High-Energy Collision Processes in Quantum Electrodynamics. III

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We obtain explicitly the asymptotic behavior of the matrix elements at high energies for Delbrück scattering, i.e., the elastic scattering of a photon by a static Coulomb field via a virtual electron-positron pair. This is the simplest nontrivial two-body elastic scattering process in quantum electrodynamics besides those discussed in the preceding paper. The considerations are limited to the lowest order, Z^2e^6, for the matrix elements, and it is found that, to this order, $\lim_{\omega\to\infty} d\sigma/dt$ exists and is nonzero for any fixed positive momentum transfer. This limiting value is expressed in terms of integrals, which are evaluated numerically. The behavior of these integrals is also studied in detail for the cases where the momentum transfer is either much larger or much smaller than the mass of the electron. Moreover, the scattered photon is significantly polarized in the scattering plane. None of the present results agree with the earlier ones of Bethe and Rohrlich using the impact-parameter approximation.

1. INTRODUCTION

THIRTY-SIX years ago, Delbrück[1] first proposed the possibility of the scattering of a photon by a static Coulomb field. In spite of the great advances in our knowledge of quantum electrodynamics in the intervening years, theoretical analysis of this process remains fragmentary. On the basis of the result obtained by Racah and by Jost, Luttinger, and Slotnick,[2] the particular case of Delbrück scattering in the forward direction, which is related to pair production by the optical theorem, has been calculated exactly, to the lowest order in the fine-structure constant α, by Toll and by Rohrlich and Gluckstern.[3] In the limit of high energies, their exact result is asymptotically

$$\mathfrak{M}_0^{(D)}\sim 4i\alpha^3Z^2\frac{\omega}{m^2}\frac{7}{9}\left(\ln\frac{2\omega}{m}-\frac{109}{42}-\frac{1}{2}i\pi\right)\delta_{ij}, \qquad (1.1)$$

where ω is the energy of the photon in the laboratory system (i.e., the frame where the Coulomb field is static), m is the mass of the electron, Ze is the charge responsible for the static Coulomb field, and δ_{ij} signifies that there is no change in photon polarization. No comparable result is so far available in the literature for nonforward scattering. By the impact-parameter approximation, Bethe and Rohrlich[4] have studied the case where $\omega\gg m$, and $\omega\theta\gtrsim m$, where θ is the scattering angle in the laboratory system. As will be discussed presently, their result is not completely correct.

It is the purpose of this paper to study Delbrück scattering by conventional relativistic perturbation theory. Before launching into this rather complicated calculation, let us discuss the various scales involved. Only one mass appears in this problem, namely, that of the electron. Let Δ be the magnitude of the momentum transfer, then m certainly is one of the scales for Δ. However, there is actually a second scale for Δ. To see this, consider the photoproduction in a Coulomb field of an electron-positron pair of equal energy and momentum.[5] In such a case, the momentum transfer is given by

$$\Delta=\omega-2[(\tfrac{1}{2}\omega)^2-m^2]^{1/2}. \qquad (1.2)$$

When $\omega\gg m$, (1.2) simplifies to

$$\Delta\sim 2m^2/\omega. \qquad (1.3)$$

Therefore, at high energies there are two scales for the momentum transfer Δ, namely, m and m^2/ω. An alternative way of expressing this is that there are two scales for θ, namely, m/ω and $(m/\omega)^2$.

A complete understanding of Delbrück scattering at high energies therefore requires the consideration of two regions:

$$\omega\gg m\gg\Delta \qquad \text{(Region A)} \qquad (1.4)$$

and

$$\omega\gg m,\quad \Delta\gg m^2/\omega \qquad \text{(Region B)}. \qquad (1.5)$$

The present paper (III) is devoted entirely to region B. More precisely, we compute here, to the order Z^2e^6,

$$\lim_{\omega\to\infty}\omega^{-1}\mathfrak{M}_0^{(D)} \qquad (1.6)$$

for any fixed nonzero momentum transfer. The existence of this limit (1.6) implies that, as $\omega\to\infty$ for fixed $=-\Delta^2$, $d\sigma/dt$ in the lowest nonvanishing order in αt approaches a finite limit. The detailed study of region A,

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[4] H. A. Bethe and F. Rohrlich, Phys. Rev. 86, 10 (1952).
[5] The matrix element for pair production happens to be zero at this point, but this fact does not affect the estimate of the order of magnitude here.

which includes the forward direction, is to be carried out in paper VIII of this series.

Without any computation, we can get some information about the region common to A and B. Let $\mathbf{\Delta}_i$ be the components of the momentum transfer, then $\mathfrak{M}_0^{(D)}$ can always be expressed in the form

$$\mathfrak{M}_0^{(D)}=4i\alpha^3Z^2\omega m^{-2}[f_1\delta_{ij}+f_2\mathbf{\Delta}_i\mathbf{\Delta}_j/\Delta^2]. \quad (1.7)$$

It follows from (1.1) that, when $\omega\gg m$ and $\Delta=0$,

$$f_1\sim\frac{7}{9}\left(\ln\frac{2\omega}{m}-\frac{109}{42}-\frac{1}{2}i\pi\right)$$
and
$$f_2\sim 0. \quad (1.8)$$

Accordingly, in region A, f_1 and f_2 are of the forms

$$f_1\sim(7/9)\ln(\omega/m)+\text{function of }(\Delta\omega/m^2)$$
and
$$f_2\sim\text{function of }(\Delta\omega/m^2); \quad (1.9)$$

while, in region B, we have instead, for $n=1,2$,

$$f_n\sim\text{function of }(\Delta/m). \quad (1.10)$$

A comparison of (1.9) and (1.10) shows that smooth connection is possible only if, in the overlapping region $\omega\gg m\gg\Delta\gg m^2/\omega$,

$$f_1\sim(7/9)\ln(m/\Delta)+C_1$$
and
$$f_2\sim C_2, \quad (1.11)$$

where C_1 and C_2 are two constants. In other words, if we are only interested in this overlapping region, it is sufficient to determine these two constants C_1 and C_2.

The present calculation shows that [Eq. (3.32) below]

$$C_1=19/27 \quad\text{and}\quad C_2=-\tfrac{1}{9}. \quad (1.12)$$

This is to be compared with the result of Bethe and Rohrlich[4]

$$C_1^{(BR)}=(7/9)(\tfrac{1}{2}+\ln 2-\gamma)\sim 0.47906$$
and
$$C_2^{(BR)}=0. \quad (1.13)$$

It is seen that there is no resemblance between (1.12) and (1.13). This discrepancy is clearly due to the inadequacy of the impact-parameter approximation that they employed.

There are several practical motivations for the present consideration. First, since the differential cross section for Delbrück scattering is extremely large at high energies, the question may be raised whether the scattered beam is useful. Second, there is a recent attempt at the Cambridge Electron Accelerator to measure Compton scattering near the forward direction using a tagged photon beam of 2–4 BeV. If and when this measurement can be pushed to sufficiently small scattering angles, a knowledge of Delbrück scattering becomes essential.

Results are summarized and discussed in Sec. 4. Readers who are not interested in the derivation can go directly to that section.

2. EXTERNAL FIELD APPROXIMATION

A. Formulation

In the external field approximation, the graphs under consideration are the two shown in Fig. 7 of paper I. Their contributions to the matrix element of Delbrück scattering are, respectively ($i,j=1,2,3$),

$$\mathfrak{M}_1=2i(2\pi)^{-7}e^6Z^2\int d^4p\, d^3q[(r_2+p)^2-m^2]^{-1}[(-r_1+p)^2-m^2]^{-1}[(p+q)^2-m^2]^{-1}$$
$$\times[(r_1+p)^2-m^2]^{-1}[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}\,\mathrm{Tr}\gamma_i(\mathbf{r}_1+\mathbf{p}+m)\gamma_0(\mathbf{p}+\mathbf{q}+m)\gamma_0(-\mathbf{r}_1+\mathbf{p}+m)\gamma_j(\mathbf{r}_2+\mathbf{p}+m) \quad (2.1)$$

and

$$\mathfrak{M}_2=i(2\pi)^{-7}e^6Z^2\int d^4p\, d^3q[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}$$
$$\times\mathrm{Tr}\gamma_i(\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{r}_2-\mathbf{p}-\tfrac{1}{2}\mathbf{q}+m)\gamma_0(-\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{r}_2-\mathbf{p}+\tfrac{1}{2}\mathbf{q}+m)$$
$$\times\gamma_j(\tfrac{1}{2}\mathbf{r}_1+\tfrac{1}{2}\mathbf{r}_2-\mathbf{p}+\tfrac{1}{2}\mathbf{q}+m)\gamma_0(-\tfrac{1}{2}\mathbf{r}_1+\tfrac{1}{2}\mathbf{r}_2-\mathbf{p}-\tfrac{1}{2}\mathbf{q}+m). \quad (2.2)$$

In (2.1) and (2.2),

$$r_1=-\tfrac{1}{2}(k_1+k_2) \quad\text{and}\quad r_2=\tfrac{1}{2}(k_1-k_2), \quad (2.3)$$

where k_1 is the four-momentum of the incident photon, and $-k_2$ that of the scattered photon, so that $k_{10}=-k_{20}=\omega$. We make the following remarks about (2.1) and (2.2): (i) In the external field approximation, the static Coulomb field can absorb any momentum transfer and cannot take up any energy. Thus, q means $(0,\mathbf{q})$. (ii) The metric $g_{\mu\nu}$ is given by $g_{00}=1$, $g_{ii}=-1$ for $i=1, 2, 3$. (iii) Since there is no confusion, all $+i\epsilon$ have been omitted in (2.1) and (2.2). (iv) A factor of 2 has been incorporated to take into account the two possible directions of drawing the internal electron loop. (v) Strictly speaking, both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are really not defined because of ultraviolet divergences associated with the p integrations. This divergence is, however, well understood and does not appear in the sum

$$\mathfrak{M}_0^{(D)}=\mathfrak{M}_1+\mathfrak{M}_2. \tag{2.4}$$

Moreover, this divergence is irrelevant in the limit of interest $\omega/m\rightarrow\infty$. (vi) Because of gauge invariance, the two graphs of Fig. 7 of I must be supplemented by a third graph involving a four-photon contact interaction. In the limit $\omega/m\rightarrow\infty$, this third graph is important only when the momentum transfer is of the order of m^2/ω. A detailed discussion of this point is therefore postponed until paper VIII of this series. (vii) A question may be raised about the range of validity of the external field approximation. More precisely, is the said approximation valid when the incident photon has a larger energy than the mass of the target? That the answer is yes is discussed in detail in Appendix A for the special case $Z=1$.

The divergent terms mentioned above in (v) come from the term $-4(p^2)^2\delta_{ij}$ contained in the trace of (2.1) and the term $4(p^2)^2\delta_{ij}$ in the trace of (2.2). If these two terms are deleted, it is found that, as $\omega\rightarrow\infty$, both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are of the order of magnitude $\omega(\ln\omega)^2$, while $\mathfrak{M}_1+\mathfrak{M}_2$ is of the order ω. Therefore, a great deal of cancellation exists between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. In Sec. 2 C, we rearrange the various terms in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ so that a number of the cancellations are taken care of in a trivial manner, and in Sec. 2 E a convenient set of variables is introduced for the other cancellations. For this purpose, we first study the traces in Sec. 2 B.

B. Traces

Let $\mathfrak{N}_1$ be the trace that appears in the integrand of (2.1), and $\mathfrak{N}_2$ that of (2.2). In this section, we obtain the leading terms of $\mathfrak{N}_1$ and $\mathfrak{N}_2$ as $\omega\rightarrow\infty$. These quantities $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are studied in greater detail in paper VIII.

We must first define leading terms. Basically, they are those terms that have contributions of the order ω or larger to the integrals as $\omega\rightarrow\infty$. Since $r_{20}=\omega$, we are therefore looking for terms of the forms

$$\omega^2,\quad \omega p_0,\quad \text{or}\quad p_0^2. \tag{2.5}$$

However, note that the integral in $\mathfrak{M}_1$ contains the denominator $(r_2+p)^2-m^2$. In the absence of this particular denominator, the integral (2.1) has no dependence on ω. Thus, this combination $(r_2+p)^2-m^2$ is negligible in $\mathfrak{N}_1$. There is no such simplification in $\mathfrak{N}_2$.

Consider $\mathfrak{N}_1$ first. By (2.1),

$$\mathfrak{N}_1=2p_0\,\mathrm{Tr}\gamma_i(r_1+p+m)\gamma_0(-r_1+p+m)\gamma_j(r_2+p+m)$$
$$-\mathrm{Tr}\gamma_i(r_1+p+m)(p+q-m)(-r_1+p+m)\gamma_j(r_2+p+m). \tag{2.6}$$

The second term in (2.6) does not contain anything of the desired form (2.5), and hence can be neglected. The first term of (2.6) leads to

$$\mathfrak{N}_1\sim 2p_0^2\,\mathrm{Tr}\gamma_i(-r_1+p+m)\gamma_j(r_2+p+m)+2p_0^2\,\mathrm{Tr}\gamma_i(r_1+p+m)\gamma_j(r_2+p+m)$$
$$+2p_0(\omega+p_0)\,\mathrm{Tr}\gamma_i(r_1+p+m)(-r_1+p-m)\gamma_j. \tag{2.7}$$

Simple, explicit evaluation of the right-hand side yields

$$\mathfrak{N}_1\sim 16p_0^2\{p_i(r_2+p)_j+p_j(r_2+p)_i+\delta_{ij}[p\cdot(r_2+p)-m^2]\}+8p_0(\omega+p_0)[2r_{1i}p_j-2p_ir_{1j}-\delta_{ij}(p^2-r_1^2-m^2)]. \tag{2.8}$$

Since $(r_2+p)^2-m^2$ is negligible, (2.8) is finally simplified to

$$\mathfrak{N}_1\sim 8p_0\omega[2r_{1i}p_j-2p_ir_{1j}-\delta_{ij}(p^2-r_1^2-m^2)]+8p_0^2[4p_ip_j-2p_i(r_1-r_2)_j+2(r_1+r_2)_ip_j+\delta_{ij}(r_1^2-r_2^2)]. \tag{2.9}$$

Attention is next turned to $\mathfrak{N}_2$ for (2.2). Instead of the three terms on the right-hand side of (2.7), there are now four terms as follows:

$$\mathfrak{N}_2\sim -2(\tfrac{1}{4}\omega^2-p_0^2)\,\mathrm{Tr}\gamma_i(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q+m)\gamma_j(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q+m)$$
$$+2(\tfrac{1}{2}\omega+p_0)^2\,\mathrm{Tr}\gamma_i\gamma_j(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q-m)(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q+m)$$
$$+2(\tfrac{1}{2}\omega-p_0)^2\,\mathrm{Tr}\gamma_i(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q-m)(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q+m)\gamma_j$$
$$-2(\tfrac{1}{4}\omega^2-p_0^2)\,\mathrm{Tr}\gamma_i(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q+m)\gamma_j(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q+m). \tag{2.10}$$

The rest of the computation is rather straightforward and leads to the desired answer

$$\mathfrak{N}_2\sim 2\omega^2[-(2p-q)_i(2p+q)_j-\delta_{ij}(4p^2-q^2-4m^2)]-8\omega p_0[-2r_{1i}p_j-r_{2i}q_j+2p_ir_{1j}+q_ir_{2j}-\delta_{ij}(2r_2p+r_1q)]$$
$$+8p_0^2[(r_1+r_2)_i(r_1-r_2)_j+(2p+q)_i(2p-q)_j+\delta_{ij}(r_1^2-r_2^2)]. \tag{2.11}$$

In (2.11), we have omitted a term $(r_1-r_2)_i(r_1+r_2)_j$ because the photons are transverse.

The approximations (2.9) and (2.11) are to be used in (2.1) and (2.2).

C. Cancellation

A few of the terms in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ cancel each other as follows:

$$2\int d^4p\, d^3q[(r_2+p)^2-m^2]^{-1}[(-r_1+p)^2-m^2]^{-1}[(p+q)^2-m^2]^{-1}[(r_1+p)^2-m^2]^{-1}$$
$$\times[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}(-8p_0\omega\delta_{ij})(p^2+r_1^2-m^2)+\int d^4p\, d^3q[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}\delta_{ij}[-8\omega^2(p^2+\tfrac{1}{4}q^2-m^2)+8\omega p_0(2r_2p-r_1q)]=0. \quad (2.12)$$

This identity is easily proved by partial fractions, since the left-hand side of (2.12) is

$$2\int d^4p\, d^3q[(r_2+p)^2-m^2]^{-1}[(p+q)^2-m^2]^{-1}[(r_1+p)^2-m^2]^{-1}[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}(-4p_0\omega\delta_{ij})$$
$$+2\int d^4p\, d^3q[(r_2+p)^2-m^2]^{-1}[(-r_1+p)^2-m^2]^{-1}[(p+q)^2-m^2][(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}(-4p_0\omega\delta_{ij})$$
$$+\int d^4p\, d^3q[(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}\delta_{ij}(-2\omega^2-4\omega p_0)+\int d^4p\, d^3q[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}\delta_{ij}(-2\omega^2-4\omega p_0)$$
$$+\int d^4p\, d^3q[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}\delta_{ij}(-2\omega^2+4\omega p_0)+\int d^4p\, d^3q[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}\delta_{ij}(-2\omega^2+4\omega p_0). \quad (2.13)$$

In writing down (2.13), we have used

$$r_1^2+r_2^2=0. \quad (2.14)$$

Linear transformations on the variables p and q then show that, in (2.13), the fourth and sixth terms taken together cancel the first terms while the remaining three terms also give zero. This proves (2.12).

It is a consequence of (2.12) that

$$\mathfrak{M}_0^{(D)}=\mathfrak{M}_1+\mathfrak{M}_2\sim\mathfrak{M}_1'+\mathfrak{M}_2', \quad (2.15)$$

where

$$\mathfrak{M}_1'=2i(2\pi)^{-7}e^6Z^2\int d^4p\, d^3q[(r_2+p)^2-m^2]^{-1}[(-r_1+p)^2-m^2]^{-1}[(p+q)^2-m^2]^{-1}$$
$$\times[(r_1+p)^2-m^2]^{-1}[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}$$
$$\times\{8p_0\omega[2r_{1i}p_j-2p_ir_{1j}+2r_1^2\delta_{ij}]+8p_0^2[4p_ip_j-2p_i(r_1-r_2)_j+2(r_1+r_2)_ip_j+\delta_{ij}(r_1^2-r_2^2)]\} \quad (2.16)$$

and

$$\mathfrak{M}_2'=i(2\pi)^{-7}e^6Z^2\int d^4p\, d^3q[(-\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1+\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}$$
$$\times[(-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p+\tfrac{1}{2}q)^2-m^2]^{-1}[(\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-p-\tfrac{1}{2}q)^2-m^2]^{-1}[(r_1+q)^2]^{-1}[(r_1-q)^2]^{-1}$$
$$\times\{2\omega^2[-(2p-q)_i(2p+q)_j+2q^2\delta_{ij}]-8\omega p_0[-2r_{1i}p_j-r_{2i}q_j+2p_ir_{1j}+q_ir_{2j}-2(r_1q)\delta_{ij}]$$
$$+8p_0^2[(r_1+r_2)_i(r_1-r_2)_j+(2p+q)_i(2p-q)_j+\delta_{ij}(r_1^2-r_2^2)]\}. \quad (2.17)$$

Note that, because of the transversality of the photon,

$$k_{1i}=k_{2j}=0, \tag{2.18}$$

or

$$r_{2i}=r_{1i} \quad \text{and} \quad r_{2j}=-r_{1j}. \tag{2.19}$$

D. Feynman Parameters

The next step is to introduce Feynman parameters for the various denominators:

$$\mathfrak{M}_1'=2i(2\pi)^{-7}e^6Z^2 5!\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)$$
$$\int d^4p\, d^3q\, D_1^{-6}\{8p_0\omega[2r_{1i}p_j-2p_ir_{1j}+\tfrac{1}{2}t\delta_{ij}]+8p_0^2[4p_ip_j-4p_ir_{1j}+4r_{1i}p_j+\tfrac{1}{2}t\delta_{ij}]\}, \tag{2.20}$$

$$\mathfrak{M}_2'=i(2\pi)^{-7}e^6Z^2 5!\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)$$
$$\int d^4p\, d^3q\, D_2^{-6}\{2\omega^2[-(2p-q)_i(2p+q)_j+2q^2\delta_{ij}]-8\omega p_0[-r_{1i}(2p+q)_j+(2p-q)_ir_{1j}-2(r_1q)\delta_{ij}]$$
$$+8p_0^2[4r_{1i}r_{1j}+(2p+q)_i(2p-q)_j+\tfrac{1}{2}t\delta_{ij}]\}, \tag{2.21}$$

where

$$D_1=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)p^2+2\alpha_3pq+(\alpha_3+\alpha_5+\alpha_6)q^2+2p[(-\alpha_2+\alpha_4)r_1+\alpha_1r_2]$$
$$+2(\alpha_5-\alpha_6)(r_1q)+[\tfrac{1}{4}(-\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6)t-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)m^2]+i\epsilon \tag{2.22}$$

and

$$D_2=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)p^2+(\alpha_1-\alpha_2-\alpha_3+\alpha_4)pq+[\tfrac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_5+\alpha_6]q^2$$
$$+p[(\alpha_1-\alpha_2+\alpha_3-\alpha_4)r_1+(-\alpha_1-\alpha_2+\alpha_3+\alpha_4)r_2]$$
$$+2q\{[\tfrac{1}{4}(\alpha_1+\alpha_2-\alpha_3-\alpha_4)+\alpha_5-\alpha_6]r_1-\tfrac{1}{4}(\alpha_1-\alpha_2+\alpha_3-\alpha_4)r_2\}+[\tfrac{1}{4}(\alpha_5+\alpha_6)t-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)m^2]+i\epsilon. \tag{2.23}$$

In (2.20) and (2.21), we have used the notation

$$t=(k_1+k_2)^2=4r_1^2=-4r_2^2<0. \tag{2.24}$$

Both D_1 and D_2 are quadratic forms in p and $\mathbf{q}$. Remembering that $q_0=0$ is not a variable, let

$$p_0=\delta p_0'+p_0'=\delta p_0''+p_0'',$$
$$\mathbf{p}=\delta\mathbf{p}'+\mathbf{p}'=\delta\mathbf{p}''+\mathbf{p}'',$$

and

$$\mathbf{q}=\delta\mathbf{q}'+\mathbf{q}'=\delta\mathbf{q}''+\mathbf{q}'', \tag{2.25}$$

so that

$$D_1=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(p_0'^2-\mathbf{p}'^2)-2\alpha_3\mathbf{p}'\cdot\mathbf{q}'-(\alpha_3+\alpha_5+\alpha_6)\mathbf{q}'^2+c_1 \tag{2.26}$$

and

$$D_2=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(p_0''^2-\mathbf{p}''^2)-(\alpha_1-\alpha_2-\alpha_3+\alpha_4)\mathbf{p}''\cdot\mathbf{q}''-[\tfrac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_5+\alpha_6]\mathbf{q}''^2+c_2. \tag{2.27}$$

In (2.25)–(2.27), the various quantities are explicitly

$$\delta p_0'=-\alpha_1(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1}\omega,$$
$$\delta\mathbf{p}'=\Lambda_1^{-1}\{[(\alpha_2-\alpha_4)(\alpha_3+\alpha_5+\alpha_6)+\alpha_3(\alpha_5-\alpha_6)]\mathbf{r}_1-\alpha_1(\alpha_3+\alpha_5+\alpha_6)\mathbf{r}_2\},$$
$$\delta\mathbf{q}'=-\Lambda_1^{-1}\{[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5-\alpha_6)+(\alpha_2-\alpha_4)\alpha_3]\mathbf{r}_1-\alpha_1\alpha_3\mathbf{r}_2\}, \tag{2.28}$$

$$\delta p_0''=\tfrac{1}{2}(\alpha_1+\alpha_2-\alpha_3-\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1}\omega,$$
$$\delta\mathbf{p}''=\tfrac{1}{2}\Lambda_2^{-1}\{-[2(\alpha_1-\alpha_2)\alpha_6+2(\alpha_3-\alpha_4)\alpha_5+\alpha_1\alpha_3-\alpha_2\alpha_4]\mathbf{r}_1+[(\alpha_1+\alpha_2-\alpha_3-\alpha_4)(\alpha_5+\alpha_6)+\alpha_1\alpha_2-\alpha_3\alpha_4]\mathbf{r}_2\},$$
$$\delta\mathbf{q}''=-\Lambda_2^{-1}\{[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5-\alpha_6)+\alpha_1\alpha_2-\alpha_3\alpha_4]\mathbf{r}_1-(\alpha_1\alpha_3-\alpha_2\alpha_4)\mathbf{r}_2\}, \tag{2.29}$$

$$c_1=\Lambda_1^{-1}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1}\alpha_1^2\alpha_3^2\omega^2-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)m^2$$
$$+t\Lambda_1^{-1}[\alpha_5\alpha_6(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_4\alpha_5(\alpha_2+\alpha_3)+\alpha_2\alpha_6(\alpha_3+\alpha_4)+\alpha_2\alpha_3\alpha_4]+i\epsilon, \tag{2.30}$$

and

$$c_2=\Lambda_2^{-1}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1}(\alpha_1\alpha_3-\alpha_2\alpha_4)^2\omega^2-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)m^2$$
$$+t\Lambda_2^{-1}[\alpha_5\alpha_6(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_1\alpha_2\alpha_6+\alpha_3\alpha_4\alpha_5]+i\epsilon, \tag{2.31}$$

where

$$\Lambda_1=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_3+\alpha_5+\alpha_6)-\alpha_3^2, \tag{2.32}$$

and

$$\Lambda_2=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5+\alpha_6)+(\alpha_1+\alpha_4)(\alpha_2+\alpha_3). \tag{2.33}$$

In order to obtain the leading terms for $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ as $\omega\to\infty$, it is permissible to replace the p_0's of (2.20) and (2.21) by $\delta p_0'$ and $\delta p_0''$, respectively. This replacement has the further advantage of removing the ultraviolet divergences mentioned in Sec. 2 A. Therefore, by (2.28) and (2.29),

$$\begin{aligned}\mathfrak{M}_1'\sim 2i(2\pi)^{-7}e^6Z^2\omega^2 5!\int_0^1 & d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6\,\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)\int d^4p'\,d^3q' \\ & \times[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(p_0'^2-\mathbf{p}'^2)-2\alpha_3\mathbf{p}'\cdot\mathbf{q}'-(\alpha_3+\alpha_5+\alpha_6)\mathbf{q}'^2+c_1]^{-6} \\ & \times\{-8\alpha_1(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1}[2r_{1i}\delta p_j'-2(\delta p_i')r_{1j}+\tfrac{1}{2}t\delta_{ij}] \\ & \quad +8\alpha_1^2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-2}[4(\delta p_i')(\delta p_j')+\tfrac{4}{3}\delta_{ij}\mathbf{p}'^2-4(\delta p_i')r_{1j}+4r_{1i}\delta p_j'+\tfrac{1}{2}t\delta_{ij}]\}\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}\mathfrak{M}_2'\sim i(2\pi)^{-7}e^6Z^2\omega^2 5!\int_0^1 & d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6\,\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)\int d^4p''\,d^3q'' \\ & \times\{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(p_0''^2-\mathbf{p}''^2)-(\alpha_1-\alpha_2-\alpha_3+\alpha_4)\mathbf{p}''\cdot\mathbf{q}''-[\tfrac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_5+\alpha_6]\mathbf{q}''^2+c_2\}^{-6} \\ & \times\{2[-(2\delta p''-\delta q'')_i(2\delta p''+\delta q'')_j-\tfrac{1}{3}(4\mathbf{p}''^2-\mathbf{q}''^2)\delta_{ij}-2(\delta\mathbf{q}'')^2\delta_{ij}-2\mathbf{q}''^2\delta_{ij}] \\ & -4(\alpha_1+\alpha_2-\alpha_3-\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1}[-r_{1i}(2\delta p''+\delta q'')_j+(2\delta p''-\delta q'')_i r_{1j}+2(\mathbf{r}_1\cdot\delta\mathbf{q}'')\delta_{ij}] \\ & +2(\alpha_1+\alpha_2-\alpha_3-\alpha_4)^2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-2}[4r_{1i}r_{1j}+(2\delta p''+\delta q'')_i(2\delta p''-\delta q'')_j+\tfrac{1}{3}(4\mathbf{p}''^2-\mathbf{q}''^2)+\tfrac{1}{2}t\delta_{ij}]\}.\end{aligned} \tag{2.35}$$

Application of the formula

$$\int d^4p\,d^3q[A(p_0^2-\mathbf{p}^2)-2B\mathbf{p}\cdot\mathbf{q}-C\mathbf{q}^2+c]^{-6}\begin{bmatrix}1\\ \mathbf{p}^2\\ \mathbf{q}^2\end{bmatrix}=\begin{bmatrix}(-c)^{-1}\\ C(AC-B^2)^{-1}\\ A(AC-B^2)^{-1}\end{bmatrix} i\pi^4A^{-1/2}(AC-B^2)^{-3/2}(-c)^{-3/2}/160 \tag{2.36}$$

to (2.34) and (2.35) yields, without additional approximation,

$$\begin{aligned}\mathfrak{M}_1'\sim -3(2\pi)^{-3}e^6Z^2\omega^2\int_0^1 & d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6\,\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-5/2}\Lambda_1^{-5/2}(-c_1)^{-5/2}\alpha_1 \\ & \times\{\alpha_1\Lambda_1^{-1}r_{1i}r_{1j}[-(2\alpha_2+\alpha_3)(2\alpha_4+\alpha_3)(\alpha_3+\alpha_5+\alpha_6)^2+\alpha_3^2(-\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_3+\alpha_5+\alpha_6) \\ & +2(\alpha_2-\alpha_4)\alpha_3(\alpha_3+\alpha_5+\alpha_6)(\alpha_5-\alpha_6)+\alpha_3^2(\alpha_5-\alpha_6)^2]-\delta_{ij}[\tfrac{1}{8}t\Lambda_1(\alpha_2+\alpha_3+\alpha_4)+\tfrac{1}{3}\alpha_1(\alpha_3+\alpha_5+\alpha_6)c_1]\}\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}\mathfrak{M}_2'\sim -\tfrac{3}{8}(2\pi)^{-3}e^6Z^2\omega^2\int_0^1 & d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6\,\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1/2}\Lambda_2^{-5/2}(-c_2)^{-5/2} \\ & \times[-4(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-2}\Lambda_2^{-1}r_{1i}r_{1j}\{(\alpha_3\alpha_5-\alpha_2\alpha_6)(\alpha_4\alpha_5-\alpha_1\alpha_6)(\alpha_1+\alpha_2-\alpha_3-\alpha_4)^2 \\ & -[(\alpha_3\alpha_5-\alpha_2\alpha_6)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+(\alpha_2+\alpha_3)(\alpha_1\alpha_3-\alpha_2\alpha_4)] \\ & \times[(\alpha_4\alpha_5-\alpha_1\alpha_6)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-(\alpha_1+\alpha_4)(\alpha_1\alpha_3-\alpha_2\alpha_4)]\} \\ & +\delta_{ij}\{\tfrac{1}{8}t\Lambda_2^{-1}[(\alpha_1+\alpha_2-\alpha_3-\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-1}\Lambda_2-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5-\alpha_6)-(\alpha_1\alpha_2-\alpha_3\alpha_4)]^2 \\ & -\tfrac{1}{8}t\Lambda_2^{-1}(\alpha_1\alpha_3-\alpha_2\alpha_4)^2+\tfrac{4}{3}(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-2}(\alpha_5+\alpha_6)c_2 \\ & -\tfrac{1}{2}\omega^2\Lambda_2^{-1}(\alpha_1\alpha_3-\alpha_2\alpha_4)^2+\tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)c_2\}].\end{aligned} \tag{2.38}$$

E. Change of Variables

As given by (2.37) and (2.38), $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ depend on ω through the presence of the combinations ω^2, c_1, and c_2. In view of the fact that the integrand of (2.37) contains an over-all factor α_1, an examination of (2.30) and (2.31) shows that the major contributions to $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ come from the regions

$$\alpha_3 \sim 0 \tag{2.39}$$

and

$$\alpha_1\alpha_3 - \alpha_2\alpha_4 \sim 0, \tag{2.40}$$

respectively. More precisely, in (2.38) we can neglect $\alpha_1\alpha_3 - \alpha_2\alpha_4$ except when it is multiplied by ω as in the next-to-last term, and in (2.37) we can neglect α_3 compared with $\alpha_5+\alpha_6$. [It is, however, incorrect in (2.37) to neglect α_3 compared with $\alpha_1+\alpha_2+\alpha_4$. A great deal of caution must be exercised in making the approximations.] The results are

$$\mathfrak{M}_1' \sim -3(2\pi)^{-3}e^6Z^2\omega^2 \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-5/2}\Lambda_1^{-5/2}(-c_1)^{-5/2}\alpha_1$$
$$\times\{-4\alpha_1\Lambda_1^{-1}r_{1i}r_{1j}[\alpha_2(\alpha_5+\alpha_6)+\alpha_3\alpha_5][\alpha_4(\alpha_5+\alpha_6)+\alpha_3\alpha_6]-\delta_{ij}[\tfrac{1}{8}t\Lambda_1(\alpha_2+\alpha_3+\alpha_4)+\tfrac{1}{3}\alpha_1(\alpha_5+\alpha_6)c_1]\} \tag{2.41}$$

and

$$\mathfrak{M}_2' \sim -\tfrac{3}{2}(2\pi)^{-3}e^6Z^2\omega^2 \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-5/2}\Lambda_2^{-5/2}(-c_2)^{-5/2}$$
$$\times\{4\Lambda_2^{-1}r_{1i}r_{1j}(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)(\alpha_3\alpha_5-\alpha_2\alpha_6)(\alpha_4\alpha_5-\alpha_1\alpha_6)$$
$$+\delta_{ij}[\tfrac{1}{8}t\Lambda_2^{-1}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2(\alpha_3\alpha_5+\alpha_4\alpha_5-\alpha_1\alpha_6-\alpha_2\alpha_6)^2+\tfrac{1}{3}(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)(\alpha_5+\alpha_6)c_2]$$
$$+\tfrac{1}{8}\delta_{ij}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)c_2-\omega^2\Lambda_2^{-1}(\alpha_1\alpha_3-\alpha_2\alpha_4)^2]\}. \tag{2.42}$$

Because of the symmetry of the graphs shown in Fig. 7 of I, it is actually sufficient to integrate over the region $\alpha_5<\alpha_6$ for $\mathfrak{M}_1$ (and $\mathfrak{M}_1'$) and the region $\alpha_5<\alpha_6$ and $\alpha_1\alpha_3>\alpha_2\alpha_4$ for $\mathfrak{M}_2$ (and $\mathfrak{M}_2'$).

The next step is to introduce a new set of variables so that the regions (2.39) and (2.40) coincide in terms of these new variables. This seems essential in order to effect the numerous cancellations between $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ all at once. A particularly convenient choice of variables is the following:

for $\mathfrak{M}_1'$:

$$\beta=\alpha_2+\alpha_3+\alpha_4,\quad \beta'=\alpha_1,\quad x=\alpha_2/(\alpha_2+\alpha_3+\alpha_4),\quad y=\alpha_3/(\alpha_3+\alpha_4); \tag{2.43}$$

and for $\mathfrak{M}_2'$:

$$\beta=\alpha_1+\alpha_2,\quad \beta'=\alpha_3+\alpha_4,\quad x=\alpha_2/(\alpha_1+\alpha_2),\quad y=\alpha_1^{-1}(\alpha_3+\alpha_4)^{-1}(\alpha_1\alpha_3-\alpha_2\alpha_4). \tag{2.44}$$

With (2.43) and (2.44), (2.41) and (2.42) lead to exactly

$$\mathfrak{M}_1' \sim -6(2\pi)^{-3}e^6Z^2\omega^2 \int_0^1 d\beta\, d\beta'\, dx\, dy \int_{\alpha_6>\alpha_5>0} d\alpha_5 d\alpha_6 \delta(1-\beta-\beta'-\alpha_5-\alpha_6)\beta^2(1-x)(\beta+\beta')^{-5/2}\Lambda_1^{-5/2}(-c_1)^{-5/2}\beta'$$
$$\times\{-4\beta'\Lambda_1^{-1}r_{1i}r_{1j}[\beta x(\alpha_5+\alpha_6)+\beta(1-x)y\alpha_5][\beta(1-x)(1-y)(\alpha_5+\alpha_6)+\beta(1-x)y\alpha_6]$$
$$-\delta_{ij}[\tfrac{1}{8}\beta t\Lambda_1+\tfrac{1}{3}\beta'(\alpha_5+\alpha_6)c_1]\} \tag{2.45}$$

and

$$\mathfrak{M}_2' \sim -6(2\pi)^{-3}e^6Z^2\omega^2 \int_0^1 d\beta\, d\beta'\, dx\, dy \int_{\alpha_6>\alpha_5>0} d\alpha_5 d\alpha_6 \delta(1-\beta-\beta'-\alpha_5-\alpha_6)\beta\beta'(1-x)(\beta+\beta')^{-5/2}\Lambda_2^{-5/2}(-c_2)^{-5/2}$$
$$\times\{4\beta\beta'\Lambda_2^{-1}r_{1i}r_{1j}[\beta'(x+y-xy)\alpha_5-\beta x\alpha_6][\beta'(1-x)(1-y)\alpha_5-\beta(1-x)\alpha_6]$$
$$+\delta_{ij}[\tfrac{1}{8}t\Lambda_2^{-1}(\beta+\beta')^2(\beta'\alpha_5-\beta\alpha_6)^2+\tfrac{1}{3}\beta\beta'(\alpha_5+\alpha_6)c_2]$$
$$+\tfrac{1}{8}\delta_{ij}(\beta+\beta')^2[(\beta+\beta')c_2-\Lambda_2^{-1}\omega^2\beta^2\beta'^2(1-x)^2y^2]\}. \tag{2.46}$$

The similarity between the right-hand sides of (2.45) and (2.46) is already striking.

Since the major contributions come from the region $y\sim 0$, we may ask whether y can be neglected when not multiplied by ω. The answer to this question is in general no, because of the complicated structures of the integrands in the corner where β, β', and α_5 are all small. However, a careful examination of this corner reveals that, in (2.45) and (2.46), we *can* neglect

$$\beta y, \quad \beta' y, \quad \text{and} \quad \alpha_5 y. \tag{2.47}$$

For example, in (2.45),

$$\beta(1-x)(1-y)(\alpha_5+\alpha_6)+\beta(1-x)y\alpha_6 \tag{2.48}$$

appears. The second term of (2.48) cannot be neglected; fortunately, it is cancelled by the first term, and by (2.47), (2.48) can be approximated by $\beta(1-x)(\alpha_5+\alpha_6)$.

After neglecting terms of the form (2.47), we get

$$\Lambda_1 \sim (\beta+\beta')(\alpha_5+\alpha_6), \tag{2.49}$$

$$\Lambda_2 \sim (\beta+\beta')[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)], \tag{2.50}$$

and the substitution of (2.45) and (2.46) into (2.15) gives

$$\begin{aligned}\mathfrak{M}_0^{(D)} \sim\ & 6(2\pi)^{-3}e^6Z^2\omega^2 \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 dx \int_0^1 dy \int_{\alpha_6>\alpha_5>0} d\alpha_5 d\alpha_6 \delta(1-\beta-\beta'-\alpha_5-\alpha_6)\beta\beta'(1-x)(\beta+\beta')^{-5} \\ & \times [\{4\beta\beta'(\beta+\beta')^{-1}x(1-x)r_{1i}r_{1j}+\tfrac{1}{8}\delta_{ij}t(\beta+\beta')\} \\ & \times\{\beta^2(\alpha_5+\alpha_6)^{-3/2}(-c_1)^{-5/2}-(\beta'\alpha_5-\beta\alpha_6)^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-7/2}(-c_2)^{-5/2}\} \\ & -\tfrac{1}{3}\delta_{ij}\beta\beta'(\alpha_5+\alpha_6)\{(\alpha_5+\alpha_6)^{-5/2}(-c_1)^{-3/2}-[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-5/2}(-c_2)^{-3/2}\} \\ & -\tfrac{1}{8}\delta_{ij}(\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-5/2}(-c_2)^{-5/2}[(\beta+\beta')c_2-\Lambda_2^{-1}\omega^2\beta^2\beta'^2(1-x)^2y^2]].\end{aligned} \tag{2.51}$$

By (2.30) and (2.31), in (2.51) c_1 and c_2 can be approximated by the forms

$$\beta^2\beta'^2(\beta+\beta')^{-2}(\alpha_5+\alpha_6)^{-1}(1-x)^2y^2\omega^2+\text{terms independent of } y \tag{2.52}$$

and

$$\beta^2\beta'^2(\beta+\beta')^{-2}[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1}(1-x)^2y^2\omega^2+\text{terms independent of } y, \tag{2.53}$$

respectively.

F. Asymptotic Behavior for Large ω

The last approximation is to replace, in (2.51), the upper limit of integration for the variable y by infinity. Once this is done, (2.52) and (2.53) show that the integration with respect to y can be easily carried out. The final result is that, as $\omega\to\infty$ for fixed $t\neq 0$, the matrix element $\mathfrak{M}_0^{(D)}$ for Delbrück scattering is given by (3.2) of paper I. The right-hand side of that expression is rather complicated, and we study some of its properties in Sec. 3.

3. SOME PROPERTIES OF RESULT FOR $\omega \gg m$

A. Formalism

Let G be defined by

$$\text{right-hand side of (3.2) of I} = \tfrac{1}{2}i(2\pi)^{-3}e^6Z^2\omega|t|^{-1}G; \tag{3.1}$$

then G is of the form

$$G=-G_1\delta_{ij}+G_2r_{1i}r_{1j}/|r_1^2|, \tag{3.2}$$

where G_1 and G_2 are functions of

$$\tau=|t|/m^2 \tag{3.3}$$

only. In this section, we study the behavior of G_1 and G_2 in the two limits of large and small $|t|$. Explicitly, the two functions under consideration are

$$\begin{aligned}G_1(\tau)=\ & \int_0^1 d\beta \int_0^1 d\beta' \int_0^1 dx \int_0^1 d\alpha_6 \int_0^{\alpha_6} d\alpha_5 \delta(1-\beta-\beta'-\alpha_5-\alpha_6)(\beta+\beta')^{-3} \\ & \times((\beta+\beta')\tau^2[\beta^2(\alpha_5+\alpha_6)\{\tau[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]+(\beta+\beta')^2(\alpha_5+\alpha_6)\}^{-2} \\ & -(\beta'\alpha_5-\beta\alpha_6)^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1} \\ & \times\{\tau[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]+(\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\}^{-2}] \\ & +\tau[4\beta\beta'\{\tau[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]+(\beta+\beta')^2(\alpha_5+\alpha_6)\}^{-1} \\ & -[4\beta\beta'(\alpha_5+\alpha_6)+(\beta+\beta')^3][(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1} \\ & \times\{\tau[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]+(\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\}^{-1}])\end{aligned} \tag{3.4}$$

and

$$G_2(\tau)=\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{\alpha_6} d\alpha_5\delta(1-\beta-\beta'-\alpha_5-\alpha_6)(\beta+\beta')^{-3}8\beta\beta' x(1-x)\tau^2$$
$$\times[\beta^2(\alpha_5+\alpha_6)\{\tau[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]+(\beta+\beta')^2(\alpha_5+\alpha_6)\}^{-2}$$
$$-(\beta'\alpha_5-\beta\alpha_6)^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1}$$
$$\times\{\tau[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]+(\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\}^{-2}]. \quad (3.5)$$

The difficulty in studying the behavior of G stems from the following fact. Let (3.4) be symbolically written as

$$G_1(\tau)=\int \text{integrand},$$

where $\int$ stands for the fivefold integration over β, β', x, α_5, and α_6. Then it turns out that

$$\lim_{\tau\to\infty}\int \text{integrand}\neq\int \lim_{\tau\to\infty}\text{integrand}, \quad (3.6)$$

even though both sides of (3.6) exist. This is perhaps one of the few natural examples in physics where an integration and a limiting process fail to commute.

We shall employ Mellin transforms to discuss the asymptotic behaviors of G_1 and G_2. The reason is that Mellin transforms are particularly convenient when various powers of $\ln\tau$ appear.[6] Let

$$\bar{G}_n(\zeta)=\int_0^\infty G_n(\tau)\tau^{-1-\zeta}d\tau \quad (3.7)$$

for $n=1,2$. Since

$$\int_0^\infty (A\tau+B)^{-1}\tau^{-\zeta}d\tau=\pi(\csc\pi\zeta)B^{-\zeta}A^{-1+\zeta}$$

and

$$\int_0^\infty (A\tau+B)^{-2}\tau^{1-\zeta}d\tau=\pi(\csc\pi\zeta)(1-\zeta)B^{-\zeta}A^{-2-\zeta}, \quad (3.8)$$

the substitution of (3.4) and (3.5) into (3.7) gives

$$\bar{G}_1(\zeta)=\pi\csc\pi\zeta\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{\alpha_6} d\alpha_5\delta(1-\beta-\beta'-\alpha_5-\alpha_6)(\beta+\beta')^{-3}$$
$$\times[(1-\zeta)(\beta+\beta')^{2-2\zeta}\{\beta^2(\alpha_5+\alpha_6)^{1-\zeta}[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]^{-2+\zeta}$$
$$-(\beta'\alpha_5-\beta\alpha_6)^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1-\zeta}[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]^{-2+\zeta}\}$$
$$+4\beta\beta'(\beta+\beta')^{-2\zeta}(\alpha_5+\alpha_6)^{-\zeta}[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]^{-1+\zeta}-[4\beta\beta'(\alpha_5+\alpha_6)+(\beta+\beta')^3](\beta+\beta')^{-2\zeta}$$
$$\times[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1-\zeta}[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]^{-1+\zeta}] \quad (3.9)$$

and

$$\bar{G}_2(\zeta)=\pi\csc\pi\zeta\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{\alpha_6} d\alpha_5\delta(1-\beta-\beta'-\alpha_5-\alpha_6)(\beta+\beta')^{-3-2\zeta}8\beta\beta' x(1-x)(1-\zeta)$$
$$\times\{\beta^2(\alpha_5+\alpha_6)^{1-\zeta}[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]^{-2+\zeta}$$
$$-(\beta'\alpha_5-\beta\alpha_6)^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1-\zeta}[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]^{-2+\zeta}\}. \quad (3.10)$$

We need to know the behavior of $\bar{G}_1(\zeta)$ and $\bar{G}_2(\zeta)$ in the vicinity of their singularities at $\zeta=$integer.

As the initial step in the reduction of (3.9) and (3.10), it is convenient to let

$$\sigma=\beta+\beta'. \quad (3.11)$$

[6] J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963).

Then

$$1-\sigma=\alpha_5+\alpha_6, \tag{3.12}$$

and (3.9) and (3.10) can be rewritten as

$$\begin{aligned}\bar{G}_1(\zeta)=\pi\csc\pi\zeta\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\int_0^1 d\sigma\,\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\sigma^{-2}(1-\sigma)\\ \times\{(1-\zeta)\sigma^{2-\zeta}(1-\sigma)^{-1}\beta^2[(1-\sigma)\alpha_5\alpha_6+\sigma\beta^2x(1-x)]^{-2+\zeta}-(1-\zeta)\sigma^{2-\zeta}(1-\sigma)^{\zeta}(\beta'\alpha_5-\beta\alpha_6)^2\\ \times[(1-\sigma)+\sigma x(1-x)]^{-1-\zeta}[(1-\sigma)\alpha_5\alpha_6+(\beta^2\alpha_6+\beta'^2\alpha_5)\sigma x(1-x)]^{-2+\zeta}+4\sigma^{1-\zeta}(1-\sigma)^{-1}\beta\beta'\\ \times[(1-\sigma)\alpha_5\alpha_6+\sigma\beta^2x(1-x)]^{-1+\zeta}-\sigma^{1-\zeta}(1-\sigma)^{-1+\zeta}[4\beta\beta'(1-\sigma)+\sigma][(1-\sigma)+\sigma x(1-x)]^{-1-\zeta}\\ \times[(1-\sigma)\alpha_5\alpha_6+(\beta^2\alpha_6+\beta'^2\alpha_5)\sigma x(1-x)]^{-1+\zeta}\}\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\bar{G}_2(\zeta)=\pi\csc\pi\zeta\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\int_0^1 d\sigma\,\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\sigma^{-2\zeta}(1-\sigma)\beta\beta'x(1-x)(1-\zeta)\\ \times\{\sigma^{\zeta}(1-\sigma)^{-1}\beta^2[(1-\sigma)\alpha_5\alpha_6+\sigma\beta^2x(1-x)]^{-2+\zeta}-\sigma^{\zeta}(1-\sigma)^{\zeta}(\beta'\alpha_5-\beta\alpha_6)^2[(1-\sigma)+\sigma x(1-x)]^{-1-\zeta}\\ \times[(1-\sigma)\alpha_5\alpha_6+(\beta^2\alpha_6+\beta'^2\alpha_5)\sigma x(1-x)]^{-2+\zeta}\}.\end{aligned} \tag{3.14}$$

A further change of variable to

$$\rho=\sigma x(1-x)/(1-\sigma) \tag{3.15}$$

makes it possible to carry out the x integration:

$$\begin{aligned}\bar{G}_1(\zeta)=\pi\csc\pi\zeta[\Gamma(\zeta)]^2[\Gamma(2\zeta)]^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\int_0^\infty d\rho\,\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\rho^{-1-\zeta}\\ \times\{(1-\zeta)[\alpha_5\alpha_6+\beta^2\rho]^{-2+\zeta}\beta^2\rho-(1-\zeta)(\beta'\alpha_5-\beta\alpha_6)^2(1+\rho)^{-1-\zeta}[\alpha_5\alpha_6+(\beta^2\alpha_6+\beta'^2\alpha_5)\rho]^{-2+\zeta}\rho\\ +4\beta\beta'[\tfrac{1}{2}\zeta(1+2\zeta)^{-1}][\alpha_5\alpha_6+\beta^2\rho]^{-1+\zeta}-[2\beta\beta'\zeta(1+2\zeta)^{-1}+\rho](1+\rho)^{-1-\zeta}[\alpha_5\alpha_6+(\beta^2\alpha_6+\beta'^2\alpha_5)\rho]^{-1+\zeta}\}\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\bar{G}_2(\zeta)=\pi\csc\pi\zeta[\Gamma(1+\zeta)]^2[\Gamma(2+2\zeta)]^{-1}(1-\zeta)\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\int_0^\infty d\rho\,\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\rho^{-\zeta}8\beta\beta'\\ \times\{\beta^2[\alpha_5\alpha_6+\beta^2\rho]^{-2+\zeta}-(\beta'\alpha_5-\beta\alpha_6)^2(1+\rho)^{-1-\zeta}[\alpha_5\alpha_6+(\beta^2\alpha_6+\beta'^2\alpha_5)\rho]^{-2+\zeta}\}.\end{aligned} \tag{3.17}$$

Use of Euler's integral representation for hypergeometric functions[7] finally yields the desired forms for $\bar{G}_1(\zeta)$ and $\bar{G}_2(\zeta)$:

$$\begin{aligned}\bar{G}_1(\zeta)=\pi\csc\pi\zeta[\Gamma(\zeta)]^2[\Gamma(2\zeta)]^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\alpha_5^{-1}\alpha_6^{-1}\\ \times\{[1-2\beta\beta'(1+2\zeta)^{-1}][\beta^{2\zeta}-\pi\zeta(\csc\pi\zeta)(1-\zeta)(\alpha_5\alpha_6)^{-1+\zeta}(\beta^2\alpha_6+\beta'^2\alpha_5)F(2-\zeta,1-\zeta;2;z)]\\ -\pi\zeta^2\csc\pi\zeta(\alpha_5\alpha_6)^{\zeta}[1-4\beta\beta'(1+2\zeta)^{-1}]F(1-\zeta,1-\zeta;2;z)\}\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\bar{G}_2(\zeta)=\pi\csc\pi\zeta[\Gamma(1+\zeta)]^2[\Gamma(2+2\zeta)]^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}\\ \times[\beta^{2\zeta}-\tfrac{1}{2}(1-\zeta^2)\pi\zeta\csc\pi\zeta(\alpha_5\alpha_6)^{-1+\zeta}(\beta'\alpha_5-\beta\alpha_6)^2F(2-\zeta,1-\zeta;3;z)],\end{aligned} \tag{3.19}$$

where

$$z=1-(\beta^2\alpha_6+\beta'^2\alpha_5)/(\alpha_5\alpha_6)=-(\beta'\alpha_5-\beta\alpha_6)^2/(\alpha_5\alpha_6). \tag{3.20}$$

In obtaining (3.18) and (3.19), the following special cases of the relations of Gauss between contiguous hypergeometric functions[8] have been used:

$$(\beta^2\alpha_6+\beta'^2\alpha_5)F(2-\zeta,1-\zeta;2;z)-\alpha_5\alpha_6F(1-\zeta,1-\zeta;2;z)-\tfrac{1}{2}(1+\zeta)(\beta'\alpha_5-\beta\alpha_6)^2F(2-\zeta,1-\zeta;3;z)=0$$

and

$$2\zeta\alpha_5\alpha_6F(1-\zeta,1-\zeta;2;z)+(1-\zeta)(\beta^2\alpha_6+\beta'^2\alpha_5)F(2-\zeta,1-\zeta;2;z)-(1+\zeta)\alpha_5\alpha_6F(1-\zeta,-\zeta;2;z)=0. \tag{3.21}$$

[7] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 60, Eq. (12).

[8] See Eqs. (38) and (33) on p. 103 of Ref. 7.

So far, no approximation has been made on the G's, and (3.18) and (3.19) are exact. The rest of this section is devoted to the approximate calculation of $\bar{G}_1(\zeta)$ and $\bar{G}_2(\zeta)$. Further properties of these functions are to be found in Appendices C–F.

B. Behavior of $\bar{G}_1(\zeta)$ near $\zeta=1$

To study the behavior of $\bar{G}_1(\zeta)$ near $\zeta=1$, let $\xi=\zeta-1$. Then, for small ξ

$$\bar{G}_1(\zeta)\sim-\xi^{-1}[\Gamma(1+\xi)]^2[\Gamma(2+2\xi)]^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\alpha_5^{-1}\alpha_6^{-1}$$
$$\times\{[1-\tfrac{2}{3}\beta\beta'][\beta^2-(\beta^2\alpha_6+\beta'^2\alpha_5)]+\xi^{-1}(1+\xi)^2(\alpha_5\alpha_6)^{1+\xi}[1-4\beta\beta'(3+2\xi)^{-1}]\}$$
$$\sim-\xi^{-2}\int_0^{1/2} d\alpha_5(\alpha_5\alpha_6)^{\xi}[1-\tfrac{2}{3}(3+2\xi)^{-1}]$$
$$\sim-(7/18)\xi^{-2}+(19/27)\xi^{-1}.$$

In other words, when ζ is near 1,

$$\bar{G}_1(\zeta)=-(7/18)(\zeta-1)^{-2}+(19/27)(\zeta-1)^{-1}+O(1). \tag{3.22}$$

C. Behavior of $\bar{G}_2(\zeta)$ near $\zeta=1$

The computation for $\bar{G}_2(\zeta)$ is even simpler, namely, as a consequence of (3.19),

$$\bar{G}_2(\zeta)=-\tfrac{1}{6}(\zeta-1)^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}[\beta^2-(\beta'\alpha_5-\beta\alpha_6)^2]+O(1)$$
$$=-\tfrac{1}{9}(\zeta-1)^{-1}+O(1). \tag{3.23}$$

D. Behavior of $\bar{G}_1(\zeta)$ near $\zeta=0$

This case is somewhat more complicated. In view of (3.18), we need the two leading terms of $F(2-\zeta,1-\zeta;2;z)$. This can be obtained as follows:

$$F(2-\zeta,1-\zeta;2;z)\sim F(2,1-\zeta;2;z)+F(2-\zeta,1;2;z)-F(2,1;2;z)$$
$$\sim(1-z)^{-1+\zeta}+\sum_{n=0}^{\infty}z^n\Gamma(2-\zeta+n)/[(n+1)!\Gamma(2-\zeta)]-(1-z)^{-1}$$
$$\sim(1-z)^{-1+\zeta}-(1-z)^{-1}+(1-\zeta)^{-1}z^{-1}[(1-z)^{-1+\zeta}-1]. \tag{3.24}$$

With z defined by (3.20), (3.24) yields immediately

$$\beta^{2\zeta}-\pi\zeta(\csc\pi\zeta)(1-\zeta)(\alpha_5\alpha_6)^{-1+\zeta}(\beta^2\alpha_6+\beta'^2\alpha_5)F(2-\zeta,1-\zeta;2;z)$$
$$\sim\beta^{2\zeta}-(1-\zeta)(\beta^2\alpha_6+\beta'^2\alpha_5)^{\zeta}+(1-\zeta)(\alpha_5\alpha_6)^{\zeta}+(\beta'\alpha_5-\beta\alpha_6)^{-2}\{\alpha_5\alpha_6(\beta^2\alpha_6+\beta'^2\alpha_5)^{\zeta}-[(\beta'\alpha_5-\beta\alpha_6)^2+\alpha_5\alpha_6](\alpha_5\alpha_6)^{\zeta}\}$$
$$\sim\{-\ln[(\beta^2\alpha_6+\beta'^2\alpha_5)\beta^{-2}]+\alpha_5\alpha_6(\beta'\alpha_5-\beta\alpha_6)^{-2}\ln[\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)]\}\zeta. \tag{3.25}$$

The substitution of (3.25) into (3.18) then shows that $\bar{G}_1(\zeta)$ has a simple pole at $\zeta=0$ given by

$$\bar{G}_1(\zeta)=\zeta^{-1}(G_3+G_4+G_5)+O(1), \tag{3.26}$$

where G_3, G_4, and G_5 are three real numbers given by[9]

$$G_3=4\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\beta\beta'(\beta'\alpha_5-\beta\alpha_6)^{-2}\ln[\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)],$$

$$G_4=4\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\beta\beta'\alpha_5^{-1}\alpha_6^{-1}\ln[\beta^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)],$$

and

$$G_5=-2\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\alpha_5^{-1}\alpha_6^{-1}\ln[\beta^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)]. \tag{3.27}$$

[9] Note that $F(1,1;2;z)=-z^{-1}\ln(1-z)$.

As a matter of fact, G_5 is the difference between the right- and left-hand sides of (3.6). These three integrals (3.27) can all be evaluated explicitly (see Appendix B):

$$G_3=2,$$
$$G_4=(2\pi^2/9)-\tfrac{4}{3},$$
and
$$G_5=-\tfrac{2}{3}\pi^2. \tag{3.28}$$

Therefore, for ζ near 0,

$$\bar{G}_1(\zeta)=[\tfrac{2}{3}-(4\pi^2/9)]\zeta^{-1}+O(1). \tag{3.29}$$

E. Behavior of $G_2(\zeta)$ near $\zeta=0$

This is again simple. Since

$$F(2,1;3;z)=-2z^{-2}[z+\ln(1-z)], \tag{3.30}$$

a comparison of (3.19) and (3.27) shows that

$$\bar{G}_2(\zeta)\sim\zeta^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5$$
$$\times\delta(1-\beta-\beta')8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}$$
$$\times[1-\tfrac{1}{2}(\alpha_5\alpha_6)^{-1}(\beta'\alpha_5-\beta\alpha_6)^2F(2,1;3;z)]$$
$$=2G_3/\zeta$$
or
$$\bar{G}_2(\zeta)=4\zeta^{-1}+O(1) \tag{3.31}$$
for ζ near 0.

F. Results

By (3.7), (3.22) and (3.23) show that, for $|t|\ll m^2$,

$$G_1\sim[-(7/18)\ln(m^2/|t|)-19/27]|t|/m^2$$
and
$$G_2\sim\tfrac{1}{9}|t|/m^2. \tag{3.32}$$

On the other hand, (3.29) and (3.31) show that

$$\lim_{|t|\to\infty}G_1=\tfrac{2}{3}-4\pi^2/9$$
and
$$\lim_{|t|\to\infty}G_2=4. \tag{3.33}$$

We have obtained much more precise information about G_1 and G_2: For $|t|\ll m^2$,

$$G_1\sim[-(7/18)\ln(m^2/|t|)-19/27]|t|/m^2+[(13/450)\ln(m^2/|t|)-91/3375](|t|/m^2)^2$$
and
$$G_2\sim\tfrac{1}{9}|t|/m^2+[(2/225)\ln(m^2/|t|)+17/3375](|t|/m^2)^2. \tag{3.34}$$

Similarly, for $|t|\gg m^2$,

$$G_1\sim(\tfrac{2}{3}-4\pi^2/9)-\{6[\ln(|t|/m^2)]^2+20\ln(|t|/m^2)+[36-8\zeta(3)]\}m^2/|t|$$
and
$$G_2\sim4-\{\tfrac{4}{3}[\ln(|t|/m^2)]^3+4[\ln(|t|/m^2)]^2+(8-16\pi^2/3)\ln(|t|/m^2)+8[1+\tfrac{2}{3}\pi^2+12\zeta(3)]\}m^2/|t|, \tag{3.35}$$

where $\zeta(3)$ is the value of the Riemann ζ function at 3. Numerically, (3.35) is

$$G_1\sim-3.719824178-\{6[\ln(|t|/m^2)]^2+20\ln(|t|/m^2)+26.383544774\}m^2/|t|,$$
and
$$G_2\sim4-\{\tfrac{4}{3}[\ln(|t|/m^2)]^3+4[\ln(|t|/m^2)]^2-44.63789014\ln(|t|/m^2)+176.03535284\}m^2/|t|. \tag{3.36}$$

The derivation of (3.34) and (3.35) is rather lengthy and given in Appendices C–F.

4. SUMMARY AND DISCUSSION

Let $\mathfrak{M}_\perp$ (or $\mathfrak{M}_{\parallel}$) be the matrix element of Delbrück scattering by a Coulomb field due to a charge Ze when the photon is linearly polarized in the direction perpendicular to the scattering plane (or in the scattering plane), then, for high energies ω of the photon and fixed momentum transfer $\Delta\neq0$, the matrix elements are given approximately by

$$\begin{bmatrix}\mathfrak{M}_\perp\\ \mathfrak{M}_{\parallel}\end{bmatrix}\sim-\tfrac{1}{2}i(2\pi)^{-3}e^6Z^2\omega\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 dx\int_0^1 d\alpha_6\int_0^{\alpha_6} d\alpha_5\delta(1-\beta-\beta'-\alpha_5-\alpha_6)(\beta+\beta')^{-3}$$
$$\times([4\beta\beta'\{\Delta^2[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]+m^2(\beta+\beta')^2(\alpha_5+\alpha_6)\}^{-1}$$
$$-[4\beta\beta'(\alpha_5+\alpha_6)+(\beta+\beta')^3][(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1}$$
$$\times\{\Delta^2[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]+m^2(\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\}^{-1}]$$
$$+\Delta^2\begin{bmatrix}(\beta+\beta')^2\\ \overline{(\beta+\beta')^2-8\beta\beta'x(1-x)}\end{bmatrix}[\beta^2(\alpha_5+\alpha_6)\{\Delta^2[(\beta+\beta')\alpha_5\alpha_6+\beta^2x(1-x)(\alpha_5+\alpha_6)]$$
$$+m^2(\beta+\beta')^2(\alpha_5+\alpha_6)\}^{-2}-(\beta'\alpha_5-\beta\alpha_6)^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]^{-1}$$
$$\times\{\Delta^2[(\beta+\beta')\alpha_5\alpha_6+x(1-x)(\beta^2\alpha_6+\beta'^2\alpha_5)]+m^2(\beta+\beta')^2[(\alpha_5+\alpha_6)+(\beta+\beta')x(1-x)]\}^{-2}]), \tag{4.1}$$

where the units are so chosen that $e^2/4\pi \sim 137.04^{-1}$ is the fine structure constant. When the incident photon beam is not polarized, the differential cross section is

$$d\sigma/d\Omega = \tfrac{1}{2}(4\pi)^{-2}[|\mathfrak{M}_\perp|^2 + |\mathfrak{M}_\parallel|^2], \tag{4.2}$$

and the polarization of the scattered beam is

$$P = [|\mathfrak{M}_\parallel|^2 - |\mathfrak{M}_\perp|^2]/[|\mathfrak{M}_\perp|^2 + |\mathfrak{M}_\parallel|^2]. \tag{4.3}$$

When $\omega \gg \Delta \gg m$,

$$\mathfrak{M}_\perp \sim \tfrac{1}{3}i(2\pi)^{-3}e^6 Z^2\omega(-1+\tfrac{2}{3}\pi^2)\Delta^{-2}, \tag{4.4}$$

$$\mathfrak{M}_\parallel \sim \tfrac{1}{3}i(2\pi)^{-3}e^6 Z^2\omega(5+\tfrac{2}{3}\pi^2)\Delta^{-2}, \tag{4.5}$$

and

$$P \sim (12+4\pi^2)/[13+(8\pi^2/3)+(4\pi^4/9)] \sim 62.3136\%. \tag{4.6}$$

When $\omega \gg m \gg \Delta \gg m^2/\omega$,

$$\mathfrak{M}_\perp \sim \tfrac{1}{2}i(2\pi)^{-3}e^6 Z^2\omega[(7/9)\ln(m/\Delta)+19/27]m^{-2}, \tag{4.7}$$

$$\mathfrak{M}_\parallel \sim \tfrac{1}{2}i(2\pi)^{-3}e^6 Z^2\omega[(7/9)\ln(m/\Delta)+22/27]m^{-2}, \tag{4.8}$$

and

$$P \sim (1/7)[\ln(m/\Delta)+41/42]/\{[\ln(m/\Delta)]^2+(41/21)\ln(m/\Delta)+845/882\}. \tag{4.9}$$

More accurate asymptotic expansions for the matrix elements can be easily found from (3.34) and (3.35). For $\omega \gg \Delta \gg m$,

$$\begin{aligned}\mathfrak{M}_\perp &\sim \tfrac{1}{3}i(2\pi)^{-3}e^6 Z^2\omega\Delta^{-2}[(-1+\tfrac{2}{3}\pi^2)+6m^2\Delta^{-2}\{6[\ln(\Delta/m)]^2+10\ln(\Delta/m)+[9-2\zeta(3)]\}]\\ &\sim \tfrac{1}{3}i(2\pi)^{-3}e^6 Z^2\omega\Delta^{-2}[5.57973627+6m^2\Delta^{-2}\{6[\ln(\Delta/m)]^2+10\ln(\Delta/m)+6.59588619\}]\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\mathfrak{M}_\parallel &\sim \tfrac{1}{3}i(2\pi)^{-3}e^6 Z^2\omega\Delta^{-2}[(5+\tfrac{2}{3}\pi^2)+6m^2\Delta^{-2}\{-(8/3)[\ln(\Delta/m)]^3+2[\ln(\Delta/m)]^2\\ &\qquad +(6+(8\pi^2/3))\ln(\Delta/m)+[7-\tfrac{4}{3}\pi^2-26\zeta(3)]\}]\\ &\sim \tfrac{1}{3}i(2\pi)^{-3}e^6 Z^2\omega\Delta^{-2}[11.57973627+6m^2\Delta^{-2}\{-(8/3)[\ln(\Delta/m)]^3+2[\ln(\Delta/m)]^2\\ &\qquad +32.31894507\ln(\Delta/m)-74.82590403\}].\end{aligned} \tag{4.11}$$

Note the appearance of $[\ln(\Delta/m)]^3$ in the second term. So far as the authors are aware, the appearance of so many logarithms all at once is very rare. Thus, the correction is not negligible even for fairly large values of $|t|/m^2$. On the other hand, for $\omega \gg m \gg \Delta \gg m^2/\omega$,

$$\begin{aligned}\mathfrak{M}_\perp \sim \tfrac{1}{2}i(2\pi)^{-3}e^6 Z^2\omega m^{-2}\{&[(7/9)\ln(m/\Delta)+19/27]\\ &+\Delta^2 m^{-2}[-(13/225)\ln(m/\Delta)+91/3375]\}\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\mathfrak{M}_\parallel \sim \tfrac{1}{2}i(2\pi)^{-3}e^6 Z^2\omega m^{-2}\{&[(7/9)\ln(m/\Delta)+22/27]\\ &+\Delta^2 m^{-2}[-(1/25)\ln(m/\Delta)+4/125]\}.\end{aligned} \tag{4.13}$$

When Δ is comparable to m, there is no longer such simple formula, and (4.1) must be used. Numerical calculations have been carried out by integrating (4.1),[10] and the results are shown in Figs. 1–3. To get the differential cross section for Delbrück scattering from a nucleus of charge Ze, it is necessary to multiply the value obtained from Fig. 1 by Z^4.

As stated in the Introduction, the present results do not agree with the earlier ones of Bethe and Rohrlich.[4] We make some further qualitative comparisons. First, as seen from Figs. 1 and 2, the differential cross section is a monotonically decreasing function of the momen-

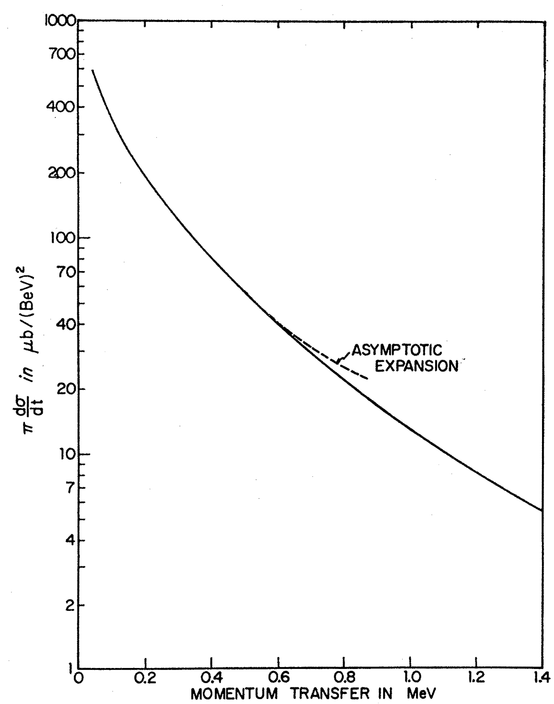

FIG. 1. Differential cross sections at high energies. (To get $d\sigma/d\Omega$ in μb/sr for a nucleus of charge Ze, multiply by $Z^4\omega^2$, where ω is the energy of the photon in BeV.)

[10] We thank Professor L. Howard for this numerical calculation.

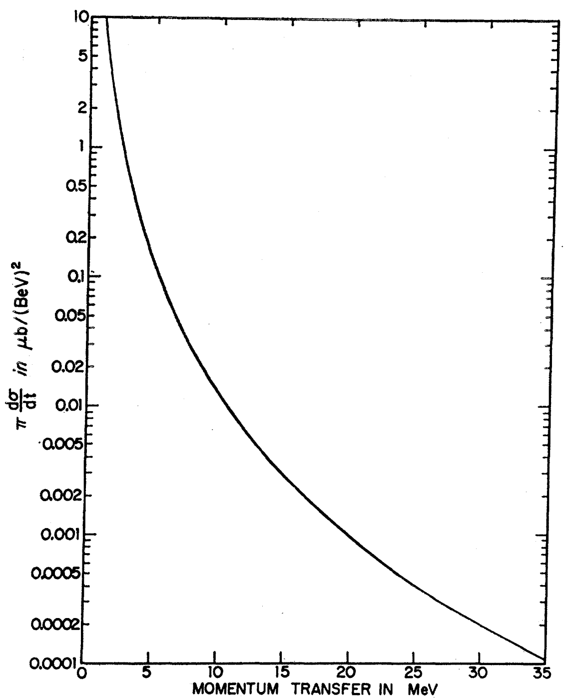

FIG. 2. Differential cross sections at high energies (continued).

tum transfer. If the result of Bethe and Rohrlich[4] is taken seriously to large momentum transfers, the differential cross section is not monotonic. This peculiarity due to their approximation was known to these authors and was probably the reason for their restriction $\Delta < m$. Secondly, Bethe and Rohrlich[4] obtained no polarization, while the polarization is actually quite large as shown in Fig. 3. Indeed, their lack of polarization was originally the reason why we were suspicious of their calculation.

Moffatt and Stringfellow[11] have measured Delbrück scattering some time ago at around 90 MeV. A comparison of their result and our asymptotic formula is

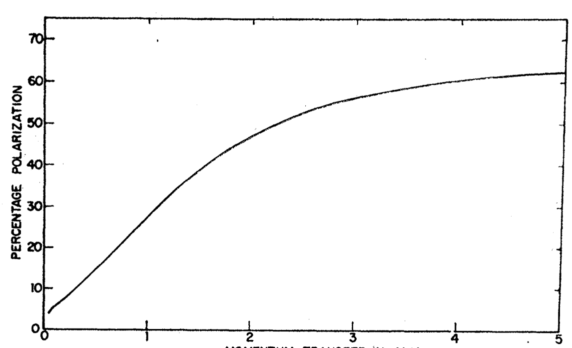

FIG. 3. Polarization of the scattered photon at high energies when the incident photon is unpolarized.

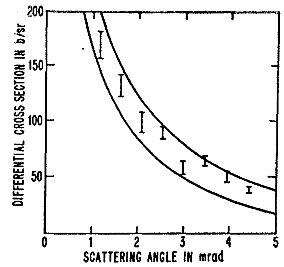

FIG. 4. Numerical comparison with the experimental data of Moffatt and Stringfellow on uranium. The upper curve is the result of the present consideration, while the lower curve is that of Bethe and Rohrlich.

shown in Fig. 4. We attribute the discrepancy to the fact that $Z\alpha$ is not small, and return to this point in paper VIII of this series.

We next discuss the physical limitations due to other processes. First, the effect of the muon can be easily obtained by reinterpreting m, and is in any case quite small. Secondly, because of the sharp decrease of the differential cross section as shown in Fig. 2, processes due to strong interactions become important at large momentum transfers. If $Z=1$, i.e., if the target is hydrogen, Compton scattering is comparable to Delbrück scattering at about $\Delta \sim 5$ MeV.[12] On the other hand, when the target is a heavy nucleus, nuclear breakup may be of importance at very roughly $\Delta \sim 30$ MeV. Thus, there is little point in extending any further the curve of Fig. 2.

Finally, we discuss briefly the experimental determination of Delbrück cross sections at high energies. At energies of a few BeV, the experiment seems feasible[13] with the most straightforward setup, namely, a well-collimated photon beam on a very thin uranium target (for example) together with a photon detecting system about 2 m away. The photon detecting system can be similar to the one used by Cronin, Kunz, Risk, and Wheeler[14] on their early experiment on $K_L \to 2\pi^0$. With intensities like that available at the Stanford linear accelerator, a counting rate of a few events per second[12] is not unreasonable. From the theoretical point of view, there are many reasons for wanting such a measure-

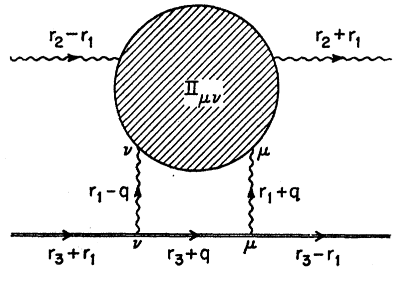

FIG. 5. Schematic Feynman-Dyson diagram for the scattering of a photon by a bare proton.

[11] J. Moffatt and M. W. Stringfellow, Phil. Mag. 3, 540 (1958); Proc. Roy. Soc. (London) A254, 242 (1960).

[12] We thank Sau Lan Wu for these estimates.

[13] We thank Professor K. W. Chen, Professor M. Deutsch, and Professor J. K. Walker for discussions on the feasibility of this experiment.

[14] J. W. Cronin, P. F. Kunz, W. S. Risk, and P. C. Wheeler, Phys. Rev. Letters 18, 25 (1967).

ment. First, the process is intrinsically of interest simply because it is large—the differential cross section at 3 BeV on uranium can be of the order of 10^5 b/sr. Second, the diagrams of importance for Delbrück scattering are of rather high order. Those considered in this paper are already of sixth order in e; those considered in paper VIII of this series are at least of tenth order. Thus, diagrams of the tenth order contribute very roughly 20%. There seems to be no compelling reason why perturbation theory need be correct to such high orders. It will be an important discovery if experimental data fail to agree with the theoretical prediction.

APPENDIX A

In this Appendix, we derive for $Z=1$ the external field approximation used in Sec. 2 A. More precisely, we show that (2.1) and (2.2) apply when the target particle is a bare proton. Consider the diagram of Fig. 5, where the details of the four-photon vertex $\Pi_{\mu\nu}(r_1-q,r_1+q)$ (with the other variables and indices not explicitly written) are of no interest here. The diagram corresponds to the integral

$$i(2\pi)^{-4}\int d^4q[\gamma_\mu(r_3+q+M)\gamma_\nu]\Pi_{\mu\nu}(r_1-q,r_1+q)[(r_3+q)^2-M^2+i\epsilon]^{-1}[(r_1-q)^2+i\epsilon]^{-1}[(r_1+q)^2+i\epsilon]^{-1}, \quad \text{(A1)}$$

where M is the mass of the proton. The four-photon vertex has the symmetry

$$\Pi_{\mu\nu}(r_1-q,r_1+q)=\Pi_{\nu\mu}(r_1+q,r_1-q). \quad \text{(A2)}$$

When the momentum transfer is small, the proton is approximately at rest both before and after scattering. Thus,

$$\langle f|(1-\gamma_0)=(1-\gamma_0)|i\rangle=0. \quad \text{(A3)}$$

Since M is large, approximately

$$r_3+q+M\sim M(1+\gamma_0). \quad \text{(A4)}$$

By (A3) and (A4),

$$\gamma_\mu(r_3+q+M)\gamma_\nu\sim 2M\delta_{\mu0}\delta_{\nu0} \quad \text{(A5)}$$

when taken between $\langle f|$ and $|i\rangle$. Moreover,

$$[(r_3+q)^2-M^2+i\epsilon]^{-1}\sim[2Mq_0+i\epsilon]^{-1}. \quad \text{(A6)}$$

Accordingly, the expression (A1) is approximately

$$\begin{aligned}
&i(2\pi)^{-4}\int d^4q\, 2M\Pi_{00}(r_1-q,r_1+q)[2Mq_0+i\epsilon]^{-1}[(r_1-q)^2+i\epsilon]^{-1}[(r_1+q)^2+i\epsilon]^{-1}\\
&=i(2\pi)^{-4}\int dq_0\int d^3q\, M[\Pi_{00}(r_1-q,r_1+q)+\Pi_{00}(r_1+q,r_1-q)][2Mq_0+i\epsilon]^{-1}[(r_1-q)^2+i\epsilon]^{-1}[(r_1+q)^2+i\epsilon]^{-1}\\
&=i(2\pi)^{-4}\int d^3q\int dq_0 M\Pi_{00}(r_1-q,r_1+q)[(2Mq_0+i\epsilon)^{-1}+(-2Mq_0+i\epsilon)^{-1}][(r_1-q)^2+i\epsilon]^{-1}[(r_1+q)^2+i\epsilon]^{-1}\\
&=i(2\pi)^{-4}\int d^3q\int dq_0 M\Pi_{00}(r_1-q,r_1+q)[-2\pi i\delta(2Mq_0)][(r_1-q)^2+i\epsilon]^{-1}[(r_1+q)^2+i\epsilon]^{-1}\\
&=\tfrac{1}{2}(2\pi)^{-3}\int d^3q\,\Pi_{00}(r_1-q,r_1+q)[(r_1-q)^2-i\epsilon]^{-1}[(r_1+q)^2-i\epsilon]^{-1}. \qquad \text{(A7)}
\end{aligned}$$

This is precisely the external field approximation used in (2.1) and (2.2).

The important point here is this: External field approximation holds when the momentum transfer Δ is small compared with M; but there is no condition on the relative magnitude of ω and M.

APPENDIX B

In this Appendix, we derive (3.28) from (3.27). First,

$$G_3=2\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^1 d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\beta\beta'(\beta'\alpha_5-\beta\alpha_6)^{-2}\ln[\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)]$$

$$=2\int_0^1 d\beta\int_0^1 d\alpha_5\beta(1-\beta)(\alpha_5-\beta)^{-2}\ln\{\alpha_5^{-1}(1-\alpha_5)^{-1}[\beta^2+(1-2\beta)\alpha_5]\}$$

$$=-2\int_0^1 d\alpha_5\ln[\alpha_5(1-\alpha_5)]\int_0^1 d\beta[-1+(1-2\alpha_5)(\beta-\alpha_5)^{-1}+\alpha_5(1-\alpha_5)(\beta-\alpha_5)^{-2}]$$
$$+2\int_0^1 d\beta\,\beta(1-\beta)\int_0^1 d\alpha_5(\alpha_5-\beta)^{-2}\ln[\beta^2+(1-2\beta)\alpha_5]$$
$$=2, \tag{B1}$$

where the contours of integration have been deformed slightly to avoid the double pole at $\alpha_5=\beta$.

Second, since $\alpha_5^{-1}\alpha_6^{-1}=\alpha_5^{-1}+\alpha_6^{-1}$,

$$G_4=4\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^1 d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\beta\beta'\alpha_5^{-1}\ln[\beta^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)]$$

$$=4\int_0^1 d\beta\int_0^1 d\alpha_5\beta(1-\beta)\alpha_5^{-1}\ln\{\beta^{-2}[\beta^2+(1-2\beta)\alpha_5]\}$$

$$=\frac{2}{3}\int_0^1 d\alpha_5\alpha_5^{-1}\left\{\ln(1-\alpha_5)+2\alpha_5\int_0^1 d\beta\,\beta(1-\beta)(3-2\beta)[\beta^2+(1-2\beta)\alpha_5]^{-1}\right\}$$

$$=-\tfrac{1}{9}\pi^2+\frac{8}{3}\int_0^1 d\beta\,\beta(1-\beta)(3-2\beta)(1-2\beta)^{-1}\ln\frac{1-\beta}{\beta}$$

$$=(2\pi^2/9)-\tfrac{4}{3}. \tag{B2}$$

Finally, very similar to (B2), we have

$$G_5=-2\int_0^1 d\alpha_5\alpha_5^{-1}\left\{\ln(1-\alpha_5)+2\alpha_5\int_0^1 d\beta(1-\beta)[\beta^2+(1-2\beta)\alpha_5]^{-1}\right\}$$

$$=\tfrac{1}{3}\pi^2-8\int_0^1 d\beta(1-\beta)\ln\frac{1-\beta}{\beta}$$

$$=-\tfrac{2}{3}\pi^2. \tag{B3}$$

APPENDIX C

In this Appendix, we study the behavior of $\bar{G}_2(\zeta)$, as given by (3.19), in the neighborhood of $\zeta=-1$. In this case, it is useful to define $\xi=\zeta+1$. Let

$$\bar{G}_2(\zeta)=-\pi\csc\pi\xi\,[\Gamma(\xi)]^2[\Gamma(2\xi)]^{-1}H_2(\xi); \tag{C1}$$

then $H_2(\xi)$ is given by

$$H_2(\xi)=\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}$$
$$\times[\beta^{-2+2\xi}-\tfrac{1}{2}\Gamma(1+\xi)\Gamma(3-\xi)(\beta'\alpha_5-\beta\alpha_6)^2(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}F(\xi,2-\xi;3;\bar{z})], \tag{C2}$$

where

$$\bar{z}=z(z-1)^{-1}=(\beta'\alpha_5-\beta\alpha_6)^2/(\beta^2\alpha_6+\beta'^2\alpha_5)\leqslant 1 \tag{C3}$$

from (3.20). In deriving (C2) from (3.19), we have used (see p. 105 of Ref. 7)

$$F(3-\xi,2-\xi;3;z)=(1-z)^{-2+\xi}F(\xi,2-\xi;3;\bar{z}). \tag{C4}$$

As an orientation, we note that the leading term of $H_2(\xi)$ is of the order of ξ^{-2}: one power comes from $\beta\sim 0$, and the other power from $\alpha_5\sim\beta^2\sim 0$. Since $\bar{G}_2(\zeta)/H_2(\xi)$ is also of the order of ξ^{-2} for small ξ, we need the terms

$$\xi^{-2},\quad \xi^{-1},\quad 1,\quad \text{and}\quad \xi$$

for $H_2(\xi)$. Thus, it is necessary to expand the F of (C2) to three terms.

1. Expansion of Hypergeometric Function

Since

$$F(\xi,2-\xi;3;\bar{z})=1+\sum_{n=1}^{\infty}\bar{z}^n\frac{\xi(1+\xi)\cdots(n-1+\xi)(2-\xi)(3-\xi)\cdots(n+1-\xi)}{n!3\cdot 4\cdots(n+2)}$$

$$\sim 1+2\xi\sum_{n=1}^{\infty}\bar{z}^n\frac{1+\xi[1-n^{-1}-(n+1)^{-1}]}{n(n+2)}, \tag{C5}$$

a straightforward calculation gives

$$F(\xi,2-\xi;3;\bar{z})\sim 1+\xi[(\bar{z}^{-2}-1)\ln(1-\bar{z})+\bar{z}^{-1}+\tfrac{1}{2}]$$
$$+\xi^2\left[-\tfrac{1}{2}\bar{z}^{-2}(\bar{z}-1)(\bar{z}+5)\ln(1-\bar{z})+\tfrac{5}{2}\bar{z}^{-1}-\tfrac{3}{4}+\int_0^{\bar{z}}d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\right] \tag{C6}$$

or

$$\tfrac{1}{2}\Gamma(1+\xi)\Gamma(3-\xi)F(\xi,2-\xi;3;\bar{z})\sim 1+\xi(1-\bar{z}^{-1})[-(1+\bar{z}^{-1})\ln(1-\bar{z})-1]$$
$$+\xi^2\left[\bar{z}^{-2}(\bar{z}-1)^2\ln(1-\bar{z})+\bar{z}^{-1}-1-\int_{\bar{z}}^{1}d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\right]. \tag{C7}$$

The substitution of (C7) into (C2) shows that

$$H_2(\xi)\sim H_{21}(\xi)+\xi H_{22}(\xi)+\xi^2 H_{23}(\xi), \tag{C8}$$

where

$$H_{21}(\xi)=\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2}d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}$$
$$\times[\beta^{-2+2\xi}-(\beta'\alpha_5-\beta\alpha_6)^2(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}], \tag{C9}$$

$$H_{22}(\xi)=\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2}d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}$$
$$\times\{-1+[1+(\beta'\alpha_5-\beta\alpha_6)^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)]\ln[\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)]\}, \tag{C10}$$

and

$$H_{23}(\xi)=-\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2}d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}$$
$$\times(\beta'\alpha_5-\beta\alpha_6)^2(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}\Bigg\{-\alpha_5^2\alpha_6^2(\beta'\alpha_5-\beta\alpha_6)^{-4}\ln[\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)]$$
$$+\alpha_5\alpha_6(\beta'\alpha_5-\beta\alpha_6)^{-2}-\int_{(\beta'\alpha_5-\beta\alpha_6)^2(\beta^2\alpha_6+\beta'^2\alpha_5)^{-1}}^{1}d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\Bigg\}. \tag{C11}$$

We need to obtain, for $H_{21}(\xi)$, terms of the order of ξ^{-2}, ξ^{-1}, 1, and ξ; for $H_{22}(\xi)$, terms ξ^{-1} and 1; and for $H_{23}(\xi)$, only terms ξ^{-1}. The remainder of this Appendix is devoted to the extraction of these terms.

2. Evaluation of $H_{21}(\xi)$

We write the $H_{21}(\xi)$ of (C9) in the form

$$H_{21}(\xi)=H_{24}(\xi)+H_{25}(\xi)\,, \tag{C12}$$

where

$$H_{24}(\xi)=\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'\alpha_5^{-1}\alpha_6^{-1}[\beta^{-2+2\xi}-(\beta^2\alpha_6+\beta'^2\alpha_5)^{-1+\xi}] \tag{C13}$$

and

$$\begin{aligned}
H_{25}(\xi)&=4\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 \delta(1-\beta-\beta')\beta\beta'\int_0^1 d\alpha_5[\beta^2-(\beta^2-\beta'^2)\alpha_5]^{-2+\xi}\\
&=4(1-\xi)^{-1}\int_0^1 d\beta\,\beta(1-\beta)(1-2\beta)^{-1}[\beta^{-2+2\xi}-(1-\beta)^{-2+2\xi}]\\
&=4(1-\xi)^{-1}2^{-2\xi}\int_0^1 d\bar\beta\,\bar\beta^{-1}[2(1-\bar\beta)^{-1+2\xi}-(1-\bar\beta)^{2\xi}-2(1+\bar\beta)^{-1+2\xi}+(1+\bar\beta)^{2\xi}]\\
&=4(1-\xi)^{-1}2^{-2\xi}[\xi^{-1}-\psi(1+2\xi)-\gamma+2\ln2-\tfrac{1}{6}\pi^2\xi+2(\ln2)^2\xi]\\
&\sim4(1-\xi)^{-1}[\xi^{-1}-2\xi\psi'(1)-\tfrac{1}{6}\pi^2\xi]\\
&=4(1-\xi)^{-1}[\xi^{-1}-\tfrac{1}{2}\pi^2\xi]\\
&\sim4\xi^{-1}[1+\xi+\xi^2(1-\tfrac{1}{2}\pi^2)]\,.
\end{aligned} \tag{C14}$$

In (C14), $\psi(x)=(d/dx)\ln\Gamma(x)$ is the logarithmic derivative of the gamma function; it has the properties that

$$\psi(1)=-\gamma\,,\quad \psi'(1)=\tfrac{1}{6}\pi^2\,,\quad\text{and}\quad \psi''(1)=-2\zeta(3)\,, \tag{C15}$$

where $\zeta(3)\sim1.2020569$ is the value of the Riemann ζ function at 3.

The calculation of $H_{24}(\xi)$ from (C13) is somewhat more involved:

$$\begin{aligned}
H_{24}(\xi)&=\int_0^1 d\beta\,8\beta^{-1+2\xi}(1-\beta)\int_0^1 d\alpha_5\alpha_5^{-1}\{1-[1-(1-(1-\beta)^2/\beta^2)\alpha_5]^{-2+\xi}\}\\
&=8[(2\xi)^{-1}-(1+2\xi)^{-1}]\int_0^1 d\alpha_5\alpha_5^{-1}[1-(1-\alpha_5)^{-2+\xi}]\\
&\qquad+16\int_0^1 d\beta[(2\xi)^{-1}-(1+2\xi)^{-1}\beta]\beta^3(1-\beta)(1-2\beta)^{-1}[\beta^{-4+2\xi}-(1-\beta)^{-4+2\xi}]\\
&=8[(2\xi)^{-1}-(1+2\xi)^{-1}][-\psi(1)+\psi(-1+\xi)]\\
&\qquad+2^{3-2\xi}\Big\{[\xi^{-1}-(1+2\xi)^{-1}]\int_0^1 d\bar\beta\,\beta^{-1}[(1-\bar\beta)^{-2+2\xi}-(1+\bar\beta)^{-2+2\xi}]\\
&\qquad+[\xi^{-1}-3(1+2\xi)^{-1}]\int_0^1 d\bar\beta\,\bar\beta[(1-\bar\beta)^{-2+2\xi}-(1+\beta)^{-2+2\xi}]\\
&\qquad+2\int_0^1 d\bar\beta[(1-\beta)^{-3+2\xi}+(1+\beta)^{-3+2\xi}][\xi^{-1}(1+\beta^2)-(1+2\xi)^{-1}(1+3\bar\beta^2)]\Big\}\\
&\sim8[(2\xi)^{-1}-(1+2\xi)^{-1}][\gamma-\xi^{-1}+\psi(1+\xi)+\xi^{-1}(1-2\xi)^{-1}(1-\xi^2)]+8[\xi^{-1}-(1+2\xi)^{-1}]\\
&\qquad\times\Big\{\tfrac{1}{2}\xi^{-1}-[\gamma+\psi(1+2\xi)+\xi+\tfrac{1}{6}\pi^2\xi][1-2\xi\ln2]-\tfrac{1}{2}-2\xi^2-2\xi^2\ln2-2\xi^2\int_0^1 dx\,x^{-1}[\ln(1+x)]^2\Big\}\\
&\sim4\xi^{-1}\{\xi^{-1}-\tfrac{5}{6}\pi^2\xi+\xi^2[2\pi^2\ln2+\tfrac{2}{3}\pi^2+6\zeta(3)]\}\,.
\end{aligned} \tag{C16}$$

In the last step of (C16), we have used the identity (G1).

Substitution of (C14) and (C16) into (C12) gives

$$H_{21}(\xi)\sim 4\xi^{-1}\{\xi^{-1}+1+(1-\tfrac{5}{6}\pi^2)\xi+\xi^2[2\pi^2\ln 2+1+\tfrac{1}{6}\pi^2+6\zeta(3)]\}. \qquad \text{(C17)}$$

3. Evaluation of $H_{22}(\xi)$

The next task is to find the two leading terms of the $H_{22}(\xi)$ of (C10):

$$H_{22}(\xi)=4\int_0^1 d\beta\int_0^1 d\alpha_5\beta(1-\beta)[(\beta-\alpha_5)^2+\alpha_5(1-\alpha_5)]^{-2+\xi} \times\{-1+[2+\alpha_5(1-\alpha_5)(\beta-\alpha_5)^{-2}]\ln[1+(\beta-\alpha_5)^2\alpha_5^{-1}(1-\alpha_5)^{-1}]\}. \qquad \text{(C18)}$$

It is convenient to use the variables

$$x=[\alpha_5(1-\alpha_5)]^{-1/2}(\beta-\alpha_5)$$

and

$$\bar{x}=[\alpha_5^{-1}(1-\alpha_5)]^{1/2}. \qquad \text{(C19)}$$

In terms of these variables, $H_{22}(\xi)$ may be expressed as the sum

$$H_{22}(\xi)=H_{26}(\xi)+H_{27}(\xi), \qquad \text{(C20)}$$

where

$$H_{26}(\xi)=16\int_0^\infty d\bar{x}\int_0^{\bar{x}} dx\,\bar{x}^{-1+2\xi}(1+\bar{x}^2)^{-1-2\xi}(\bar{x}^2-1)x(1+x^2)^{-2+\xi}[-1+(2+x^{-2})\ln(1+x^2)] \qquad \text{(C21)}$$

and

$$H_{27}(\xi)=16\int_0^\infty d\bar{x}\int_0^{\bar{x}} dx\,\bar{x}^{2\xi}(1+\bar{x}^2)^{-1-2\xi}(1-x^2)(1+x^2)^{-2+\xi}[-1+(2+x^{-2})\ln(1+x^2)]. \qquad \text{(C22)}$$

Of these two, $H_{26}(\xi)$ is the easier one to evaluate. If the $\bar{x}$ integration is carried out, and then the x integration, we get

$$H_{26}(\xi)=4\xi(1+\xi)^{-1}\{-1+2[\psi(2+\xi)-\psi(1)]+\xi^{-1}[\psi(2+\xi)-\psi(2)]\}\sim 4\xi^{-1}\{\tfrac{1}{6}\pi^2+\xi[-1+\tfrac{1}{6}\pi^2-\zeta(3)]\}, \qquad \text{(C23)}$$

where (C15) is again used.

The evaluation of $H_{27}(\xi)$ is more complicated. First, note that the integral on the right-hand side of (C22) is convergent for $\xi=0$. In fact, we can just take the limit $\xi\to 0$:

$$H_{27}(\xi)\sim 16\int_0^\infty d\bar{x}\int_0^{\bar{x}} dx(1+\bar{x}^2)^{-1}(1-x^2)(1+x^2)^{-2}[-1+(2+x^{-2})\ln(1+x^2)]$$

$$=16\int_0^\infty dx(1-x^2)(1+x^2)^{-2}[-1+(2+x^{-2})\ln(1+x^2)]\cot^{-1}x. \qquad \text{(C24)}$$

Let $x=\cot\theta$; then

$$H_{27}(\xi)\sim 16\left[\int_0^{\pi/2}\theta d\theta\cos 2\theta+2\int_0^{\pi/2}\theta d\theta\cos 2\theta\ln\sin\theta-2\int_0^{\pi/2}\theta d\theta\sec^2\theta\ln\sin\theta+4\int_0^{\pi/2}\theta d\theta\ln\sin\theta\right]. \qquad \text{(C25)}$$

A very tedious calculation gives

$$\int_0^{\pi/2}\theta d\theta\cos 2\theta\ln\sin\theta=\tfrac{1}{2}-\tfrac{1}{16}\pi^2, \qquad \text{(C26)}$$

$$\int_0^{\pi/2}\theta d\theta\sec^2\theta\ln\sin\theta=-\tfrac{1}{12}\pi^2, \qquad \text{(C27)}$$

and

$$\int_0^{\pi/2}\theta d\theta\ln\sin\theta=-\tfrac{1}{8}\pi^2\ln 2+\tfrac{7}{16}\zeta(3). \qquad \text{(C28)}$$

The substitution of (C26)–(C28) into (C25) gives

$$H_{27}(\xi)\sim 8+\tfrac{2}{3}\pi^2-8\pi^2\ln 2+28\zeta(3). \tag{C29}$$

Finally, the substitution of (C23) and (C29) into (C20) gives

$$H_{22}(\xi)\sim\tfrac{2}{3}\pi^2\xi^{-1}+4[1+\tfrac{1}{3}\pi^2-2\pi^2\ln 2+6\zeta(3)]. \tag{C30}$$

4. Evaluation of $H_{23}(\xi)$

Since we need only the leading term of $H_{23}(\xi)$ as $\xi\to 0$, it is sufficient to consider the region $\alpha_5\sim\beta^2\sim 0$ in the range of integration for the right-hand side of (C11). Thus $\alpha_6\sim 1$, $\beta'\sim 1$, and $\beta'\alpha_5-\beta\alpha_6\sim -\beta$:

$$H_{23}(\xi)\sim -\int_0^1 d\beta\int_0^1 d\alpha_5 8\beta^3\alpha_5^{-1}(\beta^2+\alpha_5)^{-2+\xi}\left\{-\alpha_5^2\beta^{-4}\ln[\alpha_5^{-1}(\beta^2+\alpha_5)]+\alpha_5\beta^{-2}-\int_{\beta^2/(\beta^2+\alpha_5)}^1 d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\right\}. \tag{C31}$$

Let $x=\beta^2+\alpha_5$ and $\bar{x}=\beta^2/(\beta^2+\alpha_5)$; then

$$\begin{aligned}H_{23}(\xi)&\sim -4\int_0^1 dx\int_0^1 d\bar{x}\,\bar{x}(1-\bar{x})^{-1}x^{-1+\xi}\left[\bar{x}^{-2}(1-\bar{x})^2\ln(1-\bar{x})+\bar{x}^{-1}(1-\bar{x})-\int_{\bar{x}}^1 d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\right]\\&=-4\xi^{-1}\int_0^1 d\bar{x}\left[\bar{x}^{-1}(1-\bar{x})\ln(1-\bar{x})+1-\bar{x}(1-\bar{x})^{-1}\int_{\bar{x}}^1 d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\right]\\&=-4\xi^{-1}[-\tfrac{1}{6}\pi^2+1+2\zeta(3)].\end{aligned} \tag{C32}$$

5. Result

It only remains to substitute (C17), (C30), and (C32) into (C8) to get

$$H_2(\xi)\sim 4\xi^{-2}+4\xi^{-1}+(4-8\pi^2/3)+4\xi[1+\tfrac{2}{3}\pi^2+10\zeta(3)]. \tag{C33}$$

It then follows from (C1) that, near $\xi=\zeta+1=0$,

$$\bar{G}_2(\zeta)\sim -2\xi^{-2}[1+2\zeta(3)\xi^3]H_2(\xi)\sim -8\xi^{-4}-8\xi^{-3}-8\xi^{-2}(1-\tfrac{2}{3}\pi^2)-8\xi^{-1}[1+\tfrac{2}{3}\pi^2+12\zeta(3)]. \tag{C34}$$

In obtaining (C34), (C15) is once more used. This is the desired result. The second equation of (3.35) then follows from (3.33), (3.7), and (C34).

APPENDIX D

In this Appendix, we study the behavior of $\bar{G}_1(\zeta)$, as given by (3.18), in the neighborhood of $\zeta=-1$. Again define $\xi=\zeta+1$, and let

$$\bar{G}_1(\zeta)=-\pi\csc\pi\xi[\Gamma(-1+\xi)]^2[\Gamma(-2-2\xi)]^{-1}H_1(\xi), \tag{D1}$$

where

$$\begin{aligned}H_1(\xi)=\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\,&\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\alpha_5^{-3+\xi}\alpha_6^{-3+\xi}\\&\times\{[1+2\beta\beta'(1-2\xi)^{-1}][\beta^{-2+2\xi}(\alpha_5\alpha_6)^{2-\xi}-\pi\csc\pi\xi(1-\xi)(2-\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)F(3-\xi,2-\xi;2;z)]\\&+\pi\csc\pi\xi(1-\xi)^2\alpha_5\alpha_6[1+4\beta\beta'(1-2\xi)^{-1}]F(2-\xi,2-\xi;2;z)\}.\end{aligned} \tag{D2}$$

The new feature of the present problem is as follows: The right-hand side of (D2) fails to converge, both at $\beta\sim 0$ and at $\alpha_5\sim\beta^2\sim 0$, unless $\mathrm{Re}\xi>\frac{1}{2}$. In other words, as it stands (D2) is meaningless for small ξ, and analytic continuation is called for. As seen from Appendix C, there is no such problem with $\bar{G}_2(\zeta)$ near $\zeta=-1$; but if further terms in the asymptotic expansion of $G_2(\tau)$ for large τ is desired, $\bar{G}_2(\zeta)$ must be studied near $\zeta=-2$, and this new feature also appears.

We shall carry out the desired analytic continuation first, and then expand the hypergeometric functions in a manner similar to that of Appendix C 1.

1. Analytic Continuation

We want to isolate from the right-hand side of (D2) a simple part which needs to be continued analytically. In the process, we use repeatedly various relations between hypergeometric functions; these relations can all be found on pp. 103 and 105 of Ref. 7. The first step is to write

$$H_1(\xi)=H_{11}(\xi)+H_{12}(\xi)\,, \tag{D3}$$

where

$$\begin{aligned}
H_{11}(\xi)=&\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\alpha_5^{-3+\xi}\alpha_6^{-3+\xi}\\
&\times[\beta^{-2+2\xi}(\alpha_5\alpha_6)^{2-\xi}-\pi\csc\pi\xi\,(1-\xi)(2-\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)F(3-\xi,2-\xi;2;z)\\
&\qquad+\pi\csc\pi\xi\,(1-\xi)^2\alpha_5\alpha_6F(2-\xi,2-\xi;2;z)]\\
=&\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\\
&\times\{\beta^{-2+2\xi}\alpha_5^{-1}\alpha_6^{-1}-\pi\csc\pi\xi\,(1-\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}[(2-\xi)F(3-\xi,\xi;2;\bar z)-(1-\xi)F(2-\xi,\xi;2;\bar z)]\}\\
=&\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\\
&\times[\beta^{-2+2\xi}\alpha_5^{-1}\alpha_6^{-1}-\pi\csc\pi\xi\,(1-\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}F(2-\xi,\xi;1;\bar z)]
\end{aligned} \tag{D4}$$

and

$$\begin{aligned}
H_{12}(\xi)=&\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)2(1-2\xi)^{-1}\beta\beta'\\
&\times\{\beta^{-2+2\xi}\alpha_5^{-1}\alpha_6^{-1}-\pi\csc\pi\xi\,(1-\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}[(2-\xi)F(3-\xi,\xi;2;\bar z)-2(1-\xi)F(2-\xi,\xi;2;\bar z)]\}\\
=&2(1-2\xi)^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\beta\beta'\\
&\times[\beta^{-2+2\xi}\alpha_5^{-1}\alpha_6^{-1}-\pi\xi\csc\pi\xi\,(1-\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)^{-2+\xi}F(2-\xi,1+\xi;2;\bar z)]\,.
\end{aligned} \tag{D5}$$

It is sufficient to analytically continue $H_{11}(\xi)$, since the right-hand side of (D5) is meaningful for $\mathrm{Re}\xi>0$. In (D4) and (D5), $\bar z$ is the quantity defined by (C3).

In $H_{11}(\xi)$, we again use the variables of (C19):

$$\begin{aligned}
H_{11}(\xi)=2\int_0^\infty \bar x d\bar x(1+\bar x^2)^{-2}\bigg\{&-(1-2\xi)^{-1}(1+\bar x^2)\\
&-\pi(1-\xi)\csc\pi\xi\,[\bar x(1+\bar x^2)^{-1}]^{-3+2\xi}\int_0^{\bar x} dx(1+x^2)^{-2+\xi}F[2-\xi,\xi;1;x^2/(1+x^2)]\bigg\}\,.
\end{aligned} \tag{D6}$$

Although by no means necessary, we split $H_{11}(\xi)$ again into two parts

$$H_{11}(\xi)=H_{13}(\xi)+H_{14}(\xi)\,, \tag{D7}$$

where

$$\begin{aligned}
H_{13}(\xi)=&2\int_0^\infty \bar x\,d\bar x(1+\bar x^2)^{-2}\bigg\{-(1-2\xi)^{-1}(1+\bar x^2)+(1-\bar x^2)(1+\bar x^2)^{-1}[\bar x(1+\bar x^2)^{-1}]^{-3+2\xi}\int_0^{\bar x} dx(1+x^2)^{-2+\xi}x^2\bigg\}\\
&+2\int_0^\infty \bar x\,d\bar x(1+\bar x^2)^{-3}(1-\bar x^2)[\bar x(1+\bar x^2)^{-1}]^{-3+2\xi}\int_0^{\bar x} dx(1+x^2)^{-2+\xi}\\
&\qquad\times\{\pi(1-\xi)\csc\pi\xi\,F[2-\xi,\xi;1;x^2/(1+x^2)]-x^2\}
\end{aligned} \tag{D8}$$

and

$$H_{14}(\xi)=-4\pi(1-\xi)\csc\pi\xi\int_0^\infty \bar{x}\,d\bar{x}(1+\bar{x}^2)^{-3}[\bar{x}(1+\bar{x}^2)^{-1}]^{-3+2\xi}\int_0^{\bar{x}} dx(1+x^2)^{-2+\xi}F[2-\xi,\xi;1;x^2/(1+x^2)]. \quad \text{(D9)}$$

Since [see Eq. (14) on p. 110 of Ref. 7]

$$\Gamma(\xi)\Gamma(2-\xi)F[2-\xi,\xi;1;x^2/(1+x^2)]=x^2+O(1) \quad \text{(D10)}$$

as $x\to\infty$, the two terms on the right-hand side of (D8) are separately convergent for $\mathrm{Re}\xi>\frac{1}{2}$. When the second of these two terms is integrated by parts, which is possible when $\mathrm{Re}\xi>\frac{1}{2}$, we get

$$H_{13}(\xi)=2\int_0^\infty \bar{x}\,d\bar{x}(1+\bar{x}^2)^{-2}\left\{-(1-2\xi)^{-1}(1+\bar{x}^2)+(1-\bar{x}^2)(1+\bar{x}^2)^{-1}[\bar{x}(1+\bar{x}^2)^{-1}]^{-3+2\xi}\int_0^{\bar{x}} dx(1+x^2)^{-2+\xi}x^2\right\}$$
$$+2(1-2\xi)^{-1}\int_0^\infty d\bar{x}\,\bar{x}^{-1+2\xi}(1+\bar{x}^2)^{-1-\xi}\{\pi(1-\xi)\csc\pi\xi\,F[2-\xi,\xi;1;\bar{x}^2/(1+\bar{x}^2)]-\bar{x}^2\}. \quad \text{(D11)}$$

In the form (D9) and (D11), the only term which still requires analytic continuation is the first one on the right-hand side of (D11). This can be done by explicit evaluation:

$$H_{13}(\xi)=2(1-2\xi)^{-2}-(1-2\xi)^{-1}[\psi(1+\xi)-\psi(1)]$$
$$+2(1-2\xi)^{-1}\int_0^\infty dx\,x^{-1+2\xi}(1+x^2)^{-1-\xi}\{\pi(1-\xi)\csc\pi\xi\,F[2-\xi,\xi;1;x^2/(1+x^2)]-x^2\}. \quad \text{(D12)}$$

The required behavior of $H_1(\xi)$ for small ξ is to be found from (D3), (D7), (D5), (D9), and (D12).

2. Expansion of Hypergeometric Functions

The expansion of the two hypergeometric functions in (D5), (D9), and (D12) can be carried out in the manner of Appendix C 1. The results are

$$F(2-\xi,\xi;1;\bar{z})\sim 1+\xi[\bar{z}(1-\bar{z})^{-1}\ln(1-\bar{z})]+\xi^2\left[\bar{z}(1-\bar{z})^{-1}+\ln(1-\bar{z})+\int_0^{\bar{z}} d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\right] \quad \text{(D13)}$$

and

$$F(2-\xi,1+\xi;2;\bar{z})=(1-\bar{z})^{-1}F(\xi,1-\xi;2;\bar{z})$$
$$\sim(1-\bar{z})^{-1}\left\{1+\xi[1+(\bar{z}^{-1}-1)\ln(1-\bar{z})]+\xi^2\left[1+(\bar{z}^{-1}-1)\ln(1-\bar{z})+\int_0^{\bar{z}} d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\right]\right\}. \quad \text{(D14)}$$

3. Evaluation of $H_{14}(\xi)$

Since, for small ξ,

$$\int_x^\infty d\bar{x}\,\bar{x}^{-2}[\bar{x}(1+\bar{x}^2)^{-1}]^{2\xi}=\int_0^{1/x} d\bar{x}'\bar{x}'^{-2\xi}[\bar{x}'^2(1+\bar{x}'^2)^{-1}]^{2\xi}$$
$$=(1-2\xi)^{-1}\left[x^{-1+2\xi}(1+x^2)^{-2\xi}-4\xi\int_0^{1/x} d\bar{x}'\bar{x}'^{2\xi}(1+\bar{x}'^2)^{-1-2\xi}\right]$$
$$\sim(1-2\xi)^{-1}\left[x^{-1+2\xi}(1+x^2)^{-2\xi}-4\xi\cot^{-1}x+8\xi^2\int_x^\infty d\bar{x}(1+\bar{x}^2)^{-1}\ln(\bar{x}+\bar{x}^{-1})\right], \quad \text{(D15)}$$

the substitution of the expansion (D13) into (D9) gives

$$H_{14}(\xi)\sim -4\pi(1-\xi)(1-2\xi)^{-1}\csc\pi\xi$$
$$\times\bigg[\int_0^\infty dx\, x^{-1+2\xi}(1+x^2)^{-2}+\xi\int_0^\infty dx(x^{1+2\xi}-4\cot^{-1}x)(1+x^2)^{-2}$$
$$+\xi^2\int_0^\infty dx\, x^{-1}(1+x^2)^{-2}\bigg\{-x^2\ln(1+x^2)-\tfrac{1}{2}[\ln(1+x^2)]^2+x^2-\ln(1+x^2)$$
$$+\int_0^{x^2/(1+x^2)} dx'x'^{-1}\ln(1-x')-4x^2[x+2\ln(1+x^2)]\cot^{-1}x+8x\int_x^\infty dx'(1+x'^2)^{-1}\ln(x'+x'^{-1})\bigg\}\bigg]. \quad \text{(D16)}$$

We calculate the following five integrals:

$$\int_0^\infty dx\, x^{-1+2\xi}(1+x^2)^{-2}=\tfrac{1}{2}\pi(1-\xi)\csc\pi\xi, \quad \text{(D17)}$$

$$\int_0^\infty dx\, x^{1+2\xi}(1+x^2)^{-2}=\tfrac{1}{2}\pi\xi\csc\pi\xi\sim\tfrac{1}{2}, \quad \text{(D18)}$$

$$\int_0^\infty dx(1+x^2)^{-2}\cot^{-1}x=\tfrac{1}{4}+\tfrac{1}{16}\pi^2, \quad \text{(D19)}$$

$$\int_0^\infty dx(1+x^2)^{-2}\bigg\{[x^2+2\ln(1+x^2)]\cot^{-1}x-2\int_x^\infty dx'(1+x'^2)^{-1}\ln(x'+x'^{-1})\bigg\}=\tfrac{1}{4}-\tfrac{1}{16}\pi^2-\tfrac{7}{8}\zeta(3), \quad \text{(D20)}$$

and

$$\int_0^\infty dx\, x^{-1}(1+x^2)^{-2}\bigg\{-x^2\ln(1+x^2)-\tfrac{1}{2}[\ln(1+x^2)]^2+x^2-\ln(1+x^2)+\int_0^{x^2/(1+x^2)}dx'x'^{-1}\ln(1-x')\bigg\}=\tfrac{1}{2}-\zeta(3). \quad \text{(D21)}$$

When the values of these five integrals are substituted into (D16), the result is

$$H_{14}(\xi)\sim -4\pi(1-\xi)(1-2\xi)^{-1}\csc\pi\xi\,\{\tfrac{1}{2}\xi^{-1}-\tfrac{1}{2}-\xi(\tfrac{1}{2}+\tfrac{1}{6}\pi^2)+\xi^2[-\tfrac{1}{2}+\tfrac{5}{6}\zeta(3)+\tfrac{1}{6}\pi^2]\}. \quad \text{(D22)}$$

4. Evaluation of $H_{13}(\xi)$

It follows from (D12) and (D13) with (C15) that

$$H_{13}(\xi)\sim 2\pi(1-\xi)(1-2\xi)^{-1}\csc\pi\xi\bigg[\xi(1+\xi)[1+2\xi-\tfrac{1}{12}\pi^2\xi]+\int_0^\infty dx\, x^{-1+2\xi}(1+x^2)^{-1-\xi}$$
$$\times\bigg\{1+\xi\ln(1+x^2)+\xi^2\bigg(-\ln(1+x^2)+\int_0^{x^2/(1+x^2)}d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\bigg)\bigg\}\bigg]$$
$$\sim 2\pi(1-\xi)(1-2\xi)^{-1}\csc\pi\xi\,\{\tfrac{1}{2}\xi^{-1}+\xi(1+\tfrac{1}{12}\pi^2)+\xi^2[3-\tfrac{1}{6}\pi^2-\zeta(3)]\}. \quad \text{(D23)}$$

Equations (D22) and (D23) can be substituted into (D7) to give

$$H_{11}(\xi)\sim 2\pi(1-\xi)(1-2\xi)^{-1}\csc\pi\xi\,\{-\tfrac{1}{2}\xi^{-1}+1+\xi(2+5\pi^2/12)+\xi^2[4-\tfrac{1}{2}\pi^2-6\zeta(3)]\}$$
$$\sim -\xi^{-2}+\xi^{-1}+(4+\tfrac{2}{3}\pi^2)+12\xi[1-\zeta(3)]. \quad \text{(D24)}$$

5. Evaluation of $H_{12}(\xi)$

The evaluation of $H_{12}(\xi)$ from (D5) is rather similar to that in Appendix C. Since (D14) can be written in the form

$$(1-\xi)\pi\xi\csc\pi\xi F(2-\xi,1+\xi;2;\bar{z})\sim(1-\bar{z})^{-1}\bigg\{1-\xi(1-\bar{z}^{-1})\ln(1-\bar{z})-\xi^2\int_{\bar{z}}^1 d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}')\bigg\}, \quad \text{(D25)}$$

the definition of $\bar{z}$ as given by (C3) can be used to simplify $H_{12}(\xi)$:

$$H_{12}(\xi)\sim H_{15}(\xi)+\xi H_{16}(\xi)+\xi^2 H_{17}(\xi), \tag{D26}$$

where

$$H_{15}(\xi)=2(1-2\xi)^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6) \times\beta\beta'\alpha_5^{-1}\alpha_6^{-1}[\beta^{-2+2\xi}-(\beta^2\alpha_6+\beta'^2\alpha_5)^{-1+\xi}], \tag{D27}$$

$$H_{16}(\xi)=2(1-2\xi)^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6) \times\beta\beta'(\beta'\alpha_5-\beta\alpha_6)^{-2}(\beta^2\alpha_6+\beta'^2\alpha_5)^{-1+\xi}\ln[\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)], \tag{D28}$$

and

$$H_{17}(\xi)=2(1-2\xi)^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6) \times\beta\beta'\alpha_5^{-1}\alpha_6^{-1}(\beta^2\alpha_6+\beta'^2\alpha_5)^{-1+\xi}\int_{\bar{z}}^1 d\bar{z}'\bar{z}'^{-1}\ln(1-\bar{z}'). \tag{D29}$$

We need the terms of the order of ξ^{-2}, ξ^{-1}, 1, and ξ from $H_{15}(\xi)$; the terms ξ^{-1} and 1 from $H_{16}(\xi)$; and only terms ξ^{-1} from $H_{17}(\xi)$.

First, a comparison of (D27) with (C13) shows that

$$H_{15}(\xi)=\tfrac{1}{4}(1-2\xi)^{-1}H_{24}(\xi). \tag{D30}$$

This has been arranged purposely. Thus, (C16) may be used immediately:

$$H_{15}(\xi)\sim\xi^{-2}+2\xi^{-1}+(4-\tfrac{5}{6}\pi^2)+\xi[8-\pi^2+2\pi^2\ln2+6\zeta(3)]. \tag{D31}$$

Secondly, the evaluation of $H_{16}(\xi)$ from (D28) is very similar to that of $H_{22}(\xi)$ from (C10), as given in Appendix C. The result is

$$H_{16}(\xi)\sim\tfrac{1}{6}\pi^2\xi^{-1}+[\pi^2-2\pi^2\ln2+6\zeta(3)]. \tag{D32}$$

Finally, the right-hand side of (D29) is simpler than that of (C11) for $H_{33}(\xi)$. The procedure given in Appendix C can be followed step by step to yield

$$H_{17}(\xi)\sim-2\zeta(3)\xi^{-1}. \tag{D33}$$

Equations (D31)–(D33) can be substituted into (D26) to yield immediately

$$H_{12}(\xi)\sim\xi^{-2}+2\xi^{-1}+(4-\tfrac{2}{3}\pi^2)+\xi[8+10\zeta(3)]. \tag{D34}$$

6. Result

By (D3), we add (D24) and (D34) to get, for $\xi\to0$,

$$H_1(\xi)=3\xi^{-1}+8+\xi[20-2\zeta(3)]+O(\xi^2). \tag{D35}$$

Note the remarkable fact that the leading terms, of the order of magnitude ξ^{-2}, cancel each other. It finally follows from (D1) that, near $\xi=\zeta+1=0$,

$$\bar{G}_1(\zeta)=-12\xi^{-3}-20\xi^{-2}-[36-8\zeta(3)]\xi^{-1}+O(1). \tag{D36}$$

Use has been made of (C34). This is the desired result. The first equation of (3.35) is a consequence of (3.33), (3.7), and (D36).

APPENDIX E

Attention is next turned to the behavior of G_1 and G_2 when $|t|\ll m^2$. In this Appendix, we treat $\bar{G}_2(\zeta)$ when ζ is close to 2. In this case, let $\xi=\zeta-2$, and it is seen from (3.19) that, when $\xi\to0$, the only singular terms are of the form of ξ^{-2} and ξ^{-1}. Thus, it is sufficient merely to take the first two terms of the hypergeometric function in (3.19):

$$F(-\xi,-1-\xi;3;z)=1+\tfrac{1}{3}\xi z+O(\xi^2). \tag{E1}$$

Accordingly,

$$\bar{G}_2(\zeta)\sim\xi^{-1}[\Gamma(3+\xi)]^2[\Gamma(6+2\xi)]^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)8\beta\beta'$$
$$\times\{\beta^4\alpha_5^{-1}\alpha_6^{-1}+\tfrac{1}{2}\xi^{-1}(1+\xi)(2+\xi)(3+\xi)(\beta'\alpha_5-\beta\alpha_6)^2[(\alpha_5\alpha_6)^\xi-\tfrac{1}{3}\xi\alpha_5^{-1}\alpha_6^{-1}(\beta'\alpha_5-\beta\alpha_6)^2]\}$$
$$\sim2\xi^{-2}[\Gamma(3+\xi)]^2[\Gamma(6+2\xi)]^{-1}(1+\xi)(2+\xi)(3+\xi)H_{28}(\xi)+(4/15)\xi^{-1}H_{29}, \tag{E2}$$

where

$$H_{28}(\xi)=\int_0^1 d\beta\int_0^1 d\alpha_5\beta(1-\beta)\alpha_5^\xi(1-\alpha_5)^\xi(\beta-\alpha_5)^2 \tag{E3}$$

and

$$H_{29}=\int_0^1 d\beta\int_0^{1/2} d\alpha_5\beta(1-\beta)\alpha_5^{-1}(1-\alpha_5)^{-1}[\beta^4-(\beta-\alpha_5)^4]. \tag{E4}$$

The value of H_{29} can be found easily by first carrying out the β integration and then the α_5 integration:

$$H_{29}=19/360. \tag{E5}$$

The computation of $H_{28}(\xi)$ from (E3) is equally straightforward:

$$H_{28}(\xi)=(20)^{-1}[\Gamma(1+\xi)]^2[\Gamma(2+2\xi)]^{-1}-\tfrac{1}{6}[\Gamma(2+\xi)]^2[\Gamma(4+2\xi)]^{-1}\sim(12-29\xi)/540. \tag{E6}$$

Since

$$[\Gamma(3+\xi)]^2/\Gamma(6+2\xi)\sim(30-47\xi)/900, \tag{E7}$$

the substitution of (E5) and (E6) into (E2) gives finally

$$\bar{G}_2(\xi)=2\xi^{-2}/225-17\xi^{-1}/3375+O(1), \tag{E8}$$

for $\xi=\zeta-2\rightarrow0$. The second equation of (3.34) then follows from (3.32), (3.7), and (E8).

APPENDIX F

It remains to study the behavior of $\bar{G}_1(\zeta)$ when ζ is close to 2. Again let $\xi=\zeta-2$, and it follows from (3.18) that for $\xi\rightarrow0$,

$$\bar{G}_1(\zeta)=\pi\csc\pi\xi[\Gamma(2+\xi)]^2[\Gamma(4+2\xi)]^{-1}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)\alpha_5^\xi\alpha_6^\xi$$
$$\times\{[1-2\beta\beta'(5+2\xi)^{-1}][\beta^{4+2\xi}(\alpha_5\alpha_6)^{-1-\xi}+\pi\csc\pi\xi(1+\xi)(2+\xi)(\beta^2\alpha_6+\beta'^2\alpha_5)F(-\xi,-1-\xi;2;z)]$$
$$-\pi\csc\pi\xi(2+\xi)^2\alpha_5\alpha_6[1-4\beta\beta'(5+2\xi)^{-1}]F(-1-\xi,-1-\xi;2;z)\}$$
$$\sim\tfrac{1}{6}\xi^{-1}(1-5\xi/3)\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^{1/2} d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)$$
$$\times\{(1-\tfrac{2}{5}\beta\beta')\beta^4\alpha_5^{-1}\alpha_6^{-1}+\xi^{-1}(2+3\xi)[1-2\beta\beta'(5+2\xi)^{-1}](\beta^2\alpha_6+\beta'^2\alpha_5)\alpha_5^\xi\alpha_6^\xi+(1-\tfrac{2}{5}\beta\beta')(\beta^2\alpha_6+\beta'^2\alpha_5)$$
$$\times[-\alpha_5^{-1}\alpha_6^{-1}(\beta'\alpha_5-\beta\alpha_6)^2]-4\xi^{-1}(1+\xi)(\alpha_5\alpha_6)^{1+\xi}[1-4\beta\beta'(5+2\xi)^{-1}][1-\tfrac{1}{2}\alpha_5^{-1}\alpha_6^{-1}(\beta'\alpha_5-\beta\alpha_6)^2]\}$$
$$=\tfrac{1}{6}\xi^{-1}(1-5\xi/3)\frac{1}{2}\int_0^1 d\beta\int_0^1 d\beta'\int_0^1 d\alpha_6\int_0^1 d\alpha_5\delta(1-\beta-\beta')\delta(1-\alpha_5-\alpha_6)$$
$$\times\{(1-\tfrac{2}{5}\beta\beta')(-\beta^2+4\beta^3)+\xi^{-1}\alpha_5^\xi\alpha_6^\xi\beta^2[(4+5\xi)-(6+7\xi)2\beta\beta'/(5+2\xi)]$$
$$-6\xi^{-1}\alpha_5^{1+\xi}\alpha_6^{1+\xi}(1+\xi)[1-4\beta\beta'/(5+2\xi)]\}$$
$$\sim\tfrac{1}{6}\xi^{-1}(1-5\xi/3)(13/75)(\xi^{-1}+13/5)$$
$$\sim(1+14\xi/15)13\xi^{-2}/450. \tag{F1}$$

The first equation of (3.34) follows from (3.32), (3.7), and (F1).

APPENDIX G

In Appendices C and D, we have several occasions to use

$$\int_0^1 dx\, x^{-1}[\ln(1+x)]^2 = \tfrac{1}{4}\zeta(3). \tag{G1}$$

To derive (G1), we begin with [see (C15)]

$$\int_0^1 dx\, x^{-1}[\ln(1-x)]^2 = -\psi''(1) = 2\zeta(3). \tag{G2}$$

Let $x=1-x'$, and $x''=(1-x')/(1+x')$; then

$$\begin{aligned}2\zeta(3) &= 4\int_0^1 dx'[(1-x')^{-1}-(1+x')^{-1}](\ln x')^2\\ &= 4\left[2\int_0^1 dx'(1-x')^{-1}(\ln x')^2 - 2\int_0^1 dx'(1-x'^2)^{-1}(\ln x')^2\right]\\ &= 4\left\{4\zeta(3) - \int_0^1 dx'' x''^{-1}[\ln(1-x'')-\ln(1+x'')]^2\right\}.\end{aligned} \tag{G3}$$

Thus,

$$\int_0^1 dx\, x^{-1}[\ln(1-x)-\ln(1+x)]^2 = \tfrac{7}{2}\zeta(3). \tag{G4}$$

On the other hand, it also follows from (G2) that

$$\int_0^1 dx\, x^{-1}[\ln(1-x)+\ln(1+x)]^2 = \frac{1}{2}\int_0^1 dx^2\, x^{-2}[\ln(1-x^2)]^2 = \zeta(3). \tag{G5}$$

The average of (G4) and (G5) is

$$\int_0^1 dx\, x^{-1}\{[\ln(1-x)]^2+[\ln(1+x)]^2\} = 9\zeta(3)/4, \tag{G6}$$

and (G1) follows immediately from (G6) and (G2).