Validity of Approximations Used in Mesonic Atom Calculations*

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The Deser-Goldberger-Baumann-Thirring (DGBT) formula $[a=\frac{1}{4}B(\Delta E_{1S}/|E_{1S}|)]$ and other various approximations commonly used to extract meson-nucleus scattering lengths from energy-level measurements on mesonic atoms are studied by means of a model calculation and are shown to be inadequate for most of the π^--mesonic atoms commonly considered. A simple empirical formula is obtained in place of the DGBT formula for the π^--mesonic atoms of the light nuclei. The DGBT formula is also examined for the case of the K^--He^4 atom and is found to be a good approximation.

I. INTRODUCTION

In recent years several measurements[1–3] of the transition energies in mesonic atoms have been made using new x-ray detectors such as bent-crystal and Ge solid-state detectors. These measurements are more accurate and include more nuclei than the ones made about a decade ago.[4] The energy spectra of the mesonic atoms obtained in these measurements contain information about the strength of the meson-nucleus strong interaction, which shows up as deviations of the measured energy spectra from the expected electromagnetic energy spectra. As a matter of fact, these deviations give the same information about the meson-nucleus interactions as is contained in the low-energy elastic π-nucleus phase shifts.

To analyze these mesonic atom data, several authors[5–8] have tried to obtain formulas relating these deviations (the energy-level shifts and spreads) to the low-energy scattering parameter, the scattering length.[9,10] The problem is hard and use of approximations is unavoidable, since it involves a Coulomb wave function which deviates only slightly from the regular Coulomb wave function. The approximations that have been used are first-order perturbation theory and Born approximation,[5] or somewhat equivalently, the boundary condition model,[7] and an approximation based on formal scattering theory.[8] Errors involved in these approximations are difficult to estimate and could be quite appreciable: In particular, the small energy-level shifts and spreads observed do not guarantee that perturbation theory is a good approximation. Small deviation in eigenenergy is only a necessary condition for validity of the approximation.

In a previous paper[11] (hereafter referred to as SC), we made a detailed optical-model calculation to obtain accurate meson-nucleus scattering lengths from the mesonic atom data. In this calculation the Coulomb wave function at an energy $E_{1S}+\Delta E_{1S}$ was expanded in powers of $\epsilon=\Delta E_{1S}/|E_{1S}|$. [Here E_{1S} is $1S$ energy level of the meson in a point Coulomb potential and ΔE_{1S} is the (complex) level shift.] This calculation can be done to any order of accuracy desired and the calculations in SC, which were done to order ϵ^3, were estimated to give errors in ΔE_{1S} well within the experimental errors. Results obtained by this procedure will be referred to as SC results and are considered to be essentially exact results for the purpose of determining the accuracy of other approximations.

It was found that the SC calculation gave values of the probability that the π^- meson exists inside the nucleus, which is only about half the value given by a perturbation calculation. This shows that perturbation theory is not a good approximation. Nevertheless, the formula (hereafter referred to as the DGBT formula)

$$a/B=\tfrac{1}{4}\Delta E_{1S}/|E_{1S}|\,, \tag{1}$$

originally obtained by Deser *et al.*,[8] has still been used to estimate the meson-nucleus scattering length a from the energy-level shift and spread ΔE_{1S}. (Here B is the

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[9] Application of effective-range theory gives a simple formula for this relation (Ref. 10). The scattering length *defined* in the effective-range theory, however, differs from the one defined in the absence of the electromagnetic interaction. The latter definition of the scattering length is used in this paper and in Ref. 11, since we are interested in extracting information about the π-nucleus interaction when the point Coulomb interaction is turned off.

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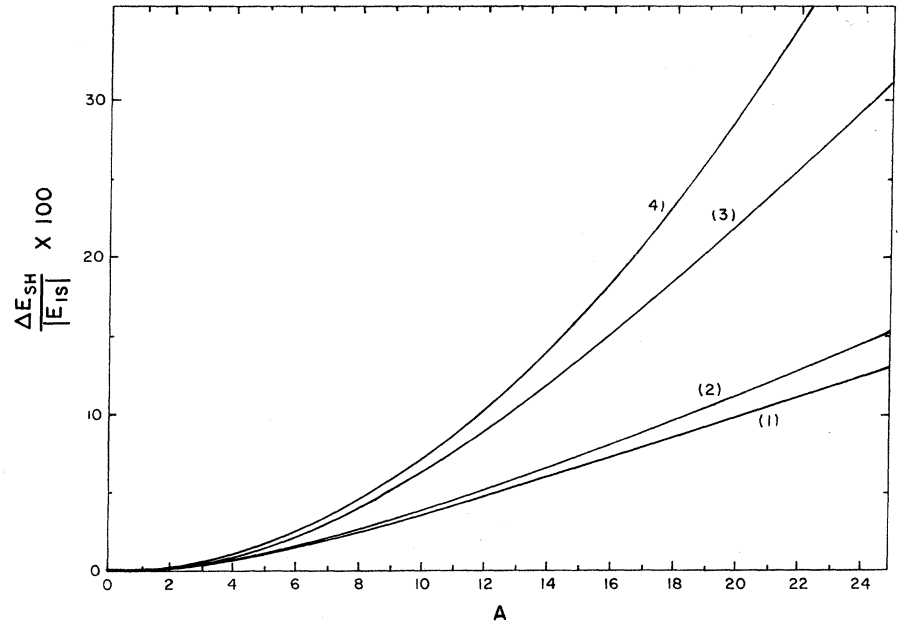

FIG. 1. Percent energy-level shift in the model calculation for $Z=N$ nuclei. The shift is calculated for a π^--nucleus strong interaction potential $V_R=10$ MeV and $R=1.3A^{1/3}$ by use of the SC method with (method 1) or without (method 2) including the effect of the finite charge distribution, and by use of perturbation theory (method 3), and the semiperturbation theory (method 4).

first Bohr radius of the system.) To establish more firmly the validity of this approximation and others mentioned before, we have made a model calculation of the π-mesonic atom in Sec. II and have also examined the K^--He^4 atom case in Sec. III. It is rather important in future analyses of the mesonic atom problems to establish the quantitative validity of these approximations, since more accurate measurements of the energy spectra of the atoms are expected to be made in the new meson-producing installations which are now planned.

II. MODEL CALCULATION FOR π^--MESONIC ATOMS

We start by generating the real energy-level shifts (i.e., with no spreads) of the π^--mesonic atoms from a fixed square-well potential,

$$V_R=10 \text{ MeV}, \quad R=1.3A^{1/3} \text{ F},$$

for $N=Z$ nuclei, using several approximations and the calculation used in SC. This model potential is chosen such that the energy-level shifts calculated from it by

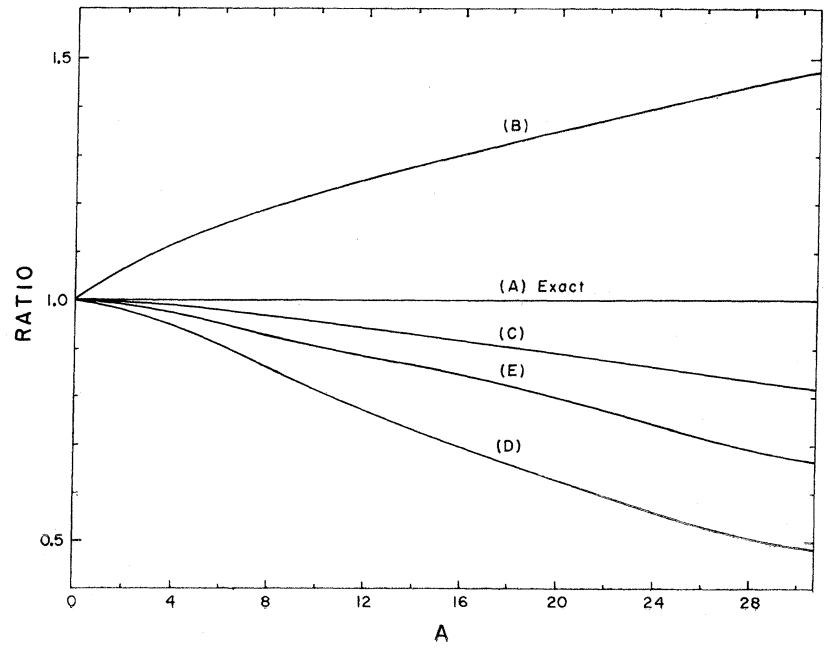

FIG. 2. Ratio of the π^--nucleus scattering lengths in the model calculation for $Z=N$ nuclei. The ratios are the scattering lengths calculated from the SC energy-level shift, method 1 in Fig. 1, in several approximations divided by the scattering length calculated exactly. These approximations are: method B, Born approximation; method C, the SC method, but neglecting the effect of the finite charge distribution; method D, the DGBT formula; method E, perturbation theory and Born approximation.

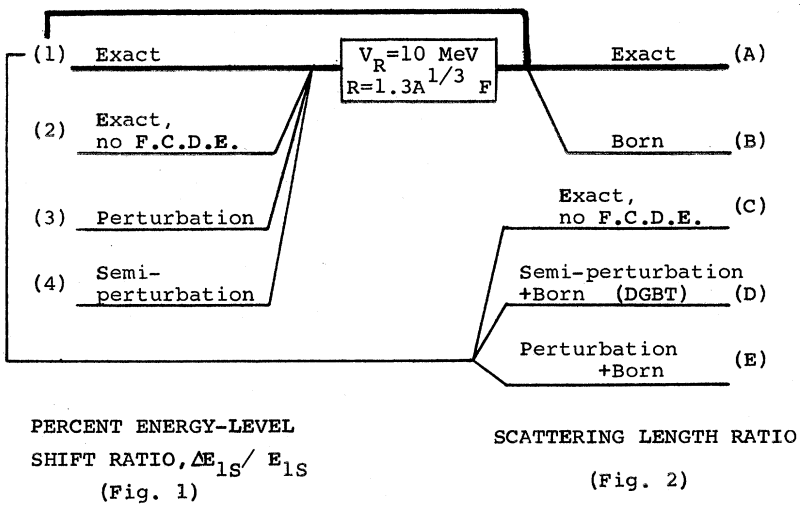

FIG. 3. Schematic diagram of the model calculation. The percent energy-level shift ratios $\Delta E_{1S}/|E_{1S}|$ with the methods 1–4 are shown in Fig. 1 and the π^--nucleus scattering-length ratios with methods A–E are shown in Fig. 2. Thick lines in the diagram imply that calculations are made using SC; the values of two quantities connected by the thick lines are obtained by a very good approximation. Thin lines imply that calculations are made with the labeled approximations. "Exact" refers to the SC method.

ıe methods of SC are in close agreement with the real ɛperimental data. The choice of the square-well shape the potential should not alter any major conclusions this model calculation, since, as was shown in SC, ɛtermination of the scattering length from the ıesonic atom data is independent of the detailed shape the potential, just as it is in the case of the low-ıergy nucleon-nucleon scattering analysis.

The calculation methods used are the following:

(1) the SC method including correctly the finite ıarge distribution of the nuclei;

(2) the SC method without the finite charge dis-ibution;

(3) first-order perturbation theory,

$$\Delta E_{1S}\sim\int\phi_{1S}^*(r)V_R\phi_{1S}(r)d^3r$$
$$=V_R\{1-[2(R/B)^2+2(R/B)+1]\times\exp(-2R/B)\};\quad(2)$$

(4) semiperturbation theory,

$$\Delta E_{1S}\sim|\phi_{1S}(0)|^2\int V_R d^3r=\tfrac{4}{3}V_R(R/B)^3.\quad(3)$$

Here $\phi_{1S}(r)=(B^3\pi)^{-1/2}\exp(-r/B)$ is the unperturbed bound-state Coulomb wave function. Note that if we neglect terms of order $(R/B)^4$ and higher in Eq. (2),

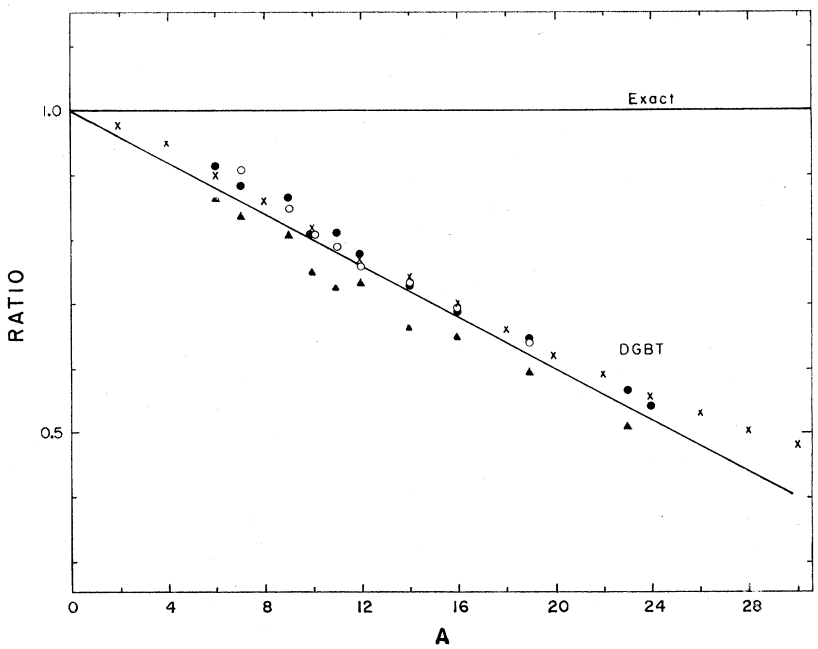

FIG. 4. Empirical relation between the DGBT formula and the SC ("exact") calculation. The ratio of the scattering lengths calculated by the DGBT formula to those calculated "exactly" are shown for the model-calculation shift (×), method 1 in Fig. 1, the Stearns and Stearns data (Ref. 4) for the level shift (open circles), and the Jenkins *et al.* data (Ref. 1) for the level shift (closed circles) and spread (▲). An empirical line $1-0.02A$ shown in the figure fits all these points reasonably well. This line yields the empirical relation between the scattering length and the energy-level shift spread, Eq. (7), in the text.

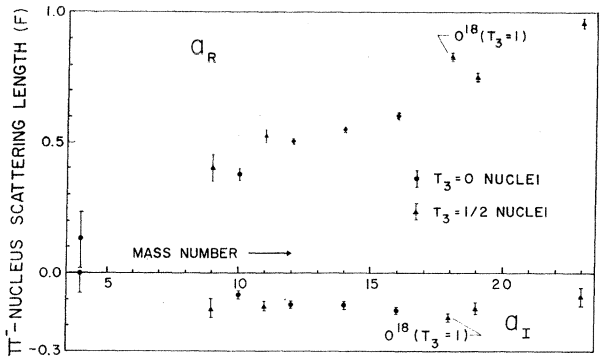

FIG. 5. An analysis of the π^--mesonic atom data from CERN (Ref. 2) and the π^--He^4 data from Virginia (Ref. 3), using the empirical relation Eq. (7). a_R and a_I are the real and imaginary part of the π^--nucleus scattering length, respectively.

methods 3 and 4 become identical. Method 4 is used to obtain the DGBT formula, Eq. (1), by combining it with Born approximation for the scattering length,

$$a\sim\frac{\mu}{2\pi\hbar^2}\int V_R d^3r, \tag{4}$$

where μ is the reduced mass of the π-nucleus system.[5]

The results of the calculations are shown in Fig. 1. From this figure we see that the perturbation calculation yields larger values of the energy-level shifts than the exact calculation by a factor of about 2. This corresponds to the conclusion obtained in SC concerning the probability of the π^- meson being inside of the nucleus, as mentioned in Sec. I. Note that the semiperturbation theory, the one used to obtain the DGBT formula, yields a larger deviation from the exact value than does the full perturbation theory.

Next, we proceed to calculate the π^--nucleus scattering lengths with various approximations from the energy-level shifts shown in Fig. 1. In this way we can examine the validity of the different approximations in various combinations.

If we start with the exact energy-level shifts (curve 1 in Fig. 1) and calculate the potential, including the finite charge distribution, using the SC method, we get back to the model potential $V_R=10$ MeV. Then using the exact relation between V_R and a,

$$a=R-(1/\beta)\tanh(\beta R), \tag{5}$$

where $\beta=(2\mu V_R/\hbar^2)^{1/2}$, we obtain the exact scattering length: model A. Alternatively, we can use Eq. (4) and calculate the Born approximation to a: model B. Also we can calculate a using these exact energy-level shifts but neglecting the finite charge distribution: model C. That is, we use the energy-level shifts of curve 1 in Fig. 1, and generate potentials V_R using the formalism of SC, but without a charged distributed over the nucleus. These potentials will differ from the model potential, $V_R=10$ MeV, and will generate a different set of scattering lengths. Next, we calculate a using the DGBT formula, Eq. (1), from the exact energy-level shifts, model D (curve 1 in the Fig. 1). Alternatively in model E, we calculate a using the formula

$$a/B\sim\tfrac{1}{3}\gamma^3[1-(1+2\gamma+2\gamma^2)e^{(-2\gamma)}]^{-1}\Delta E_{1S}/|E_{1S}|$$
$$=\tfrac{1}{4}(\Delta E_{1S}/|E_{1S}|)[1+\tfrac{3}{2}\gamma+(21/20)\gamma^2+\cdots], \tag{6}$$

where $\gamma=R/B$. This formula is obtained by combining perturbation theory, Eq. (2), and Born approximation, Eq. (4), while the DGBT formula is obtained by combining the *semiperturbation* theory, Eq. (3), and Born approximation, Eq. (4), as mentioned before.

These various model calculations of the scattering length a (A, B, C, D, E) are shown in Fig. 2. To examine the validity of the various approximations more clearly, we plot the ratio of each of these model results to A, the exact value.

To clarify the procedure explained above, we list the approximations involved in each scattering length calculation:

(A) SC with finite charge distribution, which we refer to as the "exact" calculation,

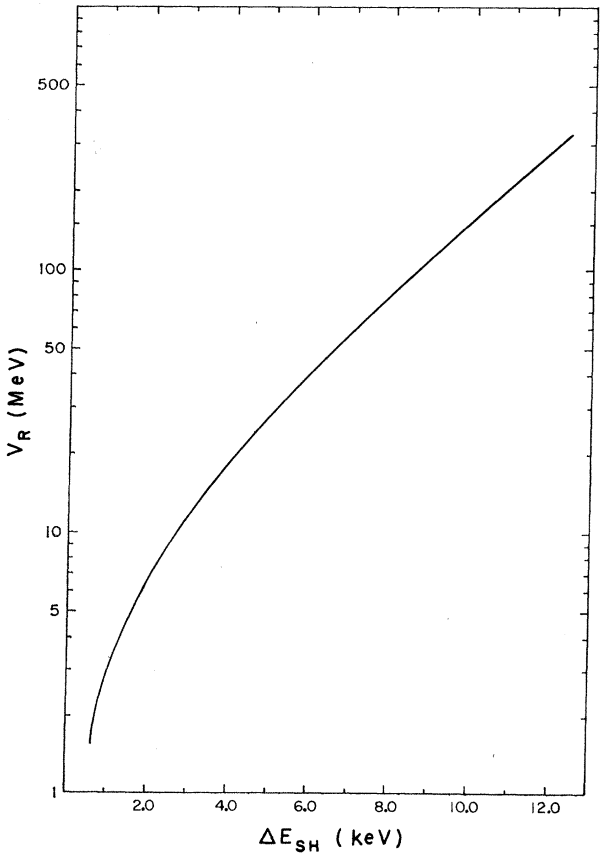

FIG. 6. The K^--He^4 strong-interaction potential as a function of the 1S energy-level shift ΔE_{SH}. The radius of the potential is taken to be 2.19 F. The curve is shown for positive level shifts, $\Delta E_{SH}>0$, corresponding to a positive potential and repulsive interaction.

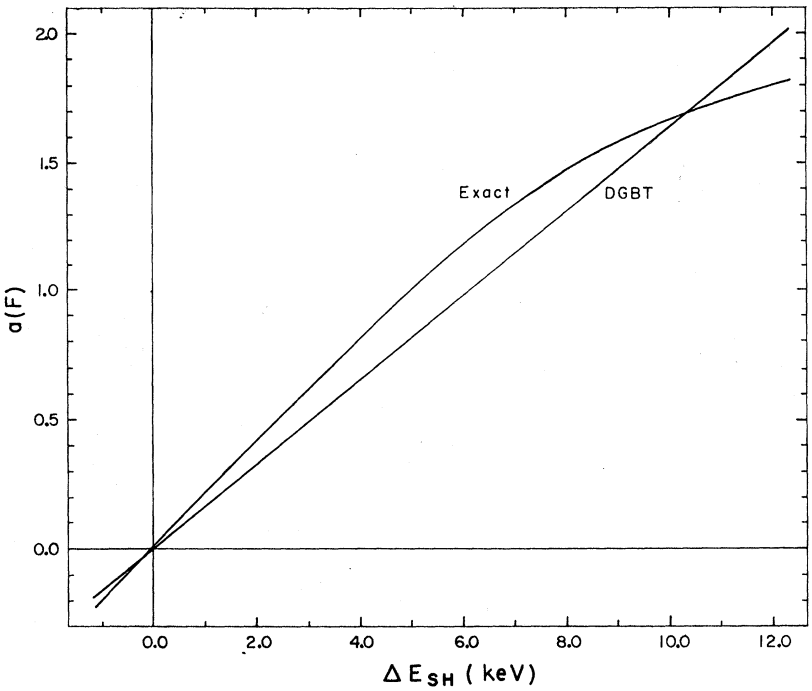

FIG. 7. K^--He^4 scattering length as a function of the $1S$ energy-level shift ΔE_{SH}. The contribution of the finite charge distribution in the nucleus to the shift is taken to be 0.17 keV, which is calculated exactly for a uniform charge distribution with a radius of 2.19 F. The scattering length is calculated exactly and by the DGBT formula.

(B) Born approximation,
(C) SC, neglecting the finite charge distribution,
(D) semiperturbation and Born approximation (the DGBT formula),
(E) perturbation theory and Born approximation.

Figure 2 shows that the DGBT result (curve D) deviates from curve A by more than 20% for nuclei with $A \geq 9$ and, thus, is inadequate for treating such nuclei. Furthermore, Fig. 2 shows that curve D gives a larger error than curve E and that this difference increases with A. Thus, calculation E (perturbation theory) is better than calculation D (DGBT), though neither gives really reliable results.

From Fig. 1, we see that the semiperturbation results (curve 4) give the worst values of the energy-level shifts, and from Fig. 2, we see that the Born approximation (curve B) gives the worst values of the scattering lengths. The DGBT formula, Eq. (1), states that these two quantities are proportional. When this formula is used, the errors in these two approximations partially cancel each other and give a smaller error than the errors of either the semiperturbation or the Born approximation, separately. As seen in Fig. 2, the DGBT formula and Eq. (6) give smaller scattering lengths than the exact calculation. Judging from this, we see that the semiperturbation and the perturbation calculations give a larger error than the Born approximation.

Summarizing this section, we show a schematic diagram of the model calculation made in this section in Fig. 3. Thick lines in the figure imply that calculations are made exactly (that is, but the method of SC); the values of two quantities combined by the thick lines are related, to a very good approximation. Thin lines imply that calculations are made with the less accurate approximations. The meanings of the abbreviated labels in the figure are clear from the content of this section.

III. APPLICATION OF THE MODEL CALCULATION

In the same way as curve E in Fig. 2 was obtained, we have also obtained the ratio of the π^--nucleus scattering length which was calculated by using the DGBT formula to those calculated using SC from the experimental data of Stearns and Stearns[4] (shift only) and of Jenkins *et al.*[1] (shift and spread). The nuclear radii were taken from the high-energy electron scattering results.[12] It is seen from the results shown in Fig. 4 that both the real and the imaginary parts of a fall along the straight line $1-0.02A$, so that we have the approximate result

$$a=(1-0.02A)^{-1}(\tfrac{1}{4}B)(\Delta E_{1S}/|E_{1S}|) \qquad (7)$$

for nuclei with $4 \leq A \leq 25$. This empirical formula should be useful for rapid analysis of future experiments of the $1S$ energy-level shifts *and* the spread in π^--mesonic atoms.

As an example of the application of this formula, we have obtained the π^--nucleus scattering lengths from the recent π-mesonic atom data from CERN[2] and Virginia.[3] The results are shown in Fig. 5.

[12] R. Herman and R. R. Hofstadter, *High Energy Electron Scattering Tables* (Stanford University Press, Stanford, 1960).

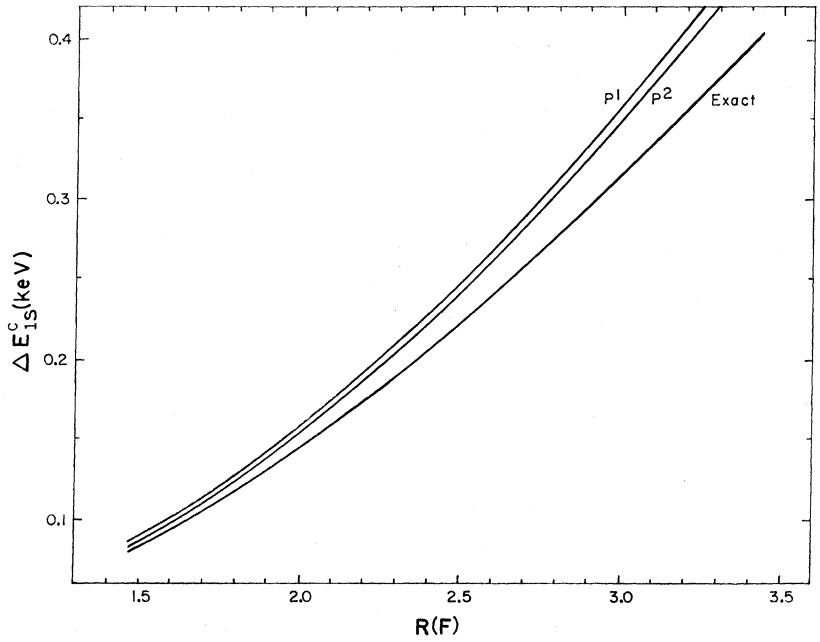

FIG. 8. The contribution to the shift of the finite charge distribution, $\Delta E_{1S}{}^c$, in the K^--He^4 atom. A uniform charge distribution is assumed. The contribution is calculated exactly, by perturbation theory, Eq. (8) in the text (the curve P1), and by the semiperturbation theory, Eq. (9) in the text (the curve P2).

IV. CASE OF THE K^--He^4 ATOM

The K^--He^4 atom is the only K^--mesonic atom whose energy spectrum has been measured. The Argonne group reported the upper limit of the $1S$ energy-level shift to be 1.0 keV and that of the spread ($\mathrm{Im}E_{1S}$) to be 0.7 keV.[13] These values correspond to an upper limit on the scattering length of $0.22-i0.17$ F, as we will show later; this is quite small considering the large values of the K-nucleon scattering lengths.[14] A new measurement is underway to clarify this situation.[15]

In this section, we compare the K^--He^4 scattering lengths calculated by the methods of SC and by the DGBT formula. Because of the somewhat ambiguous situation mentioned above, the comparison is made for energy-level shift from -0.5 keV to 12.0 keV. In the SC calculation the square-well potential is again used for the reason mentioned in Sec. II. The range of the square well is set to be 2.19 F, the value obtained from high-energy electron scattering data.[12] The effect of the finite charge distribution is calculated to be 0.17 keV for a uniform charge distribution with the same radius as the square well. The results are shown in Figs. 6 and 7. Figure 6 shows the real part of the K^--He^4 strong-interaction potential as a function of the energy-level shift for the given well radius. The height of the potential depends strongly on the level shift; a change in the energy-level shift from 1 to 5 keV corresponds to a change in the potential height from 3 to 29 MeV. In Fig. 7, we show the scattering length calculated from this potential and also calculated using the DGBT formula. We see that the DGBT result deviates by, at most, 15% from the SC result. This is similar to the π-mesonic atom case (Fig. 2), where for $A=4$ the DGBT result (curve D) deviated by only 10% from the SC result. Note that this does not guarantee that perturbation theory or Born approximation give individually a small error because the errors in these approximations partially cancel each other in the DGBT formula, as shown in Sec. II.

If the level shift and spread are as small as the one measured, electric effects other than the point Coulomb one would become important in the observed level shift. We have made a comparison of the effect of the finite charge distribution in this atom using the perturbation calculation and the SC calculation. The result is shown in Fig. 8. Perturbation theory gives the effect of the uniform charge distribution $(Ze^2/2R)\times[(r/R)^2-3]$ $(r\leq R)$ on the $1S$ energy level $\Delta E_{1S}{}^c$ to be

$$\Delta E_{1S}{}^c=(2Ze^2/5R)\eta^3[1-(5/6)\eta+3\eta^2-\cdots],$$

where $\eta=ZR/B$; for most of the mesonic atoms in light nuclei, we have $\eta<1$. Curve P1 in Fig. 8 is calculated using the leading term of the above expression,

$$\Delta E_{1S}{}^c\sim(2Ze^2/5R)\eta^3. \tag{8}$$

This was obtained by Cooper and Henley[16] and is frequently used because of its simple form. Curve P2

[13] G. R. Burleson, D. Cohen, R. C. Lamb, D. N. Michael, R. A. Schluter, and T. O. White, Phys. Rev. Letters **15**, 70 (1965); D. N. Michael, Ph.D. thesis, University of Chicago, 1966 (unpublished).

[14] The average value of the K^--proton and K^--neutron scattering lengths is about $0.6-i0.5$ F. [M. Sakitt *et al.*, Phys. Rev. **139**, B719 (1965); J. K. Kim, Phys. Rev. Letters **14**, 29 (1965). See also J. K. Kim, *ibid.* **19**, 1074 (1967)].

[15] R. A. Schluter (private communication).

[16] L. Cooper and E. Henley, Phys. Rev. **92**, 801 (1953).

is calculated with the next term taken into account,

$$\Delta E_{1S}{}^{c}\sim(2Ze^2/5R)\eta^3[1-(5\eta/6)]. \quad (9)$$

As seen in the Fig. 8, this correction is small ($\sim 3\%$) in this atom, and higher corrections within this first-order perturbation theory are not needed here. The result of the exact calculation is also shown in this figure. The error introduced by use of these approximations is about 10%, or less than 0.05 keV for 2.0 F$<R<$3.0 F, and could be neglected considering the current experimental error.

It has been shown that the vacuum-polarization effect is quite important in mesonic atoms of the light nuclei.[17] In analyses of future experiments, this effect should be examined more closely by taking into account the effect of the finite size of the nucleus on the vacuum polarization.

ACKNOWLEDGMENT

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[17] D. D. Ivanenko and G. E. Pustovalov, Usp. Fiz. Nauk **61**, 27 (1957) [English transl.: Soviet Phys.—Usp. **63**, 1043 (1961)].

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Veneziano Parametrization for Nonstrong Amplitudes*

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Veneziano-like parametrizations are found for photoproduction, Compton scattering, and current algebra.

I. INTRODUCTION

RECENTLY, Veneziano[1] gave a simple parametrization for hadronic amplitudes which is crossing-symmetric, displays Regge behavior, and satisfies duality. It is our purpose in this paper to find a similar parametrization for nonstrong amplitudes, for example, photoproduction, Compton scattering, and current algebra—thus putting the nonstrong problem on an equal footing with the strong. In general, we shall concentrate on the features that distinguish these amplitudes from purely hadronic ones—namely, double poles (gauge invariance, low-energy theorems, etc.) and fixed poles (in angular momentum, with nontrivial form factors). On the other hand, we shall not go into much detail about problems that our parametrization shares with the purely hadronic problem—i.e., parity doubling, isospin degeneracies, factorization, etc.

In Sec. II, we construct a Veneziano parametrization for a photoproduction amplitude that has Regge asymptotic behavior in all channels and gives the correct Born approximation (low-energy theorems) at low energies. In Sec. III, the same technique is used to construct the amplitudes for physical (chargeless photons) Compton scattering ($\gamma\pi\rightarrow\gamma\pi$). A natural solution yields an $M=1$ pion with a parity partner that decouples at $J=0$. Moreover, we suggest a natural scheme for introducing into the double-helicity-flip amplitude a Pomeranchukon which couples in the forward direction. Section IV treats current-algebra amplitudes and fixed poles in the t channel. We find solutions to current-algebra sum rules with form factors parametrized by ρ and ρ-satellite poles. In this parametrization, a correlation exists between fixed poles in the s channel and asymptotic behavior of form factors. The correlation may be taken to mean that form factors must fall faster than any power of t.

The problem of combining the results of Secs. III and IV into a good phenomenological description of Compton scattering is presently under investigation.

II. PHOTOPRODUCTION

Here we investigate a Veneziano parametrization for photoproduction amplitudes. In particular, we are interested in those special features associated with the zero mass of the photon and required by gauge invariance. The pole terms for soft-photon couplings (called Born terms in this paper) are in fact the main issue, since gauge invariance requires them to have a particular form.[2] As emphasized in Ref. 2, the photon, unlike a massive vector particle, can couple to a nonsense pole if the internal mass is the same as the external mass (soft-photon coupling). On the other hand, the hard-photon couplings (other poles to which the soft photon does not couple) obey the usual selection rules of angular

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[2] F. Arbab and R. C. Brower, Phys. Rev. **178**, 2470 (1969); R. C. Brower and J. Dash, *ibid.* **175**, 2014 (1968).