

The potential described by these matrix elements is nonlocal in configuration space. It is for this reason that the system does not obey the  $f$ -sum rule. This nonlocal behavior of the potential is analogous to that occurring with the BCS reduced Hamiltonian.

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## Quantum Liquids. II. Microscopic Theory of Liquid He<sup>3</sup>-He<sup>4</sup> Mixtures

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We calculate the equilibrium and transport properties of the mixtures using the overcomplete basis functions  $\varphi^k \psi_q$  discussed in I. The theory includes the enlarged correlation hole of a single He<sup>3</sup> and the backflow of He<sup>4</sup> atoms around it. The two quasiparticle scattering amplitude is the sum of two terms

$$V = -\alpha^2 \frac{m_4 s^2}{n} - \frac{\hbar^2 (\vec{k} \cdot \vec{q})(\vec{k}' \cdot \vec{q})}{m_4 q^2} \left( \frac{(1+\alpha)m_4 + \delta m}{m_3 + \delta m} \right)^2$$

the first from the interaction of the excess correlation holes and the second from the interaction of the backflows. Both terms were derived previously from macroscopic considerations by Bardeen, Baym, and Pines.

### I. INTRODUCTION

In this paper we calculate the low-temperature equilibrium and transport properties of the liquid He<sup>3</sup>-He<sup>4</sup> mixtures from first principles using the formalism developed in I.<sup>1</sup>

The work of Edwards *et al.*<sup>2</sup> on the phase-separation curve of the mixtures indicated that the mixtures would be thermodynamically stable down to  $T = 0^\circ\text{K}$  for He<sup>3</sup> concentrations less than 6%. Wheatley and co-workers<sup>3</sup> have measured the heat capacity, susceptibility, spin diffusion, and ther-

mal conductivity of 1.3 and 5% solutions down to  $T \sim 0.005^\circ\text{K}$ , where the  $\text{He}^3$  quasiparticles behave as a weakly interacting, degenerate Fermi gas.

On the theoretical side Bardeen, Baym, and Pines<sup>4</sup> (BBP) have assumed a phenomenological, local potential to describe the scattering of two  $\text{He}^3$  quasiparticles and have determined a two parameter potential by fitting the low-temperature spin diffusion for two concentrations. Their phenomenological potential is

$$\begin{aligned} V_q &= V_0 \cos(\beta q), \\ V_0 &= -0.075 m_4 s^2/n, \quad \beta = 3.16 \text{ \AA}, \end{aligned} \quad (1)$$

where  $m_4$ ,  $s$ , and  $n$  are the mass, sound velocity, and particle density of liquid  $\text{He}^4$ . Baym<sup>5</sup> has argued that the potential for small  $q$  is

$$V_0 \approx -\alpha^2 m_4 s^2/n, \quad (2)$$

where  $\alpha \approx 0.28$  can be found from the molar volume of the mixtures

$$v_{3-4} = v_4(1 + \alpha x). \quad (3)$$

Here  $x$  is the concentration of  $\text{He}^3$ . The physical argument leading to this interaction is as follows: The  $\text{He}^3$  atom has a smaller mass than a  $\text{He}^4$  and it occupies a slightly larger volume, proportional to  $1 + \alpha$ , than a  $\text{He}^4$  atom. A second  $\text{He}^3$  atom distinguishes the first  $\text{He}^3$  from the  $\text{He}^4$  background by the extra hole, proportional to  $\alpha$ . From the interaction of the holes one finds a potential proportional to  $\alpha^2$  times the characteristic energy of the liquid  $m_4 s^2$ . The momentum dependence of the interaction is stronger than one expects on dimensional grounds. The phenomenological potential goes through zero at  $q = 0.5 \text{ \AA}^{-1}$ , whereas the characteristic wave number of the liquid is  $q_c \equiv m_4 s/\hbar = 1.5 \text{ \AA}^{-1}$ .

According to the microscopic theory, which we present below, the scattering amplitude for two  $\text{He}^3$  quasiparticles with momenta  $k$  and  $k'$  scattering to states with momenta  $k - q$  and  $k' + q$  is approximately

$$\begin{aligned} V &= -\alpha^2 m_4 s^2/n \\ &\quad - \frac{\hbar^2 (\vec{k} \cdot \vec{q})(\vec{k}' \cdot \vec{q})}{n q^2 m_4} \left( \frac{(1 + \alpha) m_4 + \delta m}{m^*} \right)^2, \end{aligned} \quad (4)$$

where the first term is due to the change of zero-point energy of one  $\text{He}^3$  in the decreased particle density near a second  $\text{He}^3$  and the second term is the backflow-backflow interaction due to the change in the kinetic energy associated with the overlapping backflow patterns. For scattering on the Fermi surface the second term simplifies and we find

$$V = (m_4 s^2/n) [-\alpha^2 + \frac{4}{9} (q/q_c)^2]. \quad (5)$$

This result justifies the use of a local potential in BBP and explains the strong momentum dependence of the potential. BBP discussed the backflow-backflow interaction and derived the second term of (4) from macroscopic considerations for  $q \ll k$ . We will see that (4) is correct for  $q \ll q_c$  and that BBP derived the both terms in the scattering amplitude from macroscopic considerations, apparently without realizing that these are the only important terms.

The plan of the paper is as follows: In Sec. II we briefly sketch the formalism developed in I to treat strongly interacting quantum liquids. In Sec. III we discuss the structure of a single  $\text{He}^3$  quasiparticle and in Sec. IV we calculate the scattering amplitude for two quasiparticles. Finally in Sec. V we calculate the low-temperature transport properties from the scattering amplitude.

## II. FORMALISM

We consider a system of  $N_3$   $\text{He}^3$  atoms and  $N_4 = N - N_3$   $\text{He}^4$  atoms in a box of volume  $\Omega$ . The Hamiltonian of the system is

$$H = \sum_{i=1}^N \frac{-\hbar^2 \nabla_i^2}{2m_i} + \sum_{i<j} V(r_{ij}), \quad (6)$$

where  $V(r)$  is a realistic two-body potential. In a system of strongly interacting particles, such as liquid He, it is important to take into account at the outset the strong correlations between particles. One could do this by using the correlated wave functions

$$\Psi_0^k = \phi^k(1 \cdots N_3) \psi_0(1 \cdots N), \quad (7)$$

where  $\phi^k$  is a determinant of plane waves for the  $N_3$  fermions and  $\psi_0$  is the ground state of  $N$   $\text{He}^4$  atoms with

$$H_0 = \sum_{i=1}^N \frac{-\hbar^2 \nabla_i^2}{2m_4} + \sum_{i<j} V(r_{ij}), \quad (8)$$

$$H_0 \psi_0 = E_0 \psi_0.$$

Note that this wave function correlates each pair of atoms, not just the bosons. However it is a great advantage, as discussed in I, to include the collective modes of the system at the outset and to use the overcomplete basis set

$$\Psi_q^k = \phi^k(1 \cdots N_3) \psi_q(1 \cdots N), \quad (9)$$

where  $\psi_q$  is the ground state or an excited state of the pure Bose liquid

$$H_0 \psi_q = (E_0 + \omega_q) \psi_q. \quad (10)$$

Feynman<sup>6</sup> has argued that the low-lying excited states of liquid He<sup>4</sup> are density fluctuations with the wave function

$$\psi_q = \rho_q \psi_0 / (NS_q)^{1/2}, \quad (11)$$

$$\text{where } \rho_q = \sum_{i=1}^N e^{i\vec{q} \cdot \vec{r}_i} \quad (12)$$

and  $S_q$  is the liquid structure factor. The energy of these excitations is

$$\omega_q = \hbar^2 q^2 / 2m_4 S_q. \quad (13)$$

Note that all of the atoms, the fermions, and the bosons participate in the collective modes in zeroth order.

For the ground-state wave function one could use the Jastrow<sup>7</sup> pair function

$$\psi_0 = \exp\left(-\frac{1}{2} \sum_{i < j} u(r_{ij})\right) \quad (14)$$

which correlates each pair of particles and which has been used variationally<sup>8</sup> for liquid He<sup>4</sup>. The zero-point motion of the phonons can be included in this wave function by choosing the long-range part of  $u(r)$  properly.<sup>9</sup>

We will use this set (9) of basis functions and treat the off diagonal part of the Hamiltonian as a perturbation to calculate the properties of one and two He<sup>3</sup> atoms in liquid He<sup>4</sup>.

### III. SINGLE-PARTICLE PROPERTIES

We turn now to the calculation of the properties of one He<sup>3</sup> atom in liquid He<sup>4</sup>. We begin with the wave functions

$$\Psi_q^k = e^{i\vec{k} \cdot \vec{r}_1} \psi_q(1 \dots N). \quad (15)$$

There are two effects that we want to calculate. The He<sup>3</sup> is lighter than the He<sup>4</sup> atoms of the host liquid and its zero-point motion will be larger. In zeroth order, Eq. (15), we have given the He<sup>3</sup> atom the same size correlation hole as a He<sup>4</sup> atom in liquid He<sup>4</sup>. Crudely speaking each atom as liquid He<sup>4</sup> moves in a cage (correlation hole) of neighboring atoms and its zero-point energy  $\sim 14^\circ\text{K}$  is determined by the size of that cage. Lowering the density of the liquid increases the size of the hole and reduces the zero-point energy. However lowering the density also decreases the number of atoms sitting in the attractive well of the two-body potential and costs potential energy.

The balance of zero-point and potential energies determines the equilibrium density of liquid He<sup>4</sup>. Putting a He<sup>3</sup> atom into the same cage increases its zero-point energy by  $\frac{1}{3}$  without affecting the potential energy. It is then energetically favorable to increase the size of the He<sup>3</sup> correlation hole. This increased size of the correlation hole appears as a decreased particle density in the neighborhood of the He<sup>3</sup> and is simply described in terms of a coupling of the He<sup>3</sup> to the density fluctuations (phonons) of the system.

The second physical effect that we wish to describe is the backflow<sup>10</sup> of He<sup>4</sup> atoms around a moving He<sup>3</sup>. As the He<sup>3</sup> moves slowly through the liquid it adiabatically displaces He<sup>4</sup> atoms which then move in a dipolar velocity pattern around the He<sup>3</sup>. Since this velocity flow of the liquid is describable as a linear superposition of phonon excitations we can obtain the backflow from the He<sup>3</sup>-phonon interaction. We proceed now to a calculation of the He<sup>3</sup>-phonon vertex in order to find the size of the correlation hole and the effective mass due to backflow.

With the wave functions (15) the diagonal matrix elements of the Hamiltonian (6) are

$$\int \Psi_q^{k*} H \Psi_q^k = \epsilon_k + \omega_q + \frac{1}{3} \langle \text{KE} \rangle + E_0, \quad (16)$$

$$\epsilon_k = \hbar^2 k^2 / 2m_3,$$

where  $\langle \text{KE} \rangle$  is the kinetic energy per particle in liquid He<sup>4</sup>. The term  $\langle \text{KE} \rangle / 3$  arises from the mass difference and is the chemical potential difference  $\mu_{34} - \mu_{44}$  in first order where  $\mu_{34} (\mu_{44})$  is the chemical potential of a He<sup>3</sup> (He<sup>4</sup>) atom in liquid He<sup>4</sup>. The He<sup>3</sup>-phonon vertex is the sum of two terms

$$g_q^k \equiv \int \Psi_q^{k-q*} H \Psi_0^k = g_v^1 + g_\rho^1. \quad (17)$$

The first term is a coupling to the velocity field of the phonon and is readily evaluated

$$g_v^1 \equiv -(\hbar^2/m_3)(N/S_q)^{1/2} \\ \times \int \psi_0 e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)} \vec{k} \cdot \vec{\nabla}_1 \psi_0 d\tau_N \\ = \hbar^2 \vec{k} \cdot \vec{q} (1 - S_q) / 2m_3 (NS_q)^{1/2}, \quad (18)$$

$$\text{where } S_q = 1 + N \int \psi_0^2 e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)} d\tau_N. \quad (19)$$

The second term

$$g_\rho^1 \equiv -\frac{\hbar^2}{6m_4} \left(\frac{N}{S_q}\right)^{\frac{1}{2}} \int \psi_0 e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)} \nabla_1^2 \psi_0 \quad (20)$$

is a coupling to the density fluctuations of the phonon and can be expressed in terms of the pair function  $u(r)$  from (14) and the three-particle correlation function which is not available. We can obtain the vertex at long wavelength from the standard deformation potential argument. The energy of a He<sup>3</sup> atom (16) is a function of the local density  $\rho$  of the liquid and is equal to  $\langle \text{KE} \rangle_\rho / 3$  in first order. In the presence of a small static density variation the He<sup>3</sup> energy is

$$\Delta E_3 = \frac{\partial \langle \text{KE} \rangle}{3 \partial \rho} [\rho(r_1) - \rho_0]. \quad (21)$$

Expanding  $\rho(r_1)$  in phonon coordinates we find

$$\Delta E_3 \Psi_0^k = \sum_q \rho \frac{\partial \langle \text{KE} \rangle}{3 \partial \rho} \left( \frac{S_q}{N} \right)^{\frac{1}{2}} \Psi_q^{k-q} \quad (22)$$

$$\text{so that } g_\rho^1 = \rho \frac{\partial \langle \text{KE} \rangle}{3 \partial \rho} \left( \frac{S_q}{N} \right)^{\frac{1}{2}}, \quad q \ll q_c, \quad (23)$$

for small  $q$ . For larger  $q$  the vertex will be (23) multiplied by some form factor  $f_q$  which we do not know. In the absence of any better information we will use the form factor from (18),  $f_q = 1 - S_q$ . Equation (23) is correct to first order in the mass difference; in order to find the exact vertex we replace  $\langle \text{KE} \rangle / 3$  by the exact chemical potential difference.

$$g_\rho = \rho \frac{\partial (\mu_{34} - \mu_{44})}{\partial \rho} \left( \frac{S_q}{N} \right)^{\frac{1}{2}} f_q \quad (24)$$

which we write in terms of the dimensionless parameter  $\alpha$ .

$$g_\rho = \alpha m_4 s_4^2 (S_q / N)^{1/2} f_q. \quad (25)$$

We define  $\alpha$  in terms of the density derivative of the chemical potential difference rather than through Eq. (3).  $\alpha$  can be measured directly by measuring the phase-separation curve versus pressure at low temperature.

Treating the He<sup>3</sup>-phonon vertex in first order we find the wave function for a moving He<sup>3</sup> atom

$$\begin{aligned} \Psi^k &= \Psi_0^k + \sum_q \frac{g_q^k}{\epsilon_k - \epsilon_{k-q} - \omega_q} \Psi_q^{k-q} \\ &\approx \left[ e^{i\vec{k} \cdot \vec{r}_1} - \frac{\alpha}{2} \sum_q \frac{\rho_q}{NS} e^{i(\vec{k}-\vec{q}) \cdot \vec{r}_1} \right. \\ &\quad \left. - \frac{(1+\alpha)m_4}{m_3} \sum_q \frac{\vec{k} \cdot \vec{q} \rho}{q^2 N} e^{i(\vec{k}-\vec{q}) \cdot \vec{r}_1} \right] \psi_0, \end{aligned} \quad (26)$$

where we have assumed  $k$  and  $q$  small. The first term in brackets is the zeroth-order term and we

will see that the second term enlarges the correlation hole and the third term adds on the backflow.

In order to find the size of the correlation hole we first compute the density of He<sup>4</sup> atoms a distance  $r$  from the He<sup>3</sup>

$$\rho_{3-4}(r) \equiv \sum_{i>1}^N \int |\Psi^k|^2 \delta(\vec{r}_1 - \vec{r}_i - \vec{r}) d\tau_N \quad (27)$$

and obtain the volume of the correlation hole from

$$\begin{aligned} \mathcal{V}_3 &\equiv \int [1 - (\Omega/N) \rho_{3-4}(r)] d^3r \\ &= (\Omega/N) [1 + \alpha + O(\alpha^2)]. \end{aligned} \quad (28)$$

The molar volume of the mixture is then

$$\mathcal{V}_{34} \approx \mathcal{V}_4 (1 + \alpha x), \quad (29)$$

where  $x$  is the fraction of He<sup>3</sup> atoms, a result already derived by Baym.<sup>5</sup> From the experimental data on molar volumes<sup>11</sup> one finds  $\alpha \approx 0.28$ .

Feynman and Cohen<sup>10</sup> have argued that a moving impurity atom in liquid He<sup>4</sup> sets up a dipolar backflow velocity pattern in the host liquid. The Feynman-Cohen (FC) wave function for a mass 4 impurity with classical backflow is

$$\Psi_{\text{FC}}^k = \exp \left( i\vec{k} \cdot \vec{r}_1 - i \sum_{j>1} g(\vec{r}_j - \vec{r}_1) \right) \psi_0, \quad (30)$$

$$\text{where } g(\vec{r}) = \vec{k} \cdot \vec{r} / 4\pi\rho_0 r^3. \quad (31)$$

For small  $k$  we can expand the exponential and Fourier transform  $g(r)$  to find

$$\Psi_{\text{FC}}^k \approx \left[ e^{i\vec{k} \cdot \vec{r}_1} - \sum_q \frac{\vec{k} \cdot \vec{q}}{Nq^2} \rho_q e^{i(\vec{k}-\vec{q}) \cdot \vec{r}_1} \right] \psi_0. \quad (32)$$

We see that the third term of (26) is just the Feynman-Cohen backflow term. The prefactor in the backflow term of (26) is not exact. For a sphere of volume  $(1+\alpha)/n$  moving in a classical liquid the prefactor  $(1+\alpha)m_4/m_3$  is replaced by  $[(1+\alpha)m_4 + \delta m] / m^*$  where  $\delta m = m^* - m_3$ .

The energy of a slowly moving He<sup>3</sup> quasiparticle to second order is

$$\begin{aligned} E_k &= E_0 + \epsilon_k + \frac{\langle \text{KE} \rangle}{3} + \sum_q \frac{|g_q^k|^2}{\epsilon_k - \epsilon_{k-q} - \omega_q} \\ &= E_0 + \mu_{34} - \mu_{44} + \hbar^2 k^2 / 2m^*, \end{aligned} \quad (33)$$

where the chemical potential difference is

$$\begin{aligned} \mu_{34} - \mu_{44} &= \frac{\langle KE \rangle}{3} - \frac{\alpha^2}{2} m_4 s^2 \sum_q \frac{(1-S_q)^2}{N(1+\frac{4}{3}S_q)} \\ &= 4.7 - 0.7^\circ\text{K} \end{aligned} \quad (34)$$

and the effective mass is

$$\frac{m_3}{m^*} \approx 1 - \frac{4}{9} (1+\alpha)^2 \sum_q \frac{(1-S_q)^2}{N(1+\frac{4}{3}S_q)} = \frac{1}{2.8}. \quad (35)$$

The sum (=0.94) was evaluated using the theoretical  $S_q$  taken from the microscopic calculations<sup>8</sup> on the ground state of liquid He<sup>4</sup>. The experimental chemical potential difference<sup>12</sup> is 4.4°K and the effective mass<sup>3</sup> is 2.3 $m_3$ . The theoretical expressions in (33) and (34) involve the form factor  $f_q$ , which is unknown, and so the theoreti-

cal  $\mu_{34}$  and  $m^*$  must be regarded as rough estimates. In addition, we have used the approximate Feynman wave function (11) for the roton and neglected two-phonon vertices.

The calculation of the single-particle properties could be performed using variational methods instead of the perturbation methods used here. For a mass 3 impurity the calculations are more complicated than for a mass 4 impurity which Feynman and Cohen considered; one requires a knowledge of the ground-state wave function as well as the structure factor. The results of a variational procedure would be more reliable particularly for the chemical potential difference since we do not know the correct form factor in (34) for large  $q$ . For the calculation of the two-quasiparticle scattering amplitude in the next section we are interested only in small momentum transfers and the perturbational approach is simple and quite reliable.

#### IV. TWO QUASIPARTICLE SCATTERING

In this section we compute the scattering amplitude of two dressed He<sup>3</sup> quasiparticles moving in liquid He<sup>4</sup>. Consider first the unsymmetrized wave function

$$\Psi_0^{k, k'} = \exp(i\vec{k} \cdot \vec{r}_1 + i\vec{k}' \cdot \vec{r}_2) \psi_0. \quad (36)$$

We take matrix elements of the Hamiltonian between two such states to find the scattering amplitude in first order

$$V^{(1)}(k, k', q) \equiv \int \Psi_0^{k-q, k'+q} H \Psi_0^{k, k'} = \hbar^2 (\vec{k} \cdot \vec{q} - \vec{k}' \cdot \vec{q}) (1-S_q) / 2m_3 N + 2\alpha m_4 s^2 S_q f_q / N. \quad (37)$$

Since the wave functions are not orthogonal one must be careful either to Schmitt orthogonalize the two states or to subtract out the diagonal part of the Hamiltonian before performing the integral. The integrals are similar to those found in Sec. III for the He<sup>3</sup>-phonon vertex.

The second-order scattering amplitude is

$$V^{(2)}(k, k', q) \equiv \sum_n \int \Psi_0^{k-q, k'+q} H \Psi_n \int \Psi_n^* H \Psi_0^{k, k'} / (E_n - E_{k, k'}), \quad (38)$$

where the  $\Psi_n$  are a complete set of eigenstates. It was shown in I that the important intermediate states are the one-phonon excited states and the second-order scattering amplitude is just the familiar one-phonon exchange term

$$\begin{aligned} V^{(2)}(k, k', q) &= g_q^k g_q^{k'+q} / (\epsilon_k - \epsilon_{k-q} - \omega_q) + k \leftrightarrow k' \\ &= \frac{-2\alpha^2 m_4^2 s^4 S_q f_q^2}{N\omega_q} - \frac{\alpha m_4 s^2 \hbar^2 q^2 f_q (1-S_q)}{m_3 N\omega_q} - \frac{\hbar^4 (1-S_q)^2 [2(\vec{k} \cdot \vec{q})(\vec{k}' \cdot \vec{q}) + q^2(\vec{k} - \vec{k}') \cdot \vec{q}]}{4m_3^2 N S_q \omega_q}. \end{aligned} \quad (39)$$

For small  $k$  and  $q$  and to lowest order in the mass ratio  $m_4/m_3$  the first-order scattering amplitude cancels the second term and part of the third term in (39). We find

$$V(k, k', q) \approx -\alpha^2 m_4 s^2 / N - \hbar^2 m_4 (\vec{k} \cdot \vec{q})(\vec{k}' \cdot \vec{q}) / N m_3^2 q^2. \quad (40)$$

The two terms in the scattering amplitude have a simple physical interpretation. The liquid-particle density in the neighborhood of a He<sup>3</sup> atom is lower than that of the host liquid; about  $\frac{1}{3}$  of a particle is missing. The zero-point energy of a He<sup>3</sup> atom is proportional to the local particle density and is lower in the neighborhood of another He<sup>3</sup>. This leads to an attractive interaction at long wavelengths equal to

$-\alpha^2 m_4 s^2/N$ , a result first derived by Baym.<sup>5</sup> The second term in (40) is proportional to the velocity of both quasiparticles and is of a dipole-dipole form. This term is just the backflow-backflow interaction – the change of the kinetic energy of the host liquid due to the overlap of the backflow patterns. In a classical liquid this energy is proportional to the square of the volume of the correlation hole and to the actual velocities of the quasiparticles and is

$$-(\hbar^2/N)(\vec{k} \cdot \vec{q})(\vec{k}' \cdot \vec{q})/q^2 (1/m_4) \{[(1+\alpha)m_4 + \delta m]/(m_3 + \delta m)\}^2. \quad (41)$$

In BBP this term was derived from macroscopic considerations for the quantum liquid for  $q \ll k$ . Our result (40) is approximate because we have not used fully dressed vertices; we see, however, that the analytic expression for the backflow term is valid for  $q \ll q_c$ .

There are several corrections to (40). Each of the terms of (40) should be multiplied by a form factor, roughly  $(1 - S_q^2)$ . However, the momentum transfers in the mixtures are small enough that the form factor is unimportant. The potential (40) is not very weak on an absolute scale; the multiple scattering corrections reduce the scattering amplitude by about 10% at long wavelength. Of course, when one uses properly symmetrized wave functions one finds

$$V_{\uparrow\downarrow}(k, k', q) = V(k, k', q), \quad V_{\uparrow\uparrow}(k, k', q) = V(k, k', q) - V(k, k', k - k' - q). \quad (42)$$

The scattering amplitude of two dressed quasiparticles is actually (40) multiplied by the square of the wave-function renormalization factor.<sup>13</sup>

$$Z_{p_F} = 1 + \partial \Sigma(p_F \omega) / \partial \omega, \quad (43)$$

where  $\Sigma$  is the single-particle self-energy. For the mixtures the self-energy varies with energy over a range characteristic of a typical phonon frequency and one finds

$$Z_{p_F} = 1 + O(v_F/s). \quad (44)$$

This renormalization can be neglected.

## V. TRANSPORT PROPERTIES

In this section we will compute those transport properties of the mixtures which are dominated by He<sup>3</sup>-He<sup>3</sup> scattering. Whealey and co-workers<sup>3</sup> have measured the thermal conductivity and the spin-diffusion coefficient in the low-temperature (quantum) limit for solutions of 1.3 and 5% He<sup>3</sup>. Approximate solutions of the Boltzman equation in this limit have been given by Abrikosov and Khalatnikov<sup>14</sup> and by Hone<sup>15</sup> for arbitrary scattering amplitudes. More recently exact solutions have become available.<sup>16</sup> It is a simple matter to perform the average over the scattering probability using (40) and obtain these low-temperature transport coefficients versus concentration.

The expressions for spin-diffusion coefficient  $D$ , thermal conductivity  $k$ , and viscosity  $\eta$  at low temperature are

$$\begin{aligned} (DT^2)^{-1} &= [3m^*{}^5 k_B^2 C^{-1} (\lambda_D) / 16\pi^2 \hbar^8 k_F^2 (1 + F_0^a)] \langle W_{\uparrow\downarrow} \sin^2 \frac{1}{2} \theta (1 - \cos \varphi) / \cos \frac{1}{2} \theta \rangle, \\ (KT)^{-1} &= [3m^*{}^4 H^{-1} (\lambda_K) / 4\pi^2 \hbar^6 k_F^3] \langle W \sin^2 \frac{1}{2} \theta / \cos \frac{1}{2} \theta \rangle, \\ (\eta T^2)^{-1} &= [45k_B^2 m^*{}^4 C^{-1} (\lambda_\eta) / \hbar^8 k_F^5] \langle W \sin^4 \frac{1}{2} \theta \sin^2 \varphi / \cos \frac{1}{2} \theta \rangle, \end{aligned} \quad (45)$$

where  $k_B$  is the Boltzmann constant,  $k_F$  is the Fermi wave number, and  $(1 + F_0^a)$  is the susceptibility enhancement factor. With the correction factors  $H$  and  $C$  given by Brooker and Sykes<sup>16</sup> these solutions are exact. We will use the approximate solutions which one finds variationally<sup>17</sup>:  $H = \frac{5}{12}$ ,  $C = \frac{3}{4}$ . The scattering probabilities are

$$W_{\uparrow\downarrow}(k, k', q) = (2\pi/\hbar) |V_{\uparrow\downarrow}(k, k', q)|^2, \quad W(k, k', q) = \frac{1}{2} W_{\uparrow\downarrow}(k, k', q) + \frac{1}{4} W_{\uparrow\uparrow}(k, k', q). \quad (46)$$

At low temperatures the scattering is restricted to the Fermi surface and the average in (45) is taken over the angle  $\theta$  between the initial momenta and over the angle  $\varphi$  between the plane of the initial momenta and the plane of the final momenta. Specifically

$$\langle \dots \rangle \equiv \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} (4\pi)^{-1} d\varphi \dots \quad (47)$$

For scattering on the Fermi surface we have  $\vec{k} \cdot \vec{q} = -\vec{k}' \cdot \vec{q} = q^2/2$ , so that the scattering amplitude (40) is a function of  $q$  only. We find

$$W_{\uparrow\downarrow} = (2\pi\alpha^4 m_4^2 s^4 / \hbar n^2) [1 - \beta(q/2k_F)^2]^2, \quad (48)$$

$$\text{where } \beta = (\hbar k_F / \alpha m_3 s)^2 \quad (49)$$

$$\text{and } q^2 = 2k_F^2 \sin^2 \frac{1}{2} \theta (1 - \cos\varphi). \quad (50)$$

With this simple form for the scattering probabilities the angular averages in (45) are easily performed and we find

$$(DT^2)^{-1} = A_D \alpha^4 (1 - \frac{6}{5}\beta + \frac{3}{7}\beta^2), \quad (KT)^{-1} = A_K \alpha^4 (1 - \frac{4}{5}\beta + \frac{13}{35}\beta^2), \quad (\eta T^2)^{-1} = A_\eta \alpha^4 (1 - \frac{6}{7}\beta + \frac{1}{3}\beta^2), \quad (51)$$

$$\text{where } A_D = \frac{2m^*{}^5 k_B^2 m_4^2 s^4}{3\pi \hbar^9 k_F^2 (1+F_0) \alpha n^2}, \quad A_K = \frac{12m^*{}^4 m_4^2 s^4}{5\pi \hbar^7 k_F^3 n^2}, \quad A_\eta = \frac{32\pi k_B^2 m^*{}^4 m_4^2 s^4}{\hbar^9 k_F^5 n^2}. \quad (52)$$

Values of the  $A$ 's and the experimental dimensionless scattering probabilities [i.e.,  $(DT^2 A_D)^{-1}$ ] are given in Table I. The theoretical scattering probabilities computed from (51) with  $\alpha = 0.28$  are about a factor of 2 too large. Since  $\alpha$  is known experimentally from (29) only within say  $\pm 30\%$  we have taken the liberty of adjusting  $\alpha$  to obtain agreement with one transport coefficient. The theoretical scattering probabilities with  $\alpha = 0.24$  are given in Table I. The agreement with two transport coefficients measured for two concentrations is reasonably good but not perfect.

The low-temperature transport properties sample the scattering amplitude for  $q < 2k_F$  ( $= 0.6 \text{ \AA}^{-1}$  for a 5% solution). Ebner<sup>18</sup> has calculated the spin-diffusion coefficient at high temperature  $T \gg T_F$  and found an empirical (local) potential which fits the measured spin-diffusion coefficient. For  $T = 1^\circ\text{K}$  the important momentum transfers are of order  $1.0 \text{ \AA}^{-1}$  and we can compare our theoretical scattering amplitude with Ebner's empirical potential. If we make the local approximation ( $\vec{k} \cdot \vec{q} = -\vec{k}' \cdot \vec{q} = q^2/2$ ) in (40) and include the form factor  $1 - S_q^2$  we find

$$V_q \approx (m_4 s^2 / N) [-\alpha^2 + \frac{4}{9}(q/q_c)^2 - \frac{1}{9}(q/q_c)^4] \quad (53)$$

which is about 80% larger than Ebner's empirical potential at  $q = 1 \text{ \AA}^{-1}$ . However, for these large momenta the scattering has become so strong that multiple scattering corrections are important. We believe that the neglect of multiple scattering corrections is responsible for this discrepancy. Multiple scattering corrections will introduce an explicit velocity dependence into the scattering probability on the Fermi surface (48) so that the local potential will no longer be correct for that case.

TABLE I. A comparison of the dimensionless transport coefficients with experiment.

	1.3% solution		5% solution	
	Theory	Experiment	Theory	Experiment
$(A_D DT^2)^{-1}$	0.0015	0.0015	0.00053	0.00061
$(A_K KT)^{-1}$	0.0022	0.0023	0.0020	0.0036

## VI. CONCLUSIONS

We have applied the strong interactions formalism presented earlier to calculating the equilibrium and transport properties of dilute solutions of He<sup>3</sup> in He<sup>4</sup> at low temperature. The chemical potential difference and effective mass cannot be calculated accurately. However the two-quasiparticle

scattering amplitudes for small momentum transfer are reliable and provide reasonably good agreement with the low-temperature transport coefficients. The physical picture of a He<sup>3</sup> quasiparticle is clear: The He<sup>3</sup> carves out a correlation hole in the liquid a little larger than the He<sup>4</sup> correlation hole, and the moving He<sup>3</sup> is surrounded by a backflow of He<sup>4</sup> atoms. There are corre-

spondingly two contributions to the scattering amplitude, one arising from the interactions of the excess holes and another from the interaction of the backflows. BBP derived both these terms for small  $q$  from macroscopic considerations. There is no residual (screened)  $\text{He}^3$ - $\text{He}^3$  interaction in this picture.

We have justified the use in BBP of a local potential at low temperatures. However, one should be able to distinguish between the local and non-local forms of the scattering amplitude by comparing with spin-diffusion measurements at higher temperature. It turns out that the interesting region of temperature is  $T \sim 0.2^\circ\text{K}$  which is of the order of the Fermi temperature, a difficult regime for calculation. However, such calculations are in progress and will be reported separately.

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The author overlooked an earlier letter by Massey and Woo<sup>20</sup> which used similar techniques to those reported here. Recently Woo, Tan, and Massey<sup>21</sup> have developed the method further. The author would like to thank Professor Woo for a preprint of Ref. 21.

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