Formalism for Partial-Wave Analysis of the Reaction Meson (0^-)+Baryon $(\frac{1}{2}^+) \rightarrow$ Meson (0^-)+Baryon $(\frac{3}{2}^+)$†

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The formalism for making a partial-wave analysis of the reactions $\pi+N \rightarrow \Delta(1236)+\pi$ and $\bar{K}+N \rightarrow \Sigma(1385)+\pi$ is outlined. From such an analysis, the coupling of baryon resonances to these inelastic channels can be determined. The differential cross section, decay distribution for $\Delta \rightarrow N+\pi$, and nucleon polarization are expressed as functions of the S-matrix elements $S_{l'\frac{3}{2},l\frac{1}{2}}{}^J(k)$, which connect the initial πN state with orbital angular momentum l, spin J, and c.m. momentum k to the $\Delta\pi$ state with orbital angular momentum l'.

I. INTRODUCTION

We give the formulas for making a partial-wave analysis of the reaction sequence

$$\pi+N \rightarrow \Delta(1236)+\pi \quad (1.1a)$$
$$\hookrightarrow N+\pi \quad (1.1b)$$

and

$$\bar{K}+N \rightarrow \Sigma(1385)+\pi \quad (1.2a)$$
$$\hookrightarrow \Lambda+\pi. \quad (1.2b)$$

The decay sequence (1.1) is described by five independent variables, which may be chosen as k, Θ, θ, ϕ, $\mathbf{P}$, where k is the momentum in the πN c.m. system, $\cos\Theta = \hat{\pi}_{\text{in}} \cdot \hat{\pi}_{\text{out}}$ in the c.m. system, θ and ϕ are the polar and azimuthal angles of the decay nucleon in the Δ rest frame, and $\mathbf{P}$ is the nucleon polarization. By "partial-wave" analysis we mean that the experimental distributions are used to infer the quantities $S_{l'\frac{3}{2},l\frac{1}{2}}{}^J(k)$ which are the S-matrix elements connecting the initial πN state with orbital angular momentum l and spin J to the $\Delta\pi$ state with orbital angular momentum l' and spin J. The $S_{l'\frac{3}{2},l\frac{1}{2}}{}^J$ are complex scalar quantities which are functions only of k.

Partial-wave analyses of reactions (1.1) and (1.2) can be used to determine the decay rates of N^* and Y^* resonances into $\Delta(1236)+\pi$ and $\Sigma(1385)+\pi$, respectively, and to check the (IJ^P) values inferred from analyses of elastic-scattering data. Considerable data on these reactions are being amassed in current bubble-chamber experiments at incident momentum below 3 GeV/c. Partial-wave analyses of the reactions $\bar{K}+N \rightarrow \Sigma(1385)+\pi$, and $\bar{K}+N \rightarrow \Lambda(1520)+\pi$ have already been made for the special case that the cross section is dominated by a single amplitude.[1–3] These analyses led to a determination of the (IJ^P) quantum numbers for $\Sigma(1770)$ and $\Lambda(1815)$, and their decay rate into $\Lambda(1520)+\pi$ and $\Sigma(1385)+\pi$, respectively.

In this paper we relate the distributions in the variables k, Θ, θ, ϕ, and $\mathbf{P}$, which can be measured experimentally to the $S_{l'\frac{3}{2},l\frac{1}{2}}{}^J$ amplitudes. Previous authors have studied certain aspects of this problem (mainly the relation between the $\cos\Theta$ distribution and the $S_{l'\frac{3}{2},l\frac{1}{2}}{}^J$ amplitudes), but have not enumerated a complete set of equations which utilizes all the experimental information on the production and decay of $\Delta(\Sigma)$ to determine the $S_{l'\frac{3}{2},l\frac{1}{2}}{}^J$.[3–6]

In general, the reaction $\pi N \rightarrow \pi\pi N$ is a coherent sum of such amplitudes as $A(\pi N \rightarrow \pi N^*)$, $A(\pi N \rightarrow \rho N)$, $A(\pi N \rightarrow \pi\pi N)$, etc. The formalism outlined below is applicable only to a subset of events corresponding to the reaction $\pi N \rightarrow \pi\Delta$. In most experiments it is possible to isolate a sample of these events which are relatively free from interference of other amplitudes (in some cases the complete range of variables Θ, θ, ϕ will not be accessible due to experimental cuts).

Models which take into account all possible two-particle interactions in the $\pi\pi N$ final state have been worked out.[6,7] However, the application of these models poses practical difficulties because of their mathematical complexity and the large number of variables involved. We believe that the formalism outlined here is a useful tool for the preliminary analysis of the three-body final states (1.1) and (1.2).

II. PRODUCTION ANGULAR DISTRIBUTION

In this section the differential cross section for the reaction

$$0^- + \tfrac{1}{2}^+ \rightarrow \tfrac{3}{2} + 0^- \quad (2.1)$$

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[1] R. Armenteros, M. Ferro-Luzzi, D. W. G. Leith, R. Levi-Selti, A. Minten, R. D. Tripp, H. Filthuth, V. Hepp, E. Kluge, H. Schneider, R. Barloutaud, P. Granet, J. Meyer, and J.-P. Porte, Phys. Letters **19**, 338 (1965).

[2] R. W. Birge, R. P. Ely, G. E. Kalmus, A. Kernan, J. Louie, J. S. Sahouria, and W. M. Smart, in *Proceedings of the Second Athens Topical Conference on Recently Discovered Resonant Particles, Athens, Ohio, 1965* edited by B. A. Munir (University of Ohio Press, Athens, 1965), p. 296.

[3] R. B. Bell, R. W. Birge, Y. L. Pan, and R. T. Pu, Phys. Rev. Letters **16**, 203 (1966).

[4] R. G. Roberts, Ann. Phys. (N. Y.) **44**, 325 (1967).

[5] M. G. Olsson and G. B. Yodh, Phys. Rev. **145**, 1309 (1966); **145**, 1327 (1966).

[6] B. Deler and G. Valladas, Nuovo Cimento **45A**, 559 (1966).

[7] J. M. Namyslowski, M. S. K. Rasmi, and R. G. Roberts, Phys. Rev. **157**, 1328 (1967).

TABLE I. Notation for two-body scattering reaction.

	Entrance channel c	Exit channel c'
Momentum in c.m. system[a]	k	k_f
Total spin	$S(\frac{1}{2})$	$S'(\frac{3}{2})$
z component of spin	ν	ν'
Orbital angular momentum	l	l'
z component of orbital angular momentum	m	m'
Total angular momentum	J	J'
z component of total angular momentum	M	M'

[a] We use a system of units in which $\hbar=1$, $k=1$ λ.

is expressed as a function of the matrix elements $S_{l'\frac{3}{2},l\frac{1}{2}}{}^J$ [Eq. (2.8)]. We follow the treatment of two-body scattering reactions by Goldberger and Watson,[8] with their notation, as given in Table I.

The differential cross section in the c.m. system for the reaction (2.1) with initial state $|k,S,\nu\rangle$ and final state $|k_f,S',\nu'\rangle$ is

$$\frac{d\sigma}{d\Omega}=\left(\frac{2\pi}{k}\right)^2 |\langle c';\hat{k}_f,S',\nu'|S(k)|c;\hat{k},S,\nu\rangle|^2 \quad (2.2a)$$

$$=|\langle S',\nu'|f(\hat{k}_f,c',\hat{k},c)|S,\nu\rangle|^2, \quad (2.2b)$$

where $\langle c';\hat{k}_f,S',\nu'|S(k)|c;\hat{k},S,\nu\rangle$ is the unitary S-matrix element in the barycentric subspace on the energy and momentum shell; Eq. (2.2b) defines the scattering amplitude f for the reaction. For an unpolarized target the average cross section for any final spin orientation is obtained by averaging over the initial spin orientations ν and summing over the final spin orientations ν':

$$\frac{d\bar{\sigma}}{d\Omega}=\frac{1}{2S+1}\sum_{\nu',\nu}\frac{d\sigma}{d\Omega}(c';\hat{k}_f,S',\nu';c;\hat{k},S,\nu)$$

$$=(2S+1)^{-1}\,\mathrm{Tr}(ff^\dagger). \quad (2.3)$$

The S-matrix elements are used throughout rather than the T-matrix elements, because there is no unique convention for the normalization of the T matrix. If the T-matrix elements are defined by

$$\langle c';\hat{k}_f,S'\nu'|S(k)|c;\hat{k},S,\nu\rangle=\delta_{c',c}\delta_{S',S}\delta_{\nu',\nu}\delta_{\hat{k}_f,\hat{k}} +i\langle c';\hat{k}_f,S',\nu'|T(k)|c;\hat{k},S,\nu;\rangle,$$

S may be replaced everywhere by iT, since we are dealing with an inelastic reaction. We will omit the channel suffix c from now on.

The S-matrix element in (2.2) is transformed to the $lSJM$ representation, using the transformation matrix

$$\langle\hat{k},S,\nu|lmS\nu\rangle=Y_l{}^m(\hat{k}) \quad (2.4)$$

which resolves plane-wave states into partial waves, and with the Clebsch-Gordan coefficients $\langle l,S,m,\nu|J,M\rangle$ which resolve states $|lmS\nu\rangle$ into states of total angular momentum $|JM\rangle$

$$\langle\hat{k}_f,S',\nu'|S(k)|\hat{k},S,\nu\rangle = \sum_{l'm'J'M'lmJM}\langle\hat{k}_f,S'\nu'|l'm'S'\nu'\rangle\langle l',S';m',\nu'|J',M'\rangle \times\langle l'S'J'M'|S(k)|lSJM\rangle\langle l,S,m,\nu|J,M\rangle \times\langle lmS\nu|\hat{k},S\nu\rangle. \quad (2.5)$$

From rotational invariance,

$$\langle l'S'J'M'|S(k)|lSJM\rangle=\delta_{J,J'}\delta_{M,M'}S_{l'S',lS}{}^J(k), \quad (2.6)$$

where $S_{l'S',lS}{}^J$ is the S-matrix element in the $lSJM$ representation.

From Eqs. (2.4)–(2.6) we have

$$\langle\hat{k}_f,S',\nu'|S(k)|k,S,\nu\rangle = \sum_{l'm'lMJM} Y_{l'}{}^{m'}(\hat{k}_f)\langle l',S';m',\nu'|J,M\rangle S_{l'S',lS}{}^J(k) \times\langle l,S;m,\nu|J,M\rangle Y_l{}^{m*}(\hat{k})$$

$$=\sum_{l'lJ} Y_{l'}{}^{\nu-\nu'}(\cos\Theta)\langle l',\tfrac{3}{2};\nu-\nu',\nu'|J,\nu\rangle S_{l'\frac{3}{2},l\frac{1}{2}}{}^J \times\langle l,\tfrac{1}{2};0,\nu|J,\nu\rangle\left(\frac{2l+1}{4\pi}\right)^{1/2} \quad (2.7)$$

specifying the incident beam direction $\hat{k}$ as axis of quantization; i.e.,

$$m=0,\quad M=\nu,\quad Y_l{}^0=\left(\frac{2l+1}{4\pi}\right)^{1/2},\quad \text{and}\quad \cos\Theta=\hat{k}\cdot\hat{k}_f.$$

Inserting (2.7) into (2.3) gives

$$\frac{d\bar{\sigma}}{d\Omega}=\frac{\pi}{k^2}\sum_{\nu'}\Big|\sum_{l'J}(J+\tfrac{1}{2})^{1/2}(-)^{l'-(J-1/2)}\langle l';\tfrac{3}{2};\tfrac{1}{2}-\nu',\nu'|J,\tfrac{1}{2}\rangle \times S_{l'\frac{3}{2},l\frac{1}{2}}{}^J(k)Y_{l'}{}^{1/2-\nu'}(\cos\Theta)\Big|^2, \quad (2.8)$$

where we have dropped the summation over $\nu=\frac{1}{2}$, $-\frac{1}{2}$ since, for the chosen axis of quantization, $d\sigma(\nu',\nu)=d\sigma(-\nu',-\nu)$. The summation over l is superfluous because l is uniquely determined by J and l' and

$$\langle l,\tfrac{1}{2};0,\tfrac{1}{2}|J,\tfrac{1}{2}\rangle=(-)^{l'-(J-1/2)}\left(\frac{J+\frac{1}{2}}{2l+1}\right)^{1/2}.$$

If the relative intrinsic parities of the initial- and final-state particles are odd, then

$$\langle l,\tfrac{1}{2};0,\tfrac{1}{2}|J,\tfrac{1}{2}\rangle=(-)^{l'-(J-1/2)+1}\left(\frac{J+\frac{1}{2}}{2l+1}\right)^{1/2}.$$

The total cross section is

$$\bar{\sigma}=\sum_J \pi\lambda^2(J+\tfrac{1}{2})\sum_{l'}|S_{l'\frac{3}{2},l\frac{1}{2}}{}^J(k)|^2. \quad (2.9)$$

[8] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

TABLE II. Coefficients $B_{ll'J;\lambda\lambda'J'}{}^n$ defined by Eqs. (2.10) and (2.11) are given for all possible combinations of ll', J, $\lambda\lambda'$, and J' up to spin $\frac{7}{2}$.

$l\,l'\,2J\,\lambda\,\lambda'\,2J'$	B^0	B^1	B^2	B^3	B^4	B^5	B^6	B^7
*SD*1 *SD*1	0.25							
*PP*1		0.5						
*PP*3		−0.316						
*PF*3		0.948						
*DS*3			−0.707					
*DD*3			0.707					
*DD*5			−0.567					
*DG*5			1.388					
*FP*5				−1.161				
*FF*5				0.949				
*FF*7				−0.842				
*FH*7				1.825				
*GD*7					−1.604			
*GG*7					1.195			
*PP*1 *PP*1	0.25							
*PP*3			−0.316					
*PF*3			0.949					
*DS*3		−0.707						
*DD*3		0.707						
*DD*5				−0.567				
*DG*5				1.388				
*FP*5			−1.161					
*FF*5			0.949					
*FF*7					−0.842			
*FH*7					1.825			
*GD*7				−1.604				
*GG*7				1.195				
*PP*3 *PP*3	0.5		−0.400					
*PF*3			−0.600					
*DS*3		0.447						
*DD*3		0.358		−0.805				
*DD*5		1.506		−1.147				
*DG*5				−0.878				
*FP*5			0.735					
*FF*5			0.686		−1.286			
*FF*7			1.992		−1.476			
*FH*7					−1.155			
*GD*7				1.014				
*GG*7				1.007		−1.764		
*PF*3 *PF*3	0.5		0.4					
*DS*3				−1.342				
*DD*3		0.268		1.073				
*DD*5		−0.215		−0.861				
*DG*5		1.757		0.878				
*FP*5			−0.105		−2.10			
*FF*5			0.515		1.286			
*FF*7			−0.443		−1.106			
*FH*7			2.474		0.990			
*GD*7				−0.226		−2.817		
*GG*7				0.756		1.512		
*DS*3 *DS*3	0.5							
*DD*3			−1.0					
*DD*5			0.802					
*DG*5					−1.964			
*FP*5		1.643						
*FF*5				−1.342				
*FF*7				1.155				
*FH*7						−2.582		
*GD*7			2.268					
*GG*7					1.690			
*DD*3 *DD*3	0.5							
*DD*5			0.572		−1.375			
*DG*5			0.561		1.403			
*FP*5		−0.329		−1.315				
*FF*5		1.610		−0.268				
*FF*7				0.770		−1.925		
*FH*7				0.861		1.721		
*GD*7			−0.648		−1.620			
*GG*7			2.173		−0.483			
*DD*5 *DD*5	0.75		0.306		−0.735			
*DG*5			−0.450		−1.125			

TABLE II. (*Continued*).

$l\,l'\,2J\,\lambda\,\lambda'\,2J'$	B^0	B^1	B^2	B^3	B^4	B^5	B^6	B^7
*FP*5		0.263		1.054				
*FF*5		0.246		0.813		−2.134		
*FF*7		2.381		0.308		−1.763		
*FH*7				−0.690		−1.380		
*GD*7			0.519		1.299			
*GG*7			0.516		1.003		2.875	
*DG*5 *DG*5	0.750		0.765		0.413			
*FP*5				−0.239		−2.988		
*FF*5		0.188		0.878		1.568		
*FF*7		−0.162		−0.756		−1.350		
*FH*7		2.535		1.6905		0.846		
*GD*7			−0.035		−0.482		−3.936	
*GG*7			0.395		1.164		1.7605	
*FP*5 *FP*5	0.750		0.600					
*FF*5			−0.630		−1.575			
*FF*7			0.542		1.355			
*FH*7					−0.385		−3.857	
*GD*7		2.485		1.242				
*GG*7				−0.926		−1.852		
*FF*5 *FF*5	0.750		0.472		−0.322			
*FF*7			0.480		0.905		−2.934	
*FH*7			0.412		1.215		1.837	
*GD*7		−0.217		−1.014		−1.811		
*GG*7		2.430		0.756		−0.907		
*FF*7 *FF*7	1.000		0.794		−0.117		−1.010	
*FH*7			−0.355		−1.045		−1.581	
*GD*7		0.187		0.873		1.559		
*GG*7		0.186		0.798		0.965		−3.901
*FH*7 *FH*7	1.000		1.111		0.818		0.404	
*GD*7				−0.089		−0.751		−5.016
*GG*7				0.714		1.511		1.994
*GD*7 *GD*7	1.000		1.021		0.551			
*GG*7			−0.456		−1.344		−2.033	
*GG*7 *GG*7	1.000		0.884		0.150		−0.606	

The maximum value of $S_{l'\frac{1}{2},l\frac{1}{2}}{}^J$ is unity, so that the maximum inelastic cross section for a single partial-wave amplitude is $\pi\lambda\!\!\!^{-2}(J+\frac{1}{2})$. More generally,

$$\bar{\sigma}=\sum_J \pi\lambda\!\!\!^{-2}(J+\tfrac{1}{2})\sum_{l'}|S_{S'\frac{1}{2},S\frac{1}{2}}{}^J(k)-\delta_{c',c}\delta_{l',l}\delta_{S',S}|^2,$$

and the maximum $\bar{\sigma}$ in the elastic channel for a single amplitude is, therefore, $4\pi\lambda\!\!\!^{-}(J+\frac{1}{2})$.

The S-matrix element in the preceding equations is in general a linear combination of two isotopic-spin amplitudes. Denoting the isotopic spin of the initial- (final-) state meson and baryon by I and T (I' and T'), respectively, we have for reactions (1.1)

$$S=\langle I,T;I_3,T_3|\tfrac{3}{2},I_3+T_3\rangle S_{3/2}\langle I',T';I_3',T_3'|\tfrac{3}{2},I_3+T_3\rangle +\langle I,T;I_3,T_3|\tfrac{1}{2},I_3+T_3\rangle S_{1/2}\langle I',T';I_3',T_3'|\tfrac{1}{2},I_3+T_3\rangle,$$

and for reactions (1.2)

$$S=\langle I,T;I_3,T_3|1,I_3+T_3\rangle S_1\langle I',T';I_3',T_3'|1,I_3+T_3\rangle +\langle I,T;I_3,T_3|0,I_3+T_3\rangle S_0\langle I',T';I_3',T_3'|0,I_3+T_3\rangle.$$

The right-hand side of Eq. (2.8) can be written as an expansion in the Legendre polynomials $P_n(\cos\Theta)$

$$\frac{d\bar{\sigma}}{d\Omega}=\lambda\!\!\!^{-2}\sum_n A_nP_n(\cos\Theta), \quad (2.10)$$

where

$$A_n=\frac{(2n+1)}{4\pi\lambda\!\!\!^{-2}}\int\frac{d\bar{\sigma}}{d\Omega}P_n(\cos\Theta)d\Omega =\sum_{l',J}\sum_{\geqslant\lambda',J'}\text{Re}(S_{l'l}{}^JS_{\lambda'\lambda}{}^{J'*})B_{ll'J;\lambda\lambda'J'}{}^n. \quad (2.11)$$

The coefficients B^n, evaluated by inserting (2.8) in (2.11), are listed in Table II.

The scattering angle $\cos\Theta=\hat{k}\cdot\hat{k}_f$ is not uniquely defined. In elastic scattering the convention is that $\hat{k}$ and $\hat{k}_f$ refer to the same particle. Then the amplitude S^J always lies in the upper half of the complex plane ($\text{Im}S>0$). In inelastic scattering such as $\pi+N\rightarrow K+\Lambda$, where the outgoing baryon and boson belong to the same $SU(3)$ octets as the initial-state particles, the same convention is maintained by invoking $SU(3)$ symmetry. However, the amplitude S^J can now lie anywhere in the complex plane because of the sign of the $SU(3)$ Clebsch-Gordan coefficients.

In the reaction $\pi+N\rightarrow\pi+\Delta$, in which N and Δ belong to different $SU(3)$ multiplets, a higher symmetry is needed to make a correspondence between N and Δ. Instead, we make the simple convention that $\hat{k}$ and $\hat{k}_f$ are the directions of the initial- and final-state bosons. The formalism in Sec. II is independent of the definition of scattering angle. If $\hat{k}_f$ is replaced by $(-\hat{k}_f)$, then $S_{l'l}{}^J$ goes to $(-)^{l'}S_{l'l}{}^J$.

III. DECAY ANGULAR DISTRIBUTION

For clarity we specify reaction (1.1). The notation is as follows. θ and ϕ are the polar and azimuthal decay angles of the nucleon in the Δ rest frame, $\rho(\Theta)$ is the spin density matrix of Δ, and $f_D(\theta,\phi)$ is the amplitude for the decay (1.1b).

The differential decay distribution $W(\Theta,\theta,\phi)$ can be expressed as a function of the partial-wave amplitudes. This is most conveniently done through the density-matrix formalism

$$\rho(\Theta)=ff^\dagger/\mathrm{Tr}(ff^\dagger)\,, \qquad (3.1)$$

where f is the scattering amplitude for reaction (1.1a), defined in Eq. (2.2b). Here ρ is the $\pi\Delta$ spin-state density matrix in the over-all c.m. system. However, since the pion has zero spin, ρ is simply the Δ density matrix in the same system. The spin state of a particle is invariant under a transformation from the rest frame of the particle to a moving frame[9]; hence ρ is also the density matrix of Δ in its rest frame. Then

$$W(\Theta,\theta,\phi)=\mathrm{Tr}(f_D\rho f_D{}^\dagger)=\mathrm{Tr}(f_D ff^\dagger f_D{}^\dagger)/\mathrm{Tr}(ff^\dagger)\,. \qquad (3.2)$$

The decay amplitude for the p-wave decay $\Delta\to N+\pi$ is given by Eq. (2.7):

$$\langle\tfrac{1}{2}\nu'|f_D|\tfrac{3}{2}\nu\rangle\propto Y_1{}^{\nu-\nu'}(\cos\theta,\phi)\langle 1,\tfrac{1}{2};\nu-\nu',\nu'|\tfrac{3}{2}\nu\rangle\,. \qquad (3.3)$$

We ignore the energy-dependent part of the amplitude, since it does not affect the angular distributions. With this definition of f_D, the decay distribution (3.2) is normalized to unity.

The decay angles θ, ϕ refer to the coordinate frame in which the Z axis is the axis of quantization. The amplitude f has been calculated in Sec. II for $Z=\hat{k}$, the c.m. incident beam direction:

$$\langle\tfrac{3}{2}\nu'|f|\tfrac{1}{2}\nu\rangle=\frac{\sqrt{\pi}}{k}\sum_{l'J}(J+\tfrac{1}{2})^{1/2}(-)^{l'-(J-1/2)}Y_{l'}{}^{\nu-\nu'}(\cos\Theta)$$
$$\times\langle l',\tfrac{3}{2};\nu-\nu',\nu'|J\nu\rangle S_{l'\frac{3}{2},l\frac{1}{2}}{}^J\,. \qquad (3.4)$$

Defining a coordinate system $Z\equiv\hat{k}$, $y=\hat{k}\times\hat{k}_f$ [Fig. 1(a)] the term $Y_{l'}{}^{\nu-\nu'}(\cos\Theta)$ above becomes $(1/4\pi)^{1/2}\times[(2l'+1)(l'-|\nu-\nu'|)!/(l'+|\nu-\nu'|)!]^{1/2}P_{l'}{}^{\nu-\nu'}(\cos\Theta)$, since the production azimuthal angle is always zero in this frame. This form of f, inserted in Eq. (3.2), gives the decay angular distribution in the coordinate system of Fig. 1(a).

The decay distribution in another frame of reference is obtained by rotation of the axis of quantization of the density matrix ρ. The rotation matrices which take the quantization axis from the beam direction to the helicity direction or to the production normal are given in the Appendix.

The form of the decay distribution depends on the choice of quantization axis. For axis of quantization in the production plane [Fig. 1(a) or 1(b)], the general form for the density matrix is[10]

$$\begin{vmatrix} \rho_{33} & \rho_{31} & \rho_{3-1} & \rho_{3-3} \\ \rho_{31}{}^* & \rho_{11} & \rho_{1-1} & \rho_{3-1}{}^* \\ \rho_{3-1}{}^* & -\rho_{1-1} & \rho_{11} & -\rho_{31}{}^* \\ -\rho_{3-3} & \rho_{3-1} & -\rho_{31} & \rho_{33} \end{vmatrix}\,. \qquad (3.5)$$

Hermiticity requires that all diagonal elements are real and that ρ_{3-3} and ρ_{1-1} are purely imaginary. For a single amplitude all elements of the density matrix are real.

From Eqs. (3.2), (3.3), and (3.5), the decay distribution for the z axis in the production plane is

$$W(\Theta,\theta,\phi)=(3/4\pi)[\tfrac{1}{6}+\tfrac{2}{3}\rho_{33}+(\tfrac{1}{2}-2\rho_{33})\cos^2\theta$$
$$-\tfrac{2}{3}\sqrt{3}\,\mathrm{Re}\rho_{3-1}\sin^2\theta\cos 2\phi$$
$$-\tfrac{2}{3}\sqrt{3}\,\mathrm{Re}\rho_{31}\sin 2\theta\cos\phi]\,. \qquad (3.6)$$

For axis of quantization along the production normal $\hat{k}\times\hat{k}f$, the general form for the density matrix is[11]

$$\begin{vmatrix} \rho_{33} & 0 & \rho_{3-1} & 0 \\ 0 & \rho_{11} & 0 & \rho_{1-3} \\ \rho_{3-1}{}^* & 0 & \rho_{-1-1} & 0 \\ 0 & \rho_{1-3}{}^* & 0 & \rho_{-3-3} \end{vmatrix}\,,$$

and the corresponding decay distribution is

$$W^N(\Theta,\theta,\phi)=(1/8\pi)\{3\sin^2\theta+2(\rho_{11}{}^N+\rho_{-1-1}{}^N)$$
$$\times(2-3\sin^2\theta)-2\sqrt{3}\sin^2\theta$$
$$\times[\mathrm{Re}(\rho_{3-1}{}^N+\rho_{1-3}{}^N)\cos 2\phi$$
$$-\mathrm{Im}(\rho_{3-1}{}^N+\rho_{1-3}{}^N)\sin 2\phi]\}\,. \qquad (3.7)$$

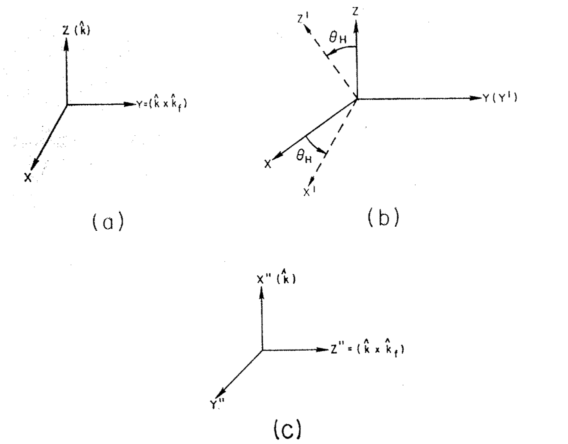

FIG. 1. Coordinate frames for decay angle of Δ. The vectors $\hat{k}$ and $\hat{k}f$ are the c.m. incident and final pion directions in the reaction $\pi+N\to\Delta+\pi$. (a) Incident beam direction in c.m. is the Z axis, production normal is the Y axis. (b) Axis Z' is $-\hat{k}_f$ (see Appendix). (c) Axis Z'' is the production normal $\hat{k}\times\hat{k}_f$ (see Appendix), X'' is the beam direction.

[9] G. C. Wick, in *High Energy Physics*, edited by C. M. Dewitt and M. Jacob (Gordon and Breach, Science Publishers, Inc., New York, 1965).

[10] See, e.g., N. Schmitz, in CERN Report No. 65-24 (unpublished).

[11] R. H. Capps, Phys. Rev. **122**, 929 (1961).

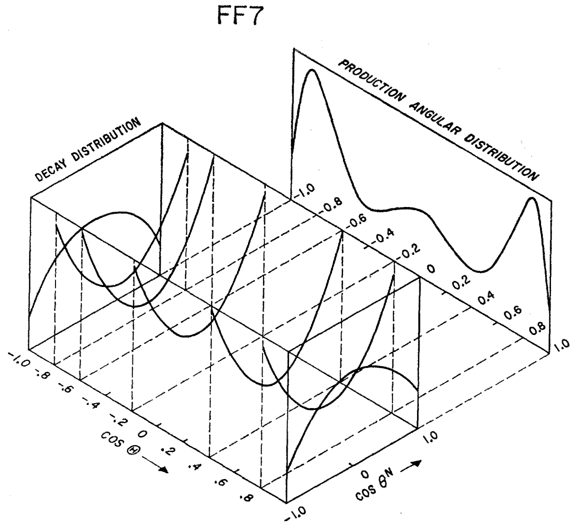

FIG. 2. Correlation between the *FF7* production angular distribution and the normalized decay distribution $W(\theta^N,\phi^N)$ integrated over ϕ^N. [Coordinate system as in Fig. 1(c).]

IV. POLARIZATION OF SPIN-½ BARYON

Whereas Δ can be polarized only along the production normal, the spin-½ baryon can be polarized in any direction. The density matrix of a spin-½ particle in its rest system can be written

$$\rho_{1/2}=\tfrac{1}{2}\begin{vmatrix} 1+P_z & P_x-iP_y \\ P_x+iP_y & 1-P_z \end{vmatrix}, \tag{4.1}$$

where P_x, P_y, and P_z are the three components of the polarization vector. The polarization can be expressed as a function of the partial-wave amplitudes through the relation

$$\rho_{1/2}=\frac{f_D\rho f_D^\dagger}{\mathrm{Tr}(f_D\rho f_D^\dagger)}=\frac{f_D\rho f_D^\dagger}{W(\Theta,\theta,\phi)}, \tag{4.2}$$

where ρ is the density matrix of the Δ. Equation (4.2) defines $\rho_{1/2}$ in the Δ rest frame with the same axis of quantization as ρ. The density matrix $\rho_{1/2}$ is unchanged by a transformation of the Δ system to the nucleon rest system. When the reaction goes by a single partial-wave amplitude, both the Δ and the nucleon are unpolarized.

For the sake of completeness, we reproduce here the expressions for the nucleon polarization derived by Jackson.[12] (They are obtained by applying the appropriate rotation to $\rho_{1/2}$ above.) It is conventional to specify the polarization in terms of P_L, the longitudinal polarization parallel to the nucleon momentum **p** in the Δ rest frame, and two transverse components $P_\perp x$ and $P_\perp y$; the positive x direction is $(\hat{q}\times\hat{p})\times\hat{p}$ and y is along $\hat{q}\times\hat{p}$, where $\hat{q}$ is the axis of quantization for the density matrix ρ.

For axis of quantization in the production plane [Fig. 1(a) and 1(b)] the polarization components are

$$\begin{aligned} P_L(\Theta,\theta,\phi)W(\Theta,\theta,\phi) &= (3/4\pi)\sin\theta\,\{[\tfrac{1}{3}\sqrt{3}(3\cos^2\theta-1)\,\mathrm{Im}\rho_{31} \\ &\quad -3(\cos^2\theta-\tfrac{1}{9})\,\mathrm{Im}\rho_{1-1}]\sin\phi+\tfrac{1}{2}\sqrt{3}\sin2\theta\,\mathrm{Im}\rho_{3-1}\sin2\phi \\ &\quad +\sin^2\theta\,\mathrm{Im}\rho_{3-3}\sin3\phi\}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} P_\perp x(\Theta,\theta,\phi)W(\Theta,\theta,\phi) &= -(3/4\pi)\{[\tfrac{1}{3}\cos\theta(9\cos^2\theta-5)\,\mathrm{Im}\rho_{1-1} \\ &\quad +2\sqrt{3}\sin^2\theta\cos\theta\,\mathrm{Im}\rho_{31}]\sin\phi-\tfrac{2}{3}\sqrt{3}\sin\theta(3\cos^2\theta-1) \\ &\quad \times\mathrm{Im}\rho_{3-1}\sin2\phi-\sin^2\theta\cos\theta\,\mathrm{Im}\rho_{3-3}\sin3\phi\}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} P_\perp y(\Theta,\theta,\phi)W(\Theta,\theta,\phi) &= -(3/4\pi)[-(\tfrac{1}{3}+\cos^2\theta)\,\mathrm{Im}\rho_{1-1}\sin\phi \\ &\quad +\tfrac{2}{3}\sqrt{3}\sin^2\theta\,\mathrm{Im}\rho_{31}\cos\phi+\tfrac{2}{3}\sqrt{3}\sin2\theta\,\mathrm{Im}\rho_{3-1}\cos2\phi \\ &\quad -\sin^2\theta\,\mathrm{Im}\rho_{3-3}\cos3\phi]. \end{aligned}$$

We note that measurement of one component of the polarization, together with the determination of the decay distribution $W(\Theta,\theta,\phi)$, [Eq. (3.6)], suffices in principle to give all elements of the density matrix $\rho(\Theta)$ [Eq. (3.5)]. Integration over ϕ causes all three polarization components to vanish.

For axis of quantization along the production normal the polarization is

$$\begin{aligned} P_L^N(\Theta,\theta,\phi)W(\Theta,\theta,\phi) &= (3/4\pi)\{[\tfrac{1}{2}(\rho_{11}^N-\rho_{-1-1}^N)(3\cos^2\theta-5/3) \\ &\quad +\tfrac{1}{2}(\rho_{33}^N-\rho_{-3-3}^N)\sin^2\theta]\cos\theta \\ &\quad -\tfrac{1}{2}\sqrt{3}\sin2\theta[\mathrm{Re}(\rho_{3-1}^N-\rho_{1-3}^N)\cos2\phi \\ &\quad -\mathrm{Im}(\rho_{3-1}^N-\rho_{1-3}^N)\sin2\phi]\times\sin\theta\}, \end{aligned} \tag{4.5}$$

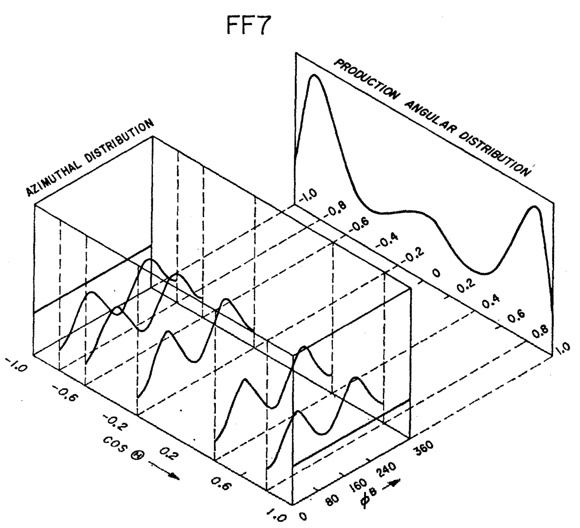

FIG. 3. Correlation between the *FF7* production angular distribution and the normalized $W(\theta^B,\phi^B)$ integrated over $\cos\theta^B$. [Coordinate system as in Fig. 1(a).]

[12] J. D. Jackson, in *High Energy Physics*, edited by C. M. Dewitt and M. Jacob (Gordon and Breach, Science Publishers, Inc., New York, 1965).

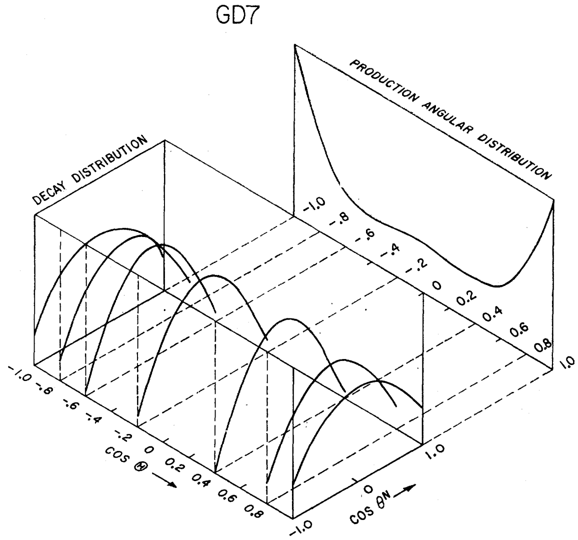

FIG. 4. Same as Fig. 2 but for the *GD*7 amplitudes.

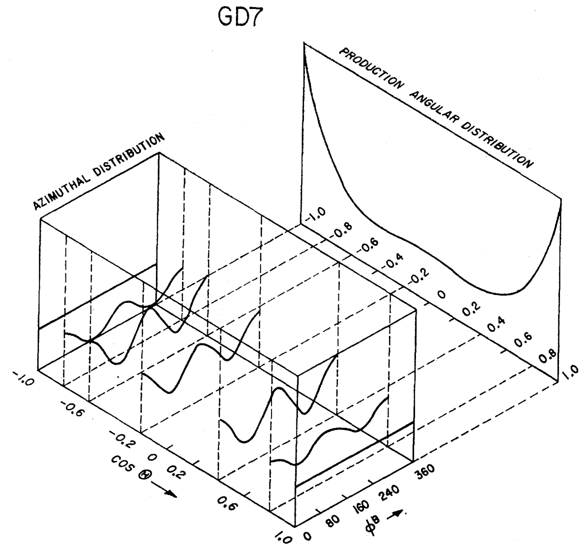

FIG. 5. Same as Fig. 3 but for the *GD*7 amplitudes.

$$\begin{aligned}P_\perp^N x(\Theta,\theta,\phi)W(\Theta,\theta,\phi) &= -(3/4\pi)\{[\tfrac{1}{2}(\rho_{11}^N-\rho_{-1-1}^N)(3\cos^2\theta-\tfrac{1}{3}) \\ &\quad+\tfrac{1}{2}(\rho_{33}^N-\rho_{-3-3}^N)\sin^2\theta]\sin\theta \\ &\quad+\tfrac{1}{3}\sqrt{3}\sin\theta(3\cos^2\theta-1)[\mathrm{Re}(\rho_{3-1}^N-\rho_{1-3}^N)\cos2\phi \\ &\quad-\mathrm{Im}(\rho_{3-1}^N-\rho_{1-3}^N)\sin2\phi]\}, \qquad (4.6)\end{aligned}$$

$$\begin{aligned}P_\perp^N y(\Theta,\theta,\phi)W(\Theta,\theta,\phi) &= +(\sqrt{3}/4\pi)\sin2\theta[\mathrm{Re}(\rho_{3-1}^N-\rho_{1-3}^N)\sin2\phi \\ &\quad+\mathrm{Im}(\rho_{3-1}^N-\rho_{1-3}^N)\cos2\phi]. \qquad (4.7)\end{aligned}$$

With the production normal as axis of quantization, the ϕ dependence takes the form $A\sin2\phi+B\cos2\phi$, and upon averaging over ϕ, the quantity $P_\perp^N y$ vanishes. Upon integrating over θ, the quantity P_L^N also vanishes, and we have

$$\begin{aligned}P_\perp^N x(\Theta) &= (3/64)[(5/3)(\rho_{11}^N-\rho_{-1-1}^N) \\ &\quad+3(\rho_{33}^N-\rho_{-3-3}^N)]. \qquad (4.8)\end{aligned}$$

The magnitude of $P_\perp^N x(\Theta)$ cannot exceed 14%.

V. ANALYSIS OF EXPERIMENTAL DATA

The experimental data at a given momentum consist of a joint distribution in four independent variables, which may be Θ, θ, and ϕ as defined in Sec. I, and the polarization **P**. The distribution in the angular variables is

$$\begin{aligned}I(\Theta,\theta,\phi) &= [d\bar{\sigma}(\Theta)/d\Omega]W(\Theta,\theta,\phi) \qquad (5.1a)\\ &= \tfrac{1}{2}\,\mathrm{Tr}(ff^\dagger)W(\Theta,\theta,\phi)\\ &= \tfrac{1}{2}\,\mathrm{Tr}f_D R f f^\dagger R^{-1} f_D^\dagger, \qquad (5.1b)\end{aligned}$$

where $d\bar{\sigma}/d\Omega$ is the differential cross section for reaction (1.1a), and $W(\Theta,\theta,\phi)$ is the decay angular distribution for (1.1b); correspondingly, f and f_D are the amplitudes for reactions (1.1a) and (1.1b), respectively, and R is a rotation matrix. The distribution $I(\Theta,\theta,\phi)$ can be written explicitly as a function of the partial-wave amplitudes $S_{l'\frac{3}{2},l\frac{1}{2}}^J$ and the observables Θ, θ, ϕ by substituting in Eq. (5.1b)

$$\begin{aligned}\langle\tfrac{3}{2}\nu'|f|\tfrac{1}{2}\nu\rangle &= \frac{1}{2k}\sum_{l'J}(J+\tfrac{1}{2})^{1/2}(-)^{l'-(J-1/2)} \\ &\quad\times[(2l'+1)(l'-|\nu-\nu'|)!/(l'+|\nu-\nu'|)!]^{1/2} \\ &\quad\times P_{l'}^{\nu-\nu'}(\cos\Theta)\langle l',\tfrac{3}{2};\nu-\nu',\nu'|J\nu\rangle S_{l'\frac{3}{2},l\frac{1}{2}}^J\end{aligned}$$

and

$$\langle\tfrac{1}{2}\nu|f_D|\tfrac{3}{2}\nu'\rangle = Y_1^{\nu'-\nu}(\cos\theta,\phi)\langle 1,\tfrac{1}{2};\nu'-\nu,\nu|\tfrac{3}{2}\nu'\rangle,$$

where the index ν runs from $\frac{1}{2}$ to $-\frac{1}{2}$ and ν' from $\frac{3}{2}$ to

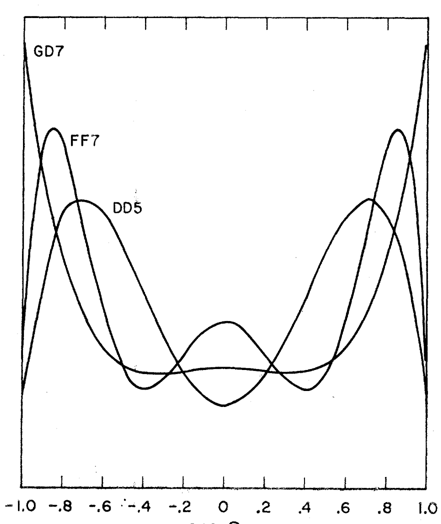

FIG. 6. Production angular distributions for *FF*7, *GD*7, and *DD*5 amplitudes.

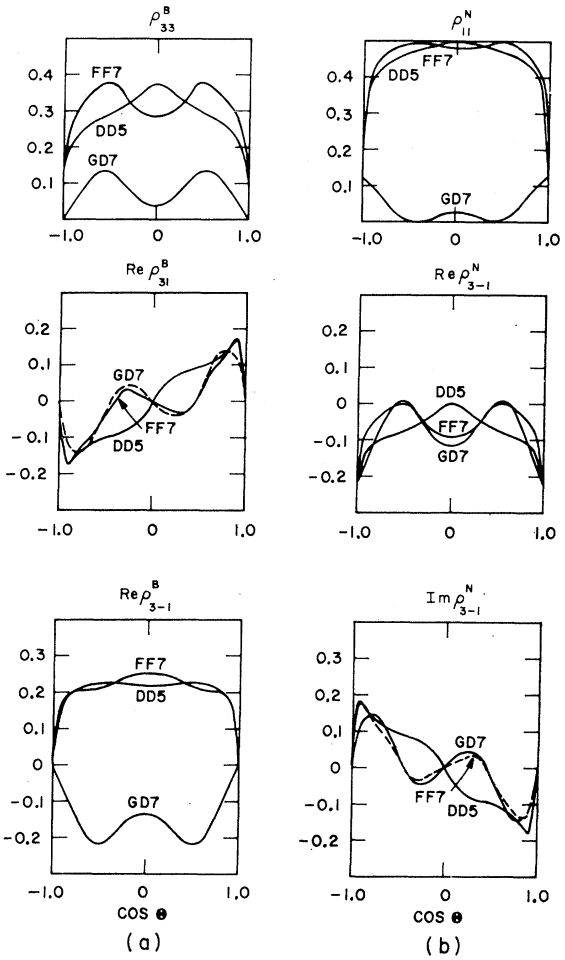

FIG. 7. The decay parameters defined in Sec. V are shown as a function of the Δ production angle Θ for quantization axis of the density matrix as (a) beam direction and (b) normal to the production plane. For a single amplitude, as shown here, $\rho_{3-1}{}^N=\rho_{1-3}{}^N$ and $\rho_{11}{}^N=\rho_{-1-1}{}^N$.

$-\frac{3}{2}$. The angle Θ is always given by $\cos\Theta=\hat{k}\cdot\hat{k}_f$. When the decay angles are measured in the coordinate system $Z=\hat{k}$, $y=\hat{k}\times\hat{k}_f$ [Fig. 1(a)], the matrix R is a unit matrix. When the decay angles are expressed in the coordinate systems of Fig. 1(b) or 1(c), the corresponding rotation matrices are R_H and R_N as given in the Appendix.

Angular distributions for the partial-wave amplitudes *FF*7 and *GD*7 are shown in Figs. 2–5 for the coordinate systems defined in Figs. 1(a) and 1(c). The correlations between the production and decay angles of the Δ are clearly sensitive to the spin and parity of the partial-wave amplitudes.

The decay distribution $W(\Theta,\theta,\phi)$ of Δ or Σ is completely specified at a given production angle by three parameters which are functions of the elements of the density matrix $\rho=ff^\dagger/\mathrm{Tr}(ff^\dagger)$ of the spin-$\frac{3}{2}$ particle: $\rho_{33}(\Theta)$, $\mathrm{Re}\rho_{3-1}(\Theta)$, and $\mathrm{Re}\rho_{31}(\Theta)$ for axis of quantization in the production plane, or $[\rho_{11}{}^N(\Theta)+\rho_{-1-1}{}^N(\Theta)]$, $\mathrm{Re}[\rho_{3-1}{}^N(\Theta)+\rho_{1-3}{}^N(\Theta)]$, and $\mathrm{Im}[\rho_{3-1}{}^N(\Theta)+\rho_{1-3}{}^N(\Theta)]$ for the quantization axis along the production normal. Then, according to Eq. (5.1a) the experimental data (excluding polarization) at a single momentum can be summarized in the form of four distributions in cosΘ—the differential cross section and the three decay parameters. These distributions are shown in Figs. 6 and 7 for the partial-wave amplitudes *DD*5, *FF*7, and *GD*7 for the coordinate systems defined in Fig. 1(a) (ρ^B) and Fig. 1(c) (ρ^N).

The experimental density-matrix elements are statistically correlated. This correlation must be taken into account if the comparison between the experimental and calculated distributions is made in terms of density-matrix elements.

The experimental data may be insufficient to determine all the correlations among Θ, θ, and φ. In that case the question arises of how best to bin the data. Also the choice of coordinate frame in which the decay angles are measured may be important. There is no simple prescription, but a study of the density-matrix elements for the hypothesis being tested will usually indicate the best procedure. For example, if one is trying to distinguish between the amplitudes *FF*7 and *GD*7, the correlations between Θ and θ^N or between Θ and ϕ^B are clearly very sensitive, as indicated by the plots of $\rho_{11}{}^N$ and $\mathrm{Re}\rho_{3-1}{}^B$ in Fig. 7.

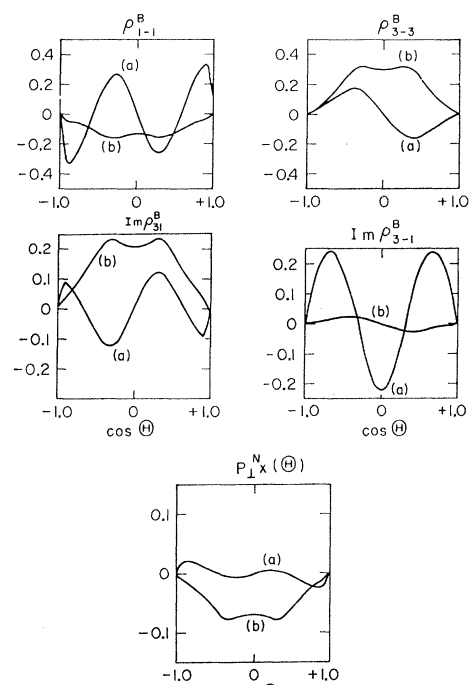

FIG. 8. The density-matrix elements $\rho_{1-1}{}^B$, $\rho_{3-3}{}^B$, $\mathrm{Im}\rho_{31}{}^B$, $\mathrm{Im}\rho_{3-1}{}^B$ are shown as a function of the Δ production angle Θ together with the quantity $P_\perp{}^Nx(\Theta)$ for interference of (a) *DD*5 and *GD*7, and (b) *DD*5 and *FF*7. The interfering amplitudes have equal magnitude and are 90° out of phase.

In reaction (1.2) the Λ polarization can readily be measured through observation of the decay $\Lambda\rightarrow p+\pi^-$. Knowledge of the polarization of the spin-$\frac{1}{2}$ baryon supplements the information on the elements of the density matrix $\rho(\Theta)$ obtainable from the decay distribution $W(\Theta,\theta,\phi)$. For example, in the coordinate system of Fig. 1(a) or 1(b), the decay distribution $W(\Theta,\theta,\phi)$ is a function of ρ_{33}, $\mathrm{Re}\rho_{3-1}$, and $\mathrm{Re}\rho_{31}$; and the polarization depends on ρ_{1-1}, ρ_{3-3}, $\mathrm{Im}\rho_{31}$, and $\mathrm{Im}\rho_{3-1}$. An alternative way of distinguishing between *FF*7 and *GD*7 amplitudes is by observing the polarization produced by their interference with the *DD*5 amplitude. The relevant density-matrix elements resulting from this interference are shown in Fig. 8. However, the statistical weight of the polarization data is down by an order of magnitude from that of the decay distribution data. Integration over the decay angles θ and ϕ leaves only $P_\perp{}^N x(\Theta)$ as given in Eq. (4.8). This quantity is also shown in Fig. 8.

ACKNOWLEDGMENT

We would like to thank Dr. Paul Söding for helpful discussions.

APPENDIX: ROTATION OF THE AXIS OF QUANTIZATION

If ρ denotes the density matrix for axis of quantization z, and ρ' the same state for axis of quantization z', ρ and ρ' are related by the unitary transformation

$$\rho'=R\rho R^{-1}.$$

The change of axis of quantization is simply a change of basis states from $|JM\rangle$ to $|JM'\rangle$, where M' is an eigenvalue of $J_{z'}$.

Below we give the transformation matrixes R_H and R_N corresponding to a rotation of the axis of quantization from the incident beam direction [Fig. 1(a)] to (1) spin-$\frac{3}{2}$ particle direction (helicity direction) [Fig. 1(b)] and to (2) the production normal [Fig. 1(c)], respectively.

(1) Rotation to the helicity direction. In the right-handed (x,y,z) coordinate frame $z=\hat{k}$ and $y=\hat{k}\times\hat{k}_f$ [Fig. 1(a)]. The Euler angles for the rotation which takes z into the helicity direction are $\alpha=0$, $\beta=\Theta_H$, and $\gamma=0$.[13] ($\Theta_H=180°-\Theta$.) The corresponding rotation matrix R_H is

$$\frac{1}{4}\left|\begin{matrix} \cos\frac{3}{2}\Theta_H+3\cos\frac{1}{2}\Theta_H & \sqrt{3}(\sin\frac{3}{2}\Theta_H+\sin\frac{1}{2}\Theta_H) & \sqrt{3}(-\cos\frac{3}{2}\Theta_H+\cos\frac{1}{2}\Theta_H) & -\sin\frac{3}{2}\Theta_H+3\sin\frac{1}{2}\Theta_H \\ -\sqrt{3}(\sin\frac{3}{2}\Theta_H+\sin\frac{1}{2}\Theta_H) & 3\cos\frac{3}{2}\Theta_H+\cos\frac{1}{2}\Theta_H & 3\sin\frac{3}{2}\Theta_H-\sin\frac{1}{2}\Theta_H & \sqrt{3}(-\cos\frac{3}{2}\Theta_H+\cos\frac{1}{2}\Theta_H) \\ \sqrt{3}(-\cos\frac{3}{2}\Theta_H+\cos\frac{1}{2}\Theta_H) & -3\sin\frac{3}{2}\Theta_H+\sin\frac{1}{2}\Theta_H & 3\cos\frac{3}{2}\Theta_H+\cos\frac{1}{2}\Theta_H & \sqrt{3}(\sin\frac{3}{2}\Theta_H+\sin\frac{1}{2}\Theta_H) \\ \sin\frac{3}{2}\Theta_H-3\sin\frac{1}{2}\Theta_H & \sqrt{3}(-\cos\frac{3}{2}\Theta_H+\cos\frac{1}{2}\Theta_H) & -\sqrt{3}(\sin\frac{3}{2}\Theta_H+\sin\frac{1}{2}\Theta_H) & \cos\frac{3}{2}\Theta_H+3\cos\frac{1}{2}\Theta_H \end{matrix}\right|.$$

(2) Rotation to production normal [see Fig. 1(c)]. The Euler angles which take z into the production normal and x into the beam direction [Fig. 1(c)] are $\alpha=90°$, $\beta=90°$, and $\gamma=180°$. The density matrix ρ^N, with axis of quantization along the production normal, is $R_N\rho R_N{}^{-1}$, where

$$R_N=\frac{1}{\sqrt{8}}\left|\begin{matrix} e^{i3\pi/4} & \sqrt{3}e^{i\pi/4} & \sqrt{3}e^{-i\pi/4} & e^{-i3\pi/4} \\ -\sqrt{3}e^{i3\pi/4} & -e^{i\pi/4} & e^{-i\pi/4} & \sqrt{3}e^{-i3\pi/4} \\ \sqrt{3}e^{i3\pi/4} & -e^{i\pi/4} & -e^{-i\pi/4} & \sqrt{3}e^{-i3\pi/4} \\ -e^{i3\pi/4} & \sqrt{3}e^{-i\pi/4} & -\sqrt{3}e^{-i\pi/4} & e^{-i3\pi/4} \end{matrix}\right|.$$

(R_N is the rotation matrix for $\alpha=90°$, $\beta=90°$, $\gamma=0$. From parity conservation the density matrix is invariant under the rotation $\gamma=180°$.)

[13] The Euler angles are as defined by M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957): a rotation α about the original z axis, followed by a rotation β about the new y axis, followed by a rotation γ about the new z axis. The rotation is performed in the positive sense in a right-handed coordinate system.