Pion-Mass Extrapolations and Partial Conservation of Axial-Vector Current*

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It is proposed, as suggested by a recent study, that the observed 10% corrections to the Goldberger-Treiman relation cannot be accounted for on the basis of unsubtracted dispersion relations. To incorporate the possibility of a small subtraction in the matrix elements of the divergence of the axial-vector current, a supersmooth pion field is defined from the weak interactions by $i\partial_\mu A_\mu^{(+)}(x)=(2Mg_A\mu^2/\sqrt{2}g)\pi^{(+)}(x)-\Delta f_\pi\Box\pi^{(+)}(x)$, with $\Delta=1-2Mg_A/\sqrt{2}gf_\pi\simeq+0.1$ representing the 10% correction. With this definition and the supersmoothness hypothesis the extrapolation of exact current-algebra threshold theorems for weak amplitudes to relate hadron amplitudes is relatively simple and eliminates a major ambiguity in performing such extrapolations in the virtual pion mass.

IN this paper, we propose a simple method for extrapolating amplitudes in the external pion mass variable by a suitable modification of the conventional definition of a smooth extrapolating pion field. It is generally useful to develop such a method, since the techniques of current algebra enable one to prove exact low-energy theorems only as the pion momentum $q_\mu\to 0$. While, in principle, it is possible to test these results by neutrino experiments, in practice one extrapolates the amplitude from $q^2=0$ to $q^2=\mu^2$, the pion mass shell, where it is simply related to a physical pion amplitude and more easily compared with existing experimental data. What is crucial to such a procedure is the smoothness assumption, which states that a suitable amplitude does not change very much from $q^2=0$ to $q^2=\mu^2$, and which provides the physical content in the application of the PCAC (partial conservation of axial-vector current) hypothesis.[1] Since we are here interested in examining precisely how much the amplitude changes, we will reexamine the usual smoothness hypothesis, which has been primarily based on the success of the Goldberger-Treiman relation for π^+ decay.

The Goldberger-Treiman relation can be obtained[2] by considering the matrix elements of the divergence of the axial-vector current between nucleon states,

$$\langle p'|i\partial_\mu A_\mu^a(0)|p\rangle=\bar{u}(p')i\gamma_5\tau^a D(q^2)u(p),$$

which at $q_\mu=(p'-p)_\mu\to 0$ are determined by the rate of Gamow-Teller transitions in β decay, $D(0)=2Mg_A$. The pion contributes a pole term to $D(q^2)$, so that $(q^2-\mu^2)D(q^2)=-\sqrt{2}gf_\pi\mu^2$ as $q^2\to\mu^2$, where g is the π-N coupling and f_π the decay amplitude. If one now assumes that $k(q^2)=(q^2-\mu^2)D(q^2)$ varies smoothly for $0\leqslant q^2\leqslant\mu^2$ [as is expected if the pion pole dominates $D(q^2)$], then $k(0)\simeq k(\mu^2)$, and we have the Goldberger-Treiman relation $2Mg_A\simeq\sqrt{2}gf_\pi$. Since this result is good to 10%, we conclude that $k(q^2)$ has changed by only 10% in extrapolating from $q^2=0$ to $q^2=\mu^2$, and the pion pole dominance is rather good. However, we are here precisely interested in the origin of this 10% correction to the pion pole term, for this is what can be expected to play an important role in the extrapolation of other amplitudes from the current-algebra point $q_\mu=0$ to the mass shell.

If it is assumed that $i\partial_\mu A_\mu^{(a)}(x)$ is a gentle operator, which is to say, its matrix elements obey unsubtracted dispersion relations in the momentum transfer, then the corrections to the Goldberger-Treiman relation can be computed from the continuum states by using the unsubtracted dispersion relation

$$\Delta=1-\frac{2Mg_A}{\sqrt{2}gf_\pi}=-\frac{1}{\sqrt{2}gf_\pi\pi}\int_{(3\mu)^2}^\infty \mathrm{Im}D(q^2)\frac{dq^2}{q^2},$$

where $\Delta^{\mathrm{expt}}=+0.105\pm0.026$ represents the 10% correction. An attempt[2] to estimate the continuum contribution represented by the integral suggests that these states fail to account for the observed value of Δ by at least an order of magnitude. The high-frequency contribution of all states with energy $\sqrt{q^2}\geqslant 2M$ (where M is the nucleon mass) could be rigorously shown to be less than $1\frac{1}{2}$% if the pion propagator

$$\Delta_\pi(q^2)=1/(\mu^2-q^2)+\int_{(3\mu)^2}^\infty \rho_\pi(q'^2)dq'^2/(q'^2-q^2)$$

at $q^2=0$ was dominated by the pion pole term,[3] as is consistent with PCAC and the conventional smoothness assumption. The $\pi\rho$ and $\pi\sigma$ states contributed negligibly. The presumably dominant 3π continuum estimated using Weinberg's treatment of $\pi\pi$ scattering[4] was severely damped by three-body phase space and was negligible. Only if there were large 3π forces, producing a heavy pion of mass $\approx 3\mu$, could one hope that the continuum would account for the observed Δ. In the experimental absence of the tripion, we must conclude that a subtraction is required in the dispersion relation

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[1] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **39**, 703 (1960) [English transl.: Soviet Phys.—JETP **12**, 492 (1961)].
[2] H. Pagels, Phys. Rev. **179**, 1337 (1969).
[3] That is, $1/\mu^2>\int\rho_\pi(q^2)dq^2/q^2$.
[4] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

for $D(q^2)$, and give up hope of calculating Δ by these means.

We propose, on these grounds, to take the possibility of a subtraction seriously.[5] Then $\partial_\mu A_\mu^{(a)}$ is not so gentle, but the nongentle part is small and proportional to Δ. To take these features of $\partial_\mu A_\mu^{(a)}$ into account, we will define the extrapolating field of the pion from the weak interactions according to

$$i\partial_\mu A_\mu^{(a)}(x)=\alpha_\pi\pi^{(a)}(x)+\beta_\pi\Box\pi^a(x)\,, \qquad (1)$$

with α_π and β_π constants. Of course, one is free to define the pion field in any consistent way one wants, but direct physical content comes with the assumption that the pion field defined by Eq. (1) is supersmooth. By "supersmooth" we mean that the matrix elements $\langle\alpha|j_\pi^{(a)}(x)|\beta\rangle$, with $j_\pi^{(a)}(x)=(\Box+\mu^2)\pi^{(a)}(x)$, exhibit a variation of no more than 1 or 2% for momentum transfers $0\leqslant(q_\alpha-q_\beta)^2\leqslant\mu^2$ [and not 10%, as required by the conventional PCAC which omits the linear term $\beta_\pi\Box\pi^a(x)$]. This supersmoothness is then consistent with the absence of any large continuum contribution to the matrix elements $\langle p'|j_\pi^{(a)}(0)|p\rangle$, which was found in Ref. 2.

The constants α_π and β_π can be determined by the relations

$$\langle 0|i\partial_\mu A_\mu^{(+)}(0)|\pi^+\rangle=\alpha_\pi-\mu^2\beta_\pi=\mu^2 f_\pi\,,$$

$$\langle p|i\partial_\mu A_\mu^{(+)}(0)|p\rangle=\frac{\alpha_\pi}{\mu^2}\langle p|j_\pi^{(+)}(0)|p\rangle$$

$$\cong\frac{\alpha_\pi}{\mu^2}\langle p'|j_\pi^{(+)}(0)|p\rangle|_{(p'-p)^2=\mu^2}\,,$$

or

$$\alpha_\pi=\frac{2Mg_A\mu^2}{\sqrt{2}g}\,,\quad \beta_\pi=\frac{2Mg_A}{\sqrt{2}g}-f_\pi=-f_\pi\Delta\simeq-f_\pi(0.1)\,,$$

where we have used supersmoothness. The nongentle piece $\Box\pi^{(a)}(x)$ in $i\partial_\mu A_\mu^{(a)}(x)$ is multiplied by $-f_\pi\Delta$ and parametrizes in a linear fashion the 10% variation between $q^2=0$ and $q^2=\mu^2$.[6]

With the pion field defined according to

$$i\partial_\mu A_\mu^{(a)}(x)=(2Mg_A\mu^2/\sqrt{2}g)\pi^a(x)-\Delta f_\pi\Box\pi^a(x)\,, \qquad (2)$$
$$\Delta=1-2Mg_A/\sqrt{2}gf_\pi\simeq+0.1\,,$$

in conjunction with the supersmoothness assumption, we may now examine the question of pion mass extrapolations of other amplitudes. Of particular interest is the problem of such extrapolations in the application of the Adler-Weisberger low-energy theorem[4] to pion-nucleon scattering, where there is considerable precision in the experimental parameters. This example will also serve to illustrate the general method of calculating extrapolations to the physical pion mass.

As is usual, we consider the weak amplitude[7]

$$T(\nu,q^2)=-i\int d^4x\, e^{iq\cdot x}\langle p|T(\partial_\mu A_\mu^{(+)}(x),\partial_\lambda A_\lambda^{(-)}(0))|p\rangle\,,$$
$$\nu M=p\cdot q\,,$$

which we wish to relate to the off-shell forward pion-nucleon amplitude

$$F(\nu,q^2)=i\int d^4x\, e^{iq\cdot x}(\Box+\mu^2)^2\langle p|T(\pi^{(+)}(x),\pi^{(-)}(0))|p\rangle,$$

so that $F(\nu,\mu^2)$ is the physical forward scattering amplitude. For the part of $T(\nu,q^2)$ odd under crossing $\nu\to-\nu$, one can derive the rigorous low-energy theorem as $q_\mu\to 0$ $(\nu=q^2=0)$, which depends only on the scaling condition for the chiral charge algebra. The part that is even under crossing can be related to the σ commutator in this limit, which is, in general, model-dependent.

Using our definition, Eq. (1), for the pion field, we can relate $T(\nu,q^2)$ and $F(\nu,q^2)$:

$$(q^2-\mu^2)^2T(\nu,q^2)=(\alpha_\pi-\beta_\pi q^2)^2F(\nu,q^2)$$
$$+(q^2-\mu^2)^2\beta_\pi E(\nu)\,. \qquad (3)$$

The additional piece $E(\nu)$ is at worst a polynomial in ν, and represents the contribution of the equal-time commutators

$$[\pi^{(+)}(\mathbf{x},0),\pi^{(-)}(0,0)] \quad\text{and}\quad [\dot\pi^{(+)}(\mathbf{x},0),\pi^{(-)}(0,0)]$$

which arise as $\beta_\pi\Box$ is pulled through the time ordering.[8] If we assume $[\pi^{(a)}(\mathbf{x},0),\pi^{(b)}(0,0)]=0$, then $[\dot\pi^{(a)}(\mathbf{x},0),\pi^{(b)}(0,0)]$ is symmetric in (a) and (b), and $E(\nu)$ does not contribute to the odd amplitude, for which we have a rigorous theorem. If one further assumes canonical commutation rules, then these commutators are C numbers and E is canceled by an identical piece in the disconnected amplitude. However, in general, the knowledge of such additional commutators precludes the estimation of extrapolation corrections. We will optimistically set $E=0$, as is consistent with field algebra for these commutators.

Finally, there remains the question of the extrapolation of the amplitude $F(\nu,q^2)$ from the point $\nu=q^2=0$ in the (ν,q^2) plane, where we have a low-energy theorem, to $q^2=\mu^2$, corresponding to π-N scattering. Fubini and Furlan[9] have extensively studied this question and found that extrapolation along the parabola $\nu^2/M^2=q^2$ leads to a simple classification of the correction factors. Included in this classification are corrections arising from normal thresholds in $F(\nu,q^2)$ in the cut q^2 plane beginning at $q^2=9\mu^2$, and from anomalous thresholds at

[5] For the implications of such a subtraction for chiral $SU(2)\times SU(2)$ breaking, see R. Dashen and M. Weinstein, Phys. Rev. (to be published).

[6] The presence of such a small slope is not ruled out by the measured rate of μ^- capture at rest in hydrogen, which gives information only on the induced pseudoscalar amplitude at $q^2=-m_\mu^2M/(M+m_\mu)$.

[7] Disconnected parts are understood to have been removed.

[8] We have dropped terms of order $\Delta^2\simeq0.01$ which involve additional equal-time commutators.

[9] S. Fubini and G. Furlan, Ann. Phys. (N.Y.) 48, 322 (1968).

$q^2 \simeq 8\mu^2$ (for $\nu=0$).[10] Now our main point is that since the pion field that we have defined is supersmooth, the contribution of these cuts to the low-q^2 region can be expected to be very small, $\sim 2\%$, and we can completely ignore such corrections. Essential to such an observation is the recognition that if the continuum states could account for the 10% correction to the Goldberger-Treiman relation (contrary to our suggestion), then we would hardly be justified in dropping these corrections to the off-shell πN amplitude—here contributing perhaps 20% (10% for each extrapolated pion). In view of the absence of any large cut contribution in $D(q^2)$, we expect the same to be true for the amplitude $T(\nu,q^2)$ in the cut q^2 plane, so instead of the continuum-cut corrections we incorporate the known small subtraction evidenced by the presence of the term $\beta_\pi q^2$ in Eq. (3).

Since the correction factor from parity doublets is small in the case of π-N scattering,[9] we have from the supersmoothness hypothesis and the method of Ref. 9 that $F(0,0) \simeq F(\mu,\mu^2)$, and $\partial F(\nu,0)/\partial\nu|_{\nu=0} \simeq \partial F(\nu,\mu^2)/\partial\nu|_{\nu=\mu}$, to an expected accuracy of the order of μ^2/M^2. Equation (3) implies $\mu^4 T(0,0) = \alpha_\pi^2 F(0,0) \simeq \alpha_\pi^2 F(\mu,\mu^2)$, as well as a similar condition on the derivative which, when translated into a statement about the S-wave πN scattering lengths, is

$$\frac{2}{3}\left(1+\frac{\mu}{M}\right)(a_1-a_2)=\frac{4f^2}{g_A{}^2}, \quad f^2=\frac{g^2}{4\pi}\left(\frac{\mu}{2M}\right)^2. \tag{4}$$

The difference between the left and the right sides is[11] $(0.203\pm0.0075)-(0.229\pm0.014)=-(0.026\pm0.022)$.

The expression (4) for a_1-a_2 is the one that is usually obtained without analysis of extrapolation corrections,[4] and one may wonder why this is so. However, the answer is clear if one recognizes that what is usually done in current-algebra calculations is to use the definition $i\partial_\mu A_\mu{}^a = \mu^2 f_\pi \pi^a$ and assume smoothness. Then the threshold theorems are expressed in terms of f_π, and sometimes, as a final step (to get better agreement with the data), the Goldberger-Treiman value for f_π is used, changing the result by 10% for each extrapolated pion. The point of our observations is that this 10% freedom for each extrapolated pion in comparing the amplitude with experiment is no longer available, and the correct choice is dictated by Eq. (2) and the supersmoothness hypothesis. It is here suggested that, except for the question of additional subtractions like $E(\nu)$ and the pieces discussed in Ref. 9 corresponding to parity doublets and other terms, the simple linear extrapolation in the pion mass according to $\alpha_\pi - \beta_\pi q^2$ suffices, and is consistent with the need for a small subtraction in the matrix elements of $i\partial_\mu A_\mu$.

It is clear that these remarks also apply to other pion processes where PCAC is used, and with suitable modification our definition (1) can be generalized to the strangeness-changing currents—where, however, the phenomenological couplings are not so precisely determined.

Note added in proof. We will here comment on the implication of the conjectured subtraction term for the Goldberger-Treiman relation and the breakdown of chiral $SU(2)\times SU(2)$ symmetry. The $SU(2)$ symmetry of the isotopic-spin current is presumably broken by the electromagnetic interaction characterized by a strength $e^2/4\pi \approx 0.007$. We do not know the dynamics of the breakdown of the axial-vector charge symmetry but, assuming that this symmetry is realized by zero-mass pions, one of the manifestations of this symmetry breakdown is the finite pion mass, the corrections to the Goldberger-Treiman relation, and all extrapolation corrections.

Now the conjecture of Gell-Mann, Oakes, and Renner[12] is that in the symmetry breakdown $SU(3)\times SU(3) \to SU(2)\times SU(2)$ the breaking of the vector charge symmetry and axial-vector charge symmetry are corollated. One might speculate that the same is true in breaking $SU(2)\times SU(2)$ if the axial-vector charge symmetry breaking is characterized by $m_\pi{}^2/m^2 \approx 0.01$, with m some baryonic mass. In this case one would introduce only one symmetry-breaking interaction characterized by strength $e^2/4\pi \sim 0.01$, which is presumably electromagnetism. Then the finite pion mass would have to be electromagnetic in origin along with the corrections to the Goldberger-Treiman relation. This speculation that the breakdown $SU(2)\times SU(2) \to U(1)$ is characterized by a single parameter seems to us implausible. First, if we adopt the usual minimal coupling of the electromagnetic field to the isovector current and hypercharge, then the π^0 would have zero mass and $\pi^\pm$ mass would be finite to first order in e^2, contrary to experience. Secondly, the electromagnetic corrections to the Goldberger-Treiman relation are only $\sim\frac{1}{2}\%$ and it is difficult to make them any larger.

We therefore conclude that the $SU(2)$-symmetry breaking of the axial-vector charges is distinct from electromagnetism and due to a dynamical mechanism of yet unknown nature. This symmetry-breaking interaction is characterized by a phenomenological parameter like the electronic charge e which appears as a subtraction constant in dispersion relations. Now, assuming an unsubtracted dispersion relation for the matrix elements of the divergence of the axial-vector current is tantamount to assuming that one can calculate this symmetry-breaking parameter. This seems to us unlikely and one recognizes in the need for a subtraction as conjectured in this paper a new independent parameter characterizing the symmetry breakdown, and which is required as input.

I would like to thank Professor Roger Dashen for discussion on these points.

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[11] We have used the data reported by V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965).

[12] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).