Momentum-Transfer-Independent Angular Relations and Solutions to the Isospin-Factored Current Algebra

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The main purpose of this paper is to construct a general class of solutions to the isospin-factored algebra of form factors at infinite momentum. Lorentz covariance imposes a very restrictive condition on these form factors, which is known as the angular condition. In this paper, we decompose this angular condition into a set of momentum-transfer-independent conditions. To further simplify our problem, we strengthen the angular conditions into three more restrictive and mutually exclusive classes of conditions (called primitive equations). These simplified angular conditions can be solved completely, and lead to three classes of primitive solutions. We find that for each of the primitive solutions there always exists an internal Lorentz group, and that these solutions are related to some very simple infinite-component wave equations. Having established the connection between the solution to the angular conditions and the wave equation, we then turn things around and construct some very general solutions from the coupled wave equations. The fact that these coupled equations represent the most general solutions to the primitive equations suggests that they may already represent the most general solution compatible with the original angular condition. Next, under very mild technical conditions, the solutions to the angular conditions are shown either to be physically trivial or to contain a spacelike ($M^2<0$) part. The possibility that the spacelike and timelike parts are not coupled by the currents is also discussed.

I. INTRODUCTION

IN the preceding paper,[1] an attempt is made to understand the structure of the current algebra at infinite momentum. In order that the current density operator $\mathcal{J}_\mu(x)$ transforms covariantly as a four-vector under Lorentz transformations, one finds that the infinite-momentum form factors must satisfy a very complicated angular condition. This angular condition is derived explicitly in I, and is found to be the necessary and sufficient condition for $\mathcal{J}_\mu$ transforming covariantly. In this paper, we first scale down the structure of current algebra from $SU(3)\otimes SU(3)$ to $SU(2)\otimes SU(2)$. Then, as a first natural step, we try to saturate the algebra of isospin charge densities by a set of states all having isospin $\frac{1}{2}$.[2] This greatly simplifies the mathematics, and may be even directly relevant for physics since all the strangeness $S=\pm1$ mesons and/or $S=-2$ baryons so far observed have isospin $\frac{1}{2}$.

To make this paper self-contained, we review some of the known solutions to these isospin-factored current algebra. There are two very interesting physical models. The first one is the two-free-quark model[3] introduced by Gell-Mann. In this model, both quarks are free particles, but only one of them is charged. Since this represents a composite system of two noninteracting particles, the current algebra is satisfied automatically. These composite two-quark systems satisfy some infinite-component wave equations which are closely related to the general solutions to the problem. These solutions have the additional interesting feature that it is the only known model which has the nice property that the spacelike solutions exist, but do *not* couple to the timelike solutions by the current. The second interesting model is the one in which one assumes not only the charge-density commutator relations, but the charge-density–current-density commutator relations as well. This model can be solved easily and yields one of the primitive class of solutions obtained later.

We then analyze the angular condition systematically. Multiplying the angular relation by the factor $F(\mathbf{k})^{-1}$, and expanding it in powers of $\mathbf{k}$, where $\mathbf{k}=(k_1,k_2)$ is the transverse momentum transfer, we find that if the expansion terminates, and if we reject solutions for which a spacelike part ($M^2<0$) exists and is definitely coupled to the timelike part by the current, the angular condition is equivalent to the set of k-independent equations given in Table I.

To solve these k-independent angular conditions, we first replace the relation

$$B[B^2-\tfrac{1}{4}(R+1)^2]=0$$

by the stronger relations $B=0$, $B=\frac{1}{2}(R+1)$, and $B=-\frac{1}{2}(R+1)$. Then, the equations of Table I can be reduced to three primitive (mutually exclusive) sets of equations. The primitive equations can be solved

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[1] S. J. Chang, R. F. Dashen, and L. O'Raifeartaigh, of preceding paper, Phys. Rev. **182**, 1805 (1969), hereafter referred to as I; a summary of results in paper I and the present paper was published in S. J. Chang, R. F. Dashen, and L. O'Raifeartaigh, Phys. Rev. Letters **21**, 1026 (1968).

[2] R. F. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966); S. Fubini, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, 1967*, edited by A. Perlmutter and B. Kursunoglu (W. H. Freeman and Co., San Francisco, 1967); H. Bebié and H. Leutwyler, Phys. Rev. Letters **19**, 618 (1967).

[3] M. Gell-Mann, in *Strong and Weak Interactions: Present Problems*, edited by A. Zichichi (Academic Press Inc., New York, 1966).

TABLE I. Momentum-transfer-independent equations for M, $\mathcal{J}_3$, $\mathbf{X}$, and $\mathfrak{J}$. The independence of the basic equations can be checked by considering the special case $M=\text{const}$. All secondary equations can be obtained from the basic equations by taking commutators with respect to $\mathbf{X}$.

Definitions	Basic equations
$X_\pm=X_1\pm iX_2,\ \mathcal{J}_\pm=\mathcal{J}_1\pm i\mathcal{J}_2$	$[X_i,[X_j,[X_k,M^2]]]=0$
$M_\pm=[X_\pm,M^2]$	$[X_+,\Lambda_+]=[X_-,\Lambda_-]=0$
$M_{++}=[X_+,[X_+,M^2]]$	$[B,\mathbf{X}]=0$
$\epsilon=(R+1)^{-1}R,\ R=-\frac{1}{4}[X_+,[X_-,M^2]]$	$[\Lambda_+,\Lambda_-]=4[M^2,B]_+-\frac{1}{4}[M_+,M_-]$
$\Lambda_\pm=\pm 2iM\mathcal{J}_\pm+\frac{1}{2}[M^2,X_\pm]_+$	$[\Lambda_\pm,M^2]=\frac{1}{2}[\ M^2,M_\pm]_+$
$iK\pm(\mathcal{J}_3-B)=\frac{1}{4}[\Lambda_\mp,X_\pm]$	$BG=B[B^2-\frac{1}{4}(R+1)^2]=0$
$G=B^2-\frac{1}{4}(R+1)^2$	$M_{++}G_+=M_{--}G_-=0$
$G_\pm=-\frac{1}{4}i[K,\mathfrak{F}_\pm]-\frac{1}{4}\mathfrak{F}_\pm$	$M_\pm G-[\Lambda_\pm+\frac{1}{2}M_\pm,G]+4[\frac{1}{2}(R+1)\mp B]G_\pm=0$
	$-2M_+G_++[\Lambda_++\frac{1}{2}M_+,\ G_+]+\frac{1}{8}i[K,M_{++}M^2]+\frac{1}{4}M_{++}M^2=0$
	$-2M_-G_-+[\Lambda_-+\frac{1}{2}M_-,\ G_-]+\frac{1}{8}i[K,\ M_{--}M^2]+\frac{1}{4}M_{--}M^2=0$

Some secondary equations

$$i[K,\mathbf{X}]=\mathbf{X},\quad [B,R]=0$$
$$M_{++}{}^2=M_{++}[B+\tfrac{1}{2}(R+1)]=[B-\tfrac{1}{2}(R+1)]M_{++}=0$$
$$i[K,R]=R(R+1)+\tfrac{1}{64}[M_{++},M_{--}]_+.$$

completely and the solutions are related to infinite-component wave equations. Having established the connection between the solution to the angular condition and the infinite-component wave equation, we then turn things around and construct some very general solutions from the coupled wave equations. The fact that these coupled equations represent the most general solution to the primitive equations, and that they are only mildly p-dependent, suggests that they may already represent the most general solution to the equations of Table I.

Under very mild technical conditions, the equations of Table I can be shown directly to either be physically trivial or admit spacelike as well as timelike solutions. The existence of the spacelike solutions obviously represents a serious difficulty for the program of saturating with timelike one-particle states. However, there still remains the question of coupling. It would be possible to saturate current algebra consistently with timelike solutions alone if the current did not couple them to the spacelike solutions. As mentioned earlier, this happens for the free-quark solution. However, it does not for the two previously known nontrivial solutions, a result which suggests that it happens only in the trivial case.

II. ISOTOPIC-SPIN-$\frac{1}{2}$ MODEL

We now introduce the isotopic-spin-$\frac{1}{2}$ model. First, for simplicity, we scale $SU(3)$ down to $SU(2)$ by letting the index a (or b, c) in the algebra of form factors

$$\begin{aligned}
[F^a(\mathbf{k}'),\,F^b(\mathbf{k})]&=if_{abc}F^c(\mathbf{k}'+\mathbf{k}),\\
[F^a(\mathbf{k}'),\,F_5{}^b(\mathbf{k})]&=if_{abc}F_5{}^c(\mathbf{k}'+\mathbf{k}),\\
[F_5{}^a(\mathbf{k}'),\,F_5{}^b(\mathbf{k})]&=if_{abc}F^c(\mathbf{k}'+\mathbf{k})
\end{aligned}\tag{2.1}$$

run from 1 to 3 instead of 1 to 8. We then make the crucial assumption, which is that *all the states necessary to saturate (2.1) have isospin* $\frac{1}{2}$. This is a strong assumption, but as pointed out in the Introduction, it may be directly relevant to physics, and in any case should yield an algebra which is nontrivial but at the same time sufficiently simple to be solved, and hence to yield some insights into the general case.

For particles of isospin $\frac{1}{2}$, we have from the Wigner-Eckart theorem

$$(N'h',\alpha'|F^a(\mathbf{k})|Nh,\alpha)=\tfrac{1}{2}\tau_{\alpha'\alpha}{}^a(N'h'|F(\mathbf{k})|Nh)\,,$$

where the $\tau_{\alpha'\alpha}{}^a$ ($\alpha=1,2$) are Pauli matrices, and $F(\mathbf{k})$ [$\mathbf{k}=(k_1,k_2)$] is a reduced charge density. Assuming now that all the states $|Nh\rangle$ have isospin $\frac{1}{2}$, the operators in (2.1) can clearly be written

$$F^a(\mathbf{k})=\tfrac{1}{2}\tau^aF(\mathbf{k})\,,\tag{2.2}$$

in which case Eqs. (2.1) reduce to

$$\begin{aligned}
F(\mathbf{k}')F(\mathbf{k})&=F(\mathbf{k}'+\mathbf{k})\,,\\
F(\mathbf{k}')F_5(\mathbf{k})&=F_5(\mathbf{k}'+\mathbf{k})\,,\\
F_5(\mathbf{k}')F_5(\mathbf{k})&=F(\mathbf{k}'+\mathbf{k})\,.
\end{aligned}\tag{2.3}$$

This is the full content of the saturated charge-density algebra for the isospin-$\frac{1}{2}$ model. The algebra (2.3) is easily solved and the general solution is

$$\begin{aligned}
F(\mathbf{k})&=e^{i\mathbf{k}\cdot\mathbf{X}}\,,\\
F_5(\mathbf{k})&=\Sigma e^{i\mathbf{k}\cdot\mathbf{X}}\,,
\end{aligned}\tag{2.4}$$

where $\mathbf{X}=(X_1,X_2)$ is any pair of *commuting* self-adjoint operators and Σ is a self-adjoint operator which commutes with $\mathbf{X}$ and satisfies the relation $\Sigma^2=1$. ($\mathbf{X}$ transforms as a two-vector and Σ a scalar under $\mathcal{J}_3$.)

Equation (2.4) is not, of course, the full solution of the saturation problem, since not every set of operators $(\mathbf{X},\Sigma)$ is allowed by Lorentz invariance. Only those solutions for which Σ is a pseudoscalar and $F(\mathbf{k})$ satisfies the angular condition are allowed. The problem, therefore, is to find those solutions.

III. SOME SIMPLE MODELS

We study some simple models which lead to various solutions to the angular condition and the algebra of the factored isocurrent. These models are either very simple in structure or can be solved easily by making use of some additional assumptions.

(1) Two-free-quark model. This simple model, introduced by Gell-Mann, is of special interest, since it gives us some insights into the structure of the general solutions. In this model, both of the quarks are free particles, but only the first one carries the isospin. Since the charged particle is a free particle, the current algebra is satisfied automatically. The wave equations obeyed by the free-quark model are simply

$$(\gamma_{(1)}\cdot p_1-m_1)\psi(p_1,p_2)=0\,, \tag{3.1}$$

$$(\gamma_{(2)}\cdot p_2-m_2)\psi(p_1,p_2)=0 \tag{3.2}$$

for the two Dirac quarks, and

$$(p_1^2-m_1^2)\phi(p_1,p_2)=0\,, \tag{3.3}$$

$$(p_2^2-m_2^2)\phi(p_1,p_2)=0 \tag{3.4}$$

for the two scalar quarks. Considering the two-quark system as a composite particle, we can reexpress (3.1) and (3.3) as

$$(\gamma_{(1)}\cdot P-\mathfrak{M})\psi(P,p_2)=0\,, \tag{3.5}$$

with

$$P=p_1+p_2\,,\quad \mathfrak{M}=m_1+\gamma_{(1)}p_2$$

and

$$(P^2-2p_2\cdot P-s)\phi(P,p_2)=0\,,\qquad s=-(p_2^2-m_1^2)=m_1^2-m_2^2\,, \tag{3.6}$$

respectively. Equations (3.5) and (3.6) can be considered as infinite-component wave equations where p_2 plays the role of a spinor index. As we shall see, the structure of Eqs. (3.5) and (3.6) is quite universal.

Let us now compute the mass operator of the free-quark model explicitly. We choose a special reference frame which is obtained from an infinite-momentum frame with a standard deceleration. This special frame is specified by

$$P_0+P_3=1\,,\quad P_1=P_2=0\,. \tag{3.7}$$

Then we have

$$P^2=M^2\,,$$

$$2p_2\cdot P=(p_0-p_3)_{(2)}+(p_0+p_3)_{(2)}M^2\,,$$

and by the use of (3.6),

$$M^2=\frac{(p_0-p_3)_{(2)}+m_1^2-m_2^2}{1-(p_0+p_3)_{(2)}}\,. \tag{3.8}$$

It is easy to see that the nominator of (3.8) is positive definite provided that $m_1^2\geq m_2^2$. However, M^2 can be both positive and negative, the negative sign corresponding to a combination of one positive-frequency and one negative-frequency quark. On the other hand, the charge density operator, which in the limit $p_z=\infty$ becomes $e^{i\mathbf{q}\cdot\mathbf{E}(2)}$, where

$$\mathbf{E}_{(2)}=(K_1+L_2,K_2-L_1)_{(2)}\,,$$

commutes with the denominator of (3.8). Thus it can never couple a spacelike solution ($M^2<0$) to a timelike solution ($M^2>0$).

These are all the nice features of the free-quark model. However, the free-quark model has its own pathologies. The most serious pathology is that it has only a continuous spectrum and that this spectrum is infinitely degenerate. It is likely that these pathologies are closely linked to the fact that in this model the timelike and spacelike parts do not couple by the current.

(2) Gell-Mann, Horn, and Weyers[4] made an important observation that the current constructed from any infinite-component wave equation of the Abers-Grodsky-Norton (AGN) type[5]

$$(\gamma\cdot p-\mathfrak{M})\psi=0 \tag{3.9}$$

satisfies the current algebra at infinite momentum automatically. The only missing link here is the connection between, on the one hand, the operators M^2, $\mathbf{X}$, and $2M\mathfrak{J}$ appearing in the angular conditions and, on the other hand, the generators of $SL(2,C)$ appearing in the wave equation. The connection between these operators is far from trivial. The main difficulty is that the particles corresponding to different solutions in general have different masses, and consequently there does not exist a common Lorentz transformation which brings all these particles from their rest frames to the frame with $p_3\to\infty$. One of the purposes of this paper is to supply this missing link.

(3) Dashen succeeded in constructing an important class of solutions by assuming that the currents also satisfy the usual charge-density–current-density commutator relations. From the results of Sec. IV of the preceding paper, we learn that these commutator relations plus current conservation lead to the following set of relations:

$$I_{\mathbf{k}}(F^a(\mathbf{k}))=i(\mathbf{k}\times\mathbf{F}^a)_3\,, \tag{3.10}$$

$$[M^2,F^a(\mathbf{k})]=\sum_{i=1}^{2}k_iF_i^a(\mathbf{k})\,, \tag{3.11}$$

and

$$[F^a(\mathbf{k}'),F_i^b(\mathbf{k})]=i\epsilon_{abc}F_i^c(\mathbf{k}'+\mathbf{k})+k_i'[F^a(\mathbf{k}'),F^b(\mathbf{k})]_++\text{S.T.} \tag{3.12}$$

(where S.T. = Schwinger terms). Making use of (3.11)

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[4] M. Gell-Mann, D. Horn, and J. Weyers, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (North-Holland Publishing Company, Amsterdam, 1968).

[5] E. Abers, I. Grodsky, and R. Norton, Phys. Rev. **159**, 1222 (1967).

and the factored isocurrent

$$F^a(\mathbf{k})=\tfrac{1}{2}\tau^a F(\mathbf{k}), \qquad (3.13)$$

we have

$$F_i(0)=-i[X_i,M^2],$$

and with the help of (3.12),

$$F_i^a(\mathbf{k})=-i(\tfrac{1}{4}\tau^a)[e^{i\mathbf{k}\cdot\mathbf{X}},[X_i,M^2]]_+. \qquad (3.14)$$

After substituting (3.14) back into (3.10) and (3.12), we have

$$I_{\mathbf{k}}(e^{i\mathbf{k}\cdot\mathbf{X}})=\tfrac{1}{2}[e^{i\mathbf{k}\cdot\mathbf{X}},[k_1X_2-k_2X_1,M^2]]_+ \qquad (3.15)$$

and

$$[e^{i\mathbf{k}'\cdot\mathbf{X}},[e^{i\mathbf{k}\cdot\mathbf{X}},[\mathbf{X},M^2]]_+]_+ =2[e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{X}},[\mathbf{X},M^2]]_+, \qquad (3.16)$$

respectively. Note that Eq. (3.16) is equivalent to

$$[X_i,[X_j,[X_k,M^2]]]=0. \qquad (3.17)$$

Next, we wish to expand (3.15) and reduce it to a set of k-independent equations. For definiteness, we choose $\mathbf{k}=(k,0)$. Hence

$$I_{\mathbf{k}}(e^{ikX_1})=\tfrac{1}{2}k[e^{ikX_1},[X_2,M^2]]_+. \qquad (3.18)$$

Multiplying (3.18) by e^{-ikX_1}, and equating the coefficients of k, we have

$$[X_1,\, 2M\mathcal{J}_1+\tfrac{1}{2}[M^2,X_2]_+]+2i\mathcal{J}_3=0. \qquad (3.19)$$

Equations (3.17) and (3.19) are equivalent to the original equations (3.9)–(3.11). Actually, they are also the sufficient conditions for the original angular condition (I2.13). As mentioned earlier, these sets of equations are in fact identical to one of our primitive set of equations. The technicalities of solving these equations are given in the Appendix B. We only summarize our solutions here

The six generators $X_\pm$, $\mathcal{J}_3$, $\bar{K}_3$, and $\mathfrak{F}_\pm$ defined by

$$\mathfrak{F}_\pm=\pm 2iM\mathcal{J}_\pm+X_\pm M^2, \qquad (3.20)$$

$$i\bar{K}_3\pm\mathcal{J}_3=\tfrac{1}{4}[\mathfrak{F}_\mp,X_\pm] \qquad (3.21)$$

from an exact $SL(2,C)$ algebra, and satisfy the pseudo-Hermitian condition

$$O^\dagger=(1-\epsilon)^{-1}O(1-\epsilon), \qquad (3.22)$$

where

$$\epsilon=(R+1)^{-1}R, \qquad R=-\tfrac{1}{4}[X_+,[X_-,M^2]]. \qquad (3.23)$$

The operator

$$\epsilon=g_0+g_3 \qquad (3.24)$$

transforms as the zeroth plus the third component of a four-vector g_μ under this $SL(2,C)$ algebra. The mass operator is given by

$$M^2=(1-\epsilon)^{-1}(g_0-g_3+s)=(M^2)^\dagger, \qquad (3.25)$$

where s is a scalar. M^2 and ϵ are Hermitian. Conversely, for a given pseudounitary representation of $SL(2,C)$ and with g_μ and M^2 given in Eqs. (3.22)–(3.24), we have a solution to the angular condition. Since the representations to the $SL(2,C)$ are well known, we have all the solutions to this simple problem, in which Eq. (3.12) is satisfied.

Next we wish to find the connection between our solutions and infinite-component wave equations. In order to write the wave equations, we use the analogy between the mass operator given in (3.25) and the mass operator (3.8) obtained in the two-free-quark model. This parallelism suggests that we should introduce a wave function ϕ in the standard frame[6]

$$(p_0+p_3)\phi=\phi, \quad p_1\phi=p_2\phi=0,$$

and express (3.25) as a wave equation

$$(p^2-2g\cdot p-s)\phi=0. \qquad (3.26)$$

This is the required wave equation, and has exactly the same form as that obtained in the two-scalar-quark model.

Let us give a final remark at this point that a special solution to the general charge-density algebra found by Leutwyler[7] is a special case of the above solutions. His solution is represented in the basis where $\mathbf{X}$ is diagonal, and corresponds to the case in which the $\mathbf{X}$ forms a complete set. From the $SL(2,C)$ point of view, this corresponds to a representation of the group in a unitary $E(2)$ basis, such that the representation of the total $SL(2,C)$ group is irreducible and pseudounitary. In fact, the six generators in Leutwyler's case are related to the six corresponding generators of the Majorana representation by the relation

$$\tilde{G}=(1-\epsilon)^{-1/2}G(1-\epsilon)^{1/2}.$$

In a unitary $E(2)$ basis,[8] the six generators G are differential operators of at most second order in $\mathbf{X}$, and it was in this differential form that Leutwyler's solution was first found.

IV. k-INDEPENDENT ANGULAR CONDITIONS

In Sec. II, we saw that the condition imposed on the vector current and the axial-vector current by the

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[6] The standard frame is described by $p_0+p_3=1$, $\mathbf{p}_\perp=0$, and can be obtained from the infinite-momentum frame $\mathbf{p}_\perp=0$, $p_3=\kappa\to\infty$ by a standard deceleration $e^{-i\lambda K_3}$, $\lambda=\ln(2\kappa)$. Note that an arbitrary state $\mathbf{p}_\perp\neq0$, $p_3=\kappa\to\infty$ in the infinite-momentum frame can be brought into a state in the special frame $p_0+p_3=1$ ($\mathbf{p}_\perp$ unchanged) by the same deceleration and vice versa. As far as practical calculations are concerned, the special frame $p_0+p_3=1$ is easier to handle than the infinite-momentum frame $p_3=\kappa\to\infty$. Since these two frames are simply related, we sometimes also refer to the special frame $p_0+p_3=1$ as the standard (decelerated) infinite-momentum frame.

[7] H. Leutwyler, Phys. Rev. Letters **20**, 561 (1968).

[8] S. J. Chang, and L. O'Raifeartaigh, J. Math. Phys. **10**, 21 (1969). The six generators of an $SL(2,C)$ in the $E(2)$ basis are $\mathbf{E}=(K_1+L_2, K_2-L_1)$, L_3, K_3, and $\mathbf{E}'=(K_1-L_2, K_2+L_1)$. In the spinor space, the generators $\mathbf{E}$ and $\mathbf{E}'$ are denoted by $\mathbf{X}$ and $\mathbf{F}$ (or $\mathfrak{F}$, $\mathbf{f}$).

saturated isospin-$\frac{1}{2}$ current algebra at $p_3=\infty$ was

$$F(\mathbf{k})=e^{i\mathbf{k}\cdot\mathbf{X}}, \tag{4.1}$$

$$F_5(\mathbf{k})=\Sigma e^{i\mathbf{k}\cdot\mathbf{X}}, \tag{4.2}$$

where Σ, $\mathbf{X}=(X_1,X_2)$ are a set of commuting Hermitian operators. In this section, we wish to combine this condition with the conditions imposed by Lorentz invariance, i.e,. the angular condition, to obtain the full set of conditions which $\mathbf{X}$ must satisfy. The restrictions imposed on $\mathbf{X}$ and Σ by the angular conditions are

$$I_{\mathbf{k}}(I_{\mathbf{k}}(I_{\mathbf{k}}(e^{i\mathbf{k}\cdot\mathbf{X}})))=I_{\mathbf{k}}(J_{\mathbf{k}}(e^{i\mathbf{k}\cdot\mathbf{X}})) \tag{4.3}$$

and

$$I_{\mathbf{k}}(I_{\mathbf{k}}(I_{\mathbf{k}}(\Sigma e^{i\mathbf{k}\cdot\mathbf{X}})))=I_{\mathbf{k}}(J_{\mathbf{k}}(\Sigma e^{i\mathbf{k}\cdot\mathbf{X}})), \tag{4.4}$$

where I and J are defined in the preceding paper [Eqs. (2.11)–(2.12)]. This is the full restriction on $\mathbf{X}$ and Σ due to the angular condition.

We shall assume henceforth that, at least for $m'\neq m$, the angular condition is analytic in $\mathbf{k}$. This is physically reasonable because $F(\mathbf{k})$ is the electromagnetic form factor and $\mathbf{k}$ the momentum transfer. With this assumption we can expand the angular condition (4.3) in powers of $\mathbf{k}$ and continue to complex values of $\mathbf{k}$. In the present section we shall make the expansion for $\mathbf{k}=(k,\pm ik)$, in which case $\mathbf{k}^2=0$.[9] We shall show that in this case the angular condition is equivalent to the two $\mathbf{k}$-independent equations

$$M_{\tau\tau\tau}=0, \tag{4.5}$$

$$[X_\tau,\Lambda_\tau]=0, \tag{4.6}$$

where $\tau=\pm$ and

$$M_{\tau\tau\tau}=[X_\tau,[X_\tau,[X_\tau,M^2]]],$$
$$X_\pm=X_1\pm iX_2,$$
$$\Lambda_\pm=\pm 2iM\mathcal{J}_\pm+\tfrac{1}{2}[M^2,X_\pm]_+.$$

In (4.5) and (4.6) the τ must have the *same* value in each position. Thus $M_{\tau\tau\tau'}\neq 0$ if $\tau=+$, $\tau'=-$. The summation convention is not used.

To establish (4.5) and (4.6) we first note that for $\mathbf{k}^2=0$, the operations with I and J simplify to

$$\begin{aligned}I_{\mathbf{k}}(\theta)&=[M^2,[\mathcal{J}_3,\theta]]-2k[M\mathcal{J}_\tau,\theta],\\ J_{\mathbf{k}}(\theta)&=[M^2,[M^2,\theta]],\end{aligned} \tag{4.7}$$

where

$$M\mathcal{J}_\tau=M\mathcal{J}_\pm=M\mathcal{J}_1\pm iM\mathcal{J}_2.$$

The plus and minus signs in (4.7) depend on whether $\mathbf{k}=(k,\pm ik)$. Inserting this result into the angular condition (4.3) and expanding in powers of k, we obtain identities for the terms in 1 and k, and the equations

$$[M^2,[M^2,([M^2,X_\tau^2]+4i\tau[M\mathcal{J}_\tau,X_\tau])]]=0 \tag{4.8}$$

and

$$[M^2,([M^2,[M^2,X_\tau^3]]+i\tau(15/4)[M^2,[M\mathcal{J}_\tau,X_\tau^2]] -3[M\mathcal{J}_\tau,[M\mathcal{J}_\tau,X_\tau]])]=0, \tag{4.9}$$

with

$$\tau=\pm,$$

for the terms in k^2 and k^3, respectively. We next note that the two over-all commutations with M^2 in (4.8) can be dropped, because when taken between states with $m'\neq m$, they yield only a nonzero over-all factor $(m'^2-m^2)^2$, and for states with $m'=m$ we have an extra "threshold" condition.[10] Dropping the double commutation with M^2 in (4.8) and rearranging the terms, we obtain (4.6).

If we now substitute (4.6) into (4.9), we obtain

$$[M^2,[M^2,([[[M^2,X_\tau],X_\tau],X_\tau])]]=0, \tag{4.10}$$

and dropping the double commutation with M^2 for the same reasons as before, this reduces to (4.5). Thus (4.5) and (4.6) are necessary conditions for the angular condition to be satisfied. It remains to show that they are also sufficient in the case $\mathbf{k}^2=0$.

To show this we note that from (4.5) and (4.6) we have by induction

$$[M\mathcal{J}_\tau,X_\tau^n]=\frac{\tau in}{2(n+1)}[M^2,X_\tau^{n+1}]. \tag{4.11}$$

Hence from (4.7) we have for $\mathbf{k}^2=0$

$$I_{\mathbf{k}}(X_\tau^n)=n[M^2,X_\tau^n]-\frac{\tau ink}{n+1}[M^2,X_\tau^{n+1}]. \tag{4.12}$$

Inserting this result into the expansion of $e^{i\mathbf{k}\cdot\mathbf{X}}$, we obtain at once

$$I_{\mathbf{k}}(e^{i\mathbf{k}\cdot\mathbf{X}})=[M^2,e^{i\mathbf{k}\cdot\mathbf{X}}]. \tag{4.13}$$

In view of the expression (4.7) for J, this result shows at once that the angular condition is satisfied.

We conclude by obtaining conditions analogous to (4.5) and (4.6) for the axial-vector current. Expanding the angular condition (4.4), we obtain the infinite sequence of equations

$$[2M\mathcal{J}_\tau,[2M\mathcal{J}_\tau,\Sigma X_\tau^{n-2}]]-2i\tau[2M\mathcal{J}_\tau,[M^2,\Sigma X_\tau^{n-1}]] -[M^2,[M^2,\Sigma X_\tau^n]]=0. \tag{4.14}$$

Using (4.5) and (4.6), it is easy to see that these equations are satisfied if and only if the two equations

$$M_{\tau\tau}[\mathfrak{F}_\tau,\Sigma]=0, \tag{4.15}$$

$$[\mathfrak{F}_\tau,[\mathfrak{F}_\tau,\Sigma]]-2M_\tau[\mathfrak{F}_\tau,\Sigma]+\tfrac{1}{2}[\Sigma,M_{\tau\tau}M^2]=0 \tag{4.16}$$

[9] In this section, we follow closely the argument of Leutwyler (Ref. 7). Note that the derivation given by Leutwyler is simply related to the derivation given by Dashen and Gell-Mann (Ref. 4) who consider, in each order k^n, only those equations which change the helicity by a maximum amount $|\Delta h|=n$.

[10] The existence of the threshold condition is related to the fact that, in deriving the angular condition (see Paper I), we have multiplied the angular relation by a numerical factor $\Delta^2=(m'^2-m^2)^2+2k^2(m'^2+m^2)+k^4$. This extra factor, in the limit of $k^2=0$, gives us exactly the threshold condition.

are satisfied, where $\tau=\pm$,

$$\mathfrak{F}_\tau=\Lambda_\tau+\tfrac{1}{2}M_\tau\,,\quad M_\tau=[X_\tau,M^2]\,,\quad M_{\tau\tau}=[X_\tau,[X_\tau,M^2]]. \tag{4.17}$$

[These equations are obtained from (4.14) by choosing $n=2$ and 3.] Equations (4.15) and (4.16) are therefore the necessary and sufficient conditions for the axial-vector current at $\mathbf{k}^2=0$.

V. GENERAL REDUCTION OF THE ANGULAR CONDITION

In this section we reduce the angular condition (4.3) into a set of $\mathbf{k}$-independent equations for all values of $\mathbf{k}$. We shall consider only the vector current and shall rely heavily on the conditions (4.5) and (4.6) already obtained for $\mathbf{k}^2=0$.

For the vector current the angular condition (4.3) may be written as

$$e^{-i\mathbf{k}\cdot\mathbf{X}}I_{\mathbf{k}}(I_{\mathbf{k}}(I_{\mathbf{k}}(e^{i\mathbf{k}\cdot\mathbf{X}})))=e^{-i\mathbf{k}\cdot\mathbf{X}}I_{\mathbf{k}}(J_{\mathbf{k}}(e^{i\mathbf{k}\cdot\mathbf{X}}))\,. \tag{5.1}$$

The first question is whether the expansion of (5.1) in powers of $\mathbf{k}$ stops at some finite power, that is to say, whether the satisfaction of the conditions for a finite number of powers of $\mathbf{k}$ is sufficient to guarantee the angular condition for all $\mathbf{k}$. From the form of (5.1) one sees that a sufficient condition for the expansion to stop is that the nth commutator of $\mathbf{X}$ with each of the six basic variables M^2, $\mathcal{J}_3$, $\mathbf{X}$ and $\mathfrak{F}$ vanishes for sufficiently large n. This is clearly true for $\mathcal{J}_3$ and $\mathbf{X}$, and from (4.5) it is true for M^2. Further, from (4.5) and (4.6), we have

$$[X_+,[X_+,[X_+,M\mathcal{J}_+]]]=0\,. \tag{5.2}$$

Hence, the question reduces to whether the nth commutator of X_+ with $M\mathcal{J}_-$ vanishes for sufficiently large n.

Henceforth we shall assume that the expansion in powers of $\mathbf{k}$ does terminate. The assumption will be justified by the self-consistency of the equations thus obtained, and by the fact that in the simplified model in which one saturates the charge-density×current-density algebra at $p_3=\infty$ the expansion can be shown to terminate. Indeed, if one commutes the angular condition for that case, namely, Eq. (3.15), three times with X_+, and uses (4.5) and (4.6), one obtains at once the relation

$$e^{-i\mathbf{k}\cdot\mathbf{X}}[[X_+,[X_+,[X_+,M\mathcal{J}_-]]],e^{i\mathbf{k}\cdot\mathbf{X}}]=0\,, \tag{5.3}$$

from which one immediately obtains, for $\mathbf{k}=(k,ik)$,

$$[X_+,[X_+,[X_+,[X_+,M\mathcal{J}_-]]]]=0\,. \tag{5.4}$$

Incidentally, in both this case and the general case, once the expansion for the vector current terminates, then so does the expansion for the axial-vector current.

Once we assume that the expansion of the angular condition in powers of $\mathbf{k}$ terminates, we can determine the power at which it terminates in general as follows From (4.5) it is clear that

$$M_5=\delta^5(X_1)M^2=0\,, \tag{5.5}$$

where

$$\delta(X_1)\theta=[X_1,\theta]\,.$$

Hence, choosing $\mathbf{k}=(k,0)$ for definiteness, we see that the leading terms in the expansion of (5.1) are

$$k^{15}M_4{}^3 \quad\text{and}\quad k^{3(n+1)}[\delta^n(X_1)2M\mathcal{J}_1]^3\,,$$

where

$$M_4=\delta^4(X_1)M^2\,.$$

It follows that $n\leq 4$.

Then, by expanding (5.1) and equating the coefficients of k, we get the required set of k-independent equations. Since the reduction is quite straightforward (though, algebraically, it is very complicated), and since it does not involve any new physical information, we shall leave the reduction to Appendix A.

The main results of the reduction are the following: If we reject the solutions for which a spacelike part exists and is definitely coupled to the timelike part by the current, then the equations obtained from the expression (5.1) in k^n for $n\geq 7$ can be summarized as

$$[X_i,[X_j,[X_k,M^2]]]=0\,, \tag{5.6}$$

$$[\mathbf{X},B]=0\,, \tag{5.7}$$

$$i[K,\mathbf{X}]=\mathbf{X}\,, \tag{5.8}$$

where $\Lambda_\pm$ are given in (4.6) and

$$iK\pm(\mathcal{J}_3-B)=\tfrac{1}{4}[\Lambda_\mp,X_\pm]\,. \tag{5.9}$$

These equations, combined with (4.6) and

$$[\Lambda_+,\Lambda_-]=4[M^2,B]_+-\tfrac{1}{4}[M_+,M_-]\,, \tag{5.10}$$

lead to a quasi-$SL(2,C)$ structure. Equation (5.10) and

$$[\Lambda_\tau,M^2]=\tfrac{1}{2}[M^2,M_\tau]_+ \tag{5.11}$$

follow from the usual angular momentum commutator relations

$$[2M\mathcal{J}_+,2M\mathcal{J}_-]=8M^2\mathcal{J}_3\,,\quad [2M\mathcal{J}_\pm,M^2]=0\,.$$

In nearly all the interesting cases, this quasi-$SL(2,$C) structure can be made exact by a proper redefinition of the generators. This $SL(2,C)$ group structure plays a central role in constructing the solutions to the angular condition.

There are several important secondary relations which can be obtained by commuting Eqs. (5.9)-(5.11) with respect to X's, giving (Appendix A)

$$[B,R]=0\,,\quad M_{++}{}^2=0\,, \tag{5.12}$$

$$M_{++}[B+\tfrac{1}{2}(R+1)]=[B-\tfrac{1}{2}(R+1)]M_{++}=0\,, \tag{5.13}$$

and

$$i[K,R]=R(R+1)+\tfrac{1}{64}[M_{++},M_{--}]_{+}, \quad (5.14)$$

where

$$R=-\tfrac{1}{4}[X_{+},[X_{-},M^2]]. \quad (5.15)$$

After this digression we return to the reduction of the angular condition. Using (5.6)–(5.14), we find that

$$e^{-ikX_1}[I_{\mathbf{k}}(I_{\mathbf{k}}(e^{ikX_1}))-J_{\mathbf{k}}(e^{ikX_1})] =4[(ik)^2G_0+(ik)^3(G_{+}+G_{-})+(ik)^4G], \quad (5.16)$$

where

$$G_0=\tfrac{1}{2}i[K,M^2]+M^2,$$
$$G_{\pm}=-\tfrac{1}{4}i[K,\mathfrak{F}_{\pm}]-\tfrac{1}{4}\mathfrak{F}_{\pm}, \quad (5.17)$$
$$G=B^2-\tfrac{1}{4}(R+1)^2,$$

and $\mathfrak{F}_{\pm}$ is defined in (4.17). Note that $G_{\pm}$ are not Hermitian conjugates.

Substituting (5.16) into the angular condition (5.1), we obtain the remaining equations for k^n, $n\leq 6$:

$$k^6: \quad BG=0; \quad (5.18)$$

$$k^5: \quad M_{++}G_{+}=M_{--}G_{-}=0,$$
$$[\mathfrak{F}_{\pm},G]-M_{\pm}G+4[\pm B-\tfrac{1}{2}(R+1)]G_{\pm}=0; \quad (5.19)$$

k^4:

$$\tfrac{1}{4}M_{++}G_0-2M_{+}G_{+}+[\mathfrak{F}_{+},G_{+}]-[X_{+},G_{+}]M^2=0,$$
$$\tfrac{1}{4}M_{--}G_0-2M_{-}G_{-}+[\mathfrak{F}_{-},G_{-}]-[X_{-},G_{-}]M^2=0. \quad (5.20)$$

The equations for other orders of k are satisfied automatically. The equations obtained in this section therefore represent the full set of k-independent equations which are imposed by the angular condition for the isospin-$\tfrac{1}{2}$ model. They are summarized in Table I.

VI. EXISTENCE OF SPACELIKE SOLUTIONS

In this section, we show that, subject to very mild technical restrictions, the nontrivial solutions of Table I have a spacelike ($M^2<0$) part. A sufficient technical condition, whose plausibility will be discussed later, is that there exist for the operators of Table I a common dense invariant domain $\mathfrak{D}$ on which M^2 and ϵ are essentially self-adjoint, M_{++} and M_{--} are essentially adjoint, and $\mathfrak{J}_3$, $\mathbf{X}$, and K generate a unitary group.

To show that this condition is sufficient for the solutions to have a spacelike part, let us assume the opposite, that is to say, let us assume that

$$M^2\geq 0. \quad (6.1)$$

Then from the expansions

$$(d,e^{-ikX_1}M^2e^{ikX_1}d)=(d,M^2d)-ik(d,[X_1,M^2]d) +k^2(d,\,R-\tfrac{1}{8}(M_{++}+M_{--})d),$$
$$(d,e^{-ikX_2}M^2e^{ikX_2}d)=(d,M^2d)-ik(d,[X_2,M^2]d) +k^2(d,\,R+\tfrac{1}{8}(M_{++}+M_{--})d), \quad (6.2)$$

which follow directly from Table I, we see that since the k^2 term dominates for large k,

$$R-\tfrac{1}{8}(M_{++}+M_{--})\geq 0,$$
$$R+\tfrac{1}{8}(M_{++}+M_{--})\geq 0, \quad (6.3)$$

whence, in particular,

$$R\geq 0. \quad (6.4)$$

Now consider the functions

$$\phi(x)=(d,e^{ixK}\epsilon e^{-ixK}d) \quad (6.5)$$

for $d\epsilon\mathfrak{D}$, $-\infty<x<\infty$, where

$$\epsilon=R/(R+1). \quad (6.6)$$

For $R\geq 0$ we have

$$0\leq\phi(x)\leq 1. \quad (6.7)$$

On the other hand, from (5.14) we have the relation

$$i[K,\epsilon]=\epsilon+\tfrac{1}{64}(1-\epsilon)[M_{++},M_{--}]_{+}(1-\epsilon) \quad (6.8)$$

on $\mathfrak{D}$. Using this result in (6.5), we have

$$\phi'(x)=\phi(x)+\tfrac{1}{64}((1-\epsilon)e^{-ixK}d,[M_{++},M_{--}]_{+} \times(1-\epsilon)e^{-ixK}d). \quad (6.9)$$

But since M_{++} and M_{--} are adjoint, $[M_{++},M_{--}]_{+}$ is positive. Hence

$$\phi'(x)\geq\phi(x). \quad (6.10)$$

From (6.7) and (6.10) we have, for $y>u>x$,

$$\phi(u)-\phi(x)=\int_x^u\phi'(t)dt\geq\int_x^u\phi(t)dt\geq 0,$$

whence

$$\phi(y)-\phi(x)=\int_x^y\phi'(u)du \geq\int_x^y\phi(u)du\geq(y-x)\phi(x). \quad (6.11)$$

But this equation is incompatible with (6.7) unless

$$\phi(x)=0, \quad (6.12)$$

in which case, from (6.5) and (6.6),

$$R=\epsilon=0. \quad (6.13)$$

Substituting this result into (6.3), we obtain finally

$$R=M_{++}+M_{--}=0. \quad (6.14)$$

But now the k^2 terms in the expansions (6.2) vanish identically and, unless they are identically zero, the k terms dominate for large k. However, since the sign of k is arbitrary, the k terms are not positive. Hence, if M^2 is to be positive, the k terms must vanish, i.e.,

$$[M^2,\mathbf{X}]=0$$

on $\mathfrak{D}$. In that case the current $e^{i\mathbf{k}\cdot\mathbf{X}}$ commutes with the mass-squared operator M^2. There are then no electromagnetic transitions between states of different mass, and the solution is trivial.

Although we have thus demonstrated the existence of a spacelike part for nontrivial solutions of Table I, it is still possible that the spacelike and timelike parts are not connected by the current, for what we have demonstrated is that the operator e^{ixK} connects the spacelike and timelike parts, and this does not necessarily imply that $e^{i\mathbf{k}\cdot\mathbf{X}}$ connects them, and, in fact, in the free-quark model it does not.

If we have the situation in which the current $e^{i\mathbf{k}\cdot\mathbf{X}}$ does not connect the spacelike and timelike parts, then the saturation proposal could be saved, because, since only $e^{i\mathbf{k}\cdot\mathbf{X}}$ enters in the charge-density algebra (2.1), the saturation of the algebra with the timelike solutions alone would be consistent.

In point of fact, however, it appears that, in general, the current $e^{i\mathbf{k}\cdot\mathbf{X}}$ does connect the spacelike and timelike parts. In orther words, the situation in the free-quark model is the exception rather than the rule. This point will be discussed at length in Sec. IX.

It remains to discuss the technical assumptions leading to the existence of spacelike solutions. The only assumption that is questionable is that concerning K, because this operator is a *bi*linear in the basic variables M, $\mathcal{J}_\alpha$, and $\mathbf{X}$ and hence might possibly be pathological. However, there is no reason to assume that it is pathological, and good reason to believe otherwise, because in Secs. VII and VIII when we express solutions of Table I as wave equations, K emerges as the generator of accelerations along the z axis in spinor space. Incidentally, it should be noted that the technical condition stated is not the weakest possible. One could, for example, use one dense domain for the expansion (6.2) and a different dense domain to define the functions $\phi(x)$. However, such refinements would appear to be unnecessary.

We conclude by discussing the connection between the results of this section and a no-go theorem proved recently by Grodsky and Streater.[11] The two results are similar, but they are by no means completely equivalent. In our case, we assume current algebra and isospin factorization, but make no *a priori* assumptions concerning the representations of $SL(2,C)$ to which the particles belong or concerning the p independence of the current. (In fact, for the Dashen solution and for the general solutions the current turns out to be linear in p.) Grodsky and Streater, on the other hand, do not assume current algebra, but assume that the current is p-independent, and restrict the representations of $SL(2,C)$ by demanding a polynomial bound on positive-frequency projection operator. Finally, our result shows that the spacelike part does not completely decouple at $p_z\to\infty$. It is coupled by the operator e^{ixK}, if not by the current.

[11] I. T. Grodsky and R. Streater, Phys. Rev. Letters **20**, 695 (1968).

VII. PRIMITIVE SOLUTIONS

In Secs. IV and V, we have decomposed the angular condition into a set of k-independent algebraic equations. The purpose of the next two sections is to construct systematically all possible solutions to this set of equations. The way in which the new solutions were found, and the Dashen solution recovered, from the equations in Table I was by strengthening the condition

$$B[B^2-\tfrac{1}{4}(R+1)^2]=0 \qquad (7.1)$$

to

$$B=0,\quad B=\tfrac{1}{2}(R+1),\quad \text{and}\quad B=-\tfrac{1}{2}(R+1), \qquad (7.2)$$

respectively. The solutions obtained for these three special cases will be called *primitive* solutions. In general, of course, Eq. (7.1) does not imply that one of the alternatives (7.2) holds for the *whole* Hilbert space H. It implies only that H consists of three parts H_0 and $H_\pm$ which are such that $BH_0=0$ and $[B\pm\tfrac{1}{2}(R+1)]H_\pm=0$. In the H_0, $H_\pm$ basis, the operators in H, M^2, for example, can be written as 3×3 matrices. The primitive solutions are those solutions given completely in those subspaces. In this section we shall concentrate on these primitive solutions only. They are classified naturally as

(*i*) *Class $B=0$.* The relations (5.13) and (5.9) reduce to

$$M_{\tau\tau}=0,\quad \tau=\pm \qquad (7.3)$$

$$iK\pm\mathcal{J}_3=\tfrac{1}{4}[\Lambda_\mp,X_\pm]. \qquad (7.4)$$

Equations (4.6) and (7.4) are, in fact, identical to Eq. (3.19) in the Dashen model. Therefore, the Dashen solution is recovered as one of our primitive solutions. The explicit structure of this solution and its relation to the infinite-component wave equations have already been given in Sec. III. Here, we wish to verify that this solution satisfies all the angular conditions given in Table I. Since Eqs. (5.6)–(5.11) and (4.6) have been used to construct the solution, they are no doubt satisfied by the solution. The only remaining equations to be verified are Eqs. (5.18)–(5.20). With the help of

$$G_\pm=-\tfrac{1}{4}i[\bar{K}_3+\tfrac{1}{2}iR,\,\mathfrak{F}_\pm]-\tfrac{1}{4}\mathfrak{F}_\pm=-\tfrac{1}{8}[\mathfrak{F}_\pm,R],$$
$$[\mathfrak{F}_\pm,R]=M_\pm(R+1),$$
$$[\mathfrak{F}_\tau,[\mathfrak{F}_\tau,R]]=2M_\tau{}^2(R+1),$$
$$[X_\tau,G_\tau]=0,\qquad \tau=\pm$$

it is straightforward to see that Eqs. (5.18)–(5.20) are indeed satisfied.

As we have shown in Sec. III, this class of solutions may be derived from an infinite-component wave equation

$$(p^2-2g\cdot p-s)\phi=0. \qquad (7.5)$$

In the following, we shall attempt to understand this equation more thoroughly and, in particular, to see from the wave equation itself why this class of solutions satisfies the current algebra.

As a first step, we find that it is very important to clarify the idea of the spinor space on which the group $SL(2,C)$ operates. From the matrix element at infinite-momentum limit, we have[12]

$$\lim_{p_3\to\infty} \langle N'h', \mathbf{p}_\perp', p_3 | \mathcal{J}_0(0) | Nh, \mathbf{p}_\perp, p_3\rangle$$
$$= {}_1\langle N'h', \mathbf{p}_\perp'=0 | (\mathcal{J}_0+\mathcal{J}_3)(0)e^{i\mathbf{k}\cdot\mathbf{E}} | Nh, \mathbf{p}_\perp=0\rangle_1$$
$$=\phi^\dagger(N',h')e^{i\mathbf{k}\cdot\mathbf{X}}\phi(N,h),$$

where $|Nh, \mathbf{p}_\perp=0\rangle$ is a standard state with $p_0+p_3=1$, $\mathbf{p}_\perp=0$. In this expression, we represent a state in the spinor space by $\phi(N,h)$ in order to distinguish it from the physical state $|N,h,\mathbf{p}\rangle$. The distinction between the generator $\mathbf{E}$ of the full Lorentz group and the corresponding operator $\mathbf{X}$ in the spinor space should be emphasized. The operator $\mathbf{E}$ is an operator in the physical Hilbert space. It has nonvanishing matrix elements between, not only states with different spinor constant (N,h), but also states with different momentum. The operator $\mathbf{X}$, on the other hand, is an operator in the spinor space alone.

One of the most frequently introduced base vectors of the spinor space is the set of spinors in all the physical states at rest. It is easy to see that these spinors form a complete set of base vectors in the spinor space, and that the helicity angular momentum operators $\mathcal{J}_3$, $\mathcal{J}_\pm$ are represented simply on these base vectors. On the other hand, the set of base vectors $\phi(N,h)$ that we introduced may be considered as the set of spinors in all the physical states with $p_0+p_3=1$, $\mathbf{p}_\perp=0$. These spinors $\phi(N,h)$ also form a complete set of base vectors in the spinor space. These two sets of base vectors are related by a mass-dependent similarity transformation $e^{i\ln mK_3}$. For the present problem of saturating the current algebra at infinite momentum, we find that our base vectors $\phi(N,h)$ are much more natural and more useful. In the following analysis, we shall use this *infinite-momentum* basis $\phi(N,h)$.

Now, we can associate with each momentum

$$p_\mu=l_{\mu\nu}(p,m)(p_0+p_3=1,\ \mathbf{p}_\perp=0)_\nu$$

a spinor $\phi(Nh,p)=L(p,m)\phi(N,h)$, where $l_{\mu\nu}(p,m)$ is the (m-dependent) Lorentz transformation which brings the standard momentum vector $(p_0+p_3=1,\ \mathbf{p}_\perp=0)$ to p_μ, and $L(p,m)$ the corresponding transformation matrix in the spinor space. One has to be careful not to mix up the spinor $\phi(p)$ with the physical state of momentum p. The spinors $\phi(p)$ with different p are, in general, not orthogonal (as can easily be understood for a Dirac spinor), while the physical states with different momentum are always orthogonal. We wish to emphasize that it is on these spinors, rather than the physical states, that the infinite-component wave equation

$$(p^2-2g\cdot p-s)\phi=0$$

is defined.

[12] This is Eq. (3.3) of Paper I. Note that $\mathbf{E}=(K_1+L_2,\ K_2-L_1)$.

Next we wish to understand directly from the wave equation why this class of solutions satisfies the current algebra at $p_z=\infty$. Since the spinor $SL(2,C)$ algebra is generated by pseudo-Hermitian operators, we know that it is

$$\bar{\phi}\phi=\phi^\dagger(1-\epsilon)^{-1}\phi$$

rather than $\phi^\dagger\phi$, which transforms as a scalar. This relation, together with the structure of (7.5), implies that the current should be

$$\bar{\phi}(p')(P-g)_\mu\phi(p)\,,\quad P=\tfrac{1}{2}(p'+p)\,.$$

Hence, at $P_3\to\infty$, the time component of the current reduces to

$$\bar{\phi}(N',h')\exp(i\mathbf{p}_\perp'\cdot\mathbf{X})(1-\epsilon)\exp(-i\mathbf{p}_\perp\cdot\mathbf{X})\phi(N,h)$$
$$=\phi^\dagger(N',h')\exp(i\mathbf{k}\cdot\mathbf{X})\phi(N,h)\quad(7.6)$$

as required. Note that the pseudo-unitary metric $(1-\epsilon)^{-1}$ is crucial in compensating the extra factor $1-\epsilon$ in the current at the limit $P_3=\infty$.

Finally, we would like to examine the relation between the six spinor generators of the $SL(2,C)$ derived from the angular relations and the helicity operators $\mathcal{J}_\pm$, $\mathcal{J}_3$. That $\mathcal{J}_\pm$ are not identical to the corresponding operators in the spinor $SL(2,C)$ generators can be checked easily from the explicit expression we have obtained in Sec. III. However, there is a simple way to relate them. Let $\mathbf{E}$, J_3, K_3, and $\mathbf{E}'$ be the generators[8] of the full Lorentz group in the physical Hilbert space; i.e., these operators change not only the spinor components of the states, but also the momentum associated with states. In particular, on the physical states $|p_0+p_3=1,\ \mathbf{p}_\perp=0;\ m^2,\ h\rangle$, we have

$$[E_i,P_j]=-i\delta_{ij}(P_0+P_3)=-i\delta_{ij}\,,\qquad(7.7)$$

$$[E_i',P_j]=-i\delta_{ij}(P_0-P_3)=-i\delta_{ij}m^2\,,\qquad(7.8)$$

where P_μ is the momentum four-vector and p_μ its corresponding eigenvalue. Hence,

$$[E_i'-E_im^2,P_j]=0\,.\qquad(7.9)$$

Making use of these relations, one finds that the operators $E_\pm'-E_\pm M^2$, where M^2 is the mass operator, when acting on a state $|p_0+p_3=1,\ \mathbf{p}_\perp=0,\ m^2,\ h\rangle$, will lead to another state with the same momentum $p_0+p_3=1$, $\mathbf{p}_\perp=0$;

$$(P_0+P_3)(E_\pm'-E_\pm M^2)|p_0+p_3=1,\ \mathbf{p}_\perp=0,\ m^2,\ h\rangle$$
$$=(E_\pm'-E_\pm M^2)|p_0+p_3=1,\ \mathbf{p}_\perp=0,\ m^2,\ h\rangle\,,$$
$$P_\perp(E_\pm'-E_\pm M^2)|p_0+p_3=1,\ p_\perp=0,\ m^2,\ h\rangle=0\,.$$

In other words, the operators $E_\pm'-E_\pm M^2$ will leave the subspace $p_0+p_3=1$, $\mathbf{p}_\perp=0$ invariant, and hence they must be operators in the helicity space alone. Since these operators raise/lower the helicity of the state by 1, they must be of the form

$$E_\pm'-E_\pm M^2=\mathrm{const}\times\mathcal{J}_\pm\,.$$

These constants can be determined either by going to the rest frame, or from the commutator relations

$$[E_+'-E_+M^2,\ E_-'-E_-M^2]=8M^2\mathcal{J}_3.$$

They both lead to

$$E_\pm'=E_\pm M^2\pm 2iM\mathcal{J}_\pm. \qquad (7.10)$$

These relations, when restricted to the spinor space, lead to

$$\mathfrak{F}_\pm=X_\pm M^2\pm 2iM\mathcal{J}_\pm$$

as expected. As we shall see in Sec. VIII, this kind of relation is very useful in constructing the general solution to the angular condition.

(ii) Class $B=\frac{1}{2}(R+1)$. For this class of solutions, we have

$$M_{\tau\tau}=[X_\tau,\Lambda_\tau]=0,\quad \tau=\pm \qquad (7.11)$$

$$[X_+,\Lambda_-]=4(\mathcal{J}_3-B)-4iK=4\bar{\mathcal{J}}_3-4iK_3-\tfrac{1}{2}R, \qquad (7.12)$$

$$[X_-,\Lambda_+]=-4(\mathcal{J}_3-B)-4iK=-4\bar{\mathcal{J}}_3-4iK_3+\tfrac{1}{2}R. \qquad (7.13)$$

It is easy to see that $\mathbf{X}$, $\bar{\mathcal{J}}_3=\mathcal{J}_3-\frac{1}{2}$, and $K_3=K$ form the generators of an $E(2)\otimes D$ algebra.[8] The problem now is to identify the remaining generators $F_\pm$ of the $SL(2,C)$. Since we can bring the extra terms, $\frac{1}{2}R$, to the left-hand sides of Eqs. (7.12) and (7.13) and absorb them into the Λ's, this leads to the following tentative identification:

$$F_+=\Lambda_+-\tfrac{1}{2}M_+=\mathfrak{F}_+-M_+,\quad F_-=\Lambda_-+\tfrac{1}{2}M_-=\mathfrak{F}_-. \qquad (7.14)$$

In terms of these $F_\pm$, we have

$$[X_+,F_+]=[X_-,F_-]=0, \qquad (7.15)$$

$$[X_+,F_-]=4\bar{\mathcal{J}}_3-4iK_3,\quad [X_-,F_+]=-4\bar{\mathcal{J}}_3-4iK_3. \qquad (7.16)$$

The remaining commutator relation to be verified is $[F_+,F_-]=0$. These relations, together with the $E(2)\otimes D$ subalgebra

$$[X_+,X_-]=[\bar{\mathcal{J}}_3,K_3]=0,$$
$$[\bar{\mathcal{J}}_3,X_\pm]=\pm X_\pm,\quad i[K_3,X_\pm]=X_\pm,$$

are precisely the $SL(2,C)$ commutator relations we wish to establish.

The evaluation of the $[F_+,F_-]$ commutator is algebraically quite involved. We only describe here some crucial steps in the derivation. By the use of a slightly different form of Eq. (5.10),

$$[\mathfrak{F}_+,\mathfrak{F}_-]=8BM^2=4(R+1)M^2, \qquad (7.17)$$

we have

$$[F_+,F_-]=[\mathfrak{F}_+-M_+,\ \mathfrak{F}_-]=8BM^2+[X_+,[\mathfrak{F}_-,M^2]]-[[X_+,\mathfrak{F}_-],M^2]=4(i[K_3,M^2]+M^2+\tfrac{1}{4}M_-M_+). \qquad (7.18)$$

Then, we are able to show from (7.18) that

$$[\mathbf{X},[F_+,F_-]]=0,$$

and consequently

$$i[K_3,\mathbf{F}]=-\mathbf{F}. \qquad (7.19)$$

Commuting (7.17) with respect to iK_3 and making use of (7.19), we obtain

$$B(i[K_3,M^2]+M^2+\tfrac{1}{4}M_-M_+)=0.$$

Since $B=\frac{1}{2}(R+1)$ is in general not zero, we have

$$i[K_3,M^2]+M^2+\tfrac{1}{4}M_-M_+=0, \qquad (7.20)$$

and consequently $[F_+,F_-]=0$.

Since all six generators $\mathbf{X}$, $\bar{\mathcal{J}}_3$, K_3, and $\mathbf{F}$ are Hermitian, they generate a unitary representation of an $SL(2,C)$. By a rather straightforward calculation similar to that in the primitive case $B=0$, we are able to show that ϵ transforms as the zeroth plus the third component of a Hermitian vector g_μ, and that

$$M_+=-2i(1-\epsilon)^{-1}g_+,$$
$$M_-=-2ig_-(1-\epsilon)^{-1}.$$

These equations, after commuting once again with F's, lead to

$$M^2=g_0-g_3+g_-(1-\epsilon)^{-1}g_+, \qquad (7.21)$$

which gives a simple structure for the mass operator. One can also verify that all the additional angular conditions are satisfied. To see this we simply note that

$$M_{++}=M_{--}=0,$$
$$G_+=-\tfrac{1}{4}i[K_3,M_+]-\tfrac{1}{4}M_+=-\tfrac{1}{2}BM_+,$$
$$G_-=G=0.$$

The verification is then very elementary.

In analogy with the primitive case $B=0$, it is also true that for any given unitary representation of an $SL(2,C)$ and a Hermitian vector operator g_μ, we can construct through these relations a solution to the original angular condition, and vice versa. The analogy between the primitive solutions does not end here. It is interesting to see that this class of solutions can also be derived from an infinite-component wave equation

$$(\gamma\cdot p-\gamma\cdot g)\psi=0, \qquad (7.22)$$

$$\gamma_5\psi=\psi, \qquad (7.23)$$

where the spinor ψ transforms as a Dirac$\otimes$unitary representation of the Lorentz group. As one would expect, the spinor space spanned by the unitary part of the spinor ψ is the representation space of our unitary $SL(2,C)$ generators. To see how everything works out, we first go to the standard frame

$$(p_0+p_3)\psi=\psi,\quad \mathbf{p}_\perp\psi=0,$$
$$(p_0-p_3)\psi=M^2\psi,$$

and then decompose the Dirac part of the spinor explicitly. Due to the additional restriction $\gamma_5\psi=\psi$, the original four-component Dirac spinor degenerates into a two-component spinor

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where ψ_1 and ψ_2 are vectors (spinors) in the unitary representation of the $SL(2,C)$ and are chosen to be the eigenstates of $\alpha_3=+1,\ -1$. Then the wave equation

$$(\alpha\cdot p-\alpha\cdot g)\psi=0\,,\quad \alpha_\mu=\gamma_0\gamma_\mu$$

reduces to

$$\begin{pmatrix}M^2-(g_0-g_3) & g_- \\ g_+ & 1-(g_0+g_3)\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0\,. \quad (7.24)$$

Since ψ_1 is the large component at $p_3=\infty$, it suggests that we should eliminate ψ_2 completely through

$$\psi_2=-(1-\epsilon)^{-1}g_+\psi_1$$

and obtain

$$[M^2-(g_0-g_3)-g_-(1-\epsilon)^{-1}g_+]\psi_1=0\,. \quad (7.25)$$

This completes the verification that our mass operator can be derived from an infinite-component wave equation at $p_3=\infty$.

The other class of primitive solution $B=-\frac{1}{2}(R+1)$ is very similar to the class $B=\frac{1}{2}(R+1)$. We can readily see that the six generators $X_\pm$, $\bar{\mathcal{J}}_3=\mathcal{J}_3+\frac{1}{2}$, $K_3=K$, and $F_\pm=\Lambda_\pm\pm\frac{1}{2}M_\pm$ generate a unitary representation of an $SL(2,C)$, and that the mass operator is

$$M^2=g_0-g_3+g_+(1-\epsilon)^{-1}g_-\,, \quad (7.26)$$

with

$$R=\epsilon/(1-\epsilon)=-\tfrac{1}{4}M_{+-}\,,\quad \epsilon=g_0+g_3\,.$$

This solution can be recovered from the infinite-component wave equation

$$(\gamma\cdot p-\gamma\cdot g)\psi=0\,,\qquad \gamma_5\psi=-\psi\,. \quad (7.27)$$

Next, we can show quite easily from the wave equation that these solutions indeed satisfy the current algebra at $p_3=\infty$. Note that the wave equations (7.22) and (7.27) are both special cases of a general type of wave equation

$$(\gamma\cdot p-\mathfrak{M})\psi=0$$

due to Abers, Grodsky, and Norton,[5] where ψ transforms as a Dirac$\otimes$unitary representation of an $SL(2,C)$, and $\mathfrak{M}$ is a p-independent scalar. It was first suggested by Gell-Mann, Horn, and Weyers[4] that the solutions to this type of equation may be used to saturate the isospin-factored current algebra at $p_3=\infty$. In the following, we shall apply the AGN equation to current algebra.

Since $\bar{\psi}\psi$ is a scalar, and since the AGN equation is linear in p, the obvious choice of current in the AGN-type solution is

$$\bar{\psi}(p')\gamma_\mu\psi(p)\,.$$

Then at $p_3\to\infty$, the time component of the current reduces to

$$\bar{\psi}(N',h')\tfrac{1}{2}(\gamma_0+\gamma_3)e^{i\mathbf{k}\cdot(\mathbf{X}+\mathbf{x})}\psi(N,h)\,,$$

where $\mathbf{k}=(\mathbf{p}'-\mathbf{p})_\perp$, $\psi(N,h)$ is a spinor at $p_0+p_3=1$, $\mathbf{p}_\perp=0$, $\mathbf{X}$ are the unitary $E(2)$ generators, and

$$\mathbf{x}=(\tfrac{1}{2}(\sigma_{01}+\sigma_{31}),\ \tfrac{1}{2}(\sigma_{02}+\sigma_{32}))$$

are the Dirac $E(2)$ generators. Using the fact that $\frac{1}{2}(1+\alpha_3)e^{i\mathbf{k}\cdot\mathbf{x}}=\frac{1}{2}(1+\alpha_3)$, we finally reduce the current operator to

$$\psi^\dagger(N',h')\tfrac{1}{2}(1+\alpha_3)e^{i\mathbf{k}\cdot\mathbf{X}}\psi(N,h)\,.$$

Since $\frac{1}{2}(1+\alpha_3)$ is a projection operator, this current indeed satisfies the factored current algebra.

Finally, we conclude this section by the following remark. As it is well known that a current is determined by the kinematical terms (p-dependent terms) alone (at least in a Lagrange foundation), we, therefore, expect that the general solution to the isospin-factored current algebra can be obtained from a coupled equation among these three primitive classes of solutions. In terms of Hilbert space language, the primitive classes of solutions are confined to the respective Hilbert spaces H_0, $H_\pm$ defined by the condition $B=0$, $B=\pm\frac{1}{2}(R+1)$. The general solution should then be a solution which couples all three Hilbert spaces H_0, $H_\pm$ together into $H=H_0+H_++H_-$ and $B[B^2-\frac{1}{4}(R+1)^2]H=0$. In the next section, we will demonstrate how the general solution is constructed from this consideration.

VIII. GENERAL SOLUTIONS

We first construct the nonprimitive solutions which couple the Hilbert space H_+ and H_-. These solutions are very useful in the sense that they share all the properties of the general solution, but at the same time are much simpler algebraically. From the result of the primitive solutions, we learn that this kind of solution can be derived from a general AGN equation

$$(\gamma\cdot p-\mathfrak{M})\psi=0\,. \quad (8.1)$$

The fact that the term $\gamma_0\mathfrak{M}$ in general does not commute with γ_5 implies that states with opposite γ_5 are coupled together or, equivalently, $H=H_++H_-$. The central problem of this section is to construct the mass operator and the generators of the spinor $SL(2,C)$ group. We have learned from the primitive solutions that (1) the $SL(2,C)$ we wish to construct is associated with the unitary part of the wave function; (2) all the generators as well as the mass operator are introduced in the subspace of the "large component" of ψ (corresponding to $\alpha_3=1$); and (3) the generators F's are related to the helicity operators $M\mathcal{J}_\pm$ in a simple way. Keeping all this information in mind, we can proceed to construct the solutions.

We first multiply Eq. (8.1) by γ_0:

$$(\alpha\cdot p-\gamma_0\mathfrak{M})\psi=0\,, \tag{8.2}$$

and decompose $\gamma_0\mathfrak{M}$ and ψ into block form according to the eigenvalues of α_3:

$$\gamma_0\mathfrak{M}=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix},\quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{8.3}$$

where m_{11} and ψ_1 are the corresponding components in the subspace $\alpha_3=1$, etc. The self-adjointness of $\gamma_0\mathfrak{M}$ implies that m_{11}, m_{22} are self-adjoint and that m_{12}, m_{21} are adjoint of each other. In the standard frame

$$(p_0+p_3)\psi=\psi\,,\quad \mathbf{p}_\perp=0\,,\quad (p_0-p_3)\psi=M^2\psi\,,$$

the wave equation can be written as

$$\begin{pmatrix} M^2-m_{11} & -m_{12} \\ -m_{21} & 1-m_{22}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0\,, \tag{8.4}$$

which implies

$$\psi_2=(1-m_{22})^{-1}m_{21}\psi_1 \tag{8.5}$$

and

$$[M^2-m_{11}-m_{12}(1-m_{22})^{-1}m_{21}]\psi_1=0\,. \tag{8.6}$$

In the space of ψ_1, the mass operator is found to be

$$M^2=m_{11}+m_{12}(1-m_{22})^{-1}m_{21}\,. \tag{8.7}$$

Note that this expression is not very useful until we know what the $SL(2,C)$ generators are and how M^2 transforms under them.

To construct the unitary generators of the $SL(2,C)$, we should remind ourselves that we have to separate out the Dirac part $\frac{1}{2}\sigma_{\mu\nu}$ from the total generators and then project onto the subspace of ψ_1. For simplicity, we choose the following Dirac generators in the $E(2)$ basis:

$$\mathbf{x}=-\begin{pmatrix}0 & 0\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad j_3=\tfrac{1}{2}\sigma_3\,,\quad \mathbf{f}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ 0 & 0\end{pmatrix},$$

where $\boldsymbol{\sigma}=(\sigma_1,\sigma_2)$ are Pauli matrices. The fact that $\mathbf{x}$ has no entry in the upper half indicates that $\mathbf{X}$ is already the correct unitary $E(2)$ generators. It is also easy to see that the rotation along the z axis is $\mathfrak{J}_3=\mathcal{J}_3-\frac{1}{2}\sigma_3$, which agrees with those in the primitive solutions. The construction of the generators F is a little tricky. We first make use of Eq. (7.10)

$$E_\pm'=E_\pm M^2\pm 2iM\mathcal{J}_\pm\,, \tag{7.10}$$

and then reduce it to the direct-product spinor space

$$(F_\pm+f_\pm)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=[(X_\pm+x_\pm)M^2\pm 2iM\mathcal{J}_\pm]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{8.8}$$

Next, we project it out on the subspace of ψ_1, giving

$$F_\pm\psi_1=(\pm 2iM\mathcal{J}_\pm+X_\pm M^2)\psi_1-2\sigma_\pm\psi_2$$
$$=[\Lambda_\pm+\tfrac{1}{2}M_\pm-2\sigma_\pm(1-m_{22})^{-1}m_{21}]\psi_1\,.$$

Therefore, we have the operator relation

$$F_\pm=\Lambda_\pm+\tfrac{1}{2}M_\pm-2\sigma_\pm(1-m_{22})^{-1}m_{21}\,, \tag{8.9}$$

which, together with $K_3=\frac{1}{8}i([X_+,F_-]+[X_-,F_+])$, completes the identification of all the generators.

One remaining problem is to work out the transformation properties for these m_{ij}'s. This information can be obtained from the fact that $\mathfrak{M}$ is a Lorentz scalar under the combined $SL(2,C)$ group. Consequently, $\mathfrak{M}$ commutes with $\mathcal{J}_{\mu\nu}=L_{\mu\nu}+\frac{1}{2}\sigma_{\mu\nu}$, giving

$$[L_{\mu\nu},\mathfrak{M}]=-\tfrac{1}{2}[\sigma_{\mu\nu},\mathfrak{M}]\,, \tag{8.10}$$

where $L_{\mu\nu}$ and $\frac{1}{2}\sigma_{\mu\nu}$ are the unitary and Dirac generators, respectively. The last equation can be translated easily into the transformation laws on m's:

$$[X_\pm,m_{11}]=2(\sigma_\pm m_{21}-m_{12}\sigma_\pm)\,,\quad [X_\pm,m_{22}]=0\,,$$
$$[X_\pm,m_{12}]=2\sigma_\pm m_{22}\,,\quad [X_\pm,m_{21}]=-2m_{22}\sigma_\pm\,, \tag{8.11}$$

$$[iK_3,m_{11}]=-m_{11}\,,\quad [iK_3,m_{12}]=[iK_3,m_{21}]=0\,,$$
$$[iK_3,m_{22}]=m_{22}\,, \tag{8.12}$$

$$[F_\pm,m_{11}]=0\,,\quad [F_\pm,m_{12}]=2m_{11}\sigma_\pm\,,$$
$$[F_\pm,m_{21}]=-2\sigma_\pm m_{11}\,,$$
$$[F_\pm,m_{22}]=2(-\sigma_\pm m_{12}+m_{21}\sigma_\pm)\,. \tag{8.13}$$

Incidentally, these relations lead to a more symmetric form of $F_\pm$,

$$F_\pm=\Lambda_\pm-\sigma_\pm(1-m_{22})^{-1}m_{21}-m_{12}(1-m_{22})^{-1}\sigma_\pm\,, \tag{8.14}$$

which makes $\mathbf{F}$ manifestly Hermitian.

Let us summarize our results here. From every AGN wave equation, we can construct a set of Hermitian $SL(2,C)$ generators (in particular, the operators $\mathbf{X}$ and $\mathfrak{J}$) in the spinor space and a mass operator through Eq. (8.7) which obeys the transformation laws Eqs. (8.11), (8.12), and (8.13). The operators, or more precisely these Hermitian matrices in the spinor space, lead to a solution to the current algebra of the factored isospin at $p_3=\infty$. Since our solution is derived from a covariant wave equation, and since the angular conditions are nothing but restrictions on the Lorentz-transformation properties of the current-density four-vector, we should expect that the angular conditions will be satisfied identically. That this is indeed true can be verified by explicit computation.

In the following, we would like to analyze some of the interesting points in this nonprimitive solution. First, we wish to emphasize the similarity between this solution and the primitive solutions. The m_{ij} transforms under the group generators quite analogously to the four-vector g_μ. Next, we shall use this nonprimitive solution to understand the mechanism of how various relations of the angular conditions are satisfied. By commuting the mass operator with $\mathbf{X}$ twice, we have

$$M_{++}=-8\sigma_+(1-m_{22})^{-1}\sigma_+\,,\quad M_{--}=-8\sigma_-(1-m_{22})^{-1}\sigma_-\,,$$
$$R=\sigma_+(m_{22}/1-m_{22})\sigma_-+\sigma_-(m_{22}/1-m_{22})\sigma_+\,,$$

while by commuting $\mathbf{F}$ with $\mathbf{X}$, we have

$$B=\tfrac{1}{2}[\sigma_+(1-m_{22})^{-1}\sigma_- - \sigma_-(1-m_{22})^{-1}\sigma_+].$$

It is now transparent why we can have relations like $M_{++}\neq 0$ but $M_{++}^2=0$—a fact which seems to be quite strange at the beginning. Note that this can happen only if there is a tensor term $\sigma_{\mu\nu}\Sigma^{\mu\nu}$ in $\mathfrak{M}$. It also becomes clear that

$$[\mathbf{X},[\mathbf{X},[\mathbf{X},M^2]]]=0,$$
$$[B-\tfrac{1}{2}(R+1)]M_{++}=M_{++}[B+\tfrac{1}{2}(R+1)]=0,$$

and

$$B^2-\tfrac{1}{4}(R+1)^2=0.$$

The Dirac matrices $\sigma_\pm$ play an important role in verifying these relations.

There remains the important question of whether all the solutions which couple the Hilbert spaces H_+ and H_- can be generated this way. At the present time we do not have a proof of this, but we conjecture that it is true. We are able to show that the mass operator can indeed be put into the standard form (8.7) with m_{ij} transforming correctly under the subgroup $E(2)$. Using the angular conditions, one should be able to verify that m_{ij} transforms correctly under the rest of the Lorentz group. We have not yet verified this, but have good reason to believe it true and, hence, to make the above conjecture.

It is now quite easy for us to figure out a general solution which couples all three primitive solutions together. The solution is simply the coupled wave equation[13]

$$(\gamma\cdot p-\mathfrak{M})\psi=\eta\phi,$$
$$(p^2-2g\cdot p-s)\phi=\bar{\eta}\psi,\quad \bar{\eta}=(1-g_0-g_3)\eta^\dagger\gamma_0, \quad (8.15)$$

where ψ transforms as a Dirac$\otimes$unitary representation, ϕ transforms as a pseudounitary representation as in the primitive case $B=0$, $\mathfrak{M}$ and g_μ transform as a scalar and a four-vector in their corresponding representations, and η transforms as a spinor which couples these two representations. As mentioned earlier, all these operators can be represented conveniently by 3×3 matrices (of course, with all their entries as operators). The construction and verification of the general solution to the angular condition are similar to those for the AGN case; we shall not repeat these calculations here. But for completeness, we write down the mass operator and the $SL(2,C)$ generators $\mathbf{X}$, $\bar{\mathfrak{J}}_3$, K_3, and $\mathbf{F}$ of this solution:

$$M^2=\begin{pmatrix} m_{11}+m_{12}(1-m_{22})^{-1}m_{21} & \eta_1+m_{12}(1-m_{22})^{-1}\eta_2 \\ \eta_1^\dagger+\eta_2^\dagger(1-m_{22})^{-1}m_{21} & (1-g_0-g_3)^{-1}(g_0-g_3+s)-\eta_2^\dagger(1-m_{22})^{-1}\eta_2 \end{pmatrix},$$

$$\bar{\mathfrak{J}}_3=\mathfrak{J}_3-\begin{pmatrix} \tfrac{1}{2}\sigma_3 & 0 \\ 0 & 0 \end{pmatrix},\mathbf{X},$$

$$F_\pm=\Lambda_\pm-\begin{pmatrix} \sigma_\pm(1-m_{22})^{-1}m_{21}+m_{12}(1-m_{22})^{-1}\sigma_\pm & \sigma_\pm(1-m_{22})^{-1}\eta_2 \\ \eta_2^\dagger(1-m_{22})^{-1}\sigma_\pm & i(1-g_0-g_3)^{-1}g_\pm \end{pmatrix},$$

$$K_3=\tfrac{1}{8}i([X_+,F_-]+[X_-,F_+)],$$

where

$$\gamma_0\mathfrak{M}=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},\quad \gamma_0\eta=\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

Of course, there still remains the question of whether all the solutions to the angular conditions can be generated this way. We have some good reasons to believe that this is indeed so.

IX. DECOUPLING PROBLEM

In Sec. VI, we have shown that any nontrivial solution to the Dashen–Gell-Mann angular conditions always possess a spacelike part. A more interesting question is whether the spacelike part is coupled to the timelike part by the current. If they are not coupled, one can still saturate the current algebra of the factored isocurrent by the timelike solution alone.[14] It is known that in the free-quark model, the timelike and spacelike solutions indeed do not couple. We shall examine in detail how and why it happens in this particular model, since this will sharpen our understanding of the general problem. In the following, we first derive (in a mathematically very nonrigorous manner) a necessary and sufficient condition for the spacelike and timelike solutions not to be coupled by the current.

Let Λ be the projection operator onto the timelike states and $\Lambda(0<\epsilon<1)$ be the projection operator onto states with $0<\epsilon<1$. A necessary and sufficient condition for the spacelike and timelike solutions not to be

[13] More generally, the coupled equation can be expressed as

$$(\gamma\cdot p-\mathfrak{M})\psi=\eta\phi,$$
$$(ap^2-2g\cdot p-s)\phi=\bar{\eta}\psi.$$

See the final remark given in Appendix B.

[14] This possibility was first raised by Gell-Mann, Horn, and Weyers. See Ref. 4.

coupled by the current $e^{i\mathbf{k}\cdot\mathbf{X}}$ is

$$\Lambda=\Lambda(0<\epsilon<1). \tag{9.1}$$

Loosely speaking, this condition is that the Hilbert space defined by $M^2>0$ coincides with that defined by $0<\epsilon<1$.

Proof. Since, on a suitable domain, $[\mathbf{X},R]=[\mathbf{X},\epsilon]=0$, and if $\Lambda=\Lambda(0<\epsilon<1)$, then

$$[\mathbf{X},\Lambda]=[\mathbf{X},\Lambda(0<\epsilon<1)]=0.$$

This means that there is no transition by the current between a timelike and a spacelike solution. Thus the condition is sufficient. To prove its necessity, we note that if the timelike and spacelike solutions do not couple through the current, then

$$[e^{i\mathbf{k}\cdot\mathbf{X}}M^2e^{-i\mathbf{k}\cdot\mathbf{X}},\Lambda]=0, \quad \text{for all } \mathbf{k}.$$

Under suitable mathematical restrictions, this implies that all three terms in the expansion

$$e^{i\mathbf{k}\cdot\mathbf{X}}M^2e^{-i\mathbf{k}\cdot\mathbf{X}}=M^2+ik_i[X_i,M^2]-\tfrac{1}{2}k_ik_j[X_i,[X_j,M^2]]$$

commute with Λ, i.e.,

$$[M^2,\Lambda]=[M_\pm,\Lambda]=[R,\Lambda]=0. \tag{9.2}$$

On the other hand, since R now commutes with Λ, and since $e^{i\mathbf{k}\cdot\mathbf{X}}M^2e^{-i\mathbf{k}\cdot\mathbf{X}}$ is dominated at large k by the leading term $-\frac{1}{2}k_1k_jM_{ij}$, the condition of decoupling also implies that

$$\langle g|\Lambda R\Lambda|g\rangle\geq 0, \tag{9.3}$$

$$\langle g|(1-\Lambda)R(1-\Lambda)|g\rangle\leq 0, \quad \text{for all } g. \tag{9.4}$$

We first choose

$$|g\rangle=|r>0\rangle$$

such that

$$R|r>0\rangle=r|r>0\rangle, \quad r>0. \tag{9.5}$$

Then we have

$$\langle g|R(1-\Lambda)|g\rangle=r\|(1-\Lambda)|g\rangle\|^2\geq 0,$$

while by (9.3)

$$\langle g|R(1-\Lambda)|g\rangle=\langle g|(1-\Lambda)R(1-\Lambda)|g\rangle\leq 0.$$

This is possible only if

$$\Lambda|r>0\rangle=|r>0\rangle. \tag{9.6}$$

Similarly, if we choose $|g\rangle=|r<0\rangle$ with

$$R|r<0\rangle=r|r<0\rangle, \quad r<0 \tag{9.7}$$

we have both

$$\langle g|R\Lambda|g\rangle=r\|\Lambda|g\rangle\|^2\leq 0$$

and

$$\langle g|R\Lambda|g\rangle=\langle g|\Lambda R\Lambda|g\rangle\geq 0.$$

This implies

$$\Lambda|r<0\rangle=0. \tag{9.8}$$

Equations (9.7) and (9.8) then imply

$$\Lambda=\Lambda(R>0)=\Lambda(0<\epsilon<1)$$

as required.

To see how the result just obtained works out in practice, we consider the free-quark model in which the mass operator can be written as

$$M^2=(1-\epsilon)^{-1}(p_0-p_3+s), \quad \epsilon=p_0+p_3>0, \quad s>0$$

where p_μ is a timelike numerical four-vector and s a numerical scalar. Since the nominator p_0-p_3+s is a positive c number, it is easy to see that the condition for $M^2>0$ coincides with the condition $0<\epsilon<1$. By the above result, we conclude that the current cannot couple the timelike and spacelike solutions. Indeed, one can verify easily that [with $\mathbf{k}=(k,0)$]

$$e^{ikX_1}M^2e^{-ikX_1}=(1-\epsilon)^{-1}[p_0-p_3+2kp_1+k^2(p_0+p_3)]$$
$$=M^2\times(\text{a positive factor}),$$

which confirms that M^2 will not change sign when transformed by the current. The verification for the decoupling in the free-quark model depends critically on the fact that p_μ and s are c-number operators and that they satisfy some positiveness requirements. As we shall see, these are the very properties which make the free-quark model exceptional.

Next, we would like to prove that for all three primitive classes of solutions, the only solution which has the nice property of decoupling is the free-quark model. We do not pretend to make our proof very rigorous mathematically. However, one can still see from the following intuitive arguments what is going on physically. For any of the primitive solutions—$B=\frac{1}{2}(R+1)$, say—let Λ be the projection on the timelike states and let the current commute with Λ. By our previous theorem, we have

$$\Lambda=\Lambda(0<\epsilon<1). \tag{9.1}$$

The condition of decoupling implies that Λ must commute with both $\mathbf{X}$ and M^2, and consequently with their commutators $M_\pm$, R (or ϵ). The fact that

$$M^2=g_0-g_3+g_-(1-\epsilon)^{-1}g_+, \qquad \epsilon=g_0+g_3,$$
$$M_+=-2i(1-\epsilon)^{-1}g_+, \qquad M_-=-2ig_-(1-\epsilon)^{-1}$$

shows that, unless there are some hidden pathologies, Λ commutes with each component of g_μ, i.e.,

$$[g_\mu,\Lambda]=0. \tag{9.9}$$

Let us assume this. Now from (9.1) and $i[K,\epsilon]=\epsilon$, Λ projects on $0<\epsilon<1$ and K dilates ϵ ($M_{++}=0$ for primitive solutions). Hence

$$\Lambda(t)=e^{itK}\Lambda e^{-itK}$$

is a spectral family for ϵ which projects on $0<\epsilon<e^t$. But e^{itK} maps g into itself,

$$e^{itK}(g_0+g_3,\,\mathbf{g}_\perp,\,g_0-g_3)e^{-itK}=(e^{-t}(g_0+g_3),\,\mathbf{g}_\perp,\,e^t(g_0-g_3)).$$

Hence

$$[g_\mu,\Lambda(t)]=0, \quad \text{for all } t.$$

The fact that t can vary continuously from $-\infty$ to

$+\infty$ implies that

$$[g_\mu,\Lambda(\epsilon,\epsilon+d\epsilon)]=0\,,\quad \text{for all } \epsilon>0 \qquad (9.10)$$

where $\Lambda(\epsilon,\epsilon+d\epsilon)$ is the projection operator onto $(\epsilon,\epsilon+d\epsilon)$. For simplicity, we now restrict ourselves to $\epsilon>0$. One can check that there is no inconsistency and practically no loss in generality in making this assumption. Under this restriction, Eq. (9.10) states that g_μ commutes with all the projection operators of ϵ. However, if an operator commutes with the spectral family of another operator, it commutes with the operator itself. Consequently, we have

$$[g_\mu,\epsilon]=[g_\mu,\,g_0+g_3]=0\,.$$

After a proper Lorentz transformation, we have

$$[g_\mu,g_\nu]=0\,,\quad \text{for all } \mu,\,\nu\,.$$

The last relation, together with the assumption that $g_0>0$, is just the content of the free-quark model, Q.E.D.

In the general, nonprimitive case, one has the somewhat looser argument: M^2, $\mathcal{J}_\pm$, $\mathcal{J}_3$ commute with Λ, and K does not. Hence if $\mathbf{X}$ commutes with Λ, the six basic variables of Table I commute with Λ, but K, which is bilinear in these variables, does not. This appears to be a rather exceptional situation and we believe that it can occur only in the case of a continuous mass spectrum. Indeed, in the free-quark model, one finds that both $\mathbf{X}$ and M^2 commute with Λ, while $[\mathbf{X},M^2]_+$ does not, and that this peculiarity is due precisely to the continuum nature of the mass spectrum. The free-quark model is, of course, very pathological in the sense that it has only continuous mass spectrum and that its mass spectrum is infinitely degenerate.

If, in general, the spacelike solutions do indeed couple, the program of saturating with states of a single isospin fails—the restriction to the $I=\frac{1}{2}$ states is apparently too strong. Note that we have made no explicit use of the discreteness of M^2, and hence can allow any amount of continuum so long as it has $I=\frac{1}{2}$. Thus, a partly continuous mass spectrum alone will not provide a cure. A continuum with increasing isospin (which is of course provided by the physical many-particle states) seems to be necessary. Physically, this is easy to understand. The inclusion of only isospin-$\frac{1}{2}$ *resonances* in the intermediate states might be reasonable because all observed $S=1$ mesons, for example, have $I=\frac{1}{2}$. However, the inclusion of only $I=\frac{1}{2}$ *continuum* is definitely objectionable. Experimentally, there is no such pure $I=\frac{1}{2}$ continuum. Any physical process which creates an $I=\frac{1}{2}$ continuum also leads, in general, to a continuum with $I\geq\frac{1}{2}$.

APPENDIX A: EXPLICIT REDUCTION OF THE ANGULAR CONDITION

In this Appendix, we reduce the angular relation

$$e^{-i\mathbf{k}\cdot\mathbf{X}}I_{\mathbf{k}}(I_{\mathbf{k}}(I_{\mathbf{k}},e^{i\mathbf{k}\cdot\mathbf{X}}))=e^{-i\mathbf{k}\cdot\mathbf{X}}I_{\mathbf{k}}(J_{\mathbf{k}}(e^{i\mathbf{k}\cdot\mathbf{X}})) \qquad (A1)$$

into a set of $\mathbf{k}$-independent equations. We assume that the expansion of (A1) in powers of $\mathbf{k}$ terminates. Once we know that the expansion terminates, we can determine the power at which it terminates in general. As pointed out in the text, each of the operators M^2, $M\mathcal{J}$ vanishes after five commutators with respect to $\mathbf{X}$.

For simplicity, we choose $\mathbf{k}=(k,0)$. Using the above result together with (4.5) and (4.6), we can make the explicit expansion

$$\tfrac{1}{2}e^{-ikX_1}I_{\mathbf{k}}(e^{ikX_1})=\tfrac{1}{2}k[X_2,M^2]+(-ik)^2A+\frac{(-ik)^3}{2!}\delta(X_1)A+\frac{(-ik)^4}{3!}\delta^2(X_1)A+\frac{(-ik)^5}{4!}\delta^3(X_1)A\,, \qquad (A2)$$

where

$$\delta(X_1)\theta=[X_1,\theta]\,, \qquad (A3)$$

$$A=B+iC\,, \qquad (A4)$$

$$B=\mathcal{J}_3+(1/2i)[X_1,\,2M\mathcal{J}_1+\tfrac{1}{2}[M^2,X_2]_+]\,, \qquad (A5)$$

$$C=\tfrac{1}{2}[X_1,[X_2,M^2]]\,. \qquad (A6)$$

Note that B and C are Hermitian. As we shall see later, the operator B plays an important role in clarifying the solutions to the angular condition. Substituting (A2) into the angular condition (A1), we find that the leading term is of order k^{15} and yields

$$\delta^3(X_1)A=0\,. \qquad (A7)$$

Thus the last term in (A2) actually vanishes. The leading term in the expansion of (A1) is then of order k^{12} and yields

$$64B_2^3+8B_2M_4[\mathcal{J}_3,B_2]+M_4[\mathcal{J}_3,\,8B_2^2+M_4[\mathcal{J}_3,B_2]]=B_2M_4^2\,, \qquad (A8)$$

where

$$M_4=\delta^4(X_1)M^2=\tfrac{3}{8}[X_+,[X_+,[X_-,[X_-,M^2]]]] \qquad (A9)$$

and

$$B_2(=A_2)=\delta^2(X_1)B=\tfrac{1}{32}([X_+,[X_+,[X_+,\Lambda_-]]]-[X_-,[X_-,[X_-,\Lambda_+]]])\,. \qquad (A10)$$

In deriving (A10), we have used Eqs. (4.5) and (4.6). From (A9) and (A10) it follows that

$$[\mathcal{J}_3,M_4]=0\,,\quad [\mathcal{J}_3,[\mathcal{J}_3,B_2]]=4B_2\,. \qquad (A11)$$

Using

$$I_k([M^2,e^{ikX_1}])=[M^2,I_k(e^{ikX_1})]\,, \qquad (A12)$$

we also have

$$[B_2,M_4]=0\,. \qquad (A13)$$

With the help of relations (A11) and (A13), Eq. (A8) can be decomposed into the two separate equations

$$(64B_2^2+3M_4^2)B_2=0\,, \qquad (A14)$$

$$M_4\{B_2[\mathcal{J}_3,B_2]+[\mathcal{J}_3,B_2^2]\}=0\,. \qquad (A15)$$

From (A13) and (A14) and the Hermiticity of M_4 and B_2, it follows that

$$B_2=0, \tag{A16}$$

and consequently

$$A_2=\delta^2(X_1)A=0. \tag{A17}$$

[Note that, conversely, for $B_2=0$, the k^{12} equation (A8), from which (A16) is derived, is automatically satisfied.] Thus the second last term in (A2) also vanishes.

We now substitute (A2) into (A1) once more. The k^{11} terms vanish identically, but for k^{10} we get

$$\begin{aligned}M_4[\mathcal{J}_3,M_3[\mathcal{J}_3,[X_1,A]]]+M_3M_4[\mathcal{J}_3,[\mathcal{J}_3,[X_1,A]]]\\+\tfrac{1}{2}M_4{}^2[\mathcal{J}_3,[\mathcal{J}_3,A]]+6M_4[\mathcal{J}_3,[X_1,A]^2]\\+6[X_1,A]M_4[\mathcal{J}_3,[X_1,A]]\\=\tfrac{1}{2}M_4{}^2A+(M_4M_3+M_3M_4)[X_1,A],\end{aligned} \tag{A18}$$

where

$$M_3=\delta^3(X_1)M^2=\tfrac{3}{8}(M_{++-}+M_{+--}),$$
$$M_{++-}=\delta^2(X_+)\delta(X_-)M^2.$$

We wish now to follow the procedure used for Eq. (A8) and decompose Eq. (A18) according to eigenvalues of $\mathcal{J}_3$. For this purpose we introduce the notation

$$[\mathcal{J}_3,\theta_{(n)}]=n\theta_{(n)}. \tag{A19}$$

From (A4) we see that in this notation A and $[X_1,A]$ have the decomposition

$$\begin{aligned}A=A_{(2)}+A_{(0)}+A_{(-2)},\\ [X_1,A]=[X_1,A]_{(1)}+[X_1,A]_{(-1)},\end{aligned} \tag{A20}$$

respectively. Clearly $M_3=M_{3(1)}+M_{3(-1)}$ and $M_4=M_{4(0)}$. Substituting (A20) into (A19), we obtain for the eigenvalue 2 of $\mathcal{J}_3$

$$\begin{aligned}3M_4{}^2A_{(2)}+24M_4[X_1,A]_{(1)}{}^2+12[X_1,A]_{(1)}M_4[X_1,A]_{(1)}\\+2M_4M_{3(1)}[X_1,A]_{(1)}=0.\end{aligned} \tag{A21}$$

Commuting the equation twice with X_-, we obtain

$$M_4{}^3=0, \tag{A22}$$

whence, since M_4 is Hermitian,

$$M_4=0. \tag{A23}$$

The k^{10}-order equation (A18), from which (A23) is derived, is then satisfied identically.

Equation (A23) is a crucial equation, for it shows that $\delta^n(X_1)M^2$ vanishes for $n\geq 4$, and not $n\geq 5$ as implied by the $\mathbf{k}^2=0$ conditions (4.5). Further, if we agree to reject as unphysical those solutions of the angular condition for which there is a spacelike ($M^2<0$) part which is definitely coupled to the timelike part by the current $e^{i\mathbf{k}\cdot\mathbf{X}}$, then (A23) can be used to show that even

$$M_3=0. \tag{A24}$$

Indeed, this result follows immediately from the expansion

$$e^{-ikX_1}M^2e^{ikX_1}=M^2-ikM_1-\frac{k^2}{2!}M_2-\frac{ik^3}{3!}M_3, \tag{A25}$$

which shows that, unless $M_3=0$, the expectation value of M^2 on the states $e^{ikX_1}f$ is negative for sufficiently large k.

The highest remaining order in the expansion of the angular condition is k^9. If we use (A24), we see at once that in this order we obtain simply

$$[X_1,A]^3=0. \tag{A26}$$

But by commuting (A24) with $\mathcal{J}_3$ we obtain

$$[X_1,C]=0.$$

Hence,

$$[X_1,A]=[X_1,B], \tag{A27}$$

and since $i[X_1,B]$ is Hermitian, it then follows from (A26) that

$$[X_1,A]=[X_1,B]=0. \tag{A28}$$

Thus Eq. (A2) finally reduces to

$$\begin{aligned}e^{-ikX_1}\mathrm{I}_k(e^{ikX_1})=k[X_2,M^2]\\+ik^2([X_1,\,2M\mathcal{J}_1+M^2X_2]+2i\mathcal{J}_3).\end{aligned} \tag{A29}$$

Note that this equation has just one more term on the right-hand side than the angular condition (3.18) which is obtained for the simple model in which the charge-density×current-density algebra is saturated. Note also that with (A29) the highest nontrivial power in the expansion of the angular condition (A1) will be k^6.

Before proceeding to obtain the equations for k^n, $n\leq 6$, it is convenient to introduce a new Hermitian operator K by the relation

$$[X_\pm,\Lambda_\mp]=\mp 4(\mathcal{J}_3-B)-4iK, \tag{A30}$$

which is consistent with the definition of B given in (A5), and resembles the commutation relations for the generators $(K_1\pm L_2)$, $(K_2\pm L_1)$ and K_3, $\mathcal{J}_3$ of the conventional $SL(2,C)$ group. One sees at once from (A30) that

$$[\mathcal{J}_3,B]=0,\quad [\mathcal{J}_3,K]=0. \tag{A31}$$

Combining the first equation in (A31) with (A28), we obtain

$$[\mathbf{X},B]=0, \tag{A32}$$

and commuting Eqs. (A30) with X_+ and X_-, respectively, we obtain

$$i[K,\mathbf{X}]=\mathbf{X}. \tag{A33}$$

This B is an $E(2)$ scalar and K is the dilation operator for $E(2)$. It is also convenient to obtain some relations from the identities

$$[M\mathcal{J}_\pm,M^2]=0,$$
$$[2M\mathcal{J}_+,2M\mathcal{J}_-]=8M^2\mathcal{J}_3,$$

which, in terms of $\Lambda_\pm$, may be written as

$$[\Lambda_\pm,M^2]=\tfrac{1}{2}[M^2,M_\pm]_+,\quad M_\pm=[X_\pm,M^2],\quad \text{(A34)}$$

and

$$[\Lambda_+,\Lambda_-]=4[M^2,B]_+-\tfrac{1}{4}[M_+,M_-].\quad \text{(A35)}$$

Operating on (A34) with $\delta^3(X_+)$, $\delta^2(X_+)\delta(X_-)$, and $\delta(X_+)\delta^2(X_-)$, we obtain

$$M_{++}{}^2=M_{--}{}^2=0,\quad \text{(A36)}$$

$$[B,M_{++}]=M_{++}+\tfrac{1}{2}[R,M_{++}]_+,\quad i[K,M_{++}]=M_{++}+[R,M_{++}]_+,\quad \text{(A37)}$$

and

$$[B,R]=0,\quad i[K,R]=R(R+1)+\tfrac{1}{64}[M_{++},M_{--}]_+,\quad \text{(A38)}$$

respectively, where

$$M_{++}=[X_+,[X_+,M^2]],\quad R=-\tfrac{1}{4}[X_+,[X_-,M^2]].\quad \text{(A39)}$$

Similarly, operating on (A35) with $\delta^2(X_+)$ and $\delta(X_+)\times\delta(X_-)$, we obtain

$$[M_{++},B]_+=\tfrac{1}{2}[R,M_{++}]\quad \text{(A40)}$$

and

$$i[K,B]=\tfrac{1}{2}[R,B]_++(1/128)[M_{++},M_{--}].\quad \text{(A41)}$$

In particular, by suitably combining these equations, we obtain

$$[B-\tfrac{1}{2}(R+1)]M_{++}=0,\quad M_{++}[B+\tfrac{1}{2}(R+1)]=0.\quad \text{(A42)}$$

Equations (A24) and (A30)–(A42) are the equations which are needed in order to make the further reduction. Since the remaining reduction has already been given explicitly in the text, we shall not reproduce it here.

APPENDIX B: DETAILED CONSTRUCTION OF THE PRIMITIVE SOLUTIONS

In this Appendix, we wish to construct the primitive solutions explicitly. Since the procedure for constructing all three classes of primitive solutions is very similar, we only carry out the construction of solutions for the class $B=0$. Interested readers are invited to reproduce the other two classes of solutions.

For the class of solution $B=0$, and under the assumption that $1+R$ has an inverse, the angular conditions reduce to

$$M_{++}=M_{--}=0,\quad \text{(B1)}$$

$$[X_+,\Lambda_+]=[X_-,\Lambda_-]=0,\quad \text{(B2)}$$

$$[X_+,\Lambda_-]=-4iK-4\mathcal{J}_3,\quad \text{(B3)}$$

$$[X_-,\Lambda_+]=-4iK+4\mathcal{J}_3,\quad \text{(B4)}$$

$$[\Lambda_+,\Lambda_-]=-\tfrac{1}{4}[M_+,M_-],\quad \text{(B5)}$$

and

$$[\Lambda_\pm,M^2]=\tfrac{1}{2}[M^2,M_\pm]_+,\quad \text{(B6)}$$

together with Eqs. (5.18)–(5.20), where the last three equations are found to be satisfied automatically in this special case. The particular case $1+R=0$ will be discussed later. It can be shown easily from (B2)–(B4) that $X_\pm$, $\mathcal{J}_3$, and K form the generators of an $E(2)\otimes D$. These four generators together with $\Lambda_\pm$ generate an algebra which is very similar to an $SL(2,C)$ group structure. Nevertheless, the $SL(2,C)$ structure cannot be exact because of $[\Lambda_+,\Lambda_-]\neq 0$. The clue of resolving this difficulty is to introduce two new generators in (B5) as

$$\mathfrak{F}_\pm=\Lambda_\pm+\tfrac{1}{2}M_\pm.\quad \text{(B7)}$$

Then Eq. (B5) simplifies to

$$[\mathfrak{F}_+,\mathfrak{F}_-]=0.\quad \text{(B8)}$$

As we have already pointed out, these operators $\mathfrak{F}_\pm$ come in naturally in the angular conditions. Once we introduce $\mathfrak{F}_\pm$, it is very simple to verify that $X_\pm$, $\mathcal{J}_3$, $\bar{K}_3=K-\tfrac{1}{2}iR$, and $\mathfrak{F}_\pm$ generate an exact $SL(2,C)$ algebra.

Having constructed the $SL(2,C)$ algebra, we only have the problem partially solved. The remaining problems are (1) to find out the Hermiticity properties of these generators and (2) to determine the transformation properties of the mass operator M^2 under the $SL(2,C)$. As we shall see, not all six generators are Hermitian. They satisfy the following pseudo-Hermitian condition:

$$G^\dagger=(1-\epsilon)^{-1}G(1-\epsilon),\quad \text{(B9)}$$

where $\epsilon=(R+1)^{-1}R$. In other words, our representation is not unitary, but it is pseudounitary. Since we can always relate our representation to a unitary representation by a similarity transformation

$$\tilde{G}=(1-\epsilon)^{-1/2}G(1-\epsilon)^{1/2},\quad \text{(B10)}$$

and since all the unitary representations of an $SL(2,C)$ are well known, we can construct all the possible representations to our $SL(2,C)$ algebra.

To determine the transformation properties of M^2, we start from the operator

$$R=-\tfrac{1}{4}[X_+,[X_-,M^2]].\quad \text{(B11)}$$

Evidently,

$$[\mathbf{X},R]=[\mathcal{J}_3,R]=0.\quad \text{(B12)}$$

Furthermore, from (B6) we have

$$[\mathfrak{F}_\pm,M^2]=M_\pm M^2,\quad \text{(B13)}$$

and, after commuting with respect to $\mathbf{X}$,

$$[\mathfrak{F}_+,M_+]=M_+{}^2,\quad [\mathfrak{F}_-,M_-]=M_-{}^2,\quad \text{(B14)}$$

$$[\mathfrak{F}_+,M_-]-[\mathfrak{F}_-,M_+]=[M_+,M_-].\quad \text{(B15)}$$

Commuting (B14) and (B15) with respect to **X** and forming proper combinations, we have

$$i[\bar{K}_3,M_\pm]=RM_\pm\,, \tag{B16}$$

$$[\mathfrak{F}_\pm,R]=M_\pm(1+R)\,, \tag{B17}$$

and

$$i[\bar{K}_3,R]=R(R+1)\,. \tag{B18}$$

Equations (B14)–(B18) imply that $\epsilon=(R+1)^{-1}R$ has a much simpler transformation property, namely,

$$[\mathbf{X},\epsilon]=[\mathcal{J}_3,\epsilon]=0\,, \tag{B19}$$

$$i[\bar{K}_3,\epsilon]=\epsilon\,, \tag{B20}$$

$$[\mathfrak{F}_\pm,\epsilon]=(1-\epsilon)M_\pm\,, \tag{B21}$$

$$[\mathfrak{F}_+,[\mathfrak{F}_+,\epsilon]]=[\mathfrak{F}_-,[\mathfrak{F}_-,\epsilon]]=0\,. \tag{B22}$$

Equations (B19)–(B22) reveal that ϵ transforms under the $SL(2,C)$ algebra defined earlier as the zeroth plus the third component of a four-vector g_μ,

$$\epsilon=g_0+g_3\,.$$

The other components of the vector g_μ can be obtained from ϵ by commuting it with respect to $\mathfrak{F}$'s, giving

$$g_\pm=g_1\pm ig_2=\tfrac{1}{2}i[\mathfrak{F}_\pm,\epsilon]=\tfrac{1}{2}i(1-\epsilon)M_\pm\,,$$
$$g_0-g_3=-\tfrac{1}{4}[\mathfrak{F}_+,[\mathfrak{F}_-,\epsilon]]\,.$$

Incidentally, knowing the transformation properties of ϵ under the $SL(2,C)$ algebra and the fact that

$$\mathbf{X}^\dagger=\mathbf{X}\,,\quad \mathcal{J}_3{}^\dagger=\mathcal{J}_3\,,\quad \bar{K}_3{}^\dagger=\bar{K}_3-iR\,,$$
$$\mathfrak{F}_\pm{}^\dagger=\mathfrak{F}_\mp-M_\mp\,,$$

one can verify easily that all six generators of the $SL(2,C)$ defined earlier are indeed pseudo-Hermitian.

We now form a new operator

$$\begin{aligned} s&=(1-\epsilon)M^2+\tfrac{1}{4}[\mathfrak{F}_+,[\mathfrak{F}_-,\epsilon]]\\ &=(1-\epsilon)M^2-(g_0-g_3)\,. \end{aligned} \tag{B23}$$

It is straightforward to verify that

$$[X_\pm,s]=(1-\epsilon)M_\pm-(-2i)g_\pm=0\,,$$
$$[\mathfrak{F}_\pm,s]=-[\mathfrak{F}_\pm,\epsilon]M^2+(1-\epsilon)[\mathfrak{F}_\pm,M^2]=0\,,$$

and then by Jacobi identities

$$[\mathcal{J}_3,s]=[\bar{K}_3,s]=0\,,$$

i.e., s commutes with all six generators of the group, and consequently it is a scalar operator. Operators s and g_μ, by construction, are also pseudo-Hermitian. By inverting (B23), we finally have

$$M^2=(1-\epsilon)^{-1}(g_0-g_3+s)=(M^2)^\dagger\,. \tag{B24}$$

This is the most general form for M^2 in the case $B=0$. It has also been shown in the text that Eq. (B24) can be derived from the infinite-component wave equation

$$(p^2-2g\cdot p-s)\phi=0\,. \tag{B25}$$

We shall conclude this Appendix by noting that the special case $1+R=0$ corresponds to the limiting case

$$M^2=-\epsilon^{-1}(g_0-g_3+s)\,. \tag{B26}$$

This special solution can be recovered from the wave equation

$$(2g\cdot p+s)\phi=0\,. \tag{B27}$$

Equations (B25) and (B27) can be combined into a single equation as

$$(ap^2-2g\cdot p-s)\phi=0\,,$$

where a is an $SL(2,C)$ scalar.