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# High-Energy Collision Processes in Quantum Electrodynamics. IV

HUNG CHENG*
*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

AND

TAI TSUN WU
*Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138*

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The rather complicated asymptotic behavior at high energies found in the preceding paper for the Delbrück-scattering matrix element is recast in a simpler form. This is accomplished by introducing a suitable form factor in the transverse plane. This factor, called the impact factor of the photon, also appears in other processes. Moreover, it is verified that the transverse momentum transfer is small in Delbrück scattering.

## 1. INTRODUCTION

In the preceding paper, we have studied the differential cross section for Delbrück scattering in the limit of infinite energy. More precisely, for any fixed nonzero momentum transfer, we have found that, to the order $\alpha^6 Z^4$ (which is the lowest nonvanishing order), $d\sigma/dt$ approaches a finite limit as the photon energy $\omega$ goes to infinity. Moreover, the limiting value is explicitly calculated in terms of an integral over Feynman parameters. It is to be shown in paper VII of this series that the existence of the limiting value also holds to the lowest order in $\alpha$ but any order in $Z\alpha$, i.e., to the order $\alpha^2(Z\alpha)^{2n}$ for $n=2, 3, 4, \cdots$.

If the interest were restricted to Delbrück scattering at high energies, there would be few further questions besides radiative corrections. However, the aim here is to have come understanding of collision processes at high energies in general, using Delbrück scattering as one of the stepping-stones. For this purpose, it is imperative to recast the result of the preceding paper in a form more susceptible to physical interpretation.

The alternative form for the limiting value of the matrix element is already summarized in (3.3) of paper I. In other words, the limit is expressed as an integral over the difference of the *transverse* momenta of the two exchanged photons, and the integrand involves as a factor the "photon impact factor" $\mathcal{G}^\gamma$. As discussed in paper I, this photon impact factor is significant because it also appears in light-light scattering, and is furthermore altered neither by the introduction of a mass for the internal photons nor by the inclusion of higher-order effects in $Z\alpha$.

It is the purpose of the present paper to discuss the origin of this photon impact factor and relate it to the result of paper III. In the preceding paper, the limit of infinite energy $\omega$ is discussed carefully in the sense that, with the exception of the well-understood and irrelevant ultraviolet divergence of the box diagram, every expression is mathematically meaningful. This entails, at a suitable point, the combination of the contributions from the two Feynman-Dyson graphs. As an example, consider (3.4) of the preceding paper, where the large braces contain four terms added together, two from each graph. If these four terms were taken separately, the resulting integrals would all be meaningless, because they each contain divergences from the neighborhood of (i) $\beta=\beta'=0$ and (ii) $\beta=\beta'=\alpha_5=0$. However, all divergences are cancelled in the sum, i.e., no logarithms of $\omega$ appear in the final results. If we take for granted that these types of divergences can cause no harm, then there is no necessity of being so careful in combining the contributions from the two graphs. In Sec. 2 here, we shall take this point of view and extract the photon impact factor directly from the Feynman integrals, or rather (2.16) and (2.17) of paper III. This direct extraction has the advantage of being easily understandable physically. On the other hand, since valid questions may be raised about the manipulation of divergent quantities, we show in Sec. 3 that the integral over the photon form factor is indeed equal to the result of the preceding paper. Although it is possible to derive the photon impact factor $\mathcal{G}^\gamma$ from the result of paper III, the process seems to us rather artificial; indeed, before $\mathcal{G}^\gamma$ was found, one of us tried in vain for a long time to get it from the result of paper III. Eventually, a guess was made and the procedure of Sec. 3 was carried out to verify its correctness.

In Sec. 4, we give another derivation of the impact factor of the photon in an alternative form without Feynman parameters.

## 2. HEURISTIC DERIVATION OF PHOTON IMPACT FACTOR

Define

$$R_1=2(2\pi)^{-4}e^4\int d^4p\,[(r_2+p)^2-m^2]^{-1}[(-r_1+p)^2-m^2]^{-1}[(p+q)^2-m^2]^{-1}[(r_1+p)^2-m^2]^{-1}$$
$$\times\{8p_0\omega[2r_{1i}p_j-2p_ir_{1j}+2r_1^2\delta_{ij}]+8p_0^2[4p_ip_j-2p_i(r_1-r_2)_j+2(r_1+r_2)_ip_j+\delta_{ij}(r_1^2-r_2^2)]\} \quad (2.1)$$

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and

$$R_2=(2\pi)^{-4}e^4\int d^4p\,[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times\{2\omega^2[-(2p-q)_i(2p+q)_j+2q^2\delta_{ij}]-8\omega p_0[-2r_{1i}p_j-r_{2i}q_j+2p_ir_{1j}+q_ir_{2j}-2(r_1q)\delta_{ij}]$$
$$+8p_0^2[(r_1+r_2)_i(r_1-r_2)_j+(2p+q)_i(2p-q)_j+\delta_{ij}(r_1^2-r_2^2)]\}\,. \quad (2.2)$$

A comparison with (2.15), (2.16), and (2.17) of paper III shows that, for $i=1, 2$,

$$\mathfrak{M}_i'=i(2\pi)^{-3}e^2Z^2\int d^3\mathbf{q}\,[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}R_i \quad (2.3)$$

and

$$\mathfrak{M}_0^{(D)}\sim i(2\pi)^{-3}e^2Z^2\int d^3\mathbf{q}\,[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}(R_1+R_2)\,. \quad (2.4)$$

As a first step, we introduce Feynman parameters for both $R_1$ and $R_2$. Since, for any $a$,

$$\int d^4p\,(p^2+a^2)^{-4}=\tfrac{1}{6}i\pi^2(a^2)^{-2}$$

and (2.5)

$$\int d^4p\,p^2(p^2+a^2)^{-4}=\tfrac{1}{3}i\pi^2(a^2)^{-1},$$

it follows from (2.1) and (2.2) together with (2.19) of III that

$$R_1\sim 2i(4\pi)^{-2}e^4\int_0^1 d\alpha_1d\alpha_2d\alpha_3d\alpha_4\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)(\bar{c}_1+i\epsilon)^{-2}(16\delta p_0')$$
$$\times\{\omega[r_{1i}\delta p_j'-\delta p_i'r_{1j}+r_1^2\delta_{ij}]+(\delta p_0')[2\delta p_i'\delta p_j'-\delta_{ij}\bar{c}_1-2\delta p'_ir_{1j}+2r_{1i}\delta p_j'+\delta_{ij}r_1^2]\} \quad (2.6)$$

and

$$R_2\sim i(4\pi)^{-2}e^4\int_0^1 d\alpha_1d\alpha_2d\alpha_3d\alpha_4\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)(\bar{c}_2+i\epsilon)^{-2}$$
$$\times\{2\omega^2[-(2\delta p''-q)_i(2\delta p''+q)_j+2\delta_{ij}\bar{c}_2+2q^2\delta_{ij}]-8\omega\delta p_0''[(2\delta p''-q)_ir_{1j}-r_{1i}(2\delta p''+q)_j-2(r_1q)\delta_{ij}]$$
$$+8(\delta p_0'')^2[4r_{1i}r_{1j}+(2\delta p''+q)_i(2\delta p''-q)_j-2\delta_{ij}\bar{c}_2+2\delta_{ij}r_1^2]\}\,, \quad (2.7)$$

where

$$\bar{c}_1=-2\alpha_1\alpha_3(r_2q)+2\alpha_3(\alpha_2-\alpha_4)(r_1\cdot q)+\tfrac{1}{4}t[4\alpha_2\alpha_4+\alpha_3(-\alpha_1+\alpha_2+\alpha_4)]+q^2\alpha_3(\alpha_1+\alpha_2+\alpha_4)-m^2\,, \quad (2.8)$$

$$\bar{c}_2=-2(\alpha_1\alpha_3-\alpha_2\alpha_4)(r_2q)+2(\alpha_1\alpha_2-\alpha_3\alpha_4)(r_1q)+\tfrac{1}{4}t(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)+q^2(\alpha_1+\alpha_4)(\alpha_2+\alpha_3)-m^2\,, \quad (2.9)$$

$$\delta p'=-\alpha_1r_2+(\alpha_2-\alpha_4)r_1-\alpha_3q\,, \quad (2.10)$$

and

$$\delta p''=\tfrac{1}{2}r_2(\alpha_1+\alpha_2-\alpha_3-\alpha_4)+\tfrac{1}{2}r_1(-\alpha_1+\alpha_2-\alpha_3+\alpha_4)+\tfrac{1}{2}q(-\alpha_1+\alpha_2+\alpha_3-\alpha_4)\,. \quad (2.11)$$

Note that the $\delta p'$ and $\delta p''$ defined here are not related to those of (2.28) and (2.29) in paper III.

Consider $R_1$ as given by (2.6). Since it is known that the dominating contribution comes from the vicinity of $\alpha_3=0$, great simplication is possible by omitting most of the $\alpha_3$'s as follows:

$$\bar{c}_1\sim-2\alpha_1\alpha_3(r_2q)+t\alpha_2\alpha_4-m^2\,, \quad (2.12)$$

$$\delta p'\sim-\alpha_1r_2+(\alpha_2-\alpha_4)r_1\,, \quad (2.13)$$

so that

$$\delta p_0'=-\alpha_1\omega\,,$$

$$\delta p_i'\sim(-\alpha_1+\alpha_2-\alpha_4)r_{1i}\,,$$

and (2.14)

$$\delta p_j'\sim(\alpha_1+\alpha_2-\alpha_4)r_{1j}\,,$$

The substitution of (2.12) and (2.14) into (2.6) gives explicitly

$$R_1 \sim 2i(4\pi)^{-2}e^4\omega^2 \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\,[-2\alpha_1\alpha_3(r_2q)+t\alpha_2\alpha_4-m^2+i\epsilon]^{-2}(16\alpha_1)\{-[2\alpha_1 r_{1i}r_{1j}+\tfrac{1}{4}t\delta_{ij}]$$
$$+\alpha_1[2(-\alpha_1+\alpha_2-\alpha_4)(\alpha_1+\alpha_2-\alpha_4)+4\alpha_1]r_{1i}r_{1j}-\alpha_1\delta_{ij}[-2\alpha_1\alpha_3(r_2\cdot q)+t\alpha_2\alpha_4-m^2-\tfrac{1}{4}t]\}$$
$$=2i(4\pi)^{-2}e^4\omega^2 \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)[-2\alpha_1\alpha_3(r_2q)+t\alpha_2\alpha_4-m^2+i\epsilon]^{-2}(16\alpha_1)$$
$$\times\{-8\alpha_1\alpha_2\alpha_4 r_{1i}r_{1j}-\delta_{ij}[-2\alpha_1^2\alpha_3(r_2q)+t\alpha_1\alpha_2\alpha_4-\alpha_1m^2+\tfrac{1}{4}t(\alpha_2+\alpha_4)]\}. \quad (2.15)$$

The corresponding formula for $R_2$ is

$$R_2 \sim i(4\pi)^{-2}e^4\omega^2 \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)(\bar{c}_2+i\epsilon)^{-2}4\{8(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)[(\alpha_2-\alpha_3)r_{1i}+(\alpha_2+\alpha_3)q_i]$$
$$\times[(\alpha_1-\alpha_4)r_{1j}+(\alpha_1+\alpha_4)q_j]+\delta_{ij}[q+(\alpha_1+\alpha_2-\alpha_3-\alpha_4)r_1]^2+4\delta_{ij}(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\bar{c}_2\}. \quad (2.16)$$

In both $\bar{c}_1$ and $\bar{c}_2$, as given by (2.8) and (2.9), $r_2$ appears only in the combination $(r_2q)$. It is therefore convenient to choose coordinate axes so that $\mathbf{r}_2$ lies in the direction of the $z$ axis. In other words,

$$r_2q=-(\omega^2+\tfrac{1}{4}t)^{1/2}q_3. \quad (2.17)$$

Note that $q_3$ can be either positive or negative, and that the photon propagators $[(r_1\pm q)^2]^{-1}$ are invariant under $q_3\to -q_3$. Furthermore, for any $A$ and $B$ with $A\neq 0$,

$$\int_0^\infty dy[(Ay+B+i\epsilon)^{-2}+(-Ay+B+i\epsilon)^{-2}]=0. \quad (2.18)$$

Therefore, the dominating contributions to $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ as expressed by (2.3) must be from the vicinity of $q_3=0$.

It is the most important point of the present paper that *the longitudinal momentum transfer is small.* This is to be discussed in detail in Sec. 5; for the moment we merely mention that this has been conjectured before[1] and is indeed incorporated in the droplet model.[2]

Since $q_3$ is small, it is natural to attempt to integrate over $q_3$. From this point on, the considerations become more formal. Let

$$\mathcal{G}_i=(2\pi)^{-1}\int_{-\infty}^{\infty} dq_3 R_i/\omega \quad (2.19)$$

for $i=1$, 2. These integrals are to be further studied in Sec. 4. Because of the form of $\bar{c}_1$ and $\bar{c}_2$, $q_3$ may be treated on the same footing as Feynman parameters. Let $A$ and $B$ be two real numbers with $A>0$; then by symmetric integration

$$\int_{-\infty}^{\infty} dq_3 \int_0^1 dy(Ayq_3+B+i\epsilon)^{-2} = -\pi i A^{-1}(B+i\epsilon)^{-1}, \quad (2.20)$$

and moreover by integrating with respect to $B$

$$\int_{-\infty}^{\infty} dq_3 \int_0^1 dy(Ayq_3+B+i\epsilon)^{-1} = \pi i A^{-1}\ln(-B-i\epsilon)+\mathfrak{C}, \quad (2.21)$$

where $\mathfrak{C}$ is a constant of integration. (Actually $\mathfrak{C}$ is not finite but, as seen below, is cancelled out.) In order to use (2.20) and (2.21) to find $\mathcal{G}_1$ and $\mathcal{G}_2$, we must apply the changes of variables (2.43) and (2.44) of paper III to (2.15) and (2.16). Thus,

$$R_1 \sim 2i(4\pi)^{-2}e^4\omega^2 \int_0^1 d\beta\, d\beta' dx\, dy\, \beta^2(1-x)\delta(1-\beta-\beta')[-2\beta\beta'(1-x)y\omega q_3+t\beta^2x(1-x)-m^2+i\epsilon]^{-2}(16\beta')$$
$$\times\{-8\beta^2\beta' x(1-x)r_{1i}r_{1j}-\delta_{ij}[-2\beta\beta'^2(1-x)y\omega q_3+t\beta^2\beta' x(1-x)-\beta' m^2+\tfrac{1}{4}t\beta]\} \quad (2.22)$$

and

$$R_2 \sim 2i(4\pi)^{-2}e^4\omega^2 \int_0^1 d\beta\, d\beta' dx\, dy\, \beta\beta'(1-x)\delta(1-\beta-\beta')(\bar{c}_1+i\epsilon)^{-2}$$
$$\times 4\{8\beta\beta' x(1-x)[(\beta-\beta')r_{1i}+q_{\perp i}][(\beta-\beta')r_{1j}+q_{\perp j}]-\delta_{ij}[(\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp]^2+4\delta_{ij}\beta\beta'\bar{c}_2\}, \quad (2.23)$$

where $\mathbf{q}_\perp$ is the projection of $\mathbf{q}$ in the $xy$ plane, and

$$\bar{c}_2=-2\beta\beta'(1-x)y\omega q_3-2(\beta-\beta')x(1-x)\mathbf{r}_1\cdot\mathbf{q}+\tfrac{1}{4}t(\beta-\beta')^2x(1-x)-x(1-x)\mathbf{q}_\perp^2-m^2. \quad (2.24)$$

[1] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).
[2] N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966).

The substitution of (2.22) and (2.23) into (2.19) then gives

$$\mathcal{I}_1 \sim (4\pi)^{-2}e^4 \int_0^1 d\beta\, d\beta'\, dx\, 8\beta\delta(1-\beta-\beta')\{[t\beta^2 x(1-x)-m^2]^{-1} \times[-8\beta^2\beta' x(1-x)r_{1i}r_{1j}-\tfrac{1}{4}t\beta\delta_{ij}]+\beta'\delta_{ij}\ln[-t\beta^2x(1-x)+m^2]+\beta'\delta_{ij}\mathcal{C}\} \quad (2.25)$$

and

$$\mathcal{I}_2 \sim (4\pi)^{-2}e^4 \int d\beta\, d\beta' dx\, 2\delta(1-\beta-\beta')[\{-x(1-x)[(\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp]^2-m^2\}^{-1} \times\{8\beta\beta' x(1-x)[(\beta-\beta')r_{1i}+q_{\perp i}][(\beta-\beta')r_{1j}+q_{\perp j}]-\delta_{ij}[(\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp]^2\} -4\delta_{ij}\beta\beta'\ln\{x(1-x)[(\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp]^2+m^2\}-4\delta_{ij}\beta\beta'\mathcal{C}]. \quad (2.26)$$

It is seen that the sum $\mathcal{I}_1+\mathcal{I}_2$ does not contain the divergent quantity $\mathcal{C}$.

We therefore define the *impact factor* of the photon as

$$\mathcal{I}^\gamma = \lim_{\omega\to\infty}(\mathcal{I}_1+\mathcal{I}_2) = \text{sum of the right-hand sides of (2.25) and (2.26)}. \quad (2.27)$$

By (2.4) and (2.19), the impact factor is related to the matrix element to the order $e^6Z^2$ of Delbrüch scattering by

$$\lim_{\omega\to\infty}\omega^{-1}\mathfrak{M}_0^{(D)} = i(2\pi)^{-2}e^2Z^2\int d\mathbf{q}_\perp[(\mathbf{r}_1+\mathbf{q}_\perp)^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_\perp)^2]^{-1}\mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp), \quad (2.28)$$

and is given explicitly by

$$\mathcal{I}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp) = 2(4\pi)^{-2}e^4\int d\beta\, d\beta' dx\, \delta(1-\beta-\beta')[[|t|\beta^2x(1-x)+m^2]^{-1}[32\beta^3\beta' x(1-x)r_{1i}r_{1j}-\beta^2|t|\delta_{ij}] -\{x(1-x)[(\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp]^2+m^2\}^{-1}\{8\beta\beta' x(1-x)[(\beta-\beta')r_{1i}+q_{\perp i}][(\beta-\beta')r_{1j}+q_{\perp j}] -\delta_{ij}[(\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp]^2\}-4\delta_{ij}\beta\beta'\ln\{[x(1-x)((\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp)^2+m^2]/[|t|\beta^2x(1-x)+m^2]\}]. \quad (2.29)$$

Alternative forms and properties of this impact factor for the photon have already been given in paper I.

## 3. COMPARISON WITH RESULT OF PAPER III

Since it seems difficult to improve significantly the heuristic procedure of Sec. 2, it is essential to verify that the result (2.28) is indeed correct. Fortunately, the left-hand side has already been studied in detail in paper III. This section is devoted to a direct evaluation of the right-hand side of (2.28), with $\mathcal{I}^\gamma$ given by (2.29).

Feynman parameters are introduced once more to combine the denominators. Let the new Feynman parameters for $[(\mathbf{r}_1+\mathbf{q}_\perp)^2]^{-1}$ and $[(\mathbf{r}_1-\mathbf{q}_\perp)^2]^{-1}$ be, respectively, $\alpha_5$ and $\alpha_6$. Also let

$$\beta=\gamma/(1-\alpha_5-\alpha_6) \quad\text{and}\quad \beta'=\gamma'/(1-\alpha_5-\alpha_6), \quad (3.1)$$

then the combined denominators are

$$[|t|\beta^2x(1-x)+m^2](1-\alpha_5-\alpha_6)+(\mathbf{r}_1+\mathbf{q}_\perp)^2\alpha_5+(\mathbf{r}_1-\mathbf{q}_\perp)^2\alpha_6 = (\alpha_5+\alpha_6)[\mathbf{q}_\perp+(\alpha_5-\alpha_6)(\alpha_5+\alpha_6)^{-1}\mathbf{r}_1]^2+(\alpha_5+\alpha_6)^{-1}(\gamma+\gamma')^{-1}c_{10} \quad (3.2)$$

and

$$\{x(1-x)[(\beta-\beta')\mathbf{r}_1+\mathbf{q}_\perp]^2+m^2\}(1-\alpha_5-\alpha_6)+(\mathbf{r}_1+\mathbf{q}_\perp)^2\alpha_5+(\mathbf{r}_1-\mathbf{q}_\perp)^2\alpha_6 = [(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]\{\mathbf{q}_\perp+[(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]^{-1}[(\alpha_5-\alpha_6)+(\gamma+\gamma')x(1-x)]\mathbf{r}_1\}^2 +(\gamma+\gamma')^{-1}[(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]^{-1}c_{20}, \quad (3.3)$$

where

$$c_{10}=|t|[\alpha_5\alpha_6(\gamma+\gamma')+\gamma^2(\alpha_5+\alpha_6)x(1-x)]+m^2(\gamma+\gamma')^2(\alpha_5+\alpha_6) \quad (3.4)$$

and

$$c_{20}=|t|[\alpha_5\alpha_6(\gamma+\gamma')+(\gamma'^2\alpha_5+\gamma^2\alpha_6)x(1-x)]+m^2(\gamma+\gamma')^2[(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]. \quad (3.5)$$

Since

$$\int_0^1 d\alpha_5\int_0^{1-\alpha_5} d\alpha_6(1-\alpha_5-\alpha_6)^{-1}\{[A\alpha_5+B\alpha_6+C(1-\alpha_5-\alpha_6)]^{-2} -[A\alpha_5+B\alpha_6+C'(1-\alpha_5-\alpha_6)]^{-2}\}=A^{-1}B^{-1}\ln(C'/C), \quad (3.6)$$

the integral on the right-hand side of (2.28) is found to be

$$
\begin{aligned}
&\int d\mathbf{q}_\perp[(\mathbf{r}_1+\mathbf{q}_\perp)^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_\perp)^2]^{-1}g^\gamma(\mathbf{r}_1,\mathbf{q}_\perp)\\
&=(4\pi)^{-1}e^4\int_0^1 d\gamma\int_0^1 d\gamma'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{\alpha_6} d\alpha_5\delta(1-\gamma-\gamma'-\alpha_5-\alpha_6)(\gamma+\gamma')^{-1}\big[[(\alpha_5+\alpha_6)^{-1}(\gamma+\gamma')^{-2}c_{10}{}^2]^{-1}\\
&\quad\times[32\gamma^3\gamma'(\gamma+\gamma')^{-4}x(1-x)r_{1i}r_{1j}-\gamma^2(\gamma+\gamma')^{-2}|t|\delta_{ij}]-\{(\gamma+\gamma')^{-2}[(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]^{-1}c_{20}{}^2\}^{-1}\\
&\quad\times[(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]^{-2}(\gamma\alpha_6-\gamma'\alpha_5)^2\{32\gamma\gamma'(\gamma+\gamma')^{-4}x(1-x)r_{1i}r_{1j}-(\gamma-\gamma')^{-2}|t|\delta_{ij}\}\\
&\quad-\{(\gamma+\gamma')^{-1}[(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]c_{20}\}^{-1}\delta_{ij}[4\gamma\gamma'(\gamma+\gamma')^{-2}x(1-x)-1]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-4\delta_{ij}\gamma\gamma'(\gamma+\gamma')^{-2}(c_{10}{}^{-1}-c_{20}{}^{-1})\big]. \qquad (3.7)
\end{aligned}
$$

Straightforward reduction then yields that

[right-hand side of (2.28)]

$$
\begin{aligned}
&=\tfrac{1}{2}i(2\pi)^{-3}e^6Z^2\int_0^1 d\gamma\int_0^1 d\gamma'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{\alpha_6} d\alpha_5\delta(1-\gamma-\gamma'-\alpha_5-\alpha_6)(\gamma+\gamma')^{-3}\\
&\quad\times\big[[32\gamma\gamma'x(1-x)r_{1i}r_{1j}-(\gamma+\gamma')^2|t|\delta_{ij}]\{\gamma^2(\alpha_5+\alpha_6)c_{10}{}^{-2}-(\gamma\alpha_6-\gamma'\alpha_5)^2[(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]^{-1}c_{20}{}^{-2}\}\\
&\qquad\qquad\qquad-\delta_{ij}\{4\gamma\gamma'c_{10}{}^{-1}-[4\gamma\gamma'(\alpha_5+\alpha_6)+(\gamma+\gamma')^3][(\alpha_5+\alpha_6)+(\gamma+\gamma')x(1-x)]^{-1}c_{20}\}\big]. \qquad (3.8)
\end{aligned}
$$

With $c_{10}$ and $c_{20}$ defined by (3.4) and (3.5), a comparison with (3.2) of paper I shows that the right-hand side of (3.8) is just the limit as $\omega\to\infty$ of $\omega^{-1}\mathfrak{M}_0{}^{(D)}$. Accordingly, (2.28) is indeed correct.

It should be emphasized that the present verification of (2.28), unlike the formal developments in Sec. 3, is completely honest and involves no divergent quantity.

## 4. IMPACT FACTOR IN MOMENTUM SPACE

Because of the importance of the impact factor, it is desirable to obtain it in as direct a manner as possible. In other words, we want to calculate

$$\lim_{\omega\to\infty}(\mathcal{I}_1+\mathcal{I}_2)$$

from (2.19), (2.1), and (2.2) without introducing Feynman parameters. Such a calculation may also facilitate physical interpretation.

For the purpose of orientation, we discuss roughly the magnitudes of various quantities in the limit where $\omega\to\infty$ but $\Delta$ remains fixed at a value different from 0. We again choose the coordinate system as in Sec. 2, where $\mathbf{r}_2$ is in the direction of the $z$ axis, and a subscript $\perp$ denotes projection into the $xy$ plane. From Bethe-Heitler formula for pair production and also (2.18), it is seen that, in the limit just mentioned, the dominating contribution to $\mathfrak{M}_0{}^{(D)}$ comes from the following region:

$$p_3\approx\omega,\qquad p_0\approx\omega, \qquad (4.1)$$

and

$$q_3\approx 0(1), \qquad (4.2)$$

while $\mathbf{p}_\perp$, $\mathbf{q}_\perp$, and $p_0-p_3$ are all at most of the order of $\max(m,\Delta)$. We shall concentrate on this region.

The polarization vector for the incident photon is perpendicular to $k_1=-r_1+r_2$, while that for the scattered photon is perpendicular to $-k_2=r_1+r_2$. Thus,

$$p_i\sim p_{\perp i}+p_3r_{1i}/\omega$$

and

$$p_j\sim p_{\perp j}-p_3r_{1j}/\omega. \qquad (4.3)$$

Define $u$, $v$, and $A$ by

$$p_0=-u-v,\quad p_3=-u+v,\quad \text{and}\quad A=u/\omega. \qquad (4.4)$$

Then it follows from (4.3) that

$$p_i\sim p_{\perp i}-Ar_{1i}\quad\text{and}\quad p_j\sim p_{\perp j}+Ar_{1j}; \qquad (4.5)$$

moreover,

$$pr_2=-(u+v)\omega+(u-v)(\omega^2-\tfrac{1}{4}|t|)^{1/2}\sim-\tfrac{1}{8}A|t|-2v\omega. \qquad (4.6)$$

The approximations (4.5) and (4.6) may be used in the numerators of (2.1) and (2.2) with the results

$$
\begin{aligned}
&8p_0\omega[2r_{1i}p_j-2p_ir_{1j}+2r_1{}^2\delta_{ij}]+8p_0{}^2[4p_ip_j-2p_i(r_1-r_2)_j+2(r_1+r_2)_ip_j+\delta_{ij}(r_1{}^2-r_2{}^2)]\\
&\qquad\sim-A\omega^2(1-A)^{-1}\{-16\delta_{ij}\mathbf{Q}_1{}^2+16(Q_{1i}p_{\perp j}-p_{\perp i}Q_{1j})-32A(1-A)(p_\perp+Q_1)_i(p_\perp-Q_1)_j\} \qquad (4.7)
\end{aligned}
$$

and

$$2\omega^2[-(2p-q)_i(2p+q)_j+2q^2\delta_{ij}]-8\omega p_0[-2r_{1i}p_j-r_{2i}q_j+2p_ir_{1j}+q_ir_{2j}-2(r_1q)\delta_{ij}]$$
$$+8p_0^2[(r_1+r_2)_i(r_1-r_2)_j+(2p+q)_i(2p-q)_j+\delta_{ij}(r_1^2-r_2^2)]$$
$$\sim\omega^2\{-16\delta_{ij}\mathbf{Q}_2^2+16(Q_{2i}p_{\perp j}-p_{\perp i}Q_{2j})-8(1-4A^2)(p_\perp+Q_2)_i(p_\perp-Q_2)_j\}\,, \quad (4.8)$$

where

$$\mathbf{Q}_1=(1-A)\mathbf{r}_1 \quad \text{and} \quad \mathbf{Q}_2=\tfrac{1}{2}\mathbf{q}_\perp-A\mathbf{r}_1. \quad (4.9)$$

Note the similarity between the right-hand sides of (4.7) and (4.8). It is convenient to define $\mathfrak{N}_{1\infty}$ and $\mathfrak{N}_{2\infty}$ as the quantities in the braces in (4.7) and (4.8). Thus, the right-hand sides of (4.7) and (4.8) are, respectively, $-A\omega^2(1-A)^{-1}\mathfrak{N}_{1\infty}$ and $\omega^2\mathfrak{N}_{2\infty}$.

The substitution of the approximation (4.7) into (2.19) with (2.1) yields

$$\mathcal{G}_1\sim 2(2\pi)^{-5}e^4\omega^{-1}\int d^4p\,\omega p_3(1-A)^{-1}\mathfrak{N}_{1\infty}[(r_2+p)^2-m^2+i\epsilon]^{-1}[(-r_1+p)^2-m^2+i\epsilon]^{-1}[(r_1+p)^2-m^2+i\epsilon]^{-1}$$
$$\times\int_{-\infty}^{\infty}dq_3[-2p_3q_3+p_0^2-p_3^2-(\mathbf{p}_\perp+\mathbf{q}_\perp)^2-m^2+i\epsilon]^{-1}$$
$$=\tfrac{1}{2}i(2\pi)^{-4}e^4\int d^4p(1-A)^{-1}\mathfrak{N}_{1\infty}[(r_2+p)^2-m^2+i\epsilon]^{-1}[(-r_1+p)^2-m^2+i\epsilon]^{-1}[(r_1+p)^2-m^2+i\epsilon]^{-1}$$
$$\sim i(2\pi)^{-4}e^4\int d\mathbf{p}_\perp du\,dv(1-A)^{-1}\mathfrak{N}_{1\infty}[-4(\omega-u)v+\tfrac{1}{4}(1-A)|t|-\mathbf{p}^2_\perp-m^2+i\epsilon]^{-1}[4uv-(-\mathbf{r}_1+\mathbf{p}_\perp)^2-m^2+i\epsilon]^{-1}$$
$$\times[4uv-(\mathbf{r}_1+\mathbf{p}_\perp)^2-m^2+i\epsilon]^{-1}$$
$$=(2\pi)^{-3}e^4\int d\mathbf{p}_\perp\int_0^\omega du(1-A)^{-1}\mathfrak{N}_{1\infty}[4(\omega-u)]^{-1}$$
$$\times\{u(\omega-u)^{-1}[\tfrac{1}{4}(1-A)|t|-\mathbf{p}_\perp^2-m^2+i\epsilon]-(-\mathbf{r}_1+\mathbf{p}_\perp)^2-m^2+i\epsilon\}^{-1}$$
$$\times\{u(\omega-u)^{-1}[\tfrac{1}{4}(1-A)|t|-\mathbf{p}_\perp^2-m^2+i\epsilon]-(\mathbf{r}_1+\mathbf{p}_\perp)^2-m^2+i\epsilon\}^{-1}$$
$$=2(4\pi)^{-3}e^4\int d\mathbf{p}_\perp\int_0^1 dA\,\mathfrak{N}_{1\infty}[\mathbf{p}_\perp^2-2(1-A)\mathbf{r}_1\cdot\mathbf{p}_\perp+(1-A)^2\mathbf{r}_1^2+m^2+i\epsilon]^{-1}$$
$$\times[\mathbf{p}_\perp^2+2(1-A)\mathbf{r}_1\cdot\mathbf{p}_\perp+(1-A)^2\mathbf{r}_1^2+m^2+i\epsilon]^{-1}$$
$$=2(4\pi)^{-3}e^4\int d\mathbf{p}_\perp\int_0^1 dA\{-16\delta_{ij}[(1-A)\mathbf{r}_1]^2-32A(1-A)$$
$$\times[p_\perp+(1-A)r_1]_i[p_\perp-(1-A)r_1]_j\}\{[\mathbf{p}_\perp-(1-A)\mathbf{r}_1]^2+m^2\}^{-1}\{[\mathbf{p}_\perp+(1-A)\mathbf{r}_1]^2+m^2\}^{-1}. \quad (4.10)$$

A similar but more complicated computation with (4.8), (2.19), and (2.2) gives

$$\mathcal{G}_2\sim(2\pi)^{-5}e^4\omega\int d^4p\,\mathfrak{N}_{2\infty}\int_{-\infty}^{\infty}dq_3[(\tfrac{1}{2}r_2-p)_3q_3+(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q_\perp)^2-m^2+i\epsilon]^{-1}$$
$$\times[-(\tfrac{1}{2}r_2-p)_3q_3+(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q_\perp)^2-m^2+i\epsilon]^{-1}[(\tfrac{1}{2}r_2+p)_3q_3+(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q_\perp)^2-m^2+i\epsilon]^{-1}$$
$$\times[-(\tfrac{1}{2}r_2+p)_3q_3+(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q_\perp)^2-m^2+i\epsilon]^{-1}$$
$$=-i(2\pi)^{-4}e^4\omega\int d^4p\,\mathfrak{N}_{2\infty}[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q_\perp)^2+(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q_\perp)^2-2m^2+i\epsilon]^{-1}$$
$$\times[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q_\perp)^2+(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q_\perp)^2-2m^2+i\epsilon]^{-1}$$
$$\times\{[(\tfrac{1}{2}r_2+p)_3(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q_\perp)^2+(\tfrac{1}{2}r_2-p)_3(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q_\perp)^2-r_{23}m^2+i\epsilon]^{-1}$$
$$+[(\tfrac{1}{2}r_2+p)_3(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q_\perp)^2+(\tfrac{1}{2}r_2-p)_3(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q_\perp)^2-r_{23}m^2+i\epsilon]^{-1}\}$$

$$\sim -\tfrac{1}{2}i(2\pi)^{-4}e^4\omega\int d\mathbf{p}_\perp du dv\, \mathfrak{N}_{2\infty}[(4u+2\omega)v+\tfrac{1}{8}(A+\tfrac{1}{2})|t|-\mathbf{p}_\perp^2-(\tfrac{1}{2}\mathbf{r}_1+\tfrac{1}{2}\mathbf{q}_\perp)^2-m^2+i\epsilon]^{-1}$$
$$\times[(4u-2\omega)v-\tfrac{1}{8}(A-\tfrac{1}{2})|t|-\mathbf{p}_\perp^2-(\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{q}_\perp)^2-m^2+i\epsilon]^{-1}$$
$$\times\{[-\omega(\mathbf{p}_\perp-\tfrac{1}{2}\mathbf{q}_\perp)^2-\tfrac{1}{4}uA|t|-u\mathbf{r}_1\cdot(2\mathbf{p}_\perp-\mathbf{q}_\perp)-m^2+i\epsilon]^{-1}$$
$$+[-\omega(\mathbf{p}_\perp+\tfrac{1}{2}\mathbf{q}_\perp)^2-\tfrac{1}{4}uA|t|+u\mathbf{r}_1\cdot(2\mathbf{p}_\perp+\mathbf{q}_\perp)-\omega m^2+i\epsilon]^{-1}\}$$
$$=2(4\pi)^{-3}e^4\omega^{-1}\int d\mathbf{p}_\perp\int_{-\omega/2}^{\omega/2} du\, \mathfrak{N}_{2\infty}\{(1+2A)[-\tfrac{1}{8}(A+\tfrac{1}{2})|t|-\mathbf{p}_\perp^2-(\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{q}_\perp)^2-m^2]$$
$$+(1-2A)[\tfrac{1}{8}(A+\tfrac{1}{2})|t|-\mathbf{p}_\perp^2-(\tfrac{1}{2}\mathbf{r}_1+\tfrac{1}{2}\mathbf{q}_\perp)^2-m^2]\}^{-1}\{[(\mathbf{p}_\perp-\tfrac{1}{2}\mathbf{q}_\perp+A\mathbf{r}_1)^2+m^2]^{-1}+[(\mathbf{p}_\perp+\tfrac{1}{2}\mathbf{q}_\perp-A\mathbf{r}_1)^2+m^2]^{-1}\}$$
$$=-2(4\pi)^{-3}e^4\int d\mathbf{p}_\perp\int_{-1/2}^{1/2} dA[-16\delta_{ij}(\tfrac{1}{2}\mathbf{q}_\perp-A\mathbf{r}_1)^2-8(1-4A^2)$$
$$\times(p_\perp+\tfrac{1}{2}q_\perp-Ar_1)_i(p_\perp-\tfrac{1}{2}q_\perp+Ar_1)_j][(\mathbf{p}_\perp-\tfrac{1}{2}\mathbf{q}_\perp+A\mathbf{r}_1)^2+m^2]^{-1}[(\mathbf{p}_\perp+\tfrac{1}{2}\mathbf{q}_\perp-A\mathbf{r}_1)^2+m^2]^{-1}. \quad (4.11)$$

Thus the impact factor for the photon is, by (2.27),

$$\mathcal{g}^\gamma=-\tfrac{1}{2}\pi^{-3}e^4\int d\mathbf{p}_\perp\int_0^1 dA\{[\delta_{ij}A^2\mathbf{r}_1^2+2A(1-A)(p_\perp+Ar_1)_i(p_\perp-Ar_1)_j][(\mathbf{p}_\perp-A\mathbf{r}_1)^2+m^2]^{-1}[(\mathbf{p}_\perp+A\mathbf{r}_1)^2+m^2]^{-1}$$
$$-[\delta_{ij}\mathbf{Q}^2+2A(1-A)(p_\perp+Q)_i(p_\perp-Q)_j][(\mathbf{p}_\perp+\mathbf{Q})^2+m^2]^{-1}[(\mathbf{p}_\perp-\mathbf{Q})^2+m^2]^{-1}\}, \quad (4.12)$$

where

$$\mathbf{Q}=\tfrac{1}{2}(\mathbf{q}+\mathbf{r}_1)-A\mathbf{r}_1. \quad (4.13)$$

This is the desired answer. It is shown in the Appendix directly that (4.12) and (2.29) are indeed equivalent.

## 5. DISCUSSION

The present paper is devoted exclusively to obtaining the impact factor of the photon, and to verifying its relation to the matrix element for Delbrück scattering. Nothing is said about other properties of the impact factor; they are already summarized in paper I.

In both Secs. 2 and 4, infinite quantities are manipulated. At least for the procedure of Sec. 4, this can be avoided by considering only the sum $\mathcal{g}_1+\mathcal{g}_2$. For the reader who is concerned with mathematical rigor, he may ignore both Secs. 2 and 4, because all the results are already contained in Sec. 3 together with paper III. However, Secs 2 and 4 are essential for future developments, the reason being that the development in paper III is by comparison much more complicated and hence too difficult to be readily generalized to higher-order diagrams. In a later paper, not to be included in this series, we shall study the electrodynamics of scalar particles to higher orders by the procedures of this paper.

One of the most important findings of the present consideration is that the longitudinal momentum transfer $q_3$ is small. However, it does not seem possible to state how small. For example, in obtaining (2.25) and (2.26), the important region of integration is

$$yq_3\approx\omega^{-1}. \quad (5.1)$$

But (5.1) can be satisfied by making $y$ and/or $q_3$ small. This point is also of great importance in connection with higher-order diagrams.

## APPENDIX

In this Appendix, we derive (2.29) from (4.12). Introducing the Feynman parameter $x$ and remembering that, for positive $a$ and $a'$,

$$\int d\mathbf{p}_\perp(\mathbf{p}_\perp^2+a)^{-2}=\pi/a \quad (A1)$$

and

$$\int d\mathbf{p}_\perp\mathbf{p}_\perp^2[(\mathbf{p}_\perp^2+a)^{-2}-(\mathbf{p}_\perp^2+a')^{-2}]=\pi\ln(a'/a), \quad (A2)$$

we obtain from (4.12) that

$$\mathcal{g}^{\gamma}=-\tfrac{1}{2}\pi^{-3}e^4\int d\mathbf{p}_\perp\int_0^1 dA\int_0^1 dx\{[\delta_{ij}A^2\mathbf{r}_1{}^2+2A(1-A)(p_\perp+Ar_1)_i(p_\perp-Ar_1)_j][\mathbf{p}_\perp{}^2-2A(1-2x)\mathbf{p}_\perp\cdot\mathbf{r}_1+A^2\mathbf{r}_1{}^2+m^2]^{-2}$$
$$-[\delta_{ij}\mathbf{Q}^2+2A(1-A)(p_\perp+Q)_i(p_\perp-Q)_j][\mathbf{p}_\perp{}^2+2(1-2x)\mathbf{p}_\perp\cdot\mathbf{Q}+\mathbf{Q}^2+m^2]^{-2}\}$$
$$=-\tfrac{1}{2}\pi^{-2}e^4\int_0^1 dA\int_0^1 dx\{[\tfrac{1}{4}\delta_{ij}A^2|t|-8A^3(1-A)x(1-x)r_{1i}r_{1j}]$$
$$\times[A^2|t|x(1-x)+m^2]^{-1}-[\delta_{ij}\mathbf{Q}^2-8A(1-A)x(1-x)Q_iQ_j][4\mathbf{Q}^2x(1-x)+m^2]^{-1}$$
$$+A(1-A)\delta_{ij}\ln[4\mathbf{Q}^2x(1-x)+m^2]/[A^2|t|x(1-x)+m^2]\}. \tag{A3}$$

By (4.13), it is seen that (A3) is identical with (2.29).

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# $\pi N$ Scattering in the Virasoro Model

YASUO HARA
*Department of Physics, Tokyo University of Education, Tokyo, Japan*
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A representation of the $\pi N$ scattering amplitude that contains Regge behavior, crossing symmetry, and analyticity is given by making use of the Virasoro model.

FOLLOWING Veneziano's discovery[1] of the scattering amplitude that contains Regge behavior, crossing symmetry, and analyticity, an alternative construction of the amplitude has been proposed by Virasoro.[2] In this paper we apply Virasoro's model to $\pi N$ scattering. That is, we construct the simplest $\pi N$ scattering amplitude that satisfies the following properties: (a) crossing symmetry between the $s$ and $u$ channels; (b) Regge behavior at asymptotic energies; (c) it satisfies all superconvergence sum rules; (d) the only singularities present (for linear trajectories, narrow-resonance approximation) are the simple poles corresponding to resonances on Regge trajectories; (e) there exist four leading trajectories, $\alpha_\rho(t)$, $\alpha_f(t)$, $\alpha_N(s)$, and $\alpha_\Delta(s)$.

We find a simple solution if these linear trajectories are parallel and if[3] $J=\alpha_\rho(t)=\alpha_f(t)$ $[=\alpha_{\mathrm{P}'}(t)]\equiv a_0+\alpha' t$ and[4] $J-\frac{1}{2}=\alpha_N(s)=\alpha_\Delta(s)\equiv\alpha(s)=\alpha_0+\alpha' t$. Our solution is as follows[5]:

$$A^{(+)}=\beta\left(-2\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u)-\frac{1}{2}a(t))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}a(t))}+\frac{\Gamma(1-\frac{1}{2}\alpha(s))\Gamma(1-\frac{1}{2}\alpha(u))}{\Gamma(1-\frac{1}{2}\alpha(u)-\frac{1}{2}a(t))\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}a(t))}\right)\frac{\Gamma(1-\frac{1}{2}a(t))}{\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))},$$

$$B^{(+)}=2\gamma\left(\frac{\Gamma(-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u)-\frac{1}{2}a(t))\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}a(t))}-\frac{\Gamma(-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}a(t))\Gamma(1-\frac{1}{2}\alpha(u)-\frac{1}{2}a(t))}\right)\frac{\Gamma(1-\frac{1}{2}a(t))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))}, \tag{1}$$

$$A^{(-)}=\beta\left(\frac{\Gamma(1-\frac{1}{2}\alpha(s))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u))}{\Gamma(1-\frac{1}{2}\alpha(u)-\frac{1}{2}a(t))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}a(t))}-\frac{\Gamma(1-\frac{1}{2}\alpha(u))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s))}{\Gamma(1-\frac{1}{2}\alpha(s)-\frac{1}{2}a(t))\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(u)-\frac{1}{2}a(t))}\right)\frac{\Gamma(\frac{1}{2}-\frac{1}{2}a(t))}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))},$$

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] M. A. Virasoro, Phys. Rev. **177**, 2309 (1969).
[3] Experimentally, $\alpha_\rho(t)\approx\alpha_{P'}(t)\approx(0.5-0.6)+1.0t$ GeV$^{-2}$.
[4] From the known nucleon resonances Barger and Cline have found $\alpha_N=-0.89+1.0s$ GeV$^{-2}$ and $\alpha_\Delta=-0.35+0.9s$ GeV$^{-2}$. See V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966); Phys. Rev. **155**, 1792 (1967).
[5] We use the notation introduced by G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).