PHYSICAL REVIEW VOLUME 182, NUMBER 5 25 JUNE 1969

Properties of Current Algebra at Infinite Momentum

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(Received 31 January 1969)

The sum rules of current algebra at infinite momentum can be considered as a system of coupled matrix equations, which we call the algebra of form factors. In addition to the equations following from sum rules, the form factors must also satisfy a complicated kinematic relation known as the angular condition. Presumably, the set of all hadron states (including the continuum) provides the basis for a representation of the algebra of form factors in which the angular condition is satisfied. It was conjectured by Dashen and Gell-Mann that a much smaller set of states containing only stable and resonant hadrons might also provide a representation. If this is true, the algebra of form factors could be used to predict many properties of hadrons. In the following paper, we attack the problem of finding a representation in which the angular condition is satisfied and in which all states have the same isospin. We obtain a large class of solutions which we suspect, but have not been able to prove, actually includes all solutions. None of our solutions has a physically acceptable mass spectrum. One purpose of the present paper is to discuss, in the proper physical context, the implications of the above-mentioned result. We discuss the algebra of form factors and the angular condition in detail, stressing those features which are general and not restricted to particular solutions. It is shown, for example, how one can incorporate additional equations following from the commutators of time components of currents with space components. We then consider the special problem of finding representations where all states have the same isospin. The relevance of this problem in the program of Dashen and Gell-Mann is discussed in detail.

I. INTRODUCTION

THREE years ago Dashen and Gell-Mann[1] suggested a program of saturating the local current algebra at infinite momentum with an infinite string of single-particle and resonant states. Roughly speaking, their idea went as follows. Assume that the Fubini–Dashen–Gell-Mann sum rule[1,2] obtained by sandwiching the local commutator of two current densities between, say, nucleons, is approximately satisfied when one keeps only resonant intermediate states. Make the further sharp-resonance approximation in which widths of the unstable states are set equal to zero. The sum rule has, of course, now become a discrete sum over a (presumably) infinite set of single-particle intermediate states. Continuing to treat the resonant states as stable, the next step is to consider the sum rules obtained by sandwiching the local commutators between resonance and nucleon and between two resonances. In this way one obtains an infinite set of equations which, in the above approximation, must be satisfied by the form factors for the transitions current+nucleon → nucleon, current+resonance → nucleon, and current+resonance → resonance. The proposal of Dashen and Gell-Mann was simply to use these equations to predict certain features of the particle spectrum and the form factors.

At first sight, it may seem surprising that the program outlined above is capable of producing any predictions. That is, it might appear that the sum rules are so unrestrictive that there would be an infinity of solutions with no way to choose between them, except by complete recourse to experiment. It is known now, however, that this is not the case, at least if one restricts the isospins of the assumed set of resonance. In fact, it now appears that the sum rules are so restrictive that in the restricted isospin case there are no physically interesting single-particle solutions to the problem. This situation will be discussed in more detail in the next paragraphs; and some later paragraphs are devoted to analyzing its physical significance.

In the following paper we consider the truncated problem of finding a solution to the equations of Dashen and Gell-Mann under the simplifying conditions that (i) only the subalgebra generated by the isospin currents is taken into account and (ii) all states are assumed to have the *same* isospin, which, in order to match experiment, we take to be $\frac{1}{2}$. We make no *a priori* restrictions on the mass spectrum; that is, the assumption of discrete states is not fed in from the start. With a few further technical assumptions, none of which seems to be serious, it is found that there are no solutions to the problem unless the mass spectrum has one or more of the following pathologies: (i) All masses are the same or, more precisely, the currents do not connect states of different mass. (ii) The mass spectrum is continuous and infinitely degenerate. (iii) The mass spectrum is not positive, i.e., there are states with $M^2<0$, which cannot be ignored. From this it would follow that it is not possible to find a solution to the isospin-$\frac{1}{2}$ problem which bears the slightest resemblance to the real world. It should be realized, however, that we have not absolutely proved (i)–(iii) and hence that there are no

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1 R. F. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966); in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1966).
2 S. Fubini and G. Furlan, Physics **1**, 229 (1964).

physically acceptable solutions. As mentioned above, our conclusions are subject to a few technical assumptions. Since the nature of these assumptions is impossible to state out of context, the reader who is interested in the possibility that there may still be a physically satisfactory solution to the truncated problem is referred to the next paper for a detailed account of the mathematics.

Since in the real world we do have states with isospin not equal to $\frac{1}{2}$, the reader may wonder why he should, after all, be interested in the above result. The answer is simply this: Suppose we wish to find a solution based on a set of mesons which look something like the experimentally observed states. Now all known mesons seem to fall into $SU(3)$ singlets and octets. In fact, if we take the quark model as a guide, we are led to the supposition that every prominent meson will belong either to a singlet or an octet. This means that all prominent strange mesons should have isospin $\frac{1}{2}$. Therefore, if one looks at the subproblem of satisfying, in the resonance approximation, the sum rules for isospin current commutators sandwiched between strange-meson states, then the situation is probably just that described above, i.e., all relevant states have $I=\frac{1}{2}$. Similarly, there can be no solution for baryons which contains only states belong to **1**, **8**, and **10** under $SU(3)$, since in this case all doubly strange states have $I=\frac{1}{2}$. What is really shown, therefore, is that there is no solution to the equations of Dashen and Gell-Mann which are like the quark model in that they contain only **1**'s and **8**'s of mesons and **1**'s, **8**'s, and **10**'s of baryons. Since the quark model agrees very well with experiment, this strongly suggests that the observed states will not fit into any simple, single-particle solution of current algebra at $P_3=\infty$.

We should also mention the relation of our results to previous work on the problem. A number of special solutions, usually based on infinite-component wave equations, have already appeared in the literature.[3–5] All of these are included in our general catalog of solutions given in the following paper and have one or more of the pathologies listed above. Also, there is the no-go theorem of Grodsky and Streater[6] which is obviously related to our results. It is, however, not the same. Grodsky and Streater choose a special way of satisfying the current-algebra equations and actually required that the resulting solution produce local currents at all momenta, not just $P_3=\infty$. Under these conditions, plus a technical assumption about polynomial boundedness of spin projection operators, Grodsky and Streater arrived at essentially the same conclusions as we do.

Our results, in contrast, are based on a straightforward algebraic reduction of Dashen and Gell-Mann's equations and do not depend on any extraneous assumptions such as the hypothesis that the solution can be derived from an infinite-component wave equation.

It seems then that the assumption that the Fubini–Dashen–Gell-Mann solutions are saturated by resonances cannot be maintained. That this is not too surprising may be seen as follows. The sum rules have the general form $\int A(q^2,q'^2,t,s)ds=F(t)$, where $F(t)$ is a form factor and A is the absorbtive part of an amplitude for (current with "mass" $|q|$+hadrons) $\rightarrow$ (current with "mass" $|q'|$+hadrons) with a momentum transfer t.[7] Because of rapidly decreasing form factors, we expect a given resonant contribution to the sum rule to vanish rapidly as q^2 or q'^2 tends to infinity. On the other hand, the right-hand side of the sum rule is independent of q^2. Presumably, the true continuum is needed to account for this behavior. Recent experiments at SLAC [8] have, in fact, strongly suggested that this is the case. In our truncated problem, we did not exclude a continuous mass spectrum. However, the true continuum which has all isospins was clearly not allowed in the truncated problem. If, then, the assumption of saturation by resonant states is not valid, is there any value remaining in the program? This question will be discussed below.

Although, as mentioned above, we obtained our catalog of solutions by starting with the Dashen–Gell-Mann equations and not with infinite-component wave equations, we can show that most, and we believe all, our solutions could have been derived from infinite-component wave equations which contain at most two space-time derivatives. Now equations involving no derivatives higher than the second are distinguished by the fact that these are the only equations for which a canonical Lagrangian-Hamiltonian formalism can be developed in a straightforward way. This suggests an interesting speculation, which we now proceed to outline. Any Lagrangian field theory with fields transforming according to definite representations of $SU(3)\otimes SU(3)$ will, insofar as it exists, provide a solution to the current algebra at infinite momentum. One can ask the converse question: Does every solution come, at least formally, from a Lagrangian field theory? It is clearly an enormous extrapolation to go from our truncated problem to the general one, but we would not be surprised to find out that the set of all solutions to current algebra at $P_3=\infty$ (we mean now complete solutions involving multiparticle intermediate states, not solutions in the resonance approximation) is identical to or not much larger than the set of all formal Lagrangian field theories built on $SU(3)\otimes SU(3)$ multiplets of fields. Now if one looks at Lagrangian

[3] H. Leutwyler, Phys. Rev. Letters **20**, 561 (1968).
[4] E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. **159**, 1222 (1967).
[5] M. Gell-Mann, D. Horn, and J. Weyers, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience, Inc., New York, 1968).
[6] I. T. Grodsky and R. Streater, Phys. Rev. Letters **20**, 695 (1968).
[7] S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).
[8] H. Kendall (private communication); in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna 1968* (CERN, Geneva 1968).

field theory in the abstract, it is not *a* theory but rather a framework in which to build a theory. That is, if all one knows is that the theory should be built on a Lagrangian containing fields which transform according to some representation of $SU(3)\otimes SU(3)$, then the theory is far from complete. However, there is not too much freedom. For example, if one wants the theory to be reasonably well behaved at high energies, then, at least as far as one can believe perturbation theory, the fields must be restricted to spins 0, $\frac{1}{2}$, and 1 and only a few interactions with dimensionless coupling constants can be employed. The point of all this is that current algebra at infinite momentum may very well provide an equally restrictive framework in which to build a theory of hadrons. If this is true, the advantages would be enormous. For one thing, the quantities entering into the theory are form factors obtained by sandwiching the currents between physical hadron scattering states, and are therefore both finite and directly measurable. Furthermore, the basic equations are simply the Fubini–Dashen–Gell-Mann sum rules which, according to present ideas about Regge poles and high-energy scattering, should be perfectly finite and convergent. The only other equations are kinematic constraints on form factors. The problem is, of course, to find a way to introduce the additional input that could change the equations of Dashen and Gell-Mann from a framework into a theory. We do not have any real idea how this should be done, except for a few small points which will be mentioned below. Nevertheless, current algebra at infinite momentum might just possibly end up following the example of dispersion relations which started as a bare framework but have been developed to the point where they could become a complete theory.

We would like to suggest, then, that the results of the following paper are both negative and positive. They are negative because the possibility of using the sum rules to create a simple model of the hadron spectrum appears to be ruled out. On the other hand, they are positive because they at least hint at the possibility that Dashen and Gell-Mann's equations might be a rather restrictive framework in which one could imagine building a complete theory.

The purpose of the present paper (Paper I) is twofold. First we want to put the results of the following paper (Paper II) into the proper physical context: hence the long introduction. Secondly, we wish to make the combination of the two papers fairly self-contained. For this reason we rederive in the present paper most of the basic equations, including the so-called angular condition due to Dashen and Gell-Mann.[1,3] In contrast to most papers on the subject, our emphasis here is on the general properties of the equations and not their restriction to a particular model solution.

Some new results are also given here. For example, we show how one can enforce the additional constraint that the $P_3\to\infty$ sum rules following from the commutator of the time component of a current with a space component be satisfied. In the real world the latter sum rules may or may not be satisfied. The additional equations one obtains in this way do, however, indicate a way in which one could put some additional input into the general framework. It turns out that in the class of model solutions given in following paper, the process of forcing the time-component–space-component sum rules to be satisfied picks out a particularly simple subclass of solutions.

One way of attacking the general problem of finding relativistic representations of current algebra at infinite momentum is to split the problem into two parts. First one looks for a promising representation of the current algebra alone, ignoring the conditions imposed by relativity, i.e., the angular condition. Then one tries to force the chosen representation to satisfy the angular condition. With this sort of procedure in mind we have given a catalog of representations of the algebra itself. We do not claim to have rigorously obtained all representations of the local current algebra, but we probably list most representations which are of physical interest.

Paper I is organized as follows. In Sec. II, we list all the basic equations and discuss their general content. The derivations, being rather long, are relegated to Secs. III and IV and may be omitted by readers who are familiar with the subject. Finally, Sec. V contains our catalog of representations of the algebra itself, before imposing the angular condition.

In the following paper (II) we concentrate entirely on the simplified problem of representing the isospin current algebra on a set of states all with $I=\frac{1}{2}$. With a few not-too-serious technical assumptions we show there how to construct the general solution to the problem. The pathologies in the mass spectrum are discussed and the connection with infinite-component wave equations is established. All these specific results have been summarized in a recent paper.[9]

II. BASIC EQUATIONS

We define a form factor[10] $\langle N'h'|F^a(\mathbf{k}_\perp)|Nh\rangle$ according to

$$\langle N'h'|F^a(\mathbf{k}_\perp)|Nh\rangle = \lim_{P_3\to\infty} \langle N'h',\tfrac{1}{2}\mathbf{k}_\perp,P_3|\mathfrak{F}_0{}^a(0)|Nh,\,-\tfrac{1}{2}\mathbf{k}_\perp,\,P_3\rangle, \quad (2.1)$$

where $\mathfrak{F}_\nu{}^a(0)$ is a current density and $|Nh,\frac{1}{2}\mathbf{k}_\perp,P_3\rangle$ is a hadron state with third component of momentum equal to P_3, perpendicular momentum equal to $\frac{1}{2}\mathbf{k}_\perp=\frac{1}{2}(k_1,k_2)$, helicity h, and internal qunatum numbers N. The meaning of N deserves further comment. For a

[9] S. J. Chang, R. F. Dashen, and L. O'Raifeartaigh, Phys. Rev. Letters **21**, 1026 (1968).

[10] We use the following notations: A three-vector is denoted by a boldface letter, such as **k**, while its transverse components $(k_1k_2,0)$ are denoted by $\mathbf{k}_\perp$. The form factor $(N'h'|F(\mathbf{k}_\perp)|Nh)$ is a reduced matrix element and can be written as $F_{N'h',Nh}(\mathbf{k}_\perp)$.

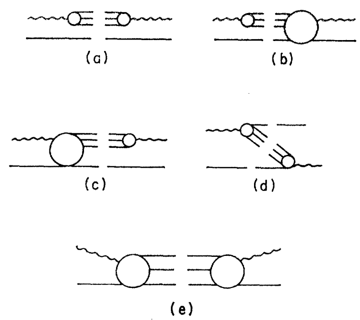

FIG. 1. Categories of intermediate states with different connectedness structure which contribute to the current commutator.

single-particle state, N includes the spin, mass, and quantum numbers such as I_3 and strangeness. For a two-particle continuum state, N is understood to specify the quantum numbers of the individual particles plus the total mass, total angular momentum (in the c.m. system), and total isospin of the combined state. For three or more particles, the submasses of various pairs are also included in N. In general, N is understood to stand for a Lorentz-invariant set of parameters which completely specify a state, apart from its total four-momentum and helicity.

The Fubini–Dashen–Gell-Mann sum rules are equivalent to the relations

$$\sum_{N''h''} [\langle N'h'|F^a(\mathbf{k}_\perp)|N''h''\rangle\langle N''h''|F^b(\mathbf{l}_\perp)|Nh\rangle - \langle N'h'|F^b(\mathbf{l}_\perp)|N''h''\rangle\langle N''h''|F^a(\mathbf{k}_\perp)|Nh\rangle] = if_{abc}\langle N'h'|F^c(\mathbf{l}_\perp+\mathbf{k}_\perp)|Nh\rangle. \quad (2.2)$$

In other words, if we think of $F^a(\mathbf{k}_\perp)$ as a matrix in Nh space, then we have the commutation rules

$$[F^a(\mathbf{k}_\perp), F^b(\mathbf{l}_\perp)] = if_{abc}F_c(\mathbf{k}_\perp+\mathbf{l}_\perp), \quad (2.3)$$

with similar relations for the matrices $F_5{}^a(\mathbf{k}_\perp)$ of axial-vector form factors obtained by replacing the vector current in Eq. (2.3) by an axial-vector current, i.e.,

$$\begin{aligned}[][F^a(\mathbf{k}_\perp), F_5{}^b(\mathbf{l}_\perp)] &= if_{abc}F_5{}^c(\mathbf{k}_\perp+\mathbf{l}_\perp), \\ [F_5{}^a(\mathbf{k}_\perp), F_5{}^b(\mathbf{l}_\perp)] &= if_{abc}F^c(\mathbf{k}_\perp+\mathbf{l}_\perp).\end{aligned} \quad (2.4)$$

It is crucial to understand at this point that Eqs. (2.3) and (2.4) are *not* operator equations of the same kind as

$$\left[\int \mathfrak{F}_a{}^0(\mathbf{x},0)\exp(i\mathbf{k}_\perp\cdot\mathbf{x})d^3x, \int \mathfrak{F}_b{}^0(\mathbf{y},0)\exp(i\mathbf{l}_\perp\cdot\mathbf{y})d^3y\right] = if_{abc}\int \mathfrak{F}_c{}^0(\mathbf{x},0)\exp i(\mathbf{k}_\perp+\mathbf{l}_\perp)\cdot\mathbf{x}d^3x, \quad (2.5)$$

which follow trivially from the local current algebra.[11] The point is that the $F(\mathbf{k}_\perp)$'s in Eqs. (2.3) and (2.4) are simply matrices defined on Nh space, whereas the operators in Eq. (2.5) are the physical local currents. These are defined on a much larger space which is spanned by the vacuum state plus states labeled not only by N and h but also by a four-momentum. We note that there is no analog of the vacuum in Nh space; the reason for this will become apparent later.

The interpretation of a matrix element $\langle N'h'|F_a(\mathbf{k}_\perp)|\times|Nh\rangle$ is that of a form factor for $Nh\to N'h'+$current where the mass squared associated with the current is $-\mathbf{k}_\perp{}^2$. It is perhaps best to think of Eqs. (2.3) and (2.4) as defining an "algebra of form factors" as opposed to the operator "algebra of currents" of Eq. (2.5). We will not, however, always keep this distinction in the remainder of these papers.

In deriving Eqs. (2.3) and (2.4) one assumes not only the local current algebra of Gell-Mann but also that a $P_3\to\infty$ limit can be taken inside a sum over states. This is equivalent to assuming that certain dispersion relations need no subtractions. The validity of this interchange of sum and limit is discussed in Ref. 7; there are several reasons to believe that the procedure is correct and, at present, no reason to doubt the validity of Eqs. (2.3) and (2.4). We will not discuss these questions here, except to point out below some special features of the $P_3\to\infty$ limit which show why there is no analog of the vacuum state in Nh space.

If we sandwich the operator equation (2.5) between single-particle states with the same P_3 and suitable perpendicular momenta, then the diagrams contributing to the commutator can be decomposed into sets having the connectedness structure shown in Fig. 1, where the break in the lines indicates the intermediate state and the blobs are connected amplitudes. The vacuum bubbles shown in Fig. 1(a) never contribute to the commutator. However, the semidisconnected terms in Figs. 1(b)–1(d) do contribute as long as P_3 is finite. But in the limit that P_3 becomes infinite these semidisconnected pieces no longer contribute, leaving only terms with the connectedness structure shown in Fig. 1(c). Another way of saying this is that the connectedness structure of the $F(\mathbf{k})$'s is that shown by either half of Fig. 1(e); i.e., the $F(\mathbf{k})$'s contain no part where the current makes hadrons out of the vacuum in a disconnected way. To avoid a possible source of confusion, we show in Fig. 2 the connectedness structure of

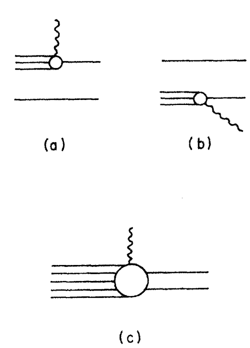

FIG. 2. Connectedness structure of the vertices whose initial and final states have four and two particles, respectively.

[11] See Chap. 6 of Ref. 7.

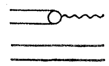

FIG. 3. A vertex which does *not* contribute at the infinite-momentum limit.

$\langle N'h'|F(\mathbf{k}_\perp)|Nh\rangle$ for the case that N is a two-particle state and N' is a four-particle state; again the blobs represent connected amplitudes. All three of the diagrams in Figs. 2(a)–2(c) do contribute to $F(\mathbf{k}_\perp)$ but the diagram in Fig. 3 where the current makes hadrons out of the vacuum does *not* contribute. A detailed discussion of why this latter type of diagram does not contribute to the sum rules at $P_3=\infty$ is given in Ref. 7; the argument need not be repeated here. We pause only to point out that the absence of diagrams like that in Fig. 3 makes the "algebra of form factors" in Eqs. (2.3) and (2.4) much simpler than the operator "algebra of currents" in Eq. (2.5). Also, it is evident that the lack of a "vacuum" state in the Nh space on which the $F(\mathbf{k}_\perp)$'s are defined is a consequence of the absence of diagrams where the currents make disconnected hadrons out of the vacuum.

We are not interested in arbitrary solutions of the "algebra of form factors." The additional constraints imposed by Lorentz invariance must also be satisfied. It is easy to see that such constraints must exist. For fixed N' and N the matrix elements $\langle N'h'|F(\mathbf{k}_\perp)|Nh\rangle$ form an $(2S_{N'}+1)\times(2S_N+1)$ matrix in helicity space, where $S_{N'}$ and S_N are the spins of N' and N. This gives $(2S_{N'}+1)(2S_N+1)$ form factors for the transition $N\to N'+$current, which cannot all be independent since there are far fewer form factors consistent with Lorentz invariance. Thus we expect Lorentz invariance to impose a number of constraints on $\langle N'h'|F(\mathbf{k}_\perp)|Nh\rangle$ for each fixed N' and N. The solution to this problem was given in Ref. 1 and will be written out below. First, however, the following point should be understood.

There is no *a priori* way of defining a Lorentz or Poincaré group on Nh space. The reason for this is that by going from the "operator algebra of currents" to our "algebra of form factors," we have lost the four-momentum labels which were associated with hadron states. It should be clear, then, that the physical Lorentz group does not act on Nh space. In the particular truncated problem solved in the following paper, we do show how to construct an "internal" Lorentz group acting on Nh space, but this is probably a very special case. In any case, even if there is always some kind of useful "internal" Lorentz group in Nh space, there is no obvious way to construct it without first solving the whole problem of the constraints on $\langle N'h'|F(\mathbf{k})|Nh\rangle$.

Since we have no *a priori* Lorentz group in Nh space, we must look for other operators which are defined *a priori* and express the constraints of relativistic invariance in terms of them. The only operators available are the $F^a(\mathbf{k})$ and the operators $\mathfrak{J}$ and M defined by

$$M|Nh\rangle=m_N|Nh\rangle\,, \tag{2.6}$$

where m_N is the mass of the state with internal quantum numbers N and

$$\begin{aligned}\mathfrak{J}&=(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3)\,,\\ \mathcal{J}_3|Nh\rangle&=h|Nh\rangle\,,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\mathcal{J}_\pm|Nh\rangle&=(\mathcal{J}_1\pm i\mathcal{J}_2)|Nh\rangle\\ &=[S_N(S_N+1)-h(h\pm1)]^{1/2}|Nh\pm1\rangle\,,\end{aligned} \tag{2.8}$$

where S_N is again the spin of the state with internal quantum numbers N. By construction, the operators $\mathfrak{J}$ and M^2 satisfy

$$\begin{aligned}[\mathcal{J}_i,\mathcal{J}_j]&=i\epsilon_{ijk}\mathcal{J}_k\,,\\ [\mathfrak{J},M]&=0\,,\end{aligned} \tag{2.9}$$

and

$$\mathfrak{J}^2|Nh\rangle=S_N(S_N+1)|Nh\rangle\,. \tag{2.10}$$

It turns out (see Sec. III) that the constraints on $\langle N'h'|F(\mathbf{k}_\perp)|Nh\rangle$ can be written as an operator equation involving M^2 and $\mathfrak{J}$ as follows. For any operator Θ in Nh space, we define new operators $I_\mathbf{k}(\Theta)$ and $J_\mathbf{k}(\Theta)$ according to

$$I_\mathbf{k}(\Theta)=[M^2,[\mathcal{J}_3,\Theta]]-2[\mathbf{k}_\perp\cdot M\mathfrak{J}_\perp,\Theta]-k^2[\mathcal{J}_3,\Theta]_+ \tag{2.11}$$

and

$$J_\mathbf{k}(\Theta)=[M^2,[M^2,\Theta]]+2k^2[M^2,\Theta]_++k^4\Theta\,, \tag{2.12}$$

where $[\;,\;]_+$ denotes the anticommutator. The condition of covariance is then the so-called angular conditions

$$I_\mathbf{k}(I_\mathbf{k}(I_\mathbf{k}(F(\mathbf{k}_\perp))))=J_\mathbf{k}(I_\mathbf{k}(F(\mathbf{k}_\perp))) \tag{2.13a}$$

and

$$[\mathcal{J}_3,F(\mathbf{k}_\perp)]=i\epsilon_{3ij}k_i\frac{\partial}{\partial k_j}F(\mathbf{k}_\perp)\,. \tag{2.13b}$$

Except for some relatively trivial "threshold conditions" which will be discussed in Sec. III, Eq. (2.13) is a necessary and sufficient condition that the form factors constructed from $\langle N'h'|F(\mathbf{k}_\perp)|Nh\rangle$ have the correct kinematic properties. More specifically, it is shown in Sec. III that, *subject to the threshold conditions, Eq. (13) is a necessary and sufficient condition that $F(\mathbf{k}_\perp)$ can be interpreted as the $P_3\to\infty$ limit of matrix elements of a vector operator $\mathfrak{F}^\mu(0)$ in the sense of Eq. (1)*. One cannot, however, generally construct a complete current from $F(\mathbf{k}_\perp)$. The reason is that two currents $\mathfrak{F}^\mu$ and $\mathfrak{F}'^\mu$ which differ only by the gradient of a scalar operator φ, i.e., $\mathfrak{F}'^\mu=\mathfrak{F}^\mu+\partial^\mu\varphi$, yield the same $F(\mathbf{k}_\perp)$. Thus, unless additional information like current conservation is available, the longitudinal part of the current is not determined. This result is derived in Sec. III.

One can work out the restrictions placed on the $F(\mathbf{k}_\perp)$'s by parity and time-reversal invariance. These restrictions, which again have not been of great practical importance, are given in Ref. 5. Finally, the $F(\mathbf{k}_\perp)$'s satisfy the obvious Hermiticity relation

$$[F(\mathbf{k}_\perp)]^\dagger=F(-\mathbf{k}_\perp)\,. \tag{2.14}$$

Equations (2.3), (2.4), (2.13), and (2.14), along with the threshold conditions and restrictions due to parity and time reversal, are the basic equations of current algebra at infinite momentum. The original suggestion of Dashen and Gell-Mann can now be phrased as follows. One should look for a model solution where the labels Nh run, not over all hadron states including the continuum, but over a discrete set of single-particle states. Physically, this is the same as assuming that all the Fubini–Dashen–Gell-Mann sum rules are saturated by single-particle and resonant states. As pointed out in the Introduction, however, we now believe that no physically interesting solution of this type exists.

At this point it is perhaps worth indicating what we do in Paper II. First we restrict ourselves to the isospin currents so that we employ only Eq. (2.3) with a and b running from 1 to 3. Then we look for a solution to Eq. (2.3) and the angular condition [Eq. (2.13)] in a space where all states $|Nh\rangle$ have isospin $\frac{1}{2}$. Apart from this restriction on isospins we make no further *a priori* assumptions about Nh space. When all states have the same isospin, the Wigner-Eckart theorem, together with Eqs. (3) and (14), implies that $F_i(\mathbf{k}_\perp)$ can be written as $\frac{1}{2}\tau_i \exp(i\mathbf{k}_\perp\cdot\mathbf{X})$, where τ_i are generators of $SU(2)$, and $\mathbf{X}_\perp=(X_1,X_2)$ is a pair of Hermitian operators which commute among themselves and with the τ_i. Current conservation also implies that M^2 and $\mathfrak{J}$ commute with the τ_i so that the angular condition becomes a constraint on $\exp(i\mathbf{k}_\perp\cdot\mathbf{X})$ itself. Then with certain specified technical assumptions, we can reduce the angular condition to a set of algebraic equations involving $\mathbf{X}$, $\mathfrak{J}$, and M^2. In certain cases, it turns out that these algebraic relations allow one to introduce an "internal" Lorentz group on this particular Nh space. This internal Lorentz group greatly facilitates the solution of the equations. Eventually, one finds that the spectrum of M^2 must have at least one of the pathologies listed in the Introduction.

It was also pointed out in the Introduction that the equations of current algebra at infinite momentum, i.e., Eqs. (2.3), (2.4), (2.13), and (2.14), might actually provide a rather restrictive framework in which one could imagine building a real theory of hadrons. We note here that these equations are, in fact, a concrete set of equations relating observable quantities. Of course, one would have to add more information to obtain a specific theory. To this end, and because it will be useful in the following paper, we turn now to the commutators of time components with space components of currents.

First some definitions and kinematic relations. We define $F_i^a(\mathbf{k}_\perp)$ for $i=1,2$ according to

$$\langle N'h'|F_i^a(\mathbf{k}_\perp)|Nh\rangle=\lim_{P_3\to\infty}[2P_3\langle N'h', P_3, +\tfrac{1}{2}\mathbf{k}_\perp|\mathfrak{F}_i^a(0)|P_3, -\tfrac{1}{2}\mathbf{k}_\perp, Nh\rangle],\quad i=1,2 \qquad (2.15)$$

with an analogous definition of the axial-vector objects $F_i^{a5}(\mathbf{k}_\perp)$. In Sec. III, it is shown that a certain component of the $F_i(\mathbf{k}_\perp)$ can be obtained from the corresponding $F(\mathbf{k}_\perp)$ through the kinematic relation

$$I_{\mathbf{k}}(F(\mathbf{k}_\perp))=i\epsilon_{3ij}k_iF_j(\mathbf{k}_\perp)\,, \qquad (2.16)$$

where the operation $I_{\mathbf{k}}(\Theta)$ is defined by Eq. (2.11). When the current $\mathfrak{F}_a{}^\mu$ is conserved, the remaining component of $F_i(\mathbf{k}_\perp)$ can be obtained from the identity

$$[M^2,F(\mathbf{k}_\perp)]=\sum_{i=l}^{2} k_iF_i(\mathbf{k}_\perp)\quad\text{(conserved current)}. \qquad (2.17)$$

We now write the equal-time commutation relation between a time component $\mathfrak{F}_a{}^0$ and a space component $\mathfrak{F}_b{}^i$ of the local current operators as

$$[\mathfrak{F}_a{}^0(\mathbf{x},0),\,\mathfrak{F}_{bi}(\mathbf{y},0)]=i\delta(\mathbf{x}-\mathbf{y})f_{abc}\mathfrak{F}_i{}^c(\mathbf{x},0)+\frac{\partial}{\partial x_j}[\delta(\mathbf{x}-\mathbf{y})\mathcal{S}_{ab}{}^{ij}(\mathbf{x},0)], \qquad (2.18)$$

where $\mathcal{S}_{ab}{}^{ij}$ is the Schwinger term.[12] The assumption implicit in Eq. (2.18) that there are no terms proportional to $\delta''(\mathbf{x}-\mathbf{y})$ or $\delta'''(\mathbf{x}-\mathbf{y})$, etc., is really not necessary, but for clarity we shall restrict ourselves to the single derivative term in Eq. (2.18). Sandwiching Eq. (2.18) between states with the same P_3 and suitable perpendicular momenta, we can take a formal $P_3\to\infty$ limit to obtain

$$[F^a(\mathbf{k}_\perp),\,F_i{}^b(\mathbf{k}'_\perp)]=if_{abc}F_i{}^c(\mathbf{k}_\perp+\mathbf{k}'_\perp)+k_i[F^a(\mathbf{k}_\perp),\,F^b(\mathbf{k}'_\perp)]_+ +ik_jS_{ab}{}^{ij}(\mathbf{k}_\perp+\mathbf{k}'_\perp)\,, \qquad (2.19)$$

where

$$\langle N'h'|S_{ab}{}^{ij}(\mathbf{k}_\perp)|Nh\rangle=\lim_{P_3\to\infty}[2P_3\langle N'h', P_3, \tfrac{1}{2}\mathbf{k}_\perp|\mathcal{S}_{ab}{}^{ij}(0)|Nh, P_3, -\tfrac{1}{2}\mathbf{k}_\perp\rangle]. \qquad (2.20)$$

Notice that the right-hand side of Eq. (2.19) contains a term proportional to the anticommutator of two F's. The origin of this term is explained in Sec. III.

It should be understood that Eq. (2.19) is on a completely different footing than Eqs. (2.3) and (2.4). Unlike the case of the commutator of two time components of currents, it is questionable whether one can interchange the $P_3\to\infty$ limit with the sum over states in the commutator of a time component and a space component of a current. If this interchange of sum and limit is not valid, then Eq. (2.19) is not valid even though Eq. (2.18) may be correct. The specific reasons why Eq. (2.19) is on a different footing than Eqs. (2.3) and (2.4) are discussed in detail in Ref. 7.

If we do assume that Eq. (2.19) is valid, either as a general principle or as an extra constraint in a particular model, then we have still not accomplished much until

[12] J. Schwinger, Phys. Rev. Letters 3, 296 (1959); the reasons why the Schwinger term is likely to be symmetric in the unitary indices can be found in S. L. Adler and C. G. Callan, CERN Report No. Th.587, 1965 (unpublished).

we know something about S_{ab}^{ij}. There are several possibilities here of which we will discuss only the simplest. In most model field theories, S_{ab}^{ij} is symmetric in a and b.[12] Thus we might suppose that in the part of Eq. (2.19) which is antisymmetric in a and b, the Schwinger term is not present and, in this way, obtain a new relation involving only F's and F_i's. Then, using Eqs. (2.16) and (2.17), one can eliminate the F_i's to obtain a relation involving F's alone. In the truncated problem discussed in the next paper, this procedure leads to a particularly simple class of solutions.

III. DERIVATION OF ANGULAR CONDITION

Let $\mathcal{J}_\mu(x)$ be any vector or axial-vector current density and consider the limits of the form

$$\lim_{\kappa\to\infty} \langle N'h', p_1'p_2', p_3'+\kappa|\mathcal{J}_0(0)|Nh, p_1p_2,p_3+\kappa\rangle, \quad (3.1)$$

where $|Nh, p_1p_2p_3\rangle$ is a hadron state of three momentum p_1, p_2, p_3, helicity h, and internal quantum number (which include mass m and spin s) N, and is normalized according to

$$\langle N'h', p_1'p_2'p_3'|Nh, p_1p_2p_3\rangle = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\delta_{N'N}\delta_{h'h}. \quad (3.2)$$

Note that the states in (3.2) are not normalized covariantly; hence states with different momentum (velocity) are related by

$$|\cdots, p'=L(p)\rangle = (p_0/p_0')^{1/2}L|\cdots, p\rangle,$$

where $p_0=(m^2+\mathbf{p}^2)^{1/2}$. L is the Lorentz transformation related to these states. The existence of the limits (3.1) is proved in Ref. 13, and in Ref. 1 it is pointed out that the limits depend only on N', N, h', h, and $\mathbf{k}_\perp$, where $\mathbf{k}_\perp=(p_1'-p_1, p_2'-p_2)$ is the transverse part of the three-momentum transfer. One finds that it is convenient to introduce the "standard decleration" $\kappa^{1/2}\exp(-i\sinh^{-1}\kappa K_3)$, where K_3 is the generator of Lorentz transformation in the z direction, and to write (3.1) in the form

$$\lim_{\kappa\to\infty} \langle N'h', p_1', p_2', p_3'+\kappa|\mathcal{J}_0(0)|Nh, p_1, p_2, p_3+\kappa\rangle = {}_1\langle N'h', p_1'p_2'|\mathcal{J}(0)|Nh, p_1p_2\rangle_1, \quad (3.3)$$

where the states

$$\begin{aligned}|Nh, p_1p_2\rangle_1 &= \lim_{\kappa\to\infty} \kappa^{1/2}\exp(-i\sinh^{-1}\kappa K_3)|Nh, p_1, p_2, p_3+\kappa\rangle \\ &= (\sqrt{m})\exp(-i\mathbf{p}_\perp\cdot\mathbf{E})\exp(-i\ln mK_3)|N, L_3'=h, \\ &\quad \times\mathbf{p}=0\rangle = \exp(-i\mathbf{p}_\perp\cdot\mathbf{E})|Nh, \mathbf{p}_\perp=0\rangle_1, \quad (3.4)\end{aligned}$$

with $\mathbf{L}$, $\mathbf{K}$ being the generators of the physical Lorentz group $E_1=K_1+L_2$, $E_2=K_2-L_1$, and the operator

$$\mathcal{J}(0) = \lim_{\kappa\to\infty} \kappa^{-1}\exp(-i\sinh^{-1}\kappa K_3)\mathcal{J}_0(0) \times\exp(i\sinh^{-1}\kappa K_3) = \mathcal{J}_0(0)+\mathcal{J}_3(0) \quad (3.5)$$

are separately finite in the limit $\kappa=\infty$. Note that the states $|Nhp_1p_2\rangle_1$, which are not to be confused with the $|Nh\rangle$ states in the Nh space of Sec. II, are states in the physical Hilbert space and span the subspace defined by $p_0+p_3=1$, and hence the subscript. Also, one should keep in mind that, unlike the states $|Nhp_1p_2p_3\rangle$, the $|Nhp_1p_2\rangle_1$ states do not have definite helicity.

From Eq. (3.3) it is clear that to every $\mathcal{J}_0(0)$ in the complete Hilbert space there corresponds a $\mathcal{J}(0)$ in the $|Nhp_1p_2\rangle_1$ space. The purpose of this section is to derive the necessary and sufficient conditions on $\mathcal{J}(0)$ in order that $\mathcal{J}_0(0)$ may transform as the fourth component of a vector. This problem is not trivial because the space of vectors $|Nhp_1p_2\rangle_1$ does not carry a representation of the Lorentz group $\mathfrak{L}$.

Some of the Lorentz conditions on $\mathcal{J}(0)$ can be obtained directly because the space $|Nhp_1p_2\rangle_1$ carries a representation of a *sub*group of $\mathfrak{L}$, namely, the three parameter subgroup $E(2)$ generated by $E_1=K_1+L_2$, $E_2=K_2-L_1$, L_3, where $\mathbf{L}$ and $\mathbf{K}$ are the conventional generators of rotations and accelerations, respectively. The subgroup $E(2)$, which is isomorphic to the Euclidean group in two dimensions, is the maximal subgroup of $\mathfrak{L}$ to respect the condition $p_0+p_3=1$, which characterizes the states $|Nhp_1p_2\rangle_1$. It is easily verified that with respect to $E(2)$, $\mathcal{J}(0)$ is a scalar.

The condition that $\mathcal{J}(0)$ be an $E(2)$ scalar is clearly not sufficient to guarantee that it has the correct properties under the rest of the Lorentz group. On the other hand, if we can impose on $\mathcal{J}(0)$ a condition in the space $|Nhp_1p_2\rangle_1$ that corresponds to rotations (other than those around the z axis) in the original space $|Nhp_1p_2p_3\rangle$, then this condition together with the $E(2)$ condition will be sufficient. This is because from $E(2)$ and the rotations about one other axis in $|Nhp_1p_2p_3\rangle$ we can generate the complete Lorentz group in $|Nhp_1p_2p_3\rangle$. The condition in the space $|Nhp_1p_2\rangle_1$ was first formulated by Dashen and Gell-Mann, and because of its connection with rotations in the original space $|Nhp_1p_2p_3\rangle$ is called the "angular condition." It is derived as follows.

We consider the matrix element

$$\langle N'h', \tfrac{1}{2}k0\kappa|\mathcal{J}_0(0)|Nh, -\tfrac{1}{2}k0\kappa\rangle \quad (3.6)$$

before taking the limit $\kappa=\infty$, and transform it to the Breit frame, i.e., the frame in which the particles N' and N have four-momenta $((m'^2+\frac{1}{4}q^2)^{1/2}00\,\frac{1}{2}q)$ and $((m^2+\frac{1}{4}q^2)^{1/2}00\,-\frac{1}{2}q)$, respectively. Because of the invariance of P^2, where $P=(\epsilon'+\epsilon, 00\,2\kappa)$,

$$\epsilon'=(m'^2+\kappa^2+\tfrac{1}{4}k^2)^{1/2}, \quad \epsilon=(m^2+\kappa^2+\tfrac{1}{4}k^2)^{1/2}$$

is the total four-momentum in the original frame; q^2 has

[13] H. Bebié and H. Leutwyler, Phys. Rev. Letters 19, 618 (1967).

necessarily the value

$$(m^2+\tfrac{1}{4}q^2)^{1/2}+(m'^2+\tfrac{1}{4}q^2)^{1/2} = [(\epsilon+\epsilon')^2-4\kappa^2]^{1/2}=E+O(1/\kappa^2),$$

where

$$E^2=2(m^2+m'^2)+k^2. \tag{3.7}$$

The reason for transforming to the Breit frame will be given below. For the moment we concentrate on the technical aspects of the transformation. It is, of course, a different transformation for each pair of particles. It is easy to see that the transformation can be expressed as the product $R^{-1}(\theta)K^{-1}(v)$, where $K^{-1}(v)$ is the deceleration with velocity

$$v=2\kappa/(\epsilon'+\epsilon)=1-E^2/8\kappa^2+O(1/\kappa^4), \tag{3.8}$$

which brings the total momentum P to its rest frame, and $R^{-1}(\theta)$ is the negative rotation through the angle

$$\theta=\tfrac{1}{2}\pi+\tan^{-1}\left(\frac{v(\epsilon'-\epsilon)}{k(1-v^2)^{1/2}}\right)=\tfrac{1}{2}\pi+\tan^{-1}\left(\frac{m'^2-m^2}{kE}\right)+O\left(\frac{1}{\kappa^2}\right), \tag{3.9}$$

which aligns the resultant momentum transfer $[(\epsilon'-\epsilon)/(1-v^2)^{1/2}, k, 0, -v(\epsilon'-\epsilon)/(1-v^2)^{1/2}]$ along the z axis.

After the transformation to the Breit frame, the three-momenta of the particles N' and N become $(00\,\tfrac{1}{2}q)$ and $(00\,-\tfrac{1}{2}q)$, respectively. However, the states do not become $|N'h'00\,\tfrac{1}{2}q\rangle$ and $|Nh00\,-\tfrac{1}{2}q\rangle$ because the transformation $R^{-1}(\theta)K^{-1}(v)$ changes the helicities, and the change has still to be calculated. The general formula for calculating the helicity change under a Lorentz transformation L has been derived in Ref. 14 and is

$$L|Nh\mathbf{p}\rangle=\sum_{h'}\mathfrak{D}_{hh'}(\phi)|Nh'\mathbf{p}(L)\rangle, \tag{3.10}$$

where $\mathbf{p}(L)$ is the transformed three-momentum, $\mathfrak{D}_{hh'}(\phi)$ is the rotation matrix in helicity space associated with the angle ϕ, and ϕ is the angle defined by

$$R(\phi)=[R(\mathbf{p}(L))K(v(L))]^{-1}L[R(\mathbf{p})K(v)], \tag{3.11}$$

where $K(v)$, for $v=p/\epsilon$, is the Lorentz transformation that accelerates the particle N to a momentum p in the 3 direction, and $R(\mathbf{p})$ rotates this momentum into the direction $\mathbf{p}$, and similarly for $K(v(L))$ and $R(\mathbf{p}(L))$.

Using (3.10), we see that in our case we may write

$$\begin{aligned} R(\theta)L(v(\kappa))|Nh-\tfrac{1}{2}k0\kappa\rangle &= \exp[-i\mathcal{J}_2\omega(\kappa)]|Nh00\,-\tfrac{1}{2}q\rangle(m^2+\tfrac{1}{4}q^2)^{1/4}/\kappa,\\ R(\theta)L(v(\kappa))|N'h'\tfrac{1}{2}0\kappa\rangle &= \exp[-i\mathcal{J}_2\omega'(\kappa)]|N'h'00\,\tfrac{1}{2}q\rangle(m'^2+\tfrac{1}{4}q^2)^{1/4}/\kappa, \end{aligned} \tag{3.12}$$

where $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ are the generators of rotations in helicity space, and $\omega(\kappa)$ and $\omega'(\kappa)$ are the angles defined by

$$\begin{aligned} R(\omega(\kappa)) &= [R(-\tfrac{1}{2}q)K(q/(q^2+4m^2)^{1/2})]^{-1}\\ &\quad\times R^{-1}(\theta)K^{-1}(v)[R(-\tfrac{1}{2}k0\kappa)K((k^2+4\kappa^2)^{1/2}/(k^2+4\kappa^2+4m^2)^{1/2})],\\ R(\omega'(\kappa)) &= [R(0)K(q/(q^2+4m'^2)^{1/2})]^{-1}\\ &\quad\times R^{-1}(\theta)K^{-1}(v)[R(\tfrac{1}{2}k0\kappa)K((k^2+4\kappa^2)^{1/2}/(k^2+4\kappa^2+4m'^2)^{1/2})]. \end{aligned} \tag{3.13}$$

In (3.12), we have generalized the definitions of $\mathcal{J}_3$ and $\mathcal{J}_\pm$ defined in (2.7) and (2.8); they are now the corresponding helicity operators on a state $|Nh,\mathbf{p}\rangle$ with the arbitrary momentum $\mathbf{p}$:

$$\begin{aligned} \mathcal{J}_3|Nh,\mathbf{p}\rangle &= h|Nh,\mathbf{p}\rangle,\\ \mathcal{J}_\pm|Nh,\mathbf{p}\rangle &= (\mathcal{J}_1\pm i\mathcal{J}_2)|Nh,\mathbf{p}\rangle\\ &= [S_N(S_N+1)-h(h\pm1)]^{1/2}|Nh,\mathbf{p}\rangle. \end{aligned}$$

Note that the $\mathcal{J}$'s are not the Lorentz generators, since they leave the momentum $\mathbf{p}$ unchanged. It is also clear that they reduce to our previously defined $\mathcal{J}$'s as $p_3\to\infty$ (i.e., $\kappa\to\infty$). Substituting (3.12) into (3.6), we obtain

$$\begin{aligned} \langle N'h',\tfrac{1}{2}k0\kappa|\mathcal{J}_0(0)|Nh,-\tfrac{1}{2}k0\kappa\rangle &= [(m^2+\tfrac{1}{4}q^2)(m'^2+\tfrac{1}{4}q^2)]^{1/4}/\kappa\,\langle N'h',00\,\tfrac{1}{2}q|\exp[+i\mathcal{J}_2\omega'(\kappa)]\\ &\quad\times R(\theta)L(v(\kappa))\mathcal{J}_0(0)L^{-1}(v(\kappa))R^{-1}(\theta)\exp[-\mathcal{J}_2\omega(\kappa)]|Nh,00\,-\tfrac{1}{2}q\rangle\\ &= \eta\langle N'h',00\tfrac{1}{2}q|\exp[+i\mathcal{J}_2\omega'(\kappa)][\mathcal{J}_0(0)+v(\mathcal{J}_3(0)\cos\theta+\mathcal{J}_1(0)\sin\theta)]\exp[-i\mathcal{J}_2\omega(\kappa)]|Nh,00\,-\tfrac{1}{2}q\rangle, \end{aligned}$$

with

$$\eta=(2/E)[(m^2+\tfrac{1}{4}q^2)(m'^2+\tfrac{1}{4}q^2)]^{1/4}, \tag{3.14}$$

whence, finally,

$$\langle N'h',\tfrac{1}{2}k0\kappa|\exp[-i\mathcal{J}_2\omega'(\kappa)]\mathcal{J}_0(0)\exp[+i\mathcal{J}_2\omega(\kappa)]|Nh,-\tfrac{1}{2}k0\kappa\rangle = \eta\langle Nh',00\,\tfrac{1}{2}q|\mathcal{J}_0(0)+v(\mathcal{J}_3(0)\cos\theta+\mathcal{J}_1(0)\sin\theta)|Nh,00\,-\tfrac{1}{2}q\rangle. \tag{3.15}$$

We come now to the reason for transforming to the Breit frame. The reason is that, in the Breit frame, scalars, vectors, two-tensors, etc., are characterized by the fact that they change the helicity by $\Delta h=0$, $\Delta h\leq1$, $\Delta h\leq2$, etc., respectively. Hence the vector character of $\mathcal{J}_0(0)$ is expressed by the fact that in Eq. (3.15) the change in helicity $\Delta h=h'-h$ is 0, ±1.[15] From the left-hand side of the equation we see that this is equivalent

[14] E. Wigner, Ann. Math. **40**, 39 (1939). This paper was included in *Symmetry Groups in Nuclear and Particle Physics*, edited by F. J. Dyson (W. A. Benjamin, Inc., New York, 1966).

[15] Strictly speaking, a four-vector is specified in the Breit frame by $|\Delta\mathcal{J}_3|\leq1$ rather than $|\Delta h|\leq1$. However, it is the $\mathcal{J}_3$, which is related simply to the helicity at $P_3\to\infty$.

to the operator condition

$$[\mathcal{J}_3,[\mathcal{J}_3,[\mathcal{J}_3, \exp(-i\mathcal{J}_2\omega'(\kappa))\mathcal{J}_0(0)\exp(i\mathcal{J}_2\omega(\kappa))]]] \\ =[\mathcal{J}_3, \exp(-i\mathcal{J}_2\omega'(\kappa))\mathcal{J}_0(0)\exp(i\mathcal{J}_2\omega(\kappa))] \quad (3.16)$$

in the space of states $|Nh, \pm\frac{1}{2}k0\kappa\rangle$.

Equation (3.16) is valid for all κ. We now take the limit $\kappa\rightarrow\infty$ and obtain

$$[\mathcal{J}_3,[\mathcal{J}_3,[\mathcal{J}_3, \exp(-i\mathcal{J}_2\omega')\mathcal{J}(0)\exp(+i\mathcal{J}_2\omega)]]] \\ =[\mathcal{J}_3, \exp(-i\mathcal{J}_2\omega')\mathcal{J}(0)\exp(+i\mathcal{J}_2\omega)], \quad (3.17a)$$

on the states $|Nh,p_1p_2\rangle$, where ω' and ω are the limits of $\omega'(\kappa)$ and $\omega(\kappa)$ as $\kappa\rightarrow\infty$. These limits are evaluated in the Appendix and for $k^2>|m'^2-m^2|>0$ turn out to be

$$\omega'=-\tan^{-1}\left(\frac{2m'k}{m^2-m'^2+k^2}\right), \qquad \omega'>\tfrac{1}{2}\pi$$
$$\omega=\tan^{-1}\left(\frac{2mk}{m'^2-m^2+k^2}\right), \qquad |\omega|<\tfrac{1}{2}\pi. \quad (3.17b)$$

The expressions for other values of m' and m can be evaluated in a similar way. Equation (3.17a) is the required angular condition on $\mathcal{J}(0)$. It is clearly a condition on the space $|Nh,p_1p_2\rangle_1$. Further it is derived directly from the condition that $\mathcal{J}_0(0)$ be the fourth component of a vector, and involves a nontrivial rotation $R(\theta)$ around the 2 axis. Hence, as explained earlier, together with $E(2)$ invariance, it is sufficient to guarantee the vector character of $\mathcal{J}_0(0)$ under the complete Lorentz group. Note that the condition is valid for both the vector and axial-vector charge densities.

Our next task is to show that Eq. (3.17a) is equivalent to the angular condition in Eq. (1.13). From the definition (1.1) of $F(\mathbf{k})$ and Eq. (3.3) it is clear that one can write

$$\langle N'h'|F(\mathbf{k})|Nh\rangle={}_1\langle N'h',\tfrac{1}{2}k_1\tfrac{1}{2}k_2|[\mathfrak{F}^0(0) \\ +\mathfrak{F}_3(0)]|Nh,-\tfrac{1}{2}k_1-\tfrac{1}{2}k_2\rangle_1. \quad (3.18a)$$

In fact, there is the more general relation

$$\langle N'h'|F(\mathbf{k})|Nh\rangle={}_1\langle N'h',p_\perp'|[\mathfrak{F}^0(0) \\ +\mathfrak{F}_3(0)]|Nh,p_\perp\rangle_1, \quad \mathbf{k}\equiv p_\perp'-p_\perp \quad (3.18b)$$

which follows from the fact that the right-hand side is independent of $p_\perp'+p_\perp$.[16] Also, it should be clear that since $\mathfrak{J}$ is defined to act only on helicity indices, it is exactly the $\mathfrak{J}$ defined in Nh space by Eqs. (2.7) and (2.8). Therefore, we can write Eq. (3.17a) as

$$[\mathcal{J}_3,[\mathcal{J}_3,[\mathcal{J}_3, \exp(-i\mathcal{J}_2\omega')F(k)\exp(i\mathcal{J}_2\omega)]]] \\ =[\mathcal{J}_3, \exp(-i\mathcal{J}_2\omega')F(k)\exp(i\mathcal{J}_2\omega)], \quad (3.19)$$

[16] This can be proved easily as

$${}_1\langle N'h',\mathbf{p}'_\perp|\mathfrak{F}^0(0)+\mathfrak{F}^3(0)|Nh,\mathbf{p}_\perp\rangle_1 \\ ={}_1\langle N'h',0|e^{i\mathbf{p}'_\perp\cdot\mathbf{E}}[\mathfrak{F}^0(0)+\mathfrak{F}^3(0)]e^{-i\mathbf{p}\cdot\mathbf{E}}|Nh,0\rangle_1 \\ ={}_1\langle N'h',0|e^{i\mathbf{k}_\perp\cdot\mathbf{E}}[\mathfrak{F}^0(0)+\mathfrak{F}^3(0)]Nh,0\rangle_1,$$

which is independent of $p'_\perp+p_\perp$. In the above derivation, we have used the fact that $\mathbf{E}$ commutes with the current $\mathfrak{F}^0(0)+\mathfrak{F}^3(0)$.

where this equation and the following equations [(3.20)–(3.22)] are understood to hold only when sandwiched between states of $|N'h'\rangle$ and $|Nh\rangle$ with definite masses m', m. Writing out Eq. (3.19) fully and multiplying to the left and right by $\exp(i\mathcal{J}_2\omega')$ and $\exp(-i\mathcal{J}_2\omega)$, respectively, we obtain

$$D^3F(k)=D(k), \quad (3.20)$$

where

$$D\theta=\mathcal{J}'\theta-\theta\mathcal{J},$$
$$\mathcal{J}'=\exp(+i\omega'\mathcal{J}_2)\mathcal{J}_3\exp(-i\omega'\mathcal{J}_2)=\mathcal{J}_3\cos\omega'-\mathcal{J}_1\sin\omega',$$
$$\mathcal{J}=\exp(i\omega\mathcal{J}_2)\mathcal{J}_3\exp(-i\omega\mathcal{J}_2)=\mathcal{J}_3\cos\omega-\mathcal{J}_1\sin\omega,$$

which, from (3.17b), can be written as

$$\tilde{D}^3F(k)=N\tilde{D}F(k), \quad (3.21)$$

where

$$\tilde{D}\theta=K'\theta-\theta K,$$
$$K'=-\mathcal{J}_3(k^2+m^2-m'^2)-2m'k\mathcal{J}_1,$$
$$K=\mathcal{J}_3(k^2-m^2+m'^2)-2mk\mathcal{J}_1,$$
$$N=k^4+2k^2(m^2+m'^2)+(m^2-m'^2)^2.$$

(Note that $\cos\omega'$ is negative, since $\omega'>\frac{1}{2}\pi$.) The point now is that since these equations hold between states of mass m' and m, the c numbers m' and m can be replaced by the mass operator M. Since

$$K'\theta=\mathcal{J}_3[M^2,\theta]-k^2\mathcal{J}_3\theta-2kM\mathcal{J}_1\theta,$$
$$\theta K=[M^2,\theta]\mathcal{J}_3+k^2\theta\mathcal{J}_3-2k\theta M\mathcal{J}_1, \quad (3.22)$$

and similary for N, and since $\mathbf{k}$ was arbitrarily chosen to be $(k,0)$, Eq. (3.31) can be written as

$$I_{\mathbf{k}}(I_{\mathbf{k}}(I_{\mathbf{k}}(F(\mathbf{k}))))=I_{\mathbf{k}}(J_{\mathbf{k}}(F(\mathbf{k}))), \quad (3.23)$$

where

$$I_k(\theta)=K'\theta-\theta K=[M^2,[\mathcal{J}_3,\theta]] \\ -k^2[\mathcal{J}_3,\theta]_+-2[\mathbf{k}_\perp\cdot M\mathcal{J},\theta], \quad (3.24)$$

$$J_k(\theta)=[M^2,[M^2,\theta]]+2k^2[M^2,\theta]_++k^4\theta. \quad (3.25)$$

Equation (3.23) is the required reduced form of the angular condition. Note that since $\mathcal{J}_i$ commutes with M^2, the operations I and J commute.

Another question which one naturally asks is to what extent the physical properties of the original current operator $\mathcal{J}_\mu(0)$ can be recovered from the form factors $F(\mathbf{k})$. We only expect a partial recovery since $F(\mathbf{k})$ is a function of the transverse momentum transfer and since $F(\mathbf{k})$ has matrix elements between connected states only. If we do not impose additional information like crossing relation, we may never obtain matrix elements between vacuum and the pair states. To give a partial answer to this question, we return to the

Breit frame

$$\langle N'h'|F(\mathbf{k})|Nh\rangle=\eta\langle N'h',00\,\tfrac{1}{2}q|\times\exp(i\mathcal{J}_2\omega')Z(0)\exp(-i\mathcal{J}_2\omega)|Nh,00\,-\tfrac{1}{2}q\rangle, \quad (3.14')$$

where

$$\mathbf{k}=(k,0),$$
$$Z(0)=\mathcal{J}_0(0)-(q_0/q)\mathcal{J}_3(0)-(k/q)\mathcal{J}_1(0).$$

It is now easy to see that only those matrix elements corresponding to some transverse parts of the current operator $Z(0)$ can be recovered. By varying the two-component vector $\mathbf{k}$, however, we find that the matrix elements of the entire transverse part of the current density $\mathcal{J}_\mu(0)$ taken between connected states can be reconstructed. This indicates that the current density operator can be recovered up to a gradient term in this connected subspace. To understand how it happens, we note that the matrix element of the longitudinal part of a current is proportional to q_0, the zeroth component of the momentum transfer. As $p_3\to\infty$, the contribution to the form factor from the longitudinal part of the current damps out as q_0/p_3, and vanishes at $p_3=\infty$. An equivalent way of seeing this is that the infinite-momentum frame can be decelerated into the standard frame $p_0+p_3=1$. Then, the contribution from the longitudinal part of the current reduces to

$$\langle N'h',p_0'+p_3'=1|(q_0+q_3)|Nh,p_0+p_3=1\rangle=0$$

as before.

For conserved vector currents, there is no ambiguity in reconstructing the current operator in the connected subspace, since the longitudinal part of the current vanishes identically. For axial-vector currents, however, we have lost all the information about the divergence at $p_3=\infty$.

IV. TIME-COMPONENT–SPACE-COMPONENT COMMUTATION RELATION

In this section we derive Eqs. (2.16), (2.17), and (2.19) in Sec. II. As a first step we write $F_i(\mathbf{k})$ introduced in (2.15) of Sec. II in the form

$$\begin{aligned}\langle N'h'|F_i(\mathbf{k})|Nh\rangle &={}_1\langle N'h',\tfrac{1}{2}\mathbf{k}|2\mathfrak{F}_i(0)|Nh,-\tfrac{1}{2}\mathbf{k}\rangle_1\\ &={}_1\langle N'h',\mathbf{p}_\perp'=0|\exp[(i/2)\mathbf{k}\cdot\mathbf{E}]2\mathfrak{F}_i(0)\\ &\qquad\times\exp[(i/2)\mathbf{k}\cdot\mathbf{E}]|Nh,\mathbf{p}_\perp=0\rangle_1.\end{aligned} \quad (4.1)$$

Briefly speaking, the factor $2P_3$ in (2.15) is included to compensate the change of normalization. More generally, we have

$$\lim_{\kappa\to\infty}\langle N'h',p_1',p_2',p_3'+\kappa|2\kappa\mathfrak{F}_i(0)|Nh,p_1,p_2,p_3+\kappa\rangle$$
$$=\langle N'h'|F_i(\mathbf{k})|Nh\rangle+(p'+p)_i\langle N'h'|F(\mathbf{k})|Nh\rangle, \quad (4.2)$$

which depends not only on the transverse momentum transfer $\mathbf{k}=(p_1'-p_1,p_2'-p_2)$, but also on the sum of the transverse momentum $(\mathbf{p}'+\mathbf{p})_\perp$. (It is still independent of p'_3 and p_3.) To see how it works, we start from l.h.s. of (4.2)

$$\begin{aligned}&={}_1\langle N'h',\mathbf{p}'_\perp=0|\exp[i\mathbf{p}'_\perp\cdot\mathbf{E}]2\mathfrak{F}_i(0)\\ &\qquad\times\exp[-i\mathbf{p}_\perp\cdot\mathbf{E}]|Nh,\mathbf{p}_\perp=0\rangle_1\\ &={}_1\langle N'h',\mathbf{p}'_\perp=0|\exp[(i/2)\mathbf{k}\cdot\mathbf{E}]2\\ &\quad\times\{\exp[(i/2)(\mathbf{p}'+\mathbf{p})_\perp\cdot\mathbf{E}]\mathfrak{F}_i(0)\\ &\quad\times\exp[-(i/2)(\mathbf{p}'+\mathbf{p})_\perp\cdot\mathbf{E}]\}\\ &\qquad\times\exp[(i/2)\mathbf{k}\cdot\mathbf{E}]|Nh,\mathbf{p}_\perp=0\rangle_1\\ &={}_1\langle N'h',\mathbf{p}'_\perp=0|\exp[(i/2)\mathbf{k}\cdot\mathbf{E}]2\\ &\quad\times\{\mathfrak{F}_i(0)+\tfrac{1}{2}(p'+p)_i[\mathfrak{F}_0(0)+\mathfrak{F}_3(0)]\}\\ &\quad\times\exp[(i/2)\mathbf{k}\cdot\mathbf{E}]|Nh,\mathbf{p}_\perp=0\rangle_1\\ &=\langle N'h'|F_i(\mathbf{k})|Nh\rangle+(p'+p)_i\langle N'h'|F(\mathbf{k})|Nh\rangle,\end{aligned}$$

which is the right-hand side of (4.2). In deriving the last equation, we have used the relations

$$[E_i,\mathfrak{F}_j(0)]=-i\delta_{ij}[\mathfrak{F}_0(0)+\mathfrak{F}_3(0)],$$
$$[E_i,(\mathfrak{F}_0+\mathfrak{F}_3)(0)]=0.$$

Next, sandwiching the commutator relation (2.18) between states of infinite momentum, and inserting a complete set of intermediate states, we have

$$\begin{aligned}\lim_{\text{all }P_3\to\infty}\sum_{N''h''}\{&\langle N'h',p'|\mathfrak{F}_a{}^0(0)|N''h'',p''\rangle\langle N''h'',p''|\mathfrak{F}_b{}^i(0)|Nh,p\rangle\\ &-\langle N'h',p'|\mathfrak{F}_b{}^i(0)|N''h'',p''\rangle\langle N''h'',p''|\mathfrak{F}_a{}^0(0)|Nh,p\rangle\}\\ &=\lim_{\text{all }P_3\to\infty}if_{abc}\langle N'h',p'|\mathfrak{F}_c{}^i(0)|Nh,p\rangle+ik_j\langle N'h',p'|\mathcal{S}_{ab}{}^{ij}(0)|Nh,p\rangle.\end{aligned} \quad (4.3)$$

By the use of Eqs. (3.2), (3.18b), and (4.2) and under the assumption that the $p_3\to\infty$ limit is valid here, Eq. (4.3) leads to the following algebra of form factors:

$$[F^a(\mathbf{k}'),F_i{}^b(\mathbf{k})]=if_{abc}F_i{}^c(\mathbf{k}'+\mathbf{k})+k'[F^a(\mathbf{k}'),F^b(\mathbf{k})]_++\text{S.T.}, \quad (4.4)$$

which is just Eq. (2.19), and where the extra term in this expression originates from the last term appearing in Eq. (4.3). (S.T. is the Schwinger term.)

Now, we wish to express the angular condition in terms of these generalized form factors. From the results of Sec. II, we learn that the infinite-momentum form factor $F(\mathbf{k})$ is related to the Breit frame matrix element through

$$\begin{aligned}\langle N'h'|\exp(-i\mathcal{J}_2\omega')F(\mathbf{k})\exp(i\mathcal{J}_2\omega)|Nh\rangle &=\eta\langle N'h'00\,\tfrac{1}{2}q|\mathfrak{F}_0\\ &\quad+v(\mathfrak{F}_3\cos\theta+\mathfrak{F}_1\sin\theta)|Nh00\,-\tfrac{1}{2}q\rangle,\end{aligned} \quad (3.15')$$

where

$$\mathbf{k}=(k,0), \quad v\sin\theta=k/q,$$
$$\eta=2[(m^2+\tfrac{1}{4}q^2)(m'^2+\tfrac{1}{4}q^2)]^{1/4}[(m'^2+\tfrac{1}{4}q^2)^{1/2}+(m^2+\tfrac{1}{4}q^2)^{1/2}]^{-1}.$$

By the use of a similar but somewhat simpler manipulation, it is verified that

$$\langle N'h'|\exp(-i\mathcal{J}_2\omega')F_2(\mathbf{k})\exp(i\mathcal{J}_2\omega)|Nh\rangle =\eta'\langle N'h'00\,\tfrac{1}{2}q|\mathfrak{F}_2(0)|Nh00\,-\tfrac{1}{2}q\rangle, \quad (4.5)$$

with

$$\eta'=2[(m'^2+\tfrac{1}{4}q^2)(m^2+\tfrac{1}{4}q^2)]^{1/4}. \quad (4.6)$$

We then obtain the following simpler angular condition:

$$[\mathcal{J}_3, \exp(-i\mathcal{J}\omega')F(\mathbf{k})\exp(i\mathcal{J}_2\omega)] =i(\eta k/\eta' q)\exp(-i\mathcal{J}_2\omega')F_2(\mathbf{k})\exp(i\mathcal{J}_2\omega),$$

which can be expressed as the operator equations

$$I_{\mathbf{k}}(F(\mathbf{k}))=i[\mathbf{k}\times\mathbf{F}(k)]_3. \quad (4.7)$$

This is, of course, Eq. (2.16). The derivation of Eq. (2.17) is simple and straightforward and need not be given here.

V. BASIC REPRESENTATIONS OF CURRENT ALGEBRA

A rather natural way to attack the problem of current algebra at infinite momentum would be to choose an appropriate solution to Eqs. (2) and (3) and then try to enforce the angular condition. The problem then splits into two parts. First, one needs a catalog of solutions to Eqs. (2) and (3); this is the topic of the present section. The second and more difficult step of enforcing the angular condition will not be discussed here.

For simplicity, we will restrict ourselves to isospin vector currents $\mathfrak{F}_a{}^V$, $a=1, 2, 3$, which lead to the $F^a(\mathbf{k}_\perp)$ for $a=1, 2, 3$. The generalization to the full algebra of F's is perfectly straightforward.

Defining $f^a(\mathbf{x}_\perp)$ through

$$f^a(\mathbf{x}_\perp)=\frac{1}{(2\pi)^2}\int\exp(-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp)F^a(\mathbf{k}_\perp)d^2\mathbf{k}_\perp, \quad (5.1)$$

Eq. (2.2) is equivalent to

$$[f^a(\mathbf{x}_\perp),f^b(\mathbf{y}_\perp)]=i\delta(\mathbf{x}_\perp-\mathbf{y}_\perp)\epsilon_{abc}f^c(\mathbf{x}_\perp), \quad (5.2)$$

where we have used the fact that $f_{abc}=\epsilon_{abc}$, where a, b, and c run from 1 to 3. For the most part, we will work with the "coordinate space" form of the algebra in Eq. (5.2). One can immediately see that a rigorous mathematical analysis of the algebra specified by Eq. (5.2) is difficult because of the singular nature of the "structure constants" $\delta(\mathbf{x}_\perp-\mathbf{y}_\perp)\epsilon_{abc}$. Evidently, the f's in Eq. (5.2) are operator-valued distributions and in a proper mathematical treatment of the problem one would have to take this into account and start with some statement about the particular kind of distributions which are allowed. We shall not attack the problem on this level. It turns out that it is very easy to carry out a rather complete analysis of Eq. (5.2) on a heuristic level. This leads to a catalog of representations of the algebra. All the representations in the catalog are mathematically respectable but, since we do not proceed on rigorously, we may not have all interesting representations.

To proceed, let us imagine that the continuum of points $\mathbf{x}_\perp$ in Eqs. (5.1) and (5.2) have been replaced by a finite set of, say, four points i_1, i_2, i_3 and i_4. We then have $f^a(i_n)$ ($n=1, 2, 3, 4$) and the commutation rules

$$[f^a(i_n),f^b(i_{n'})]=i\delta_{nn'}\epsilon_{abc}f^c(i_n), \quad n, n'=1, 2, 3, 4. \quad (5.3)$$

One immediately recognizes in Eq. (5.3) the Lie algebra of the group $SU(2)\otimes SU(2)\otimes SU(2)\otimes SU(2)$. That is, there is a separate $SU(2)$ for each point i_n, $n=1, 2, 3, 4$. It is, of course, trivial to find all the irreducible representations of $SU(2)\otimes SU(2)\otimes SU(2)\otimes SU(2)$. They are simply products of representations of $SU(2)$ and can be written as (I_1,I_2,I_3,I_4), where, for example, I_1 is the isospin associated with the group at i_1. These irreducible representations are defined on a product space $\mathcal{T}_1\otimes\mathcal{T}_2\otimes\mathcal{T}_3\otimes\mathcal{T}_4$, where $\mathcal{T}_1$ is the $(2I_1+1)$-dimensional space for a representation of $SU(2)$ with isospin I_1, etc., and the generators are sums of commuting operators, i.e.,

$$f^a(i_n)=T_1{}^a\delta_{n1}+T_2{}^a\delta_{n2}+T_3{}^a\delta_{n3}+T_4{}^a\delta_{n4}, \quad (5.4)$$

where, for example, $T_1{}^a$ is understood to be the direct product of the ath isospin generator in $\mathcal{T}_1$ with unit matrices in $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$. Actually, we are not particularly interested in irreducible representations. What is needed for current algebra at infinite momentum is a simple catalog of reducible representations. To this end, we consider the special class of reducible representations of the form $(I,0,0,0)\oplus(0,I,0,0)\oplus(0,0,I,0)\oplus(0,0,0,I)$. The space on which this representation is defined is the direct product of a $(2I+1)$-dimensional isospin space $\mathcal{T}$ with a four-dimensional space $\mathcal{S}$. Now define an operator h which is the product of the unit matrix in $\mathcal{T}$ times an operator in $\mathcal{S}$ with eigenvalues of 1, 2, 3, and 4. The f's for the representation $(I,0,0,0)\oplus(0,I,0,0)\oplus(0,0,I,0)\oplus(0,0,0,I)$ can then be written as

$$f^a(i_n)=T^a\delta_{nh}, \quad (5.5)$$

where $\mathcal{T}^a$ is the isospin operator in $\mathcal{T}$ times a unit matrix in $\mathcal{S}$. Suppose now that we want to construct the representation $(I,I',0,0)\oplus$(all permutations of I and I')$\oplus(I\otimes I',0,0,0)\oplus$(permutations). This representation is defined on a space $\mathcal{T}\otimes\mathcal{T}'\otimes\mathcal{S}\otimes\mathcal{S}'$, where $\mathcal{T}$ and $\mathcal{T}'$ are analogous to $\mathcal{T}$ above and $\mathcal{S}$ and $\mathcal{S}'$ are the analogs of $\mathcal{S}$ above. In this representation the generators are

$$f^a(i_n)=T^a\delta_{nh}+T'^a\delta_{nh'}, \quad (5.6)$$

where T^a and T'^a act in $\mathcal{T}$ and $\mathcal{T}'$, and h and h' act in $\mathcal{S}$ and $\mathcal{S}'$. By construction, we have the commutation rules

$$\begin{aligned}&[T^a,T^b]=i\epsilon_{abc}T^c,\\&[T'^a,T'^b]=i\epsilon_{abc}T'^c,\\&[T^a,T'^b]=0,\quad [h,h']=0,\\&[h,T^a]=[h',T^a]=[h,T'^a]=[h',T'^a]=0.\end{aligned}\tag{5.7}$$

It is easy to verify that the commutation relations in Eq. (5.7) guarantee that the f's of Eq. (5.7) satisfy Eq. (5.3). As a final example, suppose that we wish to construct the representation $(I,I',0,0)\oplus$(all permutations of I and I') which differs from the above representation by the absence of terms like $(I\otimes I',0,0,0)$. It is not hard to see that this representation can be obtained from Eq. (5.6) by imposing the constraint $\delta_{hh'}=0$, which is consistent since h and h' commute. The space on which h and h' act is, of course, no longer a direct product.

From the above examples, the reader should not find it hard to convince himself that the most general (reducible) representation of the algebra in Eq. (5.3) can be written as

$$f^a(i_n)=\sum_{r=1}^{N} T^{a(r)}\delta_{nh}(r)\tag{5.8}$$

in a space $\mathcal{T}^{(1)}\otimes\mathcal{T}^{(2)}\otimes\cdots\mathcal{T}^{(N)}\otimes\mathcal{S}$, where $T^{a(r)}$ is an isospin matrix acting in $\mathcal{T}^{(r)}$ and the $h^{(r)}$ are a set of commuting operators in another space $\mathcal{S}$. The commutation relations are

$$\begin{aligned}&[T^{a(r)},T^{b(s)}]=i\delta_{rs}\epsilon_{abc}T^{c(r)},\\&[h^{(r)},T^{a(s)}]=0,\quad [h^{(r)},h^{(s)}]=0.\end{aligned}\tag{5.9}$$

The fact that we have been letting n run only over 1, 2, 3, and 4 is no longer of any consequence; Eq. (5.8) gives the most general representation for any number of points. In fact, it is simple to pass to the continuum limit to obtain representations of the local algebra in Eq. (5.2); one obtains

$$f^a(\mathbf{x}_\perp)=\sum_r T^{a(r)}\delta(\mathbf{x}_\perp-\mathbf{h}_\perp^{(r)}),\tag{5.10a}$$

where the $\mathbf{h}_\perp^{(r)}$ are now commuting vector operators in two dimensions. By virtue of the commutation relations in Eq. (5.9), the above expression satisfies the local algebra. The h's are, of course, now assumed to have a continuous spectrum. It is interesting to examine the $F(k)$'s obtained from Eq. (5.10a); they are[17]

$$F^a(\mathbf{k}_\perp)=\sum_r T^{a(r)}\exp(i\mathbf{k}_\perp\cdot\mathbf{h}_\perp).\tag{5.10b}$$

In this form, one easily sees that these representations of the local algebra are mathematically respectable. That is, if the spaces $\mathcal{T}^{(r)}$ are finite-dimensional and the $\exp i\mathbf{k}_\perp\cdot\mathbf{h}_\perp$ form Abelian groups of unitary operators in $\mathcal{S}$, then the representation is rigorously defined. These representations are, however, not all the representations of the local algebra. Further representations will be discussed below.

Before looking for additional representations of the local algebra, we should mention some special cases of Eq. (5.10a). First, let us find the representation, mentioned previously, where all states have isospin $\frac{1}{2}$. Since $F^a(\mathbf{0}_\perp)=\sum_r T^{a(r)}$ is the ath component of isospin, it is clear that the sum over r must be trivial, containing only one term proportional to the Pauli matrix $\frac{1}{2}\tau^a$. Thus we have

$$F^a(\mathbf{k}_\perp)=\tfrac{1}{2}\tau^a\exp(i\mathbf{k}_\perp\cdot\mathbf{h}_\perp)\tag{5.11}$$

for the isospin-$\frac{1}{2}$ case.[18] Other special cases are Gell-Mann's "two- and three-quark" representations[19] for which the sum in Eq. (5.10a) contains two or three terms each proportional to a Pauli matrix.

To obtain further representations of the local algebra, we proceed as follows. The process which we used to find the general representation of the algebra of Eq. (5.3), which is a "local algebra" on a four-point grid, was to find the representations localized at one point, which are simply $SU(2)$ representations, and then by means of the operators h, take direct sums and products of these point representations to obtain the general representation. We can try the same trick with the local algebra. For the $f^a(\mathbf{x})$'s, a representation located at a point $\mathbf{x}_0$ is clearly one where

$$f^a(\mathbf{x}_\perp)=T^a\delta(\mathbf{x}_\perp-\mathbf{x}_0)+b_i^a\nabla^i\delta(\mathbf{x}_\perp-\mathbf{x}_0)+c_{ij}^a\nabla_i\nabla_j\delta(\mathbf{x}_\perp-\mathbf{x}_0)+\cdots\tag{5.12}$$

contains at most a finite number of derivatives of δ functions.[20] Consider, for example, a representation of the form

$$f^a(\mathbf{x}_\perp)=T^a\delta(\mathbf{x}_\perp-\mathbf{x}_0)+b_i^a\nabla^i\delta(\mathbf{x}_\perp-\mathbf{x}_0).\tag{5.13}$$

In order to satisfy Eq. (5.2), we must have

$$\begin{aligned}&[T^a,T^b]=i\epsilon_{abc}T^c,\\&[T^a,b_i^b]=i\epsilon_{abc}b_i^c,\\&[b_i^a,b_j^b]=0.\end{aligned}\tag{5.14}$$

Thus, for each fixed i, b_i^a and T^b ($a, b=1, 2, 3$) generate $E(3)$, the Euclidian group in three dimensions. From this fact, one can readily prove that all nontrivial representations of Eq. (5.14) contain arbitrarily high isospins. This is in contrast to the representations in Eq. (5.10) which, as long as the number of terms in the sum over r is finite, contain only a finite number of different isospins.

[17] See also E. H. Roffman, J. Math. Phys. 8, 1954 (1967).

[18] See Refs. 3 and 5; S. Fubini, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies 1967*, edited by A. Perlmutter and B. Kursunoglu (W. H. Freeman and Co., San Francisco, 1967).

[19] M. Gell-Mann, in *Proceedings of the International School of Physics "Ettore Majorana" Erice, Italy, 1966*, edited by Z. Zichichi (Academic Press Inc., New York, 1966).

[20] It is known that every generalized function concentrated at a single point x_0 can be represented as a (finite) linear combination of $\delta(x-x_0)$ and its derivatives. This theorem is mentioned by I. M. Gel'fand and G. E. Shilov, in *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. 1, Chap. I, Sec. A1.4.

The representation in Eq. (5.13) is localized at the point $\mathbf{x}_0$. To obtain a more general representation we may replace $\mathbf{x}_0$ by an operator $\mathbf{h}$ which commutes with T^a and b_i^a, which gives

$$f^a(\mathbf{x}_\perp)=T^a\delta(\mathbf{x}_\perp-\mathbf{h})+b_i^a\nabla^i\delta(\mathbf{x}_\perp-\mathbf{h}) \tag{5.15}$$

or

$$F^a(\mathbf{k}_\perp)=T^a\exp(i\mathbf{k}_\perp\cdot\mathbf{h})+ik_ib_i^a\exp(i\mathbf{k}_\perp\cdot\mathbf{h}). \tag{5.16}$$

Still more general representations can be obtained by taking a sum in analogy with Eq. (5.10), which yields

$$F^a(\mathbf{k}_\perp)=\sum_r \exp[i\mathbf{k}_\perp\cdot\mathbf{h}^{(r)}](T^{a(r)}+ik_ib_i^{a(r)}), \tag{5.17}$$

where $\mathbf{h}^{(r)}$, $T^{a(r)}$, and $b_i^{a(r)}$ satisfy

$$\begin{aligned}&[T^{a(r)},T^{b(s)}]=i\delta_{rs}\epsilon_{abc}T^{c(r)},\\&[T^{a(r)},b_i^{b(s)}]=i\delta_{rs}\epsilon_{abc}b_i^{c(r)},\\&[\mathbf{h}^{(r)},\mathbf{h}^{(s)}]=[b_i^{a(r)},b_j^{b(s)}]=[b_i^{a(r)},\mathbf{h}^{(s)}]\\&\qquad=[\mathbf{h}^{(r)},T^{a(s)}]=0.\end{aligned} \tag{5.18}$$

Note that some of the $b_i^{a(r)}$ could be zero; if they all vanish, we recover the representation in Eq. (5.10).

There is, of course, no reason why terms like c_{ij}^a and coefficients of higher derivatives of δ functions in Eq. (5.12) cannot be present. The most general representation at a point is, in an obvious notation,

$$f^a(\mathbf{x}_\perp)=T^a\delta(\mathbf{x}_\perp-\mathbf{x}_0)+\cdots+d_{ij\cdots pq}{}^a[\nabla_i\nabla_j\cdots\nabla_p\nabla_q\delta(\mathbf{x}_\perp-\mathbf{x}_0)], \tag{5.19}$$

where the final term multiplies an Nth-order derivative of a δ function, with N arbitrarily large but finite. The general reducible representation constructed from these representations at a point is clearly

$$F_i(\mathbf{k}_\perp)=\sum_r \exp[i\mathbf{k}_\perp\cdot\mathbf{h}^{(r)}][T^{a(r)}+\cdots+(d_{ij\cdots pq}{}^{a(r)}ik_iik_j\cdots ik_pik_q)], \tag{5.20}$$

where the $\mathbf{h}^{(r)}$ commute with everything, including each other. All quantities of a given index (r) commute with quantities for other indices (s), and for a given r one has

$$\begin{aligned}&[T^{a(r)},T^{b(r)}]=i\epsilon_{abc}T^{c(r)},\\&[T^{a(r)},d_{ij\cdots pq}{}^{b(r)}]=i\epsilon_{abc}d_{ij\cdots pq}{}^{c(r)},\\&[d_{ij\cdots pq}{}^{a(r)},d_{kl\cdots rs}{}^{b(r)}]=0,\end{aligned} \tag{5.21}$$

plus a number of commutation relations for the coefficients of lower powers of $i\mathbf{k}_\perp$ which are suppressed in Eq. (5.19). We note that Eq. (5.21) implies that the representation contains arbitrarily high isospins.

As long as the various sums in Eq. (5.20) are finite, this representation can be given a well-defined mathematical meaning. Furthermore, this representation is the most general one which can be constructed from representations at a point. For an algebra, like that in Eq. (5.3), built on a finite, discrete grid of points rather than a continuum, all representations can be constructed by taking sums and products of representations at a point. We do not know under what conditions, if any, this is true for a local algebra. To the extent that this is true, Eq. (5.20) represents the most general representation of the local algebra.

Finally, we should ask ourselves which representations are relevant for physics. We cannot, of course, really answer this question but there are a couple of points which should be made. Suppose we look at a free-field theory of nucleons. For each sector with a fixed baryon number of say, 1, the states $|Nh\rangle$ can contain one nucleon, one nucleon and one nucleon-antinucleon pair, one nucleon and two pairs, etc. Let M be the number of pairs in a state. Then from the connectedness structure of the F's discussed in Sec. II, we know that the F's do not connect states of different M, *in the free-field theory*. Furthermore, for each fixed M, the representation can be shown to be that in Eq. (5.10) where the $T^{a(r)}$ are all Pauli matrices and the sum runs over $2M+1$ terms corresponding to one nucleon and M pairs. Thus, for the free-field theory the representation can be explicitly constructed; it is simply a stack of representations like that in Eq. (5.10) with the sums running over 1, 3, 5, 7, $\cdots$, terms. Now it may or may not be true that when an interaction is turned on, the representation remains the same up to a unitary transformation. For the sake of argument, however, let us suppose that it does. What, then, is the difference between the interacting and noninteracting theories? The answer lies in the angular condition. In the free theory, each term $\frac{1}{2}\mathcal{T}^a\exp(i\mathbf{k}_\perp\cdot\mathbf{h}_\perp)$ in the sums satisfies the angular condition by itself. When the interaction is turned on the individual terms no longer satisfy the angular condition. In fact, the angular condition now couples together subspaces containing sums of 1, 3, 5, 7, $\cdots$, terms in the $F(\mathbf{k})$'s. This is simply a consequence of the fact that the old "bare particles" are no longer eigenstates of the mass operator M. In the interacting theory, then, the angular condition is satisfied only when we sum over an infinite number of terms like $\mathcal{T}^{a(r)}\exp[i\mathbf{k}\cdot\mathbf{h}^{(r)}]$.

While the above discussion about the behavior of Lagrangian field theories is, at best, heuristic, it does suggest that in the real world one needs an infinite sum of terms or representations like that of Eq. (5.10) in order to satisfy the angular condition. One may or may not have further terms like the b_i^a in Eq. (5.17).

APPENDIX

In order to calculate the limits of $\omega(k)$ and $\omega'(k)$ it is convenient to express the Lorentz transformations (3.13) by which they are defined in the *vector* representation of the Lorentz group. However, since the y axis remains inert during all of the transformations in (2.13), it is sufficient to consider only the (t,z,x) subspace of the vector space, on which space the rotations around the y axis with angle ϕ and the accelerations along the z axis with velocity v are represented by the matrices

$$R(\phi)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{pmatrix},\quad K(v)=\frac{1}{(1-v^2)^{1/2}}\begin{pmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & (1-v^2)^{1/2}\end{pmatrix}, \tag{A1}$$

respectively.

In the representation we obtain at once for the first three matrices in (3.13)

$$\left[R(-\tfrac{1}{2}q)K\left(\frac{\frac{1}{2}q}{(m^2+\frac{1}{4}q^2)^{1/2}}\right)\right]^{-1}R^{-1}(\theta)=\frac{1}{2m}\begin{pmatrix}(q^2+4m^2)^{1/2} & -q & 0\\ -q & (q^2+4m^2)^{1/2} & 0\\ 0 & 0 & 2m\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$$
$$=\frac{1}{2m}\begin{pmatrix}(q^2+2m^2)^{1/2} & q\cos\theta & -q\sin\theta\\ -q & -(q^2+4m^2)^{1/2}\cos\theta & (q^2+4m^2)^{1/2}\sin\theta\\ 0 & -2m\sin\theta & -2m\cos\theta\end{pmatrix}. \tag{A2}$$

To evaluate the last three matrices, we first make the expansions

$$\begin{aligned}&(\kappa^2+\tfrac{1}{4}k^2)^{1/2}(\kappa^2+m^2+\tfrac{1}{4}k^2)^{-1/2}=1-4m^2/8\kappa^2+O(1/\kappa^4),\\ &\cos\phi=\kappa(\kappa^2+\tfrac{1}{4}k)^{-1/2}=1-k^2/8\kappa^2+O(1/\kappa^4),\\ &\sin\phi=-k/2\kappa+O(1/\kappa^3),\\ &v=1-E^2/8\kappa^2+O(1/\kappa^4).\end{aligned} \tag{A3}$$

We then have, to order $1/\kappa^2$,

$$K^{-1}(v)R(-\tfrac{1}{2}k\;0\kappa)K((\kappa^2+\tfrac{1}{4}k^2)^{1/2}(\kappa^2+m^2+\tfrac{1}{4}k^2)^{-1/2})$$
$$=\frac{2\kappa}{E}\begin{pmatrix}1 & -(1-E^2/8\kappa^2) & 0\\ -(1-E^2/8\kappa^2) & 1 & 0\\ 0 & 0 & E/2\kappa\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1-k^2/8\kappa^2 & -k/2\kappa\\ 0 & k/2\kappa & 1-k^2/8\kappa^2\end{pmatrix}\frac{\kappa}{m}\begin{pmatrix}1 & 1-4m^2/8\kappa^2 & 0\\ 1-4m^2/8\kappa^2 & 1 & 0\\ 0 & 0 & m/\kappa\end{pmatrix}$$
$$=\frac{1}{2mE}\begin{pmatrix}3m^2+m'^2+k^2 & m'^2-m^2+k^2 & 2km\\ m'^2-m^2 & 3m^2+m'^2 & -2km\\ kE & kE & 2mE\end{pmatrix}. \tag{A4}$$

The matrix

$$R(\omega)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\omega & \sin\omega\\ 0 & -\sin\omega & \cos\omega\end{bmatrix} \tag{A5}$$

is then from (3.13) equal to the product of the matrices (A2) and (A4). However, to evaluate ω we do not have to multiply the matrices explicitly. We avoid the multiplication, and also the explicit calculation of q, by noting that from (A5)

$$\tan\omega=-a_{32}/a_{33}, \tag{A6}$$

where a_{ij} are the matrix elements of the product matrix, and that these two particular matrix elements involve only the last *row* of (A2). Calculating them by insepection from (A2) and (A4), we obtain

$$\tan\omega=-\frac{(-2m\sin\theta)(3m^2+m'^2)+(-2m\cos\theta)(kE)}{(-2m\sin\theta)(-2km)+(-2m\cos\theta)(2mE)}=-\frac{[3m^2+m'^2+kE\cot\theta}{(-2km)+2mE\cot\theta}. \tag{A7}$$

On the other hand, in the limit $\kappa\to\infty$, we obtain from (3.9)

$$\cot\theta=(-m'^2+m^2)/kE. \tag{A8}$$

Inserting this result into (A7), we obtain

$$\tan\omega=-\frac{3m^2+m'^2-(m'^2-m^2)}{-2km+2m(m'^2-m^2)/k}=\frac{k}{2m}\frac{4m^2}{k^2-m^2+m'^2}=\frac{2mk}{k^2-m^2+m'^2}.$$