Stability Limit of the Superheated Meissner State due to Three-Dimensional Fluctuations of the Order Parameter and Vector Potential*

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We have calculated near the transition temperature T_c the upper experimental magnetic field H_u at which the metastable, superheated Meissner state becomes unstable against infinitesimally small fluctuations of the order parameter and the vector potential. For $\kappa \lesssim 1.10$, the stability limit is determined by fluctuations of infinite wavelength, and the field H_u coincides with the maximum field H_{sh} at which solutions of the Ginzburg-Landau equations cease to exist. For $\kappa > 1.10$, the stability limit is determined by fluctuations of finite wavelength, and the value of H_u is smaller than H_{sh}. For example, at $\kappa \approx 4.25$, $H_u/H_c \approx 1$, and when $\kappa \to \infty$, $H_u/H_c \approx 0.745$ (H_c is the thermodynamic critical field). For $\kappa \gtrsim 1.10$, one finds within 1% accuracy $H_u/H_c = (\frac{1}{3}\sqrt{5})[1+(2\kappa)^{-1/2}]$.

I. INTRODUCTION

THE largest magnetic field H_{sh} up to which solutions of the Ginzburg-Landau (GL) equations for the Meissner state exist has been extensively investigated for a semi-infinite superconducting half-space,[1–4] finite cylinders,[5,6] and slabs[7] of various thicknesses. These calculations did not investigate the problem of stability. Because of infinitesimally small fluctuations of the order parameter and the vector potential, the superheated Meissner state, which is metastable, might become unstable at a field H_u which is smaller than H_{sh}. This problem has been investigated by Galaĭko,[8] Takács,[9] and Kramer[10,11] for $\kappa = \infty$, and they find that $H_u = 0.745 H_c$ near T_c, where H_c is the thermodynamic critical field.[8,10] Kramer[11] has also derived a set of "variational equations" which are similar to the "perturbation equations" obtained by Christiansen and Smith[12] provided one makes $k_z = 0$, $\alpha_z = 0$, and $\epsilon = 0$ in Ref. 12. We have solved Kramer's variational equations[11] for various κ values for the Meissner state of a semi-infinite superconducting half-space and have determined the field H_u at which the metastable Meissner state becomes unstable. This should be the largest magnetic field up to which the metastable, superheated Meissner state could possibly exist. Thus H_u is the upper experimental limit. In Sec. II we review and extend Kramer's[11] (and to some extent Christiansen's[12]) derivation of the variational equations. In Sec. III we describe the numerical techniques for solving the variational equations; in Sec. IV the results are discussed, and Sec. V is devoted to conclusions.

II. VARIATIONAL EQUATIONS

The GL equations in the usual GL normalization are

$$\nabla^2 F = \kappa^2 (F^2 + \mathbf{Q}^2 - 1) F, \tag{1}$$

$$\text{curl curl } \mathbf{Q} = -F^2 \mathbf{Q}, \tag{2}$$

where the order parameter $\Psi = F(x,y,z) e^{i\varphi(x,y,z)}$; $\mathbf{Q} = \nabla \varphi / \kappa - \mathbf{A}$. $\mathbf{A}$ is the vector potential, $\kappa = \lambda/\xi$, $\lambda(T)$ is the low-field penetration depth, $\xi(T)$ is the coherence length, and $\mathbf{H} = \text{curl}\mathbf{A} = -\text{curl}\mathbf{Q}$. At the boundary surface $H = H_0$ (H_0 is the external magnetic field) and $\partial F / \partial \mathbf{n} = 0$, where $\mathbf{n}$ is normal to the surface. The variations of F and $\mathbf{Q}$, namely, δF and $\delta \mathbf{Q}$, are defined by the symbols f and $\mathbf{q}$. The second variation of the Gibbs's free energy Ω is[11,12]

$$\delta^2 \Omega = \int dV \{[3F^2 + \mathbf{Q}^2 - 1] f^2 + (\nabla f/\kappa)^2 + 4 F f \mathbf{Q} \cdot \mathbf{q} + F^2 \mathbf{q}^2 + (\text{curl}\mathbf{q})^2\}, \tag{3}$$

where the integral of Eq. (3) is to be extended over all space. When $\delta^2\Omega > 0$, the solution is stable and when $\delta^2\Omega < 0$, it is unstable. Thus the stability limit is determined by $\delta^2\Omega = 0$. In order to minimize $\delta^2\Omega$ with respect to the functions f and $\mathbf{q}$, one finds the Euler-Lagrange equations from $\delta^2\Omega$ for a fixed set of the equilibrium functions F and $\mathbf{Q}$ (H_0 and κ are assumed to be constant). This is done below when F and $\mathbf{Q}$ are specialized to the semi-infinite superconducting half-space ($x \geq 0$). We consider a semi-infinite half-space which is superconducting for $x \geq 0$. The magnetic field $\mathbf{H} = -\text{curl}\mathbf{Q}$ is parallel to the z direction and is defined by the superfluid velocity $\mathbf{Q} = (0; Q_y(x); 0)$. Because of symmetry considerations we may assume that $F = F(x)$. Following

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Kramer,[11] f and $\mathbf{q}$ are expanded in Fourier series:

$$f=\sum_{k_y\geq 0}\sum_{k_z\geq 0}\tilde{f}(k_y,k_z,x)\cos(k_y y+k_z z)\,,\qquad(4)$$

$$q_x=\sum_{k_y\geq 0}\sum_{k_z\geq 0}\tilde{q}_x(k_y,k_z,x)\cos(k_y y+k_z z+\alpha)\,,\qquad(5)$$

$$q_y=\sum_{k_y\geq 0}\sum_{k_z\geq 0}\tilde{q}_y(k_y,k_z,x)\cos(k_y y+k_z z+\beta)\,,\qquad(6)$$

$$q_z=\sum_{k_y\geq 0}\sum_{k_z\geq 0}\tilde{q}_z(k_y,k_z,x)\cos(k_y y+k_z z+\gamma)\,.\qquad(7)$$

When Eqs. (4)–(7) are substituted into Eq. (3) and $\delta^2\Omega$ is minimized with respect to α, β, and γ, one obtains $\alpha=-\frac{1}{2}\pi$, $\beta=0$, and $\gamma=0$. Then Eq. (3) reduces to (C is a positive constant)

$$\begin{aligned}\delta^2\Omega=C\sum_{k_y\geq 0}\sum_{k_z\geq 0}\int_0^\infty dx\{[3F^2+Q^2-1+(k_y^2+k_z^2)/\kappa^2]\tilde{f}^2\\+\kappa^{-2}(d\tilde{f}/dx)^2+4FQ\tilde{f}\tilde{q}_y+F^2(\tilde{q}_x^2+\tilde{q}_y^2+\tilde{q}_z^2)\\+(k_y\tilde{q}_z-k_z\tilde{q}_y)^2+(k_z\tilde{q}_x-d\tilde{q}_z/dx)^2\\+(k_y\tilde{q}_x-d\tilde{q}_y/dx)^2\}\,.\qquad(8)\end{aligned}$$

With the boundary conditions $d\tilde{f}(0)/dx=0$, $\tilde{f}(\infty)=0$, $d\tilde{q}_y(0)/dx=0$, $\tilde{q}_y(\infty)=0$, $d\tilde{q}_z(0)/dx=0$, and $\tilde{q}_z(\infty)=0$, Eq. (8) is proportional to

$$\delta^2\Omega\propto\int_0^\infty dx[\tilde{f}A+\tilde{q}_xB+\tilde{q}_yC+\tilde{q}_zD]\,,\qquad(8')$$

where $A=0$, $B=0$, $C=0$, and $D=0$ are the Euler-Lagrange equations

$$\begin{aligned}-(1/\kappa^2)(d^2\tilde{f}/dx^2)+[3F^2+Q^2-1\\+(k_y^2+k_z^2)/\kappa^2]\tilde{f}+2FQ\tilde{q}_y=0\,,\qquad(9)\end{aligned}$$

$$-k_z(d\tilde{q}_z/dx)-k_y(d\tilde{q}_y/dx)+(F^2+k_y^2+k_z^2)\tilde{q}_x=0\,,\qquad(10)$$

$$\begin{aligned}-d^2\tilde{q}_y/dx^2+k_y(d\tilde{q}_x/dx)+(F^2+k_z^2)\tilde{q}_y\\-k_yk_z\tilde{q}_z+2FQ\tilde{f}=0\,,\qquad(11)\end{aligned}$$

$$-d^2\tilde{q}_y/dx^2+k_z(d\tilde{q}_x/dx)+(F^2+k_y^2)\tilde{q}_z-k_yk_z\tilde{q}_y=0\,.\qquad(12)$$

Thus if we confine the fluctuations of f and $\mathbf{q}$ to near the surface, Eqs. (9)–(12) describe the critical fluctuations which minimizes Eq. (8) and make $\delta^2\Omega=0$ with the above stated boundary conditions.

Since the following terms in Eq. (8)

$$\int_0^\infty dx\left[k_z^2\frac{\tilde{f}^2}{\kappa^2}+F^2\tilde{q}_z^2+(k_y\tilde{q}_z-k_z\tilde{q}_y)^2+\left(k_z\tilde{q}_x-\frac{d\tilde{q}_z}{dx}\right)^2\right]\qquad(13)$$

are positive definite and k_z and $\tilde{q}_z$ are not coupled to other terms in Eq. (8), the integral (13) reaches a minimum when $k_z=0$ and $\tilde{q}_z=0$. Hence, in order to find the minimum of Eq. (8) we may put k_z and $\tilde{q}_z$ in Eq. (8) equal to zero, which means that we have translational invariance in the z direction for the largest effect of the fluctuations on the stability limit of the GL solutions F and $\mathbf{Q}$. Further, since there is no coupling between the remaining modes k_y, only a single mode contributes to the ultimate instability, and the sum in front of the integral of Eq. (8) may be omitted.

The conditions for which $\delta^2\Omega(k_y,k_z)$ [Eq. (8)] is an extremum with respect to k_z and k_y are

$$(\partial/\partial k_z)[\delta^2\Omega(k_y,k_z)]=0\,,\qquad(14)$$

$$(\partial/\partial k_y)[\delta^2\Omega(k_y,k_z)]=0\,.\qquad(15)$$

By varying Eq. (8) with respect to k_z, one finds that Eq. (14) is indeed satisfied for $k_z=0$ and $\tilde{q}_z=0$. From Eqs. (8) and (15) it follows that for $k_z=0$ and $\tilde{q}_z=0$, the following condition must be satisfied:

$$k_y\int_0^\infty\left[\frac{\tilde{f}^2}{\kappa^2}-\frac{F^2}{(F^2+k_y^2)^2}\left(\frac{d\tilde{q}_y}{dx}\right)^2\right]dx=0\,.\qquad(16)$$

In order that this extremum is a stable minimum, the following relations for $k_z=0$ and $\tilde{q}_z=0$ must be obeyed [C in Eq. (8) is equated arbitrarily to unity]:

$$\frac{\partial^2}{\partial k_z^2}(\delta^2\Omega)=2\int_0^\infty dx\left[\frac{\tilde{f}^2}{\kappa^2}+\tilde{q}_y^2+\tilde{q}_x^2\right]>0\,,\qquad(17)$$

$$\frac{\partial^2}{\partial k_y^2}(\delta^2\Omega)=8k_y^2\int_0^\infty dx\frac{F^2}{(F^2+k_y^2)^3}\left(\frac{d\tilde{q}_y}{dx}\right)^2>0\,.\qquad(18)$$

In deriving Eq. (18) the condition $dH_0/dk_y=0$ was assumed to hold for the optimum value of k_y. We come back to this condition in Sec. IV. Further, the inequality (19) must be satisfied:

$$\left[\frac{\partial^2}{\partial k_z^2}(\delta^2\Omega)\right]\left[\frac{\partial^2}{\partial k_y^2}(\delta^2\Omega)\right]-\left[\frac{\partial^2}{\partial k_y\partial k_z}(\delta^2\Omega)\right]^2>0\,.\qquad(19)$$

It can be shown that $\partial^2(\delta^2\Omega)/\partial k_y\partial k_z=0$ when $dH_0/dk_y=0$. As long as solutions for $\tilde{f}$ and $\tilde{q}_y$ exist which satisfy with the above specified boundary conditions the Euler-Lagrange equations, the inequality (19) is satisfied provided $k_y\neq 0$ and $dH_0/dk_y=0$. When $k_y=0$, the minimum of $\delta^2\Omega(k_y,0)$ with respect to k_y and k_z becomes unstable.

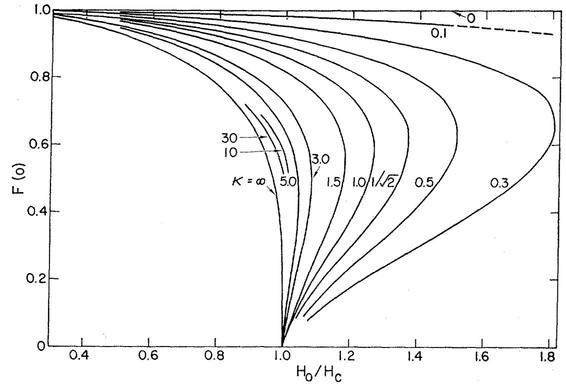

FIG. 1. Order parameter $F(0)$ at the surface of the metal is shown for the Meissner state of a semi-infinite superconducting half-space as a function of H_0/H_c, where H_0 is the applied field. The lower branch for $H_0>H_c$ is physically unstable and so is part of the upper branch near the peak for $\kappa>1.10$. The curve for $\kappa=\infty$ is calculated (Ref. 4) from $F^4(0)=1-(H_0/H_c)^2$.

With the simplifications $k_z=0$ and $\tilde{q}_z=0$, Eqs. (8)–(12) reduce to the expressions given by Kramer[11]:

$$\delta^2\Omega=\int_0^\infty dx\left\{\left[3F^2+Q^2-1+\left(\frac{k_y}{\kappa}\right)^2\right]\tilde{f}^2+\frac{1}{\kappa^2}\left(\frac{d\tilde{f}}{dx}\right)^2+4FQ\tilde{f}\tilde{q}_y+F^2\tilde{q}_y^2+\frac{F^2}{F^2+k_y^2}\left(\frac{d\tilde{q}_y}{dx}\right)^2\right\}, \quad (20)$$

$$\frac{1}{\kappa^2}\frac{d^2\tilde{f}}{dx^2}-\left[3F^2+Q^2+\left(\frac{k_y}{\kappa}\right)^2-1\right]\tilde{f}=2FQ\tilde{q}_y, \quad (21)$$

$$\frac{d}{dx}\left(\frac{F^2}{F^2+k_y^2}\frac{d\tilde{q}_y}{dx}\right)-F^2\tilde{q}_y=2FQ\tilde{f}, \quad (22)$$

$$\tilde{q}_x=\frac{k_y}{F^2+k_y^2}\frac{d\tilde{q}_y}{dx}. \quad (23)$$

It is the aim of the present investigation to find for a given value of the applied magnetic field H_0 and a constant κ value the "eigenvalue" k_y of Eqs. (21) and (22). Equations (21) and (22) are coupled equations which are linear in $\tilde{f}$ and $\tilde{q}_y$. The $F(x)$ and $Q(x)$ functions must satisfy simultaneously Eqs. (1) and (2) with the boundary conditions $dF(0)/dx=0$, $F(\infty)=1$, $dQ(0)/dx=-H_0$, and $Q(\infty)=0$. One may suspect that for a given k_y value, which we define as the optimum value k_0, the condition $dH_0/dk_y=0$ is satisfied, and that at this point $H_0(k_0)$ will have a minimum as a function of k_y. Then at $k_y=k_0$, Eq. (16) is satisfied. For a fixed κ value, the magnetic field $H_0(k_0)=H_u$ is then the smallest applied magnetic field at which the solutions of Eqs. (1) and (2) become unstable for infinitesimally small fluctuations of F and $\mathbf{Q}$. We discuss this in more detail in Sec. IV.

III. NUMERICAL PROCEDURE

The stability limit of the superconducting Meissner state was determined from solutions of the first [Eqs. (1) and (2)] and second variational equations [Eqs. (21) and (22)]. Nonlinearities in Eqs. (1) and (2) eliminated any general analytical method. Solution of these equations by numerical methods are rather laborious and time consuming owing to the sensitive two-point boundary conditions and the characteristic parameter k_y at a fixed κ value.

The method of computation used, after some preliminary studies, was a trial-and-error technique frequently used on two-point boundary and eigenvalue problems. More direct methods which make use of finite difference approximations were unsatisfactory mainly because of the zero-gradient boundary conditions required for Eqs. (21) and (22). An analog computer was used for the numerical computation. The parallel computing characteristic of the computer has an operational and economic advantage in searching for the unknown boundary conditions of F, Q, $\tilde{f}$, and $\tilde{q}_y$ and the eigenvalue k_y necessary for a unique solution at a fixed κ value.

The variational equations and the boundary conditions for the semi-infinite superconducting half-space[4] were expressed in the following form for the calculations on the computer:

$$d^2F/dx^2=\kappa^2F[F^2+Q^2-1], \quad (1')$$

$$d^2Q/dx^2=F^2Q, \quad (2')$$

$F(0)\neq0$; $F(\infty)=1$, $dF(0)/dx=0$, $Q(0)\neq0$, $Q(\infty)=0$, $dQ(0)/dx=-H_0$.

$$d^2\tilde{f}/dx^2=\kappa^2[(3F^2+Q^2+(k_y/\kappa)^2-1)\tilde{f}+2FQ\tilde{q}_y], \quad (21')$$

$$\frac{d\tilde{q}_y}{dx}=\left(1+\frac{k_y^2}{F^2}\right)\int_0^x(F^2\tilde{q}_y+2FQ\tilde{f})dt, \quad (22')$$

$\tilde{f}(0)\neq0$, $\tilde{f}(\infty)=0$, $d\tilde{f}(0)/dx=0$, $\tilde{q}_y(0)\neq0$, $\tilde{q}_y(\infty)=0$, and $d\tilde{q}_y(0)/dx=0$, where the constant of integration in Eq. (22′) is zero. The solutions of Eqs. (1′) and (2′) were obtained by selecting, through trial and error, the unknown boundary values $F(0)$ and $Q(0)$ which gave unique solutions with the above boundary conditions. For some cases the data were available for $F(0)$ and $Q(0)$ from a previous computation (Ref. 4). The second variational equations [Eqs. (21′) and (22′)] were more sensitive and difficult to solve for the unknown values $\tilde{q}_y(0)$ $\tilde{f}(0)$ and the parameter k_y for a fixed κ value. However, Eqs. (21′) and (22′) are linear in $\tilde{f}$ and $\tilde{q}_y$, so one of the unknown boundary values $\tilde{q}_y(0)$ or $\tilde{f}(0)$ was selected arbitrarily. In our computations, $\tilde{q}_y(0)$ was usually chosen to be unity and trial values of $\tilde{f}(0)$ and k_y for a given κ value were used to look for solutions.

Most of the original exploration for the solutions were made with Eqs. (21′) and (22′) expressed in the following form:

$$f=\frac{1}{3F^2+Q^2+(k_y/\kappa)^2-1}\left(\frac{1}{\kappa^2}\frac{d^2\tilde{f}}{dx^2}-2FQ\tilde{q}_y\right), \qquad (21'')$$

$$\frac{d\tilde{q}_y}{dx}=\left(1+\frac{k_y^2}{F^2}\right)\int_0^x\left[F^2\tilde{q}_y\left(1-\frac{4Q^2}{3F^2+Q^2+(k_y/\kappa)^2-1}\right)+\frac{2FQ(d^2\tilde{f}/dt^2)}{\kappa^2(3F^2+Q^2+(k_y/\kappa)^2-1)}\right]dt. \qquad (22'')$$

This reduced the survey time considerably by eliminating the sensitive boundary condition $\tilde{f}(0)$ from the trial-and-error routine. However, accuracy was somewhat impaired because integration can be performed with greater precision than differentiation on an analog computer. For relative large κ values, this error was probably insignificant since $\kappa^{-2}(d^2\tilde{f}/dx^2)$ and $\kappa^{-2}(d^2\tilde{f}/dt^2)$ were less important in comparison with the other terms in Eqs. (21″) and (22″). The calculations were repeated with Eqs. (21′) and (22′) for greater accuracy.

IV. RESULTS OF VARIATIONAL EQUATIONS

In order to find the solutions of Eqs. (21) and (22) the solutions of Eqs. (1′) and (2′) have to be known. Some of the latter solutions were computed previously.[4]

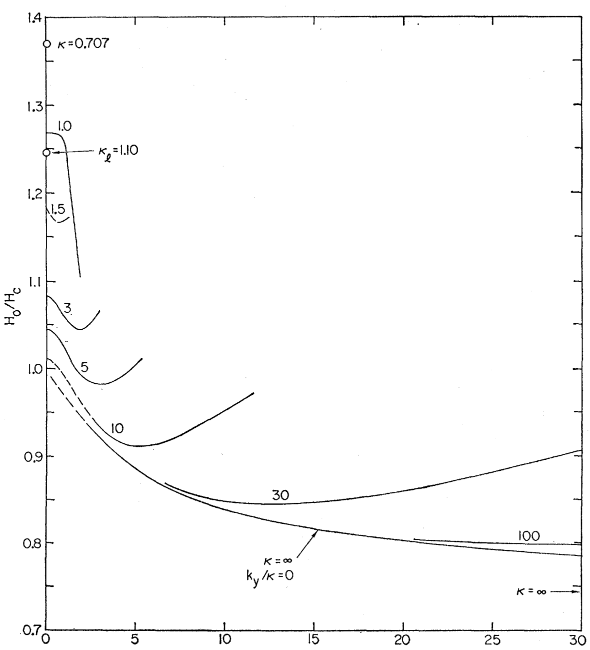

FIG. 2. "Eigenvalue" k_y of Eqs. (21) and (22) as a function of the applied field H_0 for various κ values. Uncertainties are indicated by dashed lines.

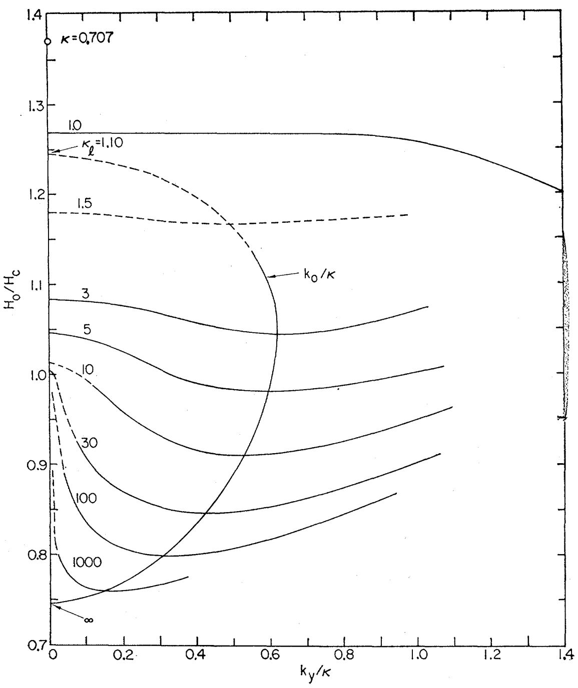

FIG. 3. Same data as in Fig. 2 except that k_y is divided by κ. In real units (unnormalized) k_y/κ becomes $k_y\xi$. The curve through the minima for κ=const determines H_u the lowest field at which the GL equations [(1′) and (2′)] become unstable for infinitesimally small fluctuations of F and Q. Uncertainties are indicated by dashed lines.

In Fig. 1 of Ref. 4 the δ/λ values are shown which in the present notation relate to the boundary value of Q by $Q(0)=H_0\delta/\lambda$, where H_0 is written in the GL normalization $(H_0/\sqrt{2}H_c)$. The corresponding $F(0)$ values have also been calculated previously[4] or can be obtained directly from Eq. (4) of Ref. 4. They are shown in Fig. 1 as a function of the unnormalized magnetic field. For magnetic fields larger than H_c there are two branches, the lower of which, one would expect, corresponds to a physically unstable situation. The magnetic field H_u which we are looking for is therefore of physical significance only if the corresponding $F(0)$ value is located on the upper branch of Fig. 1.

Figure 2 shows the "eigenvalue" k_y which satisfies Eqs. (21) and (22). For a given value of H_0 the F and Q functions were computed from Eqs. (1′) and (2′) while simultaneously Eqs. (21) and (22) were solved as described in Sec. III. Only for certain k_y values (at a fixed κ value) solutions of $\tilde{f}$ and $\tilde{q}_y$ exist which satisfy the above equations with the above boundary conditions. The curve for $\kappa=\infty$ is calculated when one assumes that Eq. (1′) reduces to $F^2+Q^2=1$, that $k_y/\kappa=0$, and that for $k_y\rightarrow\infty$ the lowest field is reached which corresponds to the minima of the curves for finite κ values. When $k_y\rightarrow\infty$ we seem to approach the limit $H_0/H_c=0.745$ which was calculated by Galaiko and by Kramer. We discuss this limit in more detail in the conclusions. For finite κ values the H_0/H_c curves have

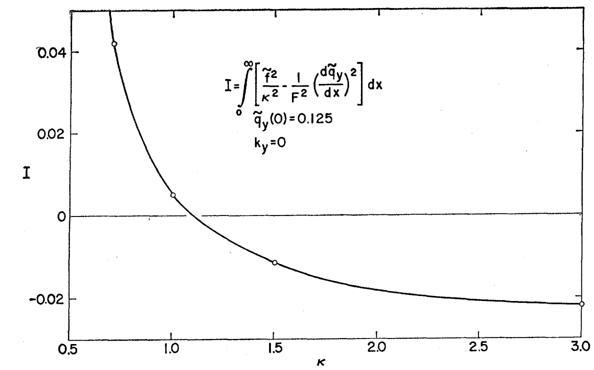

FIG. 4. Integral of Eq. (16) with $k_y=0$ for $q_y(0)=0.125$. At $\kappa=1.10$, Eq. (16) is satisfied. At this κ value $H_u=H_{\rm sh}$, where $H_{\rm sh}$ is the maximum field at which mathematical solutions of Eqs. (1′) and (2′) exist, and H_u is the lowest field at which the physical solutions of Eqs. (1′) and (2′) become unstable.

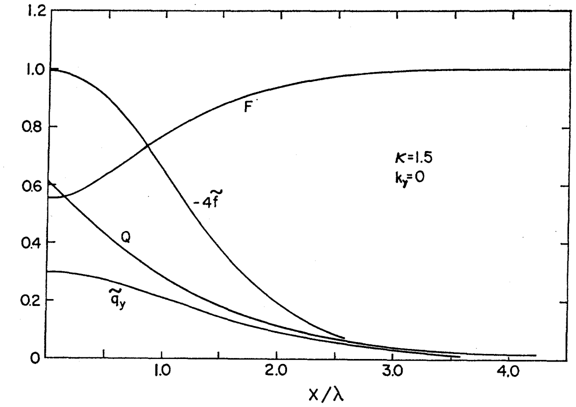

FIG. 6. $F(x)$, $Q(x)$, $\tilde{f}(k_y,x)$, and $\tilde{q}_y(k_y,x)$ for $k_y=0$ and $\kappa=1.5$. $\tilde{q}_y$ and $\tilde{f}$ are always of opposite sign.

a minimum for a finite value of k_y, where k_y is written in the GL normalization $(k_y\lambda)$. The k_y value of this minimum, k_0, becomes smaller the smaller the corresponding κ value is. When $k_y=0$, the corresponding H_0 value is $H_{\rm sh}$ as calculated in Refs. 1–7. Thus a fluctuation whose wave vector k_y is larger than zero decreases the field H_0 at which a stable solution may exist. A fluctuation with the wave vector $k_y=k_0$ determines the largest magnetic field at which a metastable solution of F and Q may exist without becoming unstable. For $\kappa=1.5$ we found solutions of $\tilde{f}$ and $\tilde{q}_y$ in the magnetic field range as indicated in Fig. 2. In this particular instance the relative error in k_y was large when matching the boundary conditions of $\tilde{f}$ and $\tilde{q}_y$ at large value of x. This is sufficient to determine H_u at $\kappa=1.5$, but the corresponding k_0 value has a large uncertainty associated with it. For $\kappa=1$ the solutions of Eqs. (21)

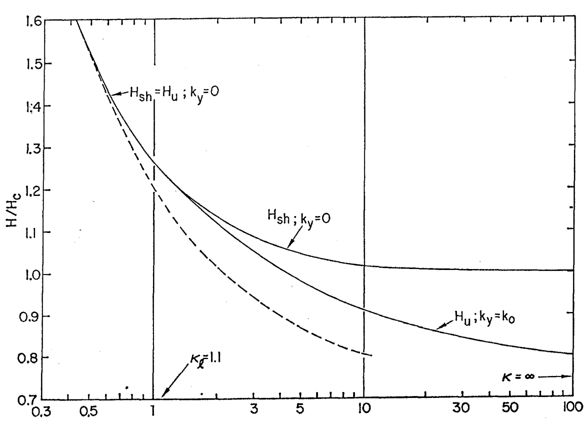

FIG. 5. $H_{\rm sh}$ is the maximum field for which Eqs. (1′) and (2′) have solutions for the Meissner state of a semi-infinite half-space. H_u is the field at which these solutions become unstable for infinitesimally small fluctuations of F and Q. At $\kappa=\kappa_l=1.10$ the value of $H_u=H_{\rm sh}$. For $\kappa\leq 1.10$, the wave vector $k_y=0$ determines the stability limit. For $\kappa>1.10$, the value of H_u is determined by the wave vector k_0. The dashed curve is Kramer's (Ref. 11) "estimate" of H_u.

and (22) are associated with the $F(0)$ values on the lower branch of Fig. 1. Hence this solution is unstable and not of physical interest. The point at $\kappa=\kappa_l=1.1$ was derived from Figs. 4 and 5 and thus when $\kappa=\kappa_l$ the magnetic field $H_u=H_{\rm sh}$.

Figure 3 shows the same data as Fig. 2, but they are plotted as a function of k_y/κ. When we write k_y/κ in real units, it becomes $k_y\xi$. At the minima the value of $k_y=k_0$ defines $H_u=H_0(k_0)$. This is the lowest field at which solutions of the GL equations [(1′) and (2′)] become unstable. The values $H_0(k_0/\kappa)/H_c$ as a function of k_0/κ are shown by the curve which connects the minima for $\kappa=$ const. This latter curve extrapolates to Kramer's value $H_0/H_c=0.745$ for $\kappa\to\infty$ $(k_y/\kappa\to 0)$ and to 1.245 for $\kappa=\kappa_l=1.10$ $(k_y\to 0)$. The latter value was determined from Eq. (16) and Figs. 4 and 5.

In Fig. 4 we show the results when the integral of Eq. (16) is solved with $k_y=0$:

$$I=\int_0^\infty\left[\left(\frac{\tilde{f}^2}{\kappa^2}\right)-\frac{1}{F^2}\left(\frac{d\tilde{q}_y}{dx}\right)^2\right]dx. \qquad (16')$$

I was solved for four κ values as indicated in Fig. 4. A smooth curve through the four points makes $I=0$ at $\kappa=\kappa_l=1.10$, which is in good agreement with Christiansen's[13] calculation of κ_l by another method. He obtains $\kappa_l=1.13$. At this κ value the field $H_u=H_{\rm sh}$.

H_u and $H_{\rm sh}$ as a function of κ are plotted in Fig. 5. The corresponding field for $\kappa=1.1$ is read from Fig. 5 and is plotted in Figs. 2 and 3. The curve $H_u(\kappa)$ is located at larger magnetic fields than that "estimated" by Kramer.[11]

To give the reader a feeling for the general shapes of the functions $F(x)$, $Q(x)$, $\tilde{f}(k_y,x)$, and $\tilde{q}_y(k_y,x)$, we show these functions in Fig. 6 for $k_y=0$ and $\kappa=1.5$ as an example. For the simplest kind of fluctuations, as shown in Fig. 6, the functions $\tilde{q}_y$ and $\tilde{f}$ must always be of opposite sign in order to satisfy Eqs. (21) and (22) with the above boundary conditions.

[13] P. F. Christiansen (private communication).

V. CONCLUSIONS

Infinitesimally small fluctuations of the order parameter and the vector potential decrease the stability limit of the superheated, metastable Meissner state for κ values larger than 1.10 below that which is calculated from the GL equations without taking fluctuations into account (Fig. 5). These fluctuations are assumed to be localized near the surface. Depending on the wave vector of the fluctuations, the field corresponding to the stability limit varies and it is reduced to its lowest value H_u when $k_y=k_0$ (see Fig. 3). Fluctuations with this wave vector k_0 determine the upper experimental field H_u which one could measure while investigating the superheated Meissner state. When $F(0)$ is located on the upper branch in Fig. 1, no solutions to Eqs. (21) and (22) were found when $\kappa\leq 1.10$ except at $H_0=H_{\mathrm{sh}}$. At this field the value of k_y is zero and therefore $H_u=H_{\mathrm{sh}}$ for $\kappa\leq 1.10$. When $\kappa\ll 1$, $H_{\mathrm{sh}}/H_c=H_u/H_c=1/(\sqrt{2}\kappa)^{1/2}$.

When $\kappa\to\infty$, the field H_u can be estimated. It follows from Eq. (1′) that $F^2(x)+Q^2(x)=1$. Then Eq. (20) with the help of Eq. (22) becomes ($k_y\to\infty$, $\kappa\to\infty$)

$$\delta^2\Omega=\int_0^\infty dx\left[2-6Q^2+\left(\frac{k_y}{\kappa}\right)^2\right]\tilde{f}^2. \qquad (20')$$

Because $\tilde{f}$ is an arbitrary fluctuation and the last term on the right-hand side of Eq. (20′) is positive definite, $\delta^2\Omega$ reaches a minimum with respect to the parameter k_y/κ when $k_y/\kappa=0$. Then Eq. (20′) ceases to be positive definite for all permissible functions $\tilde{f}$ as soon as $Q^2\geq\frac{1}{3}$ somewhere. Since $Q(x)$ has its largest value at the surface, the stability limit is reached when $Q^2(0)=\frac{1}{3}$. Then it follows from $H_u^2=Q^2(0)-\frac{1}{2}Q^4(0)$ [Eq. (16) of Ref. 11] that $H_u/H_c=\frac{1}{3}\sqrt{5}=0.745$.

For $\kappa\geq 1.10$ one can approximate our results within 1% accuracy by

$$H_u/H_c=\tfrac{1}{3}\sqrt{5}(1+(2\kappa)^{-1/2}). \qquad (24)$$

Experiments[14–23] which were performed on materials with κ values smaller than 4–5 do not contradict the above calculated fields H_{sh} for $\kappa\leq 1.1$ and H_u for $\kappa>1.10$. The ultimate stability limit of the giant vortex state[24] (for example, superconducting surface sheath on a large cylinder for $H_0>H_{c2}$) will be determined by similar considerations as discussed above, and one might expect that one will obtain similar results as one has obtained from the postulate of the critical state of the surface sheath.[25]

After completion of this work, Galaĭko[26] calculated, with a number of approximations and the assumption $k_y=k_z=0$, the stability limit of the superheated Meissner state for $\kappa\ll 1$. In this limit he reaches the same conclusions as Ref. 11 and this paper.

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