig. 3 for a Sn-O-Sn junction.

The physical origin of the first term in (7.3) involving the pair tunneling current can be understood and derived simply as follows. When the oscillator is in a steady state, each time a photon is dissipated in the cavity a pair must tunnel to replace it. Assuming that the dissipation is a random process, the pair tunneling current will also be random. We can then apply the

[21] All the quantities appearing in (7.4) are measured independently of the linewidth in Refs. 4 and 5. A detailed comparison of (7.4) with experiment has been made by Dr. W. H. Parker with satisfactory agreement.

[22] The zero-temperature linewidth given in I differs from that predicted by (7.5) by a factor 2 owing to the incorrect noise source (Ref. 19).

standard formulas of shot noise to it.[23] The current fluctuations in the pair current are

$$\langle \Delta J_s(t_1)\Delta J_s(t_2)\rangle = 2eJ_s\delta(t_1-t_2)\,, \qquad (7.9)$$

where J_s is the pair current, the factor of $2e$ comes from the charge on the pairs, and we have set the bandwidth of the noise equal to infinity. At finite temperatures when there is blackbody radiation in the cavity, pairs may tunnel in both directions with the emission or absorption of a photon. Adding the fluctuations from both currents multiplies (7.9) by $2\bar{n}+1$. The voltage fluctuations arising out of (7.9) are

$$\langle \Delta V(t_1)\Delta V(t_2)\rangle = 2eJ_sR_s^2(2\bar{n}+1)\delta(t_1-t_2)\,, \qquad (7.10)$$

where R_s is $(dJ_s/dV)^{-1}$ and we have included the factor $2\bar{n}+1$. The phase difference θ of the superconductors from (4.1) now diffuses under the influence of the voltage fluctuations and the diffusion constant is

$$\begin{aligned} D_0 &= \langle[\theta(t)-\theta(0)]^2\rangle/t \\ &= \left(\frac{2e}{\hbar}\right)^2 \frac{1}{t}\int_0^t dt_1dt_2\,\langle \Delta V(t_1)\Delta V(t_2)\rangle \\ &= (8e^3/\hbar^2)J_sR_s^2(2\bar{n}+1)\,, \end{aligned} \qquad (7.11)$$

in agreement with (7.5).

8. SPECTRUM OF VOLTAGE FLUCTUATIONS

From (6.1)–(6.5) we can also obtain the spectrum of voltage fluctuations. Eliminating all the variables except V_1 from these equations, we obtain

$$\begin{aligned} F\left(-\frac{d}{dt}\right)\frac{dV_1}{dt} &= -B_{41}\left(-\frac{d}{dt}\right)\mathrm{Im}[(f/b_0)e^{i\omega_0 t}] \\ &\quad + B_{43}\left(-\frac{d}{dt}\right)\mathrm{Re}[(f/b_0)e^{i\omega_0 t}] + B_{44}\left(-\frac{d}{dt}\right)\frac{F}{C}\,. \end{aligned} \qquad (8.1)$$

The power spectrum of V_1 is given by

$$\langle V_1(\omega)V_1(-\omega)\rangle = \int_{-\infty}^{\infty} e^{-i\omega t}\langle V_1(t)V_1(0)\rangle dt\,,$$

and we find from (8.1) for low frequencies

$$\langle V_1(\omega)V_1(-\omega)\rangle = \frac{R^2}{1+\omega^2\tau^2}\left(2eJ_s(2\bar{n}+1) + eJ_n(V_0)\coth\frac{eV_0}{2kT} + \frac{2kT_e}{R_e}\right), \qquad (8.2)$$

[23] S. O. Rice, in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, Inc., New York, 1954).

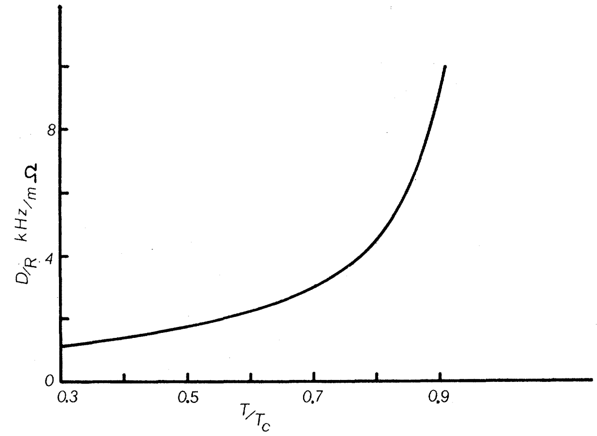

FIG. 3. Temperature dependence of D/R for a Sn-O-Sn junction. We have used the BCS value of Δ_g, $V=20\,\mu$V, $R_0=800$ mΩ, $\gamma=10^9$ sec^{-1}, $\epsilon=5$, $l=10^{-7}$ cm, $L_xL_Y=10^{-2}$ cm², and $\Delta=\frac{1}{2}\gamma$.

where the relaxation time τ is given by

$$\tau^2 = \frac{R_N^2}{(R_s+R_N)^2}\left[(\gamma^{-1}+R_sC)^2 + \frac{R_s^2\gamma^2}{R_N^2(\Delta^2+\frac{1}{4}\gamma^2)^2} - \frac{2R_sC\gamma}{\Delta^2+\frac{1}{4}\gamma^2}\left(1+\frac{R_s}{\gamma R_N^2 C}\right)\right]. \qquad (8.3)$$

This reduces to the result given in I when $R_N=\infty$. In the limit that $\gamma \gg (RC)^{-1}$, τ reduces simply to RC, the time constant of the circuit.

In this limit we can adiabatically eliminate the electromagnetic field from the problem and obtain a single circuit equation. From (4.1) and (4.5), putting $b=b'e^{-i\theta(t)}$, we have

$$db'/dt = [i(\omega-\Omega)-\tfrac{1}{2}\gamma]b' + T + fe^{i\theta}\,, \qquad (8.4)$$

where $\omega = 2eV(t)/\hbar$. For large γ we have approximately

$$b' = (T+fe^{i\theta(t)})/[i(\Omega-\omega)+\tfrac{1}{2}\gamma]. \qquad (8.5)$$

Substituting in (4.2), we find the circuit equation

$$dQ/dt = (V_b-V)/R_e - J_s(V) - J_n(V) + G(t)\,, \qquad (8.6)$$

where

$$J_s(V) = 2eT^2\gamma/[(\Omega-\omega)^2+\tfrac{1}{4}\gamma^2]$$

and depends on the instantaneous voltage. The noise source $G(t)$ in (8.6) has the property

$$\langle G(t_1)G(t_2)\rangle = 2(D_c+D_Q)\delta(t_1-t_2)\,, \qquad (8.7)$$

where D_Q is given by (4.4) and the cavity part is given by

$$2D_c = 2eJ_s(V)(2\bar{n}+1) \qquad (8.8)$$

and is appropriate for shot noise. It should be noted that the condition $\gamma \gg (RC)^{-1}$ is not always met, and then (8.6) can only be used to describe low-frequency fluctuations when $\omega\tau < 1$.

9. INTENSITY FLUCTUATIONS

The fluctuations in the number of photons in the cavity are determined by the variable

$$p=b^\dagger b-|b_0|^2\simeq 2|b_0|^2u. \tag{9.1}$$

The power spectrum of p is given by

$$\langle p(\omega)p(-\omega)\rangle=\int_{-\infty}^{\infty}e^{-i\omega t}\langle p(t)p(0)\rangle dt. \tag{9.2}$$

We will not give the complete spectrum of p but only consider the result at low temperatures, when we neglect the quasiparticle noise. Then

$$\langle p(\omega)p(-\omega)\rangle =\frac{\gamma|b_0|^2}{|F(i\omega)|^2}\{[\omega^2(\Delta^2+\tfrac{1}{4}\gamma^2)+(\omega^2-4e\Delta\alpha/\hbar)^2] \times(2\bar{n}+1)+2\Delta\omega(\omega^2-4e\Delta\alpha/\hbar)\}. \tag{9.3}$$

At low frequencies this reduces to

$$\langle p(\omega)p(-\omega)\rangle=|b_0|^2(2\bar{n}+1)/\gamma(1+\omega^2\tau^2), \tag{9.4}$$

where τ is given by (8.3).

The mean-square fluctuations in the photon number are obtained by integrating (9.3) over all frequencies. For simplicity we choose the detuning $\Delta=\frac{1}{2}\gamma$, since it is the most interesting situation. Then

$$\langle p^2(t)\rangle=\frac{|b_0|^2}{2\pi}\times\int_{-\infty}^{\infty}dx\frac{(x^2-\eta^2)^2+\frac{1}{2}x^2}{(x^2-\eta^2)^2+x^2(x^2-\frac{1}{2}-\eta^2)^2}(2\bar{n}+1), \tag{9.5}$$

when $\eta^2=(2\gamma R_sC)^{-1}$. The integral in (9.5) can be evaluated approximately for large and small η, and it is found that

$$\begin{aligned}\langle p^2(t)\rangle&=|b_0|^2(2\bar{n}+1), \quad \eta\ll1\\ &=\tfrac{1}{4}|b_0|^2(2\bar{n}+1), \quad \eta\gg1.\end{aligned} \tag{9.6}$$

This indicates that the radiation has a second moment close to that appropriate to coherent radiation.[24] For pure coherent radiation the second moment would simply be $|b_0|^2$. The intensity fluctuations calculated above would probably be difficult to measure because of the small amount of power available.

ACKNOWLEDGMENT

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[24] The second result in (9.6) was incorrectly given in I owing to the incorrect noise source (Ref. 19).