is calculated with the next term taken into account,

$$\Delta E_{1S}^{c} \sim (2Ze^2/5R)\eta^3[1-(5\eta/6)]. \qquad (9)$$

As seen in the Fig. 8, this correction is small ($\sim 3\%$) in this atom, and higher corrections within this first-order perturbation theory are not needed here. The result of the exact calculation is also shown in this figure. The error introduced by use of these approximations is about 10%, or less than 0.05 keV for 2.0 F$<R<$3.0 F, and could be neglected considering the current experimental error.

It has been shown that the vacuum-polarization effect is quite important in mesonic atoms of the light nuclei.[17] In analyses of future experiments, this effect should be examined more closely by taking into account the effect of the finite size of the nucleus on the vacuum polarization.

ACKNOWLEDGMENT

I would like to thank Professor Alan H. Cromer for constant advice and encouragement during this work.

[17] D. D. Ivanenko and G. E. Pustovalov, Usp. Fiz. Nauk **61**, 27 (1957) [English transl.: Soviet Phys.—Usp. **63**, 1043 (1961)].

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Veneziano Parametrization for Nonstrong Amplitudes*

R. C. Brower and M. B. Halpern

Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California 94720

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Veneziano-like parametrizations are found for photoproduction, Compton scattering, and current algebra.

I. INTRODUCTION

Recently, Veneziano[1] gave a simple parametrization for hadronic amplitudes which is crossing-symmetric, displays Regge behavior, and satisfies duality. It is our purpose in this paper to find a similar parametrization for nonstrong amplitudes, for example, photoproduction, Compton scattering, and current algebra—thus putting the nonstrong problem on an equal footing with the strong. In general, we shall concentrate on the features that distinguish these amplitudes from purely hadronic ones—namely, double poles (gauge invariance, low-energy theorems, etc.) and fixed poles (in angular momentum, with nontrivial form factors). On the other hand, we shall not go into much detail about problems that our parametrization shares with the purely hadronic problem—i.e., parity doubling, isospin degeneracies, factorization, etc.

In Sec. II, we construct a Veneziano parametrization for a photoproduction amplitude that has Regge asymptotic behavior in all channels and gives the correct Born approximation (low-energy theorems) at low energies. In Sec. III, the same technique is used to construct the amplitudes for physical (chargeless photons) Compton scattering ($\gamma\pi \to \gamma\pi$). A natural solution yields an $M=1$ pion with a parity partner that decouples at $J=0$. Moreover, we suggest a natural scheme for introducing into the double-helicity-flip amplitude a Pomeranchukon which couples in the forward direction. Section IV treats current-algebra amplitudes and fixed poles in the t channel. We find solutions to current-algebra sum rules with form factors parametrized by ρ and ρ-satellite poles. In this parametrization, a correlation exists between fixed poles in the s channel and asymptotic behavior of form factors. The correlation may be taken to mean that form factors must fall faster than any power of t.

The problem of combining the results of Secs. III and IV into a good phenomenological description of Compton scattering is presently under investigation.

II. PHOTOPRODUCTION

Here we investigate a Veneziano parametrization for photoproduction amplitudes. In particular, we are interested in those special features associated with the zero mass of the photon and required by gauge invariance. The pole terms for soft-photon couplings (called Born terms in this paper) are in fact the main issue, since gauge invariance requires them to have a particular form.[2] As emphasized in Ref. 2, the photon, unlike a massive vector particle, can couple to a nonsense pole if the internal mass is the same as the external mass (soft-photon coupling). On the other hand, the hard-photon couplings (other poles to which the soft photon does not couple) obey the usual selection rules of angular

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[1] G. Veneziano, Nuovo Cimento **57**, 190 (1968).

[2] F. Arbab and R. C. Brower, Phys. Rev. **178**, 2470 (1969); R. C. Brower and J. Dash, *ibid.* **175**, 2014 (1968).

momentum conservation (no zero-to-zero transitions); consequently, these terms will not be discussed in detail. Indeed, only little attention will be given to the problems encountered in purely hadronic processes, such as parity doubling, factorization, and isospin symmetry.

For convenience we consider pion photoproduction from spinless nucleons (s-channel $\gamma N \to \pi N$). By restricting ourselves to spinless hadrons with parities appropriate for a nonzero coupling $g=\langle\pi|\bar{N}N\rangle$, there is only one helicity amplitude $\mathfrak{M}(\lambda_\gamma=1)=\mathfrak{M}(\lambda_\gamma=-1)$, with the kinematical singularity $\phi^{1/2}$, where ϕ is the Kibble function

$$\phi(s,t)=4tq_\gamma^2(t)p_N^2(t)\sin^2\theta_t. \tag{2.1}$$

The standard decomposition for $A=\mathfrak{M}_1/\phi^{1/2}$ into pure isotopic spin in the t channel, $\gamma\pi_\alpha \to \bar{N}_1 N_2$, is

$$A_\alpha=\tau_\alpha A^{(0)}+\tfrac{1}{2}\{\tau_3,\tau_\alpha\}A^{(+)}+\tfrac{1}{2}[\tau_3,\tau_\alpha]A^{(-)}, \tag{2.2}$$

where $A^{(0)}$ ($I_t=1$, $G=+1$, even in s-u) involves isoscalar photons, and both $A^{(+)}$ ($I_t=0$, $G=-1$, even in s-u) and $A^{(-)}$ ($I_t=1$, $G=-1$, odd in s-u) involve the isovector photon.

For the time being, let us consider only the amplitude $A^{(-)}$, which has the pion pole. We seek a function $A^{(-)}(s,t)$ with the following properties:

(i) It is odd under $s\leftrightarrow u$ crossing.

(ii) It has the correct Born term

$$A^{(-)}(s,t)=\tfrac{1}{2}eg\{[(t-m_\pi^2)(s-m^2)]^{-1} -[(t-m_\pi^2)(u-m^2)]^{-1}\}+\text{background}, \tag{2.3}$$

where the background is regular at $t=m_\pi^2$. This will guarantee the full content of the low-energy theorems (Kroll-Ruderman for spin-zero nucleons).

(iii) The remainder of the amplitude (background) is built out of poles with polynomial residues corresponding to physical partial waves ($J\geqq 1$) in the expansion

$$A^{(-)}(s,t)=\sum_{J=1}^{\infty}(2J+1)a^J(t)e_{10}{}^J(z_t),\quad e_{10}{}^J\sim P_J{}'(z_t) \tag{2.4}$$

and similarly for s- and u-channel poles.

(iv) The pion lies on a Regge trajectory with appropriate signature and helicity flip. That is, as $s\to\infty$, for fixed t,

$$A^{(-)}(s,t)\sim\tfrac{1}{4}eg\Gamma(-\alpha_\pi(t))(1+e^{-i\pi\alpha_\pi(t)})s^{\alpha_\pi(t)-1}. \tag{2.5}$$

Similar Regge limits must exist for fixed s and u.

It is remarkable that only a slight modification of the β function introduces the double pole for the Born term without introducing any other $J=0$ poles or "ancestor" trajectories. Moreover, this function $\tilde{B}$ automatically has one unit of helicity flip:

$$\begin{aligned}\tilde{B}(-\alpha_\pi(t),\,-\alpha_N(s)) &=\Gamma(-\alpha_\pi(t))\Gamma(-\alpha_N(s))/\Gamma(1-\alpha_\pi(t)-\alpha_N(s))\\ &=-[\alpha_\pi(t)+\alpha_N(s)]^{-1}B(-\alpha_\pi(t),\,-\alpha_N(s)).\end{aligned} \tag{2.6}$$

Conditions (i)–(iv) are completely satisfied by

$$A^{(-)}(s,t)=\tfrac{1}{2}egb^2\{\tilde{B}(-\alpha_\pi(t),\,-\alpha_N(s)) -\tilde{B}(-\alpha_\pi(t),\,-\alpha_N(u))+\cdots\}, \tag{2.7}$$

where b is the universal slope of the trajectories. The odd symmetry under $s\leftrightarrow u$ crossing automatically gives the correct signature. Having thus gotten the soft-photon poles correctly, other terms $(\cdots)$, in general involving other trajectories, can now be added in with ordinary β functions. For example, a term $B(1-\alpha_{A_2}(t),\,1-\alpha_N(s))-B(1-\alpha_{A_2}(t),\,1-\alpha_N(u))$ can be added to $A^{(-)}$ without affecting the Born terms. All the trajectories that do not contribute to the Born term should be introduced with this standard form, just as in the original paper[1] on $\pi\pi\to\pi\omega$.

Similar expressions can be written for $A^{(0)}$ and $A^{(+)}$:

$$\begin{aligned}A^{(+)}(s,t)&=\tfrac{1}{2}egb^2[\tilde{B}(-\alpha_N(s),\,-\alpha_N(u)) +C_+S(-\alpha_\pi(t),\,-\alpha_N(s),\,-\alpha_N(u))+\cdots],\\ A^{(0)}(s,t)&=\tfrac{1}{2}egb^2[\tilde{B}(-\alpha_N(s),\,-\alpha_N(u)) +C_0S(-\alpha_\pi(t),\,-\alpha_N(s),\,-\alpha_N(u))+\cdots],\end{aligned} \tag{2.8}$$

where we have introduced the symmetric function

$$S=\tilde{B}(-\alpha_\pi(t),\,-\alpha_N(s))+\tilde{B}(-\alpha_\pi(t),\,-\alpha_N(u)) +\tilde{B}(-\alpha_N(s),\,-\alpha_N(u)) \tag{2.9}$$

with arbitrary weight. It does not contribute to the Born term because the identity $\alpha_\pi(t)+\alpha_N(s)+\alpha_N(u) =b(s+t+u-m_\pi^2-2m_N^2)=0$ removes the apparent double poles from S. We have checked that the Born terms in these amplitudes are the most general ones compatible with charge conservation for the various physical processes (e.g., $\gamma p\to\pi^+n$, etc.).

To develop amplitudes of phenomenological value, one has to extend this analysis to the four invariant amplitudes (A,B,C,D) for the spin-$\frac{1}{2}$ nucleon and add on the contribution from the lower trajectories. However, some of the difficulties can be understood in this scalar model. Clearly, C_+ should be set to zero to avoid a trajectory degenerate with the pion having $I=0$ and negative signature. However, if one calculates $A(I_s=\frac{3}{2}) =A^{(+)}-A^{(-)}$, one discovers that this has introduced an $I=\frac{3}{2}$ negative-signature trajectory (Δ) degenerate with the nucleon. A nonzero C_0 may be reasonable in order to introduce the exchange-degenerate B meson onto the pion trajectory; this, of course, will affect the recurrence of the nucleon in $A(I_s=\frac{1}{2}) =A^{(+)}+2A^{(-)}+A^{(0)}$.

One can anticipate that spin for the nucleon will bring in difficulties with parity doublets similar to the πN problem.[3] We do not want to pursue further such problems encountered already in the purely strong problem. It seems to us, however, that with the $\tilde{B}$ function, the formalism is sufficiently flexible to develop

[3] K. Igi, Phys. Letters **28B**, 330 (1968); M. A. Virasoro, University of Wisconsin (report of work prior to publication).

a reasonable phenomenological amplitude for photoproduction off nucleons.

III. COMPTON SCATTERING

The considerations presented here very closely parallel those of Sec. II on photoproduction. Again, we pick the simplest possible spin configuration: Compton scattering on a spinless target. This restricts us to two helicity amplitudes, with the following kinematical factors[4]:

$$\begin{aligned}\mathfrak{M}_{1-1}{}^{t}(s,t)&=\mathfrak{M}_{1,1}{}^{s}(s,t)=(\phi/-t)A_{1-1}\,,\\ \mathfrak{M}_{1+1}{}^{t}(s,t)&=\mathfrak{M}_{1,-1}{}^{s}(s,t)=-tm_\pi{}^2A_{1+1}.\end{aligned}\tag{3.1}$$

For purposes of isospin analysis, we pick the pion as target and split the photon up into its isoscalar and isovector parts. The G parity of the pion restricts the transition to pure isoscalar-isoscalar amplitudes ($A_{\lambda_1\lambda_2}{}^{(s)}$) and isovector-isovector amplitudes ($A_{\lambda_1\lambda_2}{}^{(I_t)}$). Again, we are primarily interested in the Born terms (in this case, the pion pole terms) which, because of G parity, contribute only to the isovector amplitudes $A_{\lambda_1\lambda_2}{}^{(I_t)}$, where we have chosen amplitudes with pure isotopic spin in the t channel. The gauge-invariant Born terms[4] for these amplitudes are given by

$$\begin{aligned}A_{1\pm1}{}^{(0)}(s,t)&=4e^2/(s-m_\pi{}^2)(u-m_\pi{}^2)\,,\\ A_{1\pm1}{}^{(2)}&=-2e^2/(s-m_\pi{}^2)(u-m_\pi{}^2)\,,\\ A_{1\pm1}{}^{(1)}&=-(2/t)[e^2/(s-m_\pi{}^2)-e^2/(u-m_\pi{}^2)].\end{aligned}\tag{3.2}$$

The pion pole terms have been introduced so that they are pure $I_s=1$ and $I_u=1$ in the s and u channels, respectively. The amplitude $I_t=1$ has been introduced for completeness at this point; it does not contribute at all to Compton scattering for physical (chargeless) photons. In Sec. IV, the problem of fixed poles in the helicity-flip-2 isospin-1 t-channel amplitudes will be considered in detail.

For simplicity, we assume that the amplitudes contributing to the chargeless-photon Compton scattering

$$[\gamma\gamma\to\pi^\pm\pi^\mp\propto\tfrac{1}{3}(A^{(0)}-A^{(2)})+A^{(s)}\quad\text{and}\quad \gamma\gamma\to\pi^0\pi^0\propto\tfrac{1}{3}(A^{(0)}+2A^{(2)})+A^{(s)}]$$

have pure Regge asymptotic behavior. Amplitudes with fixed poles are discussed in Sec. IV. Let us see how such Reggeized poles can be introduced for the Born terms in the $I_t=0$, 2 amplitudes.

Since the kinematical factor ϕ/t and t behave like one power of s (or u) at fixed u (or s), we can use the same function $\tilde{B}$ introduced for photoproduction [Eq. (2.6)]. The Reggeized amplitudes are therefore given by

$$\begin{aligned}A_{1\pm1}{}^{(0)}&=4e^2b^2[\tilde{B}(-\alpha_\pi(s),\,-\alpha_\pi(u))+\cdots],\\ A_{1\pm1}{}^{(2)}&=-2e^2b^2[\tilde{B}(-\alpha_\pi(s),\,-\alpha_\pi(u))+\cdots],\end{aligned}\tag{3.3}$$

where again these satisfy the proper decomposition of the amplitudes into Born terms plus background. This decomposition and the proper kinematical singularities are the content of the low-energy theorems.[5] The dots $(\cdots)$ indicate other trajectories (ω, ϕ, A_2, etc.) added on in the standard way. These terms will certainly break the helicity independence of the Born term. Similarly, terms should be included in the isoscalar-isoscalar amplitudes $A_{1\pm1}{}^{(s)}$ to represent B, ρ, etc., in the s and u channels.

It is interesting to note that this solution[6] is an $M=1$ parity-doublet pion trajectory (i.e., the leading Lorentz pole is $M=1$). The introduction of parity doublets is a common feature of the Veneziano model, but here the solution may be physically interesting, since the parity partner to the pion has no pole at $J=0$: Defining the s-channel parity-conserving amplitudes

$$F^{s\pm}(s,t)=\tfrac{1}{2}(sA_{1-1}\mp m_\pi{}^2A_{11})\,,\tag{3.4}$$

we observe that only the pion has a pole at $J=0$, since

$$F^{s\pm}\sim\tfrac{1}{2}e^2(s\mp m_\pi{}^2)b^2\Gamma(-\alpha_\pi(s))(-bu)^{\alpha_\pi(s)-1},\tag{3.5}$$

for $u\to\infty$ at fixed s. Note that the zero at $s=-m_\pi{}^2$ in the $\gamma_{\pi\pi}$ vertex is the zero considered in fits to photoproduction (see Ref. 2). The odd-signature pole has the interesting interpretation as an A_1 (at $J=1$) exchange-degenerate with the pion, but again there may be difficulty with parity doubling. There may also be $I=0$, 2 particles at the A_1 mass. This, of course, cannot be decided until models for $A^{(1)}$ are included (see Sec. IV).

It is amusing that there is a natural way of introducing the Pomeranchuk trajectory that couples at $t=0$ in the double-helicity-flip amplitude $A_{1-1}{}^{(0)}$. Provided that $1-\alpha_P(t)+\alpha_\pi(s)+\alpha_\pi(u)=0$ [i.e., $\alpha_P(0)=1$, where α_P is the Pomeranchukon], the symmetric function defined in Sec. II may be added to $A_{1-1}{}^{(0)}$ in the form $C_0S(1-\alpha_P(t),\,-\alpha_\pi(s),\,-\alpha_\pi(u))$ without changing the Born terms. This model for the Pomeranchuk trajectory closely resembles the model of Abarbanel *et al.*[7] except that we have as yet no determination of C_0. Finally, let us emphasize that we have not yet solved the problem of obtaining a sensible particle spectrum in all channels, but further work is proceeding on this problem and there is indication that bootstrap conditions[8] will emerge to fix the value of C_0, as well as other parameters.

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[4] D. Horn, California Institute of Technology Report No. CALT-68-131, 1967 (unpublished).

[5] H. D. I. Abarbanel and M. L. Goldberger, Phys. Rev. **165**, 1594 (1968); F. Arbab and R. C. Brower, Phys. Rev. **181**, 2124 (1969).

[6] R. C. Brower and J. Weis, University of California Radiation Laboratory Report No. UCRL-19222 (unpublished). The solution presented here yields an $M=0$ pion.

[7] H. D. I. Abarbanel, F. E. Low, I. J. Muzinich, S. Nussinov, and J. H. Schwarz, Phys. Rev. **160**, 1329 (1967). Our model has the additive fixed pole in the t channel at $J=1$ in $A_{1-1}{}^{(0)}$ which comes from the term $(1+C_0)B(-\alpha_\pi(s),\,-\alpha_\pi(u))$, where $-C_0$ is the strength of the Pomeranchuk coupling.

[8] For example, with the simplest parametrization of $A_{1-1}{}^{(I)t}$ [$I_t=0$ (Eq. (3.3) plus C_0S), $I_t=1$ (Eq. (4.5)), and $I_t=2$ (Eq. (3.3))], the condition of no $I=2$ resonances on leading trajectories leads to the determination $C_0=-\frac{3}{4}$. This predicts a reasonable total cross section $\sigma_{\gamma\pi}(\infty)$, with $\alpha_P'(0)=b\approx1$. On the other hand, this solution has an $I=0$ particle degenerate with the A_1 on the pion trajectory.

IV. CURRENT ALGEBRA AND FIXED POLES

In this section, we set ourselves the task of finding a Veneziano parametrization for the amplitudes involved in current-algebra sum rules, namely, helicity-flip-2 isospin-1 t-channel helicity amplitudes for scattering of massive charged photons from arbitrary targets. Explicitly, the s channel is $\gamma_a(q_1)+T(J)\to\gamma_b(q_2)+T(J')$, where a, b are the internal-symmetry indices of the currents of mass q_1^2 and q_2^2, and J is the spin of the target T. As in Secs. II and III, we shall pay relatively little attention in this discussion to problems encountered already at the purely hadronic level, that is, parity doubling, etc. Instead, we shall concentrate on the characteristic feature of these amplitudes—that they are not purely Regge.[9] In fact, they are usually assumed to involve a fixed pole[10] at the nonsense point $J_t=1$, the residue of which is a form factor.

We begin with the simplest case, namely, charged photons scattering off pions (or spinless nucleons). We seek, then, a function $T(s,t,q_1^2,q_2^2)$ with the following properties[11]:

(i) It is odd under $(s\leftrightarrow u)$ crossing.

(ii) It satisfies the Fubini–Dashen–Gell-Mann sum rule[12]

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}ds'A(s',t,q_1^2,q_2^2)=F(t)\,, \tag{4.1}$$

where A is the absorptive part of T in s. That is to say, as $s\to\infty$,

$$T\to\beta(q_1^2,q_2^2,t)(-s)^{\alpha_\rho(t)-2}+2F(t)/s\,, \tag{4.2}$$

where we have suppressed the odd-signature factor for the ρ trajectory, and $F(t)$ is the pion form factor. Moreover, at $q_1^2=0$, $q_2^2=t$, and at $q_2^2=0$, $q_1^2=t$, only the pion pole can contribute to the sum rule.

(iii) It is constructed entirely out of poles in s, t, and u (and the q's), with the correct polynomial residues. In particular, it must have no poles with spin less than 2 in the t channel. The spin of t-channel poles can be read from the partial-wave expansion (suppressing the q's)

$$T(t,z_t)=\sum_{J=2}^{\infty}e_{20}{}^J(z_t)(2J+1)F^J(t)\,,\quad e_{20}{}^J\sim P_J'' \tag{4.3}$$

while a pole of spin J in s or u carries a residue t^J.

At first it is convenient to specialize even further, to the case of $q_1^2=q_2^2=0$. One might try to guess such a function along the following line: If we were at $q_1^2=q_2^2=m_\rho^2$, that is, the purely hadronic process $\pi+\rho\to\pi+\rho$, a satisfactory functional form might be

$$B(2-\alpha_\rho(t),\ -\alpha_\pi(s))-(s\leftrightarrow u)\,. \tag{4.4}$$

But we cannot use this function directly at $q_1^2=q_2^2=0$, because (a) it has no fixed poles, and (b) much more than the pion pole contributes at $t=0$. We could try fixing these up by considering $tB(2-\alpha_\rho(t),\ 1-\alpha_\pi(s))+1/\alpha_\pi(s)$, which eliminates objections (a) and (b), and yields a constant form factor. On the other hand, adding a term like $F(t)/\alpha_\pi(s)$ would give pion "ancestors" for any nontrivial form factor. The path we choose is somewhat different. Consider (with obvious $s\leftrightarrow u$ symmetrization)

$$T(s,t)=b^2\left[\frac{t\Gamma(1-\alpha_\rho(t))\Gamma(-\alpha_\pi(s))}{\Gamma(2-\alpha_\rho(t)-\alpha_\pi(s))}+\frac{m_\rho^2}{\alpha_\pi(s)[1-\alpha_\rho(t)]}\right]$$
$$\sim\left[\frac{t}{t-m_\rho^2}B(2-\alpha_\rho(t),\ -\alpha_\pi(s))-\frac{m_\rho^2}{t-m_\rho^2}B(1,\ -\alpha_\pi(s))\right]. \tag{4.5}$$

Relative to (4.4), we have simply inserted a ρ pole and then subtracted it out. This T satisfies conditions (i)–(iii) above.[13] In particular, it has no double poles, and contains no spin-1 poles in t at all, not even ρ satellites. It is already normalized to unity at the pion poles, and yields

$$F(t)=[1-\alpha_\rho(0)]/[1-\alpha_\rho(t)]\,, \tag{4.6}$$

which is the (normalized) ρ-dominance pion form factor. Asymptotically, (4.5) has precisely the form of Eqs. (19) and (20) in Bronzan *et al.*[10]

What other functions can we add to (4.5)? Of course, we can add pure Regge terms like

$$\frac{t\Gamma(2-\alpha_\rho(t))\Gamma(1-\alpha_\pi(s))}{\Gamma(3-\alpha_\rho(t)-\alpha_\pi(s))}=tB(2-\alpha_\rho(t),\ 1-\alpha_\pi(s))\,, \tag{4.7}$$

which couples the pion trajectory except for the pion itself. This form may also be used to incorporate other s-channel trajectories, and such terms do not alter the form factor. Moreover, we can, in fact, write functions with much more general form factors. Consider the set of functions

$$B_{m0}=\frac{t\Gamma(m-\alpha_\rho(t))\Gamma(-\alpha_\pi(s))}{\Gamma(m+1-\alpha_\rho(t)-\alpha_\pi(s))}+\frac{(\rho_m)^2}{\alpha_\pi(s)[m-\alpha_\rho(t)]} \tag{4.8}$$
$$m\geqq 1$$

[9] K. Bardakci, M. B. Halpern, and G. Segrè, Phys. Rev. **158**, 1544 (1967).

[10] J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters **18**, 32 (1967); V. Singh, *ibid.* **18**, 36 (1967).

[11] Our normalization in this section is $T(q_1^2=q_2^2=0)=-(t/2e^2)\times A_{1-1}{}^{(1)}$, where $A_{1,-1}{}^{(1)}$ is the similar amplitude introduced in Sec. III. At $q_1^2=q_2^2=0$, then, our pion poles have unit residue.

[12] R. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1966); S. Fubini, Nuovo Cimento **43**, 1 (1966).

[13] After symmetrization, the amplitude (4.5) has fixed poles at all odd J_t below and including $J_t=1$. Notice that the pure Regge part of (4.5) is the $\tilde{B}$ function of Secs. II and III.

where ρ_m is the mass of the mth ρ satellite. Physically, these functions are constructed by inserting, then subtracting off, ρ satellites, just as we did for ρ itself in (4.5). Any linear combination of these functions can be taken for T, leading to a general form factor parametrized with ρ and ρ-satellite poles,

$$F(t)=\sum_m f_m \frac{m-\alpha_\rho(0)}{m-\alpha_\rho(t)}. \tag{4.9}$$

The f_m must then be determined by factorization (see below). Nor does this exhaust the set of interesting functions. Consider (with $m\geqq 1$, $n\geqq 1$)

$$B_{mn}=t\left[\frac{\Gamma(m-\alpha_\rho(t))\Gamma(n-\alpha_\pi(s))}{\Gamma(m+n+1-\alpha_\rho(t)-\alpha_\pi(s))}+\frac{1}{m-\alpha_\rho(t)}\frac{1}{\alpha_\pi(s)-n}\right], \tag{4.10}$$

in which the pion trajectory does not couple but which can be used to change $F(t)$. The form factors associated with each of these functions go like t^0 at large t, but linear combinations can easily be constructed with any desired asymptotic behavior. The leading trajectory can also be reinstated if desired. For example, instead of B_{m1}, take

$$t\left[\frac{t\Gamma(m-\alpha_\rho(t))\Gamma(1-\alpha_\pi(s))}{\Gamma(m+2-\alpha_\rho(t)-\alpha_\pi(s))}+\frac{(\rho_m)^2}{m-\alpha_\rho(t)}\frac{1}{\alpha_\pi(s)-1}\right]. \tag{4.11}$$

Such construction gives functions which affect both the leading couplings and the form factor, but are different from (4.8).

$q^2\neq 0$ and Factorization

If there is to be hope of eventually finding restrictions on form factors, we must extend our scheme both (a) to q_1^2 and q_2^2 nonzero, and (b) to, say, the entire π trajectory as target. Whether factorization will really determine the F's is beyond the scope of this paper, but we can suggest how one might begin to ask such questions.

We first discuss the $q^2\neq 0$ problem. We can easily write a solution to the sum rule for nonzero q_1^2, q_2^2 which has the ρ-dominance pion form factor F [Eq. (4.6)] consistently. Consider, for example,

$$T(s,t,q_1^2,q_2^2)=\{T(s,t)+b[1-F(q_1^2)F(q_2^2)]\times B(2-\alpha_\rho(t),\ -\alpha_\pi(s))\}-(s\leftrightarrow u), \tag{4.12}$$

in which a pure Regge model for the hadronic amplitude $\pi\rho\to\pi\rho$ has been added to the $q^2=0$ solution (4.5). The residue at the pion pole is then $F(q_1^2)F(q_2^2)$, consistent with $F(t)$ in the fixed-pole term, and so on. The solution for general $F(t)$ [Eq. (4.9)] is somewhat more involved. Correction terms (without the pion pole) are required to eliminate ancestors for $\alpha_\pi(s)=J>0$, and to satisfy the gauge-invariance condition at $q_1^2=0$, $q_2^2=t$ (and vice versa). One possible solution is

$$b^{-1}T(s,t,q_1^2,q_2^2)=T(s,t)+[1-F(q_1^2)F(q_2^2)]B(2-\alpha_\rho(t),\ -\alpha_\pi(s))+\sum_{m=1}^{\infty} f_m[F_m(q_1^2)F_m(q_2^2)-1]D_m(s,t), \tag{4.13}$$

where $\sum f_m=1$, and

$$T(s,t)=b\sum_{m=1}^{\infty} f_m B_{m0}(s,t),\quad F(t)=\sum_{m=1}^{\infty} f_m F_m(t),$$
$$F_m(t)=[m-\alpha_\rho(0)]/[m-\alpha_\rho(t)], \tag{4.14}$$
$$D_m(s,t)=B(2-\alpha_\rho(t),\ -\alpha_\pi(s))-B(m+1-\alpha_\rho(t),\ -\alpha_\pi(s)),$$

and many other forms may be guessed (see below). A more transparent form for this amplitude is

$$b^{-1}T(s,t,q_1^2,q_2^2)=[F(t)-F(q_1^2)F(q_1^2)]B(2-\alpha_\rho(t),\ -\alpha_\pi(s))+F(t)/\alpha_\pi(s)+\sum_{m=1}^{\infty} f_m[F_m(q_1^2)F_m(q_2^2)-F_m(t)]D_m(s,t). \tag{4.15}$$

Our method of construction of these forms may be interesting in its own right. Break T up into three terms $b^{-1}T=T_D(s,t)+F(t)/\alpha_\pi(s)+T_H(s,t,q_1^2,q_2^2)$, where T_H contains the hadronic scattering terms (ρ_n,ρ_n') and is purely Regge. At $q_1^2=q_2^2=0$, assume that only the first two terms have the pion pole so that the pionic residue of T_H must be $F(q_1^2)F(q_2^2)-1$ for factorization. T_D is defined by $T_D(s,t)\equiv -T_H(s,t,0,t)$, thus guaranteeing gauge invariance. This also guarantees no ancestors as long as $T_H(s,t,0,t)$ has no singular residues above $\alpha_\pi=0$. Starting, then, with

$$T_H(s,t,q_1^2,q_2^2)=[1-F(q_1^2)F(q_2^2)]B(2-\alpha_\rho(t),\ -\alpha_\pi(s))+\sum_{m=1}^{\infty} f_m F_m(q_1^2)F_m(q_2^2)D_m(s,t), \tag{4.16}$$

we recover the above forms. Other terms may be added to T_H as long as (a) they have no pion pole, and (b) their $q_1^2=0$, $q_2^2=t$ form has no singular residues.

On the other hand, the function (4.15) does not yet factorize at higher pion recurrences. This can be fixed systematically by hand. Suppose we consider the first recurrence of the π, say, π', and we are given some $F_{\pi\pi'}$. Then we could add to (4.15) another term $(t-q_1^2-q_2^2)\times f(q_1^2,q_2^2)B_H'$, where B_H' is an (ordinary) β function which couples π' and higher, but not π [e.g., $B_H'=B(2-\alpha_\rho(t),\ 1-\alpha_\pi(s))$]. Thus, we do not disturb the pion pole. Now $f(q_1^2,q_2^2)$ can be solved for trivially by requiring that the π' residue is $F_{\pi\pi'}(q_1^2)F_{\pi\pi'}(q_2^2)$. This procedure can be continued to any finite number of particles, for any given set of form factors. Of course, we have not yet used π', etc., as targets.

Another interesting feature of (4.15) is that, even with $F(t)$ rapidly falling, the large q^2 dependence of the ρ-trajectory residue is independent of q^2; that is, it does not behave as a form factor.[14] It is not clear how seriously to take this; it may change if complete factorization is required. In order to muse further over factorization, we need models for the whole trajectory as target.

Going back to $q_1^2=q_2^2=0$, we can imagine constructing such models just as we did above. Suppose we have the helicity-flip-2 isospin-1 part of the six-point function[15] for a (hadronic) process like $\rho+\pi+\sigma\to\rho+\pi+\sigma$, say, $B_6(2-\alpha_\rho,\cdots)$. Then we could construct our model by going to the poles in the $\pi\sigma$ channels of the function

$$C_1\left[\frac{t}{t-m_\rho^2}B_6(2-\alpha_\rho,\ \cdots)-\frac{m_\rho^2}{t-m_\rho^2}B(1,\cdots)\right]. \quad (4.17)$$

The second term is a "fixed pole" for the six-point function proportional to $\langle\pi\sigma|J_{a\mu}|\pi\sigma\rangle$. This procedure will yield ρ-dominance form-factor models for all form factors $F_{JJ'}$ in all the reactions $T(J)+\gamma_a\to T(J')+\gamma_b$, where J and J' are the angular momenta along the pion trajectory. Now, of course, the parameter C_1 will not be enough even to normalize all the F_{JJ}, and the factorization problems begin. A set of B_6 functions analogous to the B_{mn} can evidently be constructed, and one can try to tackle the $q^2\neq0$ problem.[16,17]

Fixed Poles in s Channel

One last comment is of interest. The functions constructed for the simplest example above have fixed poles in s (as well as t). For example (4.5) has a fixed pole at $J_s=-1$. Using the B_{mn}, it is a simple matter to construct models with the fixed pole at $J_s=-n$ (and even to push n to infinity). In this model, there is a curious correlation between this fixed pole and the asymptotic behavior of the form factor: If we push J_s back to $-n$, then the associated form factor falls off like t^{-n}.

Mandelstam[18] has given two separate arguments of relevance to this situation, namely, (a) that there should be no fixed poles in s for such processes, and (b) that the form factor should fall faster than any power. Here the two arguments are correlated. If we wanted to take these arguments very seriously, we might insist on parametrizing the problem only with $F(t)$ that decrease faster than any power. Even demanding that the form-factor sums are absolutely convergent, an infinite number of such functions can be constructed.[19] On the other hand, we feel it would be nicer to see all this come about through factorization.

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[14] For the opposite view, see H. Harari, Weizmann Institute Report (unpublished). Harari is led to conjecture that ordinary trajectory residues (like ρ) fall off rapidly (like form factors) for large q^2. Thus he must introduce q^2-dependent modifications of various sum rules, including that of Fubini, Dashen, and Gell-Mann.

[15] H. M. Chan, CERN Report (unpublished); C. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969); K. Bardakci and H. Ruegg, University of California, Berkeley, Report (unpublished).

[16] In particular, note that we cannot use the $\langle\pi\sigma|J_{a\mu}|\pi\sigma\rangle$ term directly as we did in obtaining (4.12). The square of this fixed-pole term would only be expected as residue in a two-particle ($\pi\sigma$) elastically unitary approximation to the six-point function. By the same token, it may be useful in some more sophisticated (unitary) model.

[17] The factorization scheme outlined around Eq. (4.15) changes the purely hadronic amplitude $\pi\rho\to\pi\rho$ (at the ρ poles when $q_1^2=q_2^2=m_\rho^2$), and does not seem to determine form factors very well. We might imagine another scheme along the following lines. Again we limit ourselves to the pion and its first few recurrences (say, π and π'). Suppose now that the purely hadronic processes $\pi\rho\to\pi\rho$, $\pi\rho\to\pi'\rho$, and $\pi'\rho\to\pi'\rho$ are already known. Then we cannot add further terms like $f(q_1^2,q_2^2)$, and we would have three equations like (4.15) with only $F_{\pi\pi}$, $F_{\pi\pi'}$, and $F_{\pi'\pi'}$ unknown. Factorization will then put severe restrictions on the f_m of the form factors. Such a scheme is like that of R. F. Dashen and S. C. Frautschi [Phys. Rev. **145**, 1287 (1966)] in assuming the hadronic process known, but differs in that local current algebra would be required (via the sum rules) as a constraint. In any case, we would probably not want to use any satellites as target, considering them as a mockup of continuum. With this philosophy, one might even allow the satellites to have high isospin.

[18] S. Mandelstam, *1966 Tokyo Summer Lectures in Theoretical Physics* (W. A. Benjamin, Inc., New York, 1967).

[19] D. Atkinson and M. B. Halpern, Phys. Rev. **163**, 1611 (1967).