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Equivalence of the Coulomb Gauge and the Reformulated Lorentz Gauge

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A reformulated version of quantum electrodynamics in the Lorentz gauge is shown to be identical, in all of its physical consequences, to quantum electrodynamics in the Coulomb gauge. The reformulated Lorentz gauge has previously been shown to differ from the usual version of the Lorentz gauge in some important aspects.

I. INTRODUCTION

In an earlier paper[1] we called attention to the fact that the subsidiary condition which is generally used to define the physical states in the Lorentz gauge[2] cannot properly be applied to a theory of photons interacting with charged particles. This condition,[3]

$$\chi^{(+)}(x)|n\rangle=0\,, \tag{1}$$

in which the frequency is defined in the interaction picture, has the property that its validity at one time, together with the equations of motion, is inconsistent with its subsequent validity at later times. In the course of time, outgoing scattering states that were chosen to be in the physical space in the remote past, drift into the unphysical space where unphysical photons exist and where Maxwell's equations do not hold; however, asymptotically as $t\rightarrow\infty$, these scattering states withdraw into the physical space again.

Apart from defining a set of states which do not always remain in the physical space, Eq. (1) in the presence of interactions is not an invariant equation, because the operator $\chi^{(+)}$ is not the invariant positive-frequency part of $\partial A_\mu/\partial x_\mu$. In spite of these two difficulties with the usual subsidiary condition, the S-matrix elements for quantum electrodynamics that follow from it have been shown to be correct,[4] except perhaps in the case of a certain class of strongly interacting charged particles.

In Ref. 1 we developed a scattered formalism for states chosen by the subsidiary condition[5]

$$\Omega^{(+)}(x)|\nu\rangle=0\,, \tag{2}$$

where $\Omega^{(+)}(x)$ is given by

$$\Omega^{(+)}(x)=\chi^{(+)}(x)+\tfrac{1}{2}i\int d\mathbf{y}\;\mathfrak{D}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y},x_0)\,,$$

and where

$$\mathfrak{D}(\mathbf{x}-\mathbf{y})=(2\pi)^{-3}\int d\mathbf{k}\;e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}/|\mathbf{k}|\,.$$

$\Omega^{(+)}$ is the invariant positive-frequency part of the operator $\partial A_\mu/\partial x_\mu$ and retains this role in the presence of any additional interactions among the various species of charged particles[6] or between the charged particles and other neutral ones, provided $\partial j_\mu/\partial x_\mu=0$ is preserved by these interactions. This subsidiary condition is therefore preserved, so that if it holds at any one time it must hold forever after.

The states $|\nu\rangle$, which obey the subsidiary condition $\Omega^{(+)}(x)|\nu\rangle=0$, are not eigenstates of the usual "free-field" Hamiltonian H_0 but of $\mathcal{H}_0$ given by

$$\mathcal{H}_0=e^{-D}H_0e^{D},$$

where D is given by

$$D=-\tfrac{1}{2}\sum_{\mathbf{k}}k^{-3}[a_{k,R}\rho(-k)-a_{k,Q}{}^{\dagger}\rho(k)]$$
$$=i\int d\mathbf{x}d\mathbf{y}\;\rho(\mathbf{y})[\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{y})+i\Pi_4(\mathbf{y})](8\pi|\mathbf{x}-\mathbf{y}|)^{-1}\,,$$

so that

$$\mathcal{H}_0|\nu\rangle=E_n|\nu\rangle \quad\text{and}\quad |\nu\rangle=e^{-D}|n\rangle\,.$$

The interaction Hamiltonian which governs the time translations of the states in *their own* interaction picture is given by $\mathcal{H}_1=H_1+H_0-\mathcal{H}_0$.

The application of iteration methods to the solution of quantum electrodynamics problems in this theory leads to the same formal series solution as is the case in the conventional Lorentz gauge, except that the unperturbed-state vectors $|\nu\rangle$ and the perturbing Hamiltonian $\mathcal{H}_1$ appear in place of $|n\rangle$ and H_1, respectively. Moreover, the typical matrix element $\langle\nu'^{\star}|\mathcal{H}_1|\nu\rangle$

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[1] K. Haller and L. F. Landovitz, Phys. Rev. **171**, 1749 (1968).
[2] See, for example, A. I. Akhieser and C. B. Berestetskii, *Quantum Electrodynamics* (Wiley-Interscience, Inc., New York, 1965).
[3] The notation here is as that of Ref. 1.
[4] See Ref. 1, Sec. V.
[5] K. Bleuler, Helv. Phys. Acta **23**, 167 (1950); for other references see Ref. 1, footnote 2.
[6] See Ref. 1, Sec. III.

can be written

$$\langle n'^{\star}| e^{D}\mathcal{H}_1 e^{-D}| n\rangle,$$

which in turn can be written[7]

$$\langle n'^{\star}| e^{D}\mathcal{H}_1 e^{-D}| n\rangle = \langle n'^{\star}|\hat{H}_1| n\rangle,$$

where

$$\hat{H}_1 = e^{D}\mathcal{H}_1 e^{-D} = H_1 - [H_1, D] - [H_0, D]. \quad (3)$$

From the foregoing, it follows that the sole difference in the dynamical content between the two formulations of the Lorentz gauge lies in the appearance of the interaction Hamiltonian $\hat{H}_1$ in the new formulation, instead of H_1 which appears in the usual formulation. This difference, however, is not trivial and has observable consequences.

It has previously been shown[8] that the substitution of Eq. (2) for Eq. (1) changes the values of the off-shell scattering transition amplitudes in quantum electrodynamics, although the on-shell values remain unaltered. Although scattering phenomena depend upon on-shall transition amplitudes only, other temporally nonadiabatic effects do involve off-shell scattering transition amplitudes and the two formulations of the Lorentz gauge would lead to different predictions in these cases.[9]

II. LORENTZ AND COULOMB GAUGES

Since the two formulations of the Lorentz gauge differ from each other in some important respects, it is crucial to discover which of the two is equivalent to the Coulomb gauge and thus preserves gauge invariance. We will demonstrate that it is the new reformulation of the Lorentz gauge which gives results identical to those of the Coulomb gauge.

We note that the new reformulated Lorentz-gauge theory is characterized by the "free-field" Hamiltonian $\mathcal{H}_0$, the interaction Hamiltonian $\mathcal{H}_1$, and the basis set of states $|\nu\rangle$ for which $(\mathcal{H}_0 - E_n)|\nu\rangle = 0$. A trivial pseudo-unitary transformation carried out on this theory gives us a set of states $|n\rangle = |\nu\rangle$, a "free-field" Hamiltonian $\hat{H}_0 = e^{D}\mathcal{H}_0 e^{-D} = H_0$, and an interaction Hamiltonian $\hat{H}_1 = e^{D}\mathcal{H}_1 e^{-D}$, where[10] $\hat{H}_1 \neq H_1$. This pseudo-unitary transformation returns us to a represenstation in which the "unperturbed" states are the usual noninteracting "bare" electron and photon states (the latter including transverse, "Q"- and "R"-type photons).

The interaction Hamiltonian $\hat{H}_1$, in this representation, is given by

$$\hat{H}_1 = H_{1,T} + H_C + H_{QR}, \quad (4)$$

where

$$H_{1,T} = -\sum_{\mathbf{k}} (2k)^{-1/2} [a_{\mathbf{k},\epsilon(i)} \mathbf{J}(-\mathbf{k}) \cdot \boldsymbol{\varepsilon}(i) + a_{\mathbf{k},\epsilon(i)}^{\dagger} \mathbf{J}(\mathbf{k}) \cdot \boldsymbol{\varepsilon}(i)],$$

$$H_C = \int d\mathbf{x} d\mathbf{y}\, \rho(\mathbf{x}) \rho(\mathbf{y}) (8\pi |\mathbf{x}-\mathbf{y}|)^{-1},$$

$$H_{QR} = -\sum_{\mathbf{k}} \tfrac{1}{2} (k)^{-1/2} \{ a_{\mathbf{k},Q} [\rho(-\mathbf{k}) + \mathbf{k}\cdot\mathbf{J}(-\mathbf{k})/|\mathbf{k}|] + a_{\mathbf{k},R}^{\dagger} [\rho(\mathbf{k}) + \mathbf{k}\cdot\mathbf{J}(\mathbf{k})/|\mathbf{k}|] \}.$$

Of these, $H_{1,T} + H_C$ is precisely what the interaction Hamiltonian in the Coulomb (or radiation) gauge would be.[11] In the Coulomb gauge, the transversely polarized photon states would be the only ones in the Hilbert space, and longitudinal or timelike photons would neither arise in initial or final states, nor would they appear in intermediate states as parts of the unit operator $|\lambda\rangle\langle\lambda|$, where $|\lambda\rangle$ denotes a complete set of states in the space.

In the reformulated Lorentz gauge, H_{QR} mediates some unphysical transitions, such as scattering to "R"-type ket states. These photon states are not transverse, yet they obey the subsidiary condition; they carry neither probability value nor energy momentum, since they have zero norms in the indefinite metric space.

H_{QR}, however, can have *no effect on any physical process*. This is because matrix elements of products of any number of H_{QR} vertices, and any number of other terms that are free of nontransverse photon operators, vanish when evaluated between physically observable states (i.e., states that do not contain any nontransverse photons). For example, a term such as

$$\mathcal{V} = U_1 H_{QR} U_2 H_{QR} U_3 \cdots U_{n-1} H_{QR} U_n,$$

where the U's contain no Q- or R-type photon operators, satisfies $\langle b'^{\star}|\mathcal{V}|b\rangle = 0$, where $|b\rangle$ and $|b'\rangle$ are physically observable states.

That this is the case follows immediately from the fact that $\mathcal{V}$-type terms already are normally ordered in Q- and R-photon operators. They can therefore *never* contribute to any virtual process in which longitudinal or timelike photons appear only in intermediate states but not in initial or final states. The only process that H_{QR} can mediate when acting on physically observable states is the creation of R-type photons. Other H_{QR} vertices can create further R-type but never any Q-type photons. Most importantly, no H_{QR} vertex can ever annihilate R-type photons created by other H_{QR} vertices. The matrix elements of $\mathcal{V}$-type terms, between physically observable states, must therefore vanish.

Note that even if the set of states $|b\rangle$ is enlarged from the set of physically observable states to the set of all

[7] See Ref. 1, Sec. IV.

[8] See Ref. 1, Sec. IV.

[9] K. Haller and L. F. Landovitz, Phys. Rev. Letters **22**, 245 (1969).

[10] See Ref. 1, footnote 6.

[11] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1954), 3rd ed., Sec. 13.

physical states (i.e., all those states $|n\rangle$ for which $e^{D}\Omega^{(+)}e^{-D}|n\rangle=\chi^{(+)}|n\rangle=0$), matrix elements such as $\langle b'^{\star}|\mathcal{V}|b\rangle$ still vanish. This result, though true, is not important to the question under discussion in this paper.

In any matrix element that arises in the calculation of scattering transition amplitudes or bound-state energy shifts, only the $H_{1,T}$ or H_C part of $\hat{H}$ can appear as a vertex, since the appearance of only a single $H_{Q,R}$ vertex in a matrix element guarantees that the matrix element vanishes. Therefore, in the new formulation of the Lorentz gauge, based upon the subsidiary condition Eq. (2), the interaction Hamiltonian is composed of a part $H_{1,T}+H_C$, which is identical to the interaction Hamiltonian in the Coulomb gauge, and a part H_{QR} which in principle can never contribute to any physical process. In fact, in calculations of physical quantities H_{QR} can be entirely dropped. Once the interaction Hamiltonian (for physical processes) has been reduced to $H_{1,T}+H_C$, not even virtual transitions from observable states to states containing nontransverse photons are possible, and these latter may be entirely eliminated from the spectrum of unperturbed intermediate states. The result is that the computational procedure in the new formulation of the Lorentz gauge is wholly identical to that of the Coulomb gauge.

It is interesting to note that this result is made possible by the fact that the photon annihilation and creation parts of H_{QR} commute with each other. This is normally not allowed since it keeps the Hamiltonian from being Hermitian. This is indeed the case here, and $\hat{H}_1^{\dagger}\neq\hat{H}_1$. However, in this indefinite metric space, $\hat{H}_1$ should not be Hermitian but self-adjoint in the indefinite metric space; i.e., it should satisfy $\hat{H}_1^{\star}=\hat{H}_1$, which it does.

Erratum

Decays of Odd-Parity Baryon Resonances, J. C. CARTER AND M. E. M. HEAD [Phys. Rev. **176**, 1808 (1968)]. On p. 1810 the following entries in Table I should read:

	Observed resonance		Calculated branching ratios
$Y=1,\ I=\frac{1}{2}$	$N^*(1518)$	$N_{\frac{3}{2}}^*\pi/N\pi$	2
$Y=0,\ I=1$	$Y^*(1660)$	$\Sigma\pi/\Lambda\pi$	20
		$Y_1^*\pi/\Lambda\pi$	6.2
$Y=0,\ I=0$		$\bar{K}N/Y_1^*\pi$	1.3
		$\bar{K}N/\Lambda\eta$	21
$Y=-1,\ I=\frac{1}{2}$		$\bar{K}\Lambda/\bar{K}\Sigma$	0.28
		$\Xi^*\pi/\bar{K}\Lambda$	1.8

The ratios $N^*\pi/N\eta$ for the $N^*(1518)$ are in better agreement with experiment with these corrections.

In column 3 of Table II, $\Lambda^*\pi/\Lambda\pi$ should be $\Sigma^*\pi/\Lambda\pi$, and $\Lambda^*\pi/\Sigma\pi$ should be $\Sigma\pi/\Sigma^*\pi$.