PHYSICAL REVIEW VOLUME 182, NUMBER 5 25 JUNE 1969

Regge-Pole Inelasticity, Partial Waves, and a Model for the S Matrix

ROBERT W. CHILDERS
Department of Physics, University of Tennessee, Knoxville, Tennessee 37916

AND

ARTHUR W. MARTIN*
Department of Physics, Rutgers, The State University, New Brunswick, New Jersey 08903
(Received 18 October 1968)

The asymptotic behavior of partial-wave amplitudes in the physical region, as deduced from Regge theory, is used to calculate the inelasticity functions and partial-wave driving forces at large energies for pion-pion and pion-nucleon scattering. The results provide useful information for N/D calculations. The Regge-pole inelasticity functions possess a special property; they are asymptotically (in energy) independent of l. This implies that inelastic unitarity at large energies closely resembles the elastic-unitarity condition and suggests an approximate model for the two-body S matrix. The model incorporates Mandelstam analyticity, inelastic unitarity, and crossing symmetry, and leads to integral equations for the absorptive parts of the invariant amplitudes.

I. INTRODUCTION

It has become evident over the last few years that inelastic effects can be of major importance in partial-wave calculations. It is also clear that the larger problem of determining the invariant amplitudes $A(s,t,u)$ through the requirements of analyticity, unitarity, and crossing symmetry is not well defined without a statement of inelastic unitarity. In this paper we wish to point out that the asymptotic information provided by Regge-pole theory leads to models for inelastic effects that can be useful in both partial-wave and invariant-amplitude calculations.

In the N/D formalism[1,2] the inelasticity function must be given as input. In the low-energy region this information can be obtained from experiment, when available, or from models for the effect of coupled two-body channels.[3] At high energies such methods are impractical, but the Regge-pole phenomenology steps in to provide a simple prescription for the inelasticity function. We will use the statement of partial-wave inelastic unitarity

$$\operatorname{Im}A_l(s)=k(s)R_l(s)|A_l(s)|^2, \tag{1}$$

where $k(s)$ is the appropriate kinematical factor, depending upon the spins of the particles, and $R_l(s)$ is the ratio of the total to the elastic cross section in the lth partial wave.

The method for determining the asymptotic behavior of $R_l(s)$ from Regge-pole theory is transparent; one obtains the asymptotic form for $A_l(s)$ and plugs it into (1). This is done, with the necessary assumptions about Regge-pole behavior, for the cases of pion-pion and pion-nucleon scattering in Sec. II. The asymptotic behavior of $A_l(s)$, together with the assumed validity of partial-wave dispersion relations, also allows us to calculate the asymptotic behavior of the partial-wave "driving forces" in the physical region. This analysis is presented in Sec. III.

One interesting property of the Regge-pole prediction, and one that is easily understood, is that $R_l(s)$ is asymptotically (in energy) independent of l. This fact suggests a model for inelastic unitarity that, when coupled with analyticity assumptions and crossing symmetry, leads to integral equations for the invariant amplitudes (more precisely, the absorptive parts of the invariant amplitudes). We do not know at this point whether consistant solutions of the equations exist. Nevertheless, the structure of the integral equations is sufficiently simple that it should be possible to extract the basic properties of the theory without enormous computational labor. The development of this approximate S-matrix formulation is dealt with in Sec. IV. Section V is devoted to conclusions, and some mathematical details are presented in an Appendix.

II. REGGE-POLE INELASTICITY FUNCTIONS

As noted in the Introduction, the asymptotic behavior of the inelasticity function $R_l(s)$ follows immediately from partial-wave unitarity once the asymptotic behavior of $A_l(s)$ is known. The Regge-pole results for $A_l(s)$ were stated by Chew,[4] and derived with more attention to details by Squires[5] and more recently by Warnock,[6] so we will not go through the full calculation here. We do wish to outline the method, however, to emphasize the assumptions made and the physics behind the result.

In the case of pion-pion scattering we work with the invariant amplitudes $A^I(s,z)$, where $I=0$, 1, 2 denotes the isospin and z is the cosine of the c.m. scattering angle. The amplitudes are related to the partial-wave

* Research supported in part by the National Science Foundation.
[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).
[3] See, for example, J. J. Brehm and L. F. Cook, Phys. Rev. **170**, 1381 (1968).
[4] G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[5] E. J. Squires, Nuovo Cimento **34**, 1277 (1964).
[6] R. L. Warnock (to be published).

amplitudes through

$$A^I(s,z)=\sum(2l+1)P_l(z)A_l{}^I(s)\,,$$
$$A_l{}^I(s)=\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)A^I(s,z)\,. \quad (2)$$

The optical theorem reads[7]

$$\mathrm{Im}A^I(s,\,z=1)=(16\pi)^{-1}[s(s-1)]^{1/2}\sigma_{\rm tot}{}^I(s)\,,$$

and the partial-wave amplitudes satisfy the unitarity condition (1) with $k(s)=[(s-1)/s]^{1/2}$.

We assume that the Pomeranchuk trajectory dominates the elastic scattering at large energies, and we write the Regge amplitudes in the form

$$A^I(s,z)\xrightarrow[s\to\infty]{} C^I(t)\frac{1+\exp[-i\pi\alpha(t)]}{\sin\pi\alpha(t)}s^{\alpha(t)} +(-1)^I C^I(u)\frac{1+\exp[-i\pi\alpha(u)]}{\sin\pi\alpha(u)}s^{\alpha(u)}\,, \quad (3)$$

where t and u are the usual momentum-transfer variables. The optical theorem then gives

$$C^I(0)=-(16\pi)^{-1}\sigma_{\rm tot}{}^I(\infty)\,,$$

with the standard assumption $\alpha(0)=1$.

The asymptotic behavior of the partial-wave amplitudes follows from inserting (3) into (2) and carrying out the partial-wave projection. The result is[4–6]

$$A_l{}^I(s)\xrightarrow[s\to\infty]{}\frac{[1+(-1)^{l+I}]\sigma_{\rm tot}{}^I(\infty)}{16\pi\alpha'(0)\ln(s)}\times\left[i-\frac{\pi}{2\ln(s)}+\frac{i16\pi C'^I(0)}{\alpha'(0)\sigma_{\rm tot}{}^I(\infty)\ln(s)}+O[\ln^{-2}(s)]-\frac{il(l+1)}{\alpha'(0)s\ln(s)}\right], \quad (4)$$

where we have included the first l-dependent term to show that it is reduced by the factor $[s\ln(s)]^{-1}$ in comparison with the dominant term. It has been assumed that $\alpha'(0)>0$ for the Pomeranchuk trajectory, and that $\alpha(t)<1-\epsilon$ for $-(s-1)\leqslant t\leqslant-\delta(\epsilon)$, where ϵ and $\delta(\epsilon)$ are positive. This last assumption, that the amplitude at large energies is dominated by the forward and backward peaks, is in agreement with present experiment and with many theoretical considerations.[8]

The inelasticity function for the nonvanishing pion-pion partial-wave amplitudes is then found from (1) and (4) to be

$$R_l{}^I(s)\xrightarrow[s\to\infty]{}\frac{8\pi\alpha'(0)\ln(s)}{\sigma_{\rm tot}{}^I(\infty)}+O(\text{const})\,, \quad (5)$$

where the constant term is also independent of l. The first l-dependent term in the inelasticity goes as s^{-1}. The physics underlying the l independence of the dominant terms in (4) and (5) is easily seen. The Regge formula with its shrinking diffraction peaks becomes at large energies essentially δ functions in the forward and backward directions, $z=\pm1$, and the partial-wave projections of these "δ functions" are independent of l. Finally, we note that the l independence of the inelasticity function provides a particularly simple derivation of the result

$$\sigma_{\rm elas}{}^I(s)\xrightarrow[s\to\infty]{}\frac{[\sigma_{\rm tot}{}^I(\infty)]^2}{8\pi\alpha'(0)\ln(s)}\,.$$

The calculation for the case of pion-nucleon scattering is quite similar to the spinless case and we simply state the results. We work with the standard partial-wave amplitudes[9] $f_{l\pm}{}^I(w)$ which satisfy the unitarity condition (1) with $k(s)=q$, where q is the magnitude of the three-momentum in the c.m. system. The assumption of the dominance of the Pomeranchuk trajectory then gives

$$f_{l\pm}{}^I(w)\xrightarrow[w\to\infty]{}\frac{i\sigma_{\rm tot}{}^I(\infty)}{16\pi\alpha'(0)w\ln(w)}+O[w^{-1}\ln^{-2}(w)]\,, \quad (6)$$

where $\sigma_{\rm tot}{}^I(\infty)$ in (6) now refers to the total pion-nucleon cross sections in the isospin states $I=\frac{1}{2},\frac{3}{2}$. In this pion-nucleon case the first l-dependent term is smaller by a factor $[w\ln(w)]^{-1}$ than the leading term.

The asymptotic form of the inelasticity function is

$$R_{l\pm}{}^I(w)\xrightarrow[w\to\infty]{}\frac{32\pi\alpha'(0)\ln(w)}{\sigma_{\rm tot}{}^I(\infty)}+O(\text{const})\,,$$

and the first l-dependent term vanishes as w^{-1}. The inclusion of secondary trajectories with $\alpha(0)<1$ does not enhance the l dependence of the asymptotic formulas, nor does the inclusion of Regge branch cuts.[5,6] These asymptotic expressions for the inelasticity functions are readily incorporated into partial-wave N/D calculations. In setting up a specific model there remains the question of blending the intermediate-energy inelasticity functions with the asymptotic forms. But this problem can be handled in reasonable ways and without the introduction of large numbers of arbitrary parameters.

III. ASYMPTOTIC BEHAVIOR OF DRIVING FORCES

The results presented in Sec. II hold only for $s=w^2\to+\infty$. The asymptotic behavior of partial-wave amplitudes as $s\to-\infty$ requires knowledge of Regge trajectories and residue functions outside the physical region,[10] and the theoretical situation on this point is

[7] The units $\hbar=c=4\mu^2=1$, where μ is the pion mass, are used.

[8] See, for example, T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. **135**, B1464 (1964); G. Tiktopoulos and S. B. Treiman, *ibid.* **167**, 1437 (1968).

[9] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[10] R. Omnes, Phys. Rev. **133**, B1543 (1964). See also the comments of Ref. 5.

presently in a state of flux. Omnes[10] has suggested, on the basis of certain assumptions about Regge-pole behavior, that unsubtracted dispersion relations hold for the partial-wave amplitudes. Kinoshita,[11] with assumptions of a quite different nature, suggests that at most one subtraction is necessary.

In this section we assume that partial-wave dispersion relations hold and calculate the asymptotic behavior of the driving forces, as $s\to+\infty$, implied by Regge theory. Consider pion-pion scattering. It is clear from (4) that an unsubtracted dispersion relation for $A_l(s)$ does not hold; the integral along the physical cut is divergent. So we consider the once-subtracted relation

$$A_l(s)=A_l(s_0)+B_l(s)+\frac{s-s_0}{\pi}\int_1^\infty\frac{ds'\,\mathrm{Im}A_l(s')}{(s'-s_0)(s'-s)},\quad(7)$$

with the partial-wave driving force $B_l(s)$ given by

$$B_l(s)=\frac{s-s_0}{\pi}\int_{-\infty}^0\frac{ds'\,\mathrm{Im}A_l(s')}{(s'-s_0)(s'-s)}.\quad(8)$$

To evaluate the asymptotic behavior of $B_l(s)$ from (7), we first break the integral over the physical cut into a finite integral, with limits 1 to s_1, say, with $s_1<s$, and the remaining infinite integral.

In the limit of large s, the finite integral contributes a constant plus vanishing terms. We then have, from (7),

$$B_l(s)\xrightarrow[s\to\infty]{}\mathrm{Re}A_l(s)-\frac{s-s_0}{\pi}P\int_{s_1}^\infty\frac{ds'\,\mathrm{Im}A_l(s')}{(s'-s_0)(s'-s)}+\text{const}.\quad(9)$$

Supposing s_1 to be so large that the Regge asymptotic form can be used, we insert (4) into (9) and encounter the integral

$$I(s)=(s-s_0)P\int_{s_1}^\infty ds'(s'-s_0)^{-1}(s'-s)^{-1}\ln^{-1}(s').$$

The asymptotic behavior of this integral is shown in the Appendix to be

$$I(s)\xrightarrow[s\to\infty]{}-\ln[\ln(s)]+O(\text{const}).$$

It is also shown in the Appendix that the smaller terms in the asymptotic expansion (4), those proportional to $\ln^{-k}(s)$ with $k=2,3,\cdots$, contribute only constant and smaller terms to (9).

As a consequence, we find that Regge-pole theory together with the assumption of partial-wave dispersion relations determines the asymptotic behavior of the once-subtracted driving force [Eq. (8)] to be

$$B_l(s)\xrightarrow[s\to\infty]{}\frac{\sigma_{\mathrm{tot}}(\infty)}{8\pi^2\alpha'(0)}\ln[\ln(s)]+\text{const}.$$

[11] T. Kinoshita, Phys. Rev. **154**, 1438 (1967).

We note in passing that this result suggests the asymptotic behavior for the discontinuity on the unphysical cut

$$\mathrm{Im}A_l(s+i\epsilon)\xrightarrow[s\to-\infty]{}\frac{\sigma_{\mathrm{tot}}(\infty)}{8\pi\alpha'(0)\ln(-s)}.$$

The analysis for pion-nucleon partial-wave driving forces in the w plane is quite similar. The main differences are the existence of the two physical cuts, easily handled by MacDowell symmetry,[12] and the additional convergence factor supplied by the definition of the partial-wave amplitudes. We write the unsubtracted dispersion relation

$$f_{l\pm}(w)=B_{l\pm}(w)+\frac{1}{\pi}\int_{w_0}^\infty\frac{dw'\,\mathrm{Im}f_{l\pm}(w'+i\epsilon)}{w'-w}-\frac{1}{\pi}\int_{w_0}^\infty\frac{dw'\,\mathrm{Im}f_{l\pm1,\mp}(w'+i\epsilon)}{w'+w},\quad(10)$$

where $w_0=M+\mu$ is the physical threshold and the driving force $B_{l\pm}(w)$ is given by the contour integral around the unphysical cuts

$$B_{l\pm}(w)=\frac{1}{2\pi i}\oint_U\frac{dw'f_{l\pm}(w')}{w'-w}.$$

The last integral on the right-hand side of (10) follows from MacDowell symmetry.

As in the pion-pion case, we separate the integrals in (10) into finite and infinite integrals, the first of which vanish as w^{-1} and in the second of which we can use the Regge asymptotic forms [Eq. (6)]. The results of the Appendix then give for the driving forces

$$B_{l\pm}(w)\xrightarrow[w\to\infty]{}\frac{\sigma_{\mathrm{tot}}(\infty)\ln[\ln(w)]}{8\pi^2\alpha'(0)w}+O(w^{-1}).$$

If the partial-wave dispersion relations hold, which must remain an assumption at this time, then the Regge-pole model leads to precise predictions for the asymptotic behavior of the driving forces in the physical region.

Since the N/D method is based upon the validity of the dispersion relations, it follows that N/D calculations incorporating Regge behavior should employ phenomenological driving forces with the asymptotic behavior found above. It is amusing (but not at all surprising) that the Regge theory determines completely the asymptotic behavior of the kernel function in the N/D equations, both in the R_l formalism and the η_l formalism.[6]

IV. APPROXIMATE MODEL FOR THE S MATRIX

The results presented in the previous sections are rather straightforward consequences of Regge-pole

[12] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

asymptotic behavior. They shed some light on the "boundary conditions" of partial-wave calculations with inelasticity, but there is clearly a great deal of physics yet to be supplied. In particular, the basic ingredient of crossing symmetry is missing. In this section we propose an approximate model for the two-body S matrix that overcomes this deficiency and that may be close enough to the real world to be interesting.

Consider the mathematical problem delineated by the conditions: crossing symmetry, elastic unitarity in the three crossed channels, and the validity of the n-times-subtracted Mandelstam representation. It is well known that this problem is underdetermined; there are presumably an infinite number of solutions. The reason is, of course, that a statement of inelastic unitarity is required to make the mathematics well defined. It is also clear that an arbitrary statement will not do; inelastic unitarity is too closely connected with the crossing relations.[13,14]

Our model is suggested by the Regge-pole inelasticity functions of Sec. II and the fact that they are asymptotically independent of l. For the sake of definiteness and simplicity we consider the scattering of neutral, spinless particles. The generalizations of the model to more complicated processes will be clear. We write the unitarity condition in the s channel in the form

$$\mathrm{Im}A(s,z)=\left(\frac{s-1}{s}\right)^{1/2}R(s)\times\int\frac{d\Omega_n}{4\pi}A^*(s,z_{fn})A(s,z_{ni})+F(s,z)\,,\quad(11)$$

where $R(s)=1$ and $F(s,z)=0$ for s in the elastic region $1\leqslant s<4$. (We suppose that the first inelastic channel is the four-meson state.)

It is obvious that inelastic unitarity can be written in the form (11), since $F(s,z)$ is completely unspecified. But the Regge theory suggests that $F(s,z)$ vanishes rapidly for large s [with the appropriate choice for $R(s)$], as is seen by taking the partial-wave projection of (11)

$$\mathrm{Im}A_l(s)=[(s-1)/s]^{1/2}R(s)\,|A_l(s)|^2+F_l(s)\,.$$

We give $R(s)$ the asymptotic behavior of (5) and see from (4) that $F_l(s)\propto s^{-1}\ln^{-2}(s)$ for large s. Our model consists of neglecting $F(s,z)$ in the unitarity condition (11) at first. $F(s,z)$ cannot remain zero since that would violate crossing symmetry, but it will in fact be determined in a first approximation through crossing symmetry.

Before proceeding to the mathematics, it is helpful to point out some of the general features of this approach. First, it is clear that we are relying upon the Regge-pole partial-wave results for all angular momentum states. But it is known[15] that one cannot blithely interchange the large-s and the large-l limits, so we are mistreating the very high angular-momentum components of the invariant amplitude. We do not expect this to have a serious effect upon the model. Second, it is evident from (11) with $F=0$ that the amplitude will have s-channel double spectral functions with the Landau curves of the "first wings" alone [$t=u=4s/(s-1)$ in our symmetric case]. The "second wings" required by crossing must come from $F(s,z)$. Note that this implies that at large energies the s-channel absorptive part receives a dominant contribution from the first wings of the double spectral functions.

The model then consists of applying the approximate unitarity condition (11) with $F=0$, obtaining a result that is not crossing-symmetric, and reinstating crossing by adding the second-wing double spectral functions that are determined by crossing symmetry. There is an immediate consistency check; the second-wing spectral functions so determined must not modify the asymptotic behavior of the amplitude. If they do, the model fails. If they do not, then, depending upon the complexity of the mathematics, one could contemplate using the first approximation for $F(s,z)$ obtained and recycling. But even after one cycle one should have a crossing-symmetric amplitude with a fair approximation to unitarity.

Finally, there is the question of parametrizing $R(s)$ in the region between the first inelastic threshold and the asymptotic domain. It is known,[14] for example, that $R_l(s)$ for very large l exhibits a rather sudden rise from values near unity to very large values in the region about $s=5$, the crossover point for the leading Landau curves. This is a consequence of the second wing of the spectral function, which is missing in our first approximation. We would suggest, instead of such behavior, that $R(s)$ should climb smoothly and slowly to its logarithmic asymptotic behavior. This would appear to be a better approximation for the lower partial waves that should dominate the amplitude in the intermediate energy range.

We turn to the integral equations that result from our model. The mathematics is familiar; it is just like applying elastic unitarity to the Mandelstam representation, but with the crucial difference that "elastic" unitarity now holds throughout the physical region. We will present the results for the assumptions of an unsubtracted and once-subtracted Mandelstam representation. Further subtractions can be accommodated in an obvious manner. For neutral, spinless particles the invariant amplitude $A(s,t,u)$ is symmetric in its three variables. This implies that the double spectral function $\rho(s,t)$ is symmetric in its two variables.

We define the function (the absorptive parts) for the case of the unsubtracted Mandelstam repre-

[13] A. J. Dragt, Phys. Rev. **156**, 1588 (1967).

[14] A. W. Martin, Phys. Rev. **173**, 1439 (1968).

[15] See, in this regard, C. Goebel, Phys. Rev. Letters **21**, 383 (1968); M. Kugler, *ibid.* **21**, 570 (1968).

sentation

$$B(s,t)=\frac{1}{\pi}\int_1^\infty \frac{dx\,\rho(s,x)}{x-t}+\frac{1}{\pi}\int_1^\infty \frac{dx\,\rho(s,x)}{x+s+t-1}. \quad (12)$$

In the s channel we have $\mathrm{Im}A(s,t)=B(s,t)$, and because of our simple crossing relations we have the fixed-s dispersion relation

$$A(s,t,u)=\frac{1}{\pi}\int_1^\infty \frac{dx\,B(x,s)}{x-t}+\frac{1}{\pi}\int_1^\infty \frac{dx\,B(x,s)}{x-u}. \quad (13)$$

We now insert (13) into the unitarity condition (11), carry out the angular integrations, and obtain the integral equation

$$B(s,t)=\frac{4R(s)}{[s(s-1)]^{1/2}\pi^2}\times\int\int_1^\infty dx\,dy\,B^*(x,s)B(y,s)H(x,y;s,t), \quad (14)$$

where the kernel enjoys the integral representation

$$H(x,y;s,t)=\int_{t_+}^\infty \frac{dz\,(2z+s-1)}{[(z-t_+)(z-t_-)]^{1/2}(z-t)(z+t+s-1)}, \quad (15)$$

with

$$t_\pm=(s-1)^{-1}[x^{1/2}(y+s-1)^{1/2}\pm y^{1/2}(x+s-1)^{1/2}]^2.$$

The kernel can be expressed in terms of Legendre functions (equivalently, logarithms) as

$$H(x,y;s,t)=\frac{1}{[(t-t_+)(t-t_-)]^{1/2}}Q_0\left(\frac{\frac{1}{2}(t_++t_-)-t}{[(t-t_+)(t-t_-)]^{1/2}}\right)+\frac{1}{[(t+s-1+t_+)(t+s-1+t_-)]^{1/2}}\times Q_0\left(\frac{\frac{1}{2}(t_++t_-)+(t+s-1)}{[(t+s-1+t_+)(t+s-1+t_-)]^{1/2}}\right). \quad (16)$$

We note some features of the integral equation. First, it is complex; the kernel (15) develops an imaginary part for $t\geqslant t_+\geqslant 4s/(s-1)$, and the imaginary part of $B(s,t)$, which is the first wing of the double spectral function, is given by the familiar expression

$$\rho_1(s,t)=\frac{4R(s)}{[s(s-1)]^{1/2}\pi}\times\int\int_1^\theta \frac{dx\,dy\,B^*(x,s)B(y,s)}{[(t-t_+)(t-t_-)]^{1/2}}\theta(t-t_+), \quad (17)$$

which now holds for all positive s. Results known previously for the double spectral function in the elastic region[16] can therefore be extended throughout the physical region in this model.

Second, because the kernel is symmetric in x and y, the only surviving part of the expression $B^*(x,s)B(y,s)$ is the real part, namely, $\mathrm{Re}B(x,s)\,\mathrm{Re}B(y,s)+\mathrm{Im}B(x,s)\times\mathrm{Im}B(y,s)$. This indicates that the range of integration in (14) (roughly the first quadrant in the xy plane) can be halved by cutting along the diagonal. This should help to simplify the analysis. Finally, the most important feature of (14) is the "reversed" structure of the variables. All of the t dependence on the right-hand side is in the kernel. Suppose, for example, one chose a form for $B(x,s)$. The integral in (14) then yields the t dependence of $B(s,t)$ which translates back into the s dependence of $B(x,s)$, and so forth. In this way the problem is more like an integral equation in a single variable than one in two variables. The dependence upon the two variables is strongly coupled in a self-consistent manner.

Because of the nonlinearity of (14), we are unaware of possible existence or uniqueness theorems for the solutions. To conclude the formulation of the model, however, let us suppose that a complex function $B(s,t)$, with $1\leqslant s,\ t<\infty$, is found which satisfies (14). Note that the t-u symmetry ($t\to u=1-s-t$) of the amplitude is preserved by the kernel (15). This means that $B(s,t)$ is defined for all t by the continuation of (14) and is symmetric in t and u. But since the second wings of the s-channel double spectral functions are missing, it is impossible at this stage to have s-t or s-u crossing symmetry.

We therefore construct the symmetric double spectral function

$$\rho(s,t)=\theta(st-4s-t)\rho_1(s,t)+\theta(st-4t-s)\rho_1(t,s), \quad (18)$$

where $\rho_1(s,t)$ is given by (17) and is known from the solution of the integral equation. This "corrected" spectral function is inserted in (12) and makes an inelastic region contribution to the absorptive part $B(s,t)$. As mentioned above, the large-s behavior of $B(s,t)$ must be essentially unaffected by this modification. Otherwise the model is internally inconsistent. The last step involves inserting the modified $B(s,t)$ into (13) to obtain a manifestly crossing-symmetric amplitude.

In reinstating crossing symmetry in this way we have inevitably tampered with unitarity. In the inelastic domain (our region of almost complete ignorance) this does not bother us. In fact, it is an advantage because the close interconnection between crossing and inelasticity has been exploited in a first approximation. But we have also tampered with elastic unitarity, and this is a more serious matter. We observe that there is a simple check on the degree to which elastic unitarity has been affected. Equation (17) is an exact statement for s in

[16] See, for example, A. W. Martin, Phys. Rev. **162**, 1534 (1967).

the elastic region [with $R(s)=1$]. If $\rho_1(s,t)$, with $1\leqslant s<4$, changes only slightly when the modified $B(s,t)$ is inserted in (17), then elastic unitarity is fulfilled to a good approximation. If the converse is true, then one must consider recycling. We note in this regard that the first approximation to $F(s,z)$ in the unitarity condition (11) is

$$F(s,z)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'\rho_1(t',s)}{t'-t}+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du'\rho_1(u',s)}{u'-u},$$

where $t_0=u_0=s/(s-4)$ and $\rho_1(t,s)$ has been determined from the integral equation.

We are aware that our approximate formulation of the two-body S-matrix problem is not a simple one; it requires the solution of a rather unconventional integral equation without even the guarantee of existence or uniqueness. Nevertheless, so little is known about inelastic unitarity or how to combine crossing symmetry and unitarity in S-matrix calculations that we feel this model provides a necessary first step. We are also convinced that the determination of the asymptotic behavior of $B(s,t)$ in both variables from (14) will be relatively straightforward, even if finding exact solutions is not. As a consequence, we think that it will be possible to judge the validity and usefulness of the model at an early stage. If Regge-pole behavior emerges, as we fully expect, then one could begin to think about further approximations to simplify the mathematics.

To conclude this section we state the integral equation following from the once-subtracted Mandelstam representation. It is almost certain that subtractions will be necessary to obtain solutions with asymptotically constant total cross sections. Consider the Mandelstam representation, again for neutral, spinless particles, subtracted at the symmetry point $s_0=t_0=u_0=\frac{1}{3}$. The s-channel absorptive part is given by

$$B(s,t)=\sigma(s)+\frac{t-t_0}{\pi}\int_1^{\infty}\frac{dt'\rho(s,t')}{(t'-t_0)(t'-t)}+\frac{u-u_0}{\pi}\int_1^{\infty}\frac{du'\rho(s,u')}{(u'-u_0)(u'-u)},\qquad(19)$$

where $\sigma(s)$ is a single spectral function. The fixed-s dispersion relation reads

$$A(s,t,u)=C(s)+\frac{t-t_0}{\pi}\int_1^{\infty}\frac{dt'B(t',s)}{(t'-t_0)(t'-t)}+\frac{u-u_0}{\pi}\int_1^{\infty}\frac{du'B(u',s)}{(u'-u_0)(u'-u)},\qquad(20)$$

with

$$C(s)=\lambda+\frac{s-s_0}{\pi}\int_1^{\infty}\frac{ds'\sigma(s')}{(s'-s_0)(s'-s)}-\frac{s-s_0}{\pi^2}\iint_1^{\infty}\frac{dt'du'\rho(t',u')}{(t'-t_0)(u'-u_0)(t'+u'+s-1)},$$

where λ is the value of the amplitude at the symmetry point. Note that $C(s)$ contributes only to the s-wave scattering.

When the fixed-s dispersion relation is inserted into the unitarity condition (11), with $F(s,z)=0$ as before, the integral equation results

$$B(s,t)=\mathrm{Im}A_0(s)+\frac{4R(s)}{[s(s-1)]^{1/2}\pi^2}\times\iint_1^{\infty}dx\,dy\,B^*(x,s)B(y,s)\tilde{H}(x,y;s,t),\qquad(21)$$

where $A_0(s)$ is the s-wave partial-wave amplitude and the kernel enjoys the integral representation

$$\tilde{H}(x,y;s,t)=\int_{t_+}^{\infty}\frac{dz}{[(z-t_+)(z-t_-)]^{1/2}}\left[\frac{2z+s-1}{(z-t)(z+t+s-1)}-\frac{4}{s-1}Q_0\left(1+\frac{2z}{s-1}\right)\right].$$

Explicitly the kernel is given by

$$\tilde{H}(x,y;s,t)=H(x,y;s,t)-\frac{4}{s-1}Q_0\left(1+\frac{2x}{s-1}\right)Q_0\left(1+\frac{2y}{s-1}\right),$$

where $H(x,y;s,t)$ is the unsubtracted kernel given by (16).

We note the following features of the once-subtracted integral equation. First, and quite important, the new kernel has different asymptotic behavior in the variables x and y; it vanishes more rapidly. This suggests that the asymptotic behavior of the solution $B(s,t)$ in the variable s will be greater than in the unsubtracted case [Eq. (14)]. Second, the s-wave piece of the absorptive part has been isolated in the "inhomogeneous" term $\mathrm{Im}A_0(s)$. This implies that the s-wave projection of the kernel vanishes, as can be checked by direct calculation. Third, the "inhomogeneous" term is not independent. Martin has shown[17] that the absorptive part fixes the s-wave amplitude up to a real constant. From the point of view of obtaining approximate solutions of (21), however, $\mathrm{Im}A_0(s)$ will probably not play an important role. It is bounded by unitarity and should be given the asymptotic behavior of (4) in a first (and hopefully consistent) approximation.

The model then proceeds essentially as in the unsubtracted case. One solves (21) for $B(s,t)$ and determines the first wing of the double spectral function, the imaginary part of $B(s,t)$. Note that (17) holds independent of the number of subtractions. Equation (19) then determines the single spectral function in a

[17] A. Martin, Nuovo Cimento 47, 265 (1967).

first approximation. Finally, one reinstates crossing by symmetrizing the double spectral function as in (18), computing the modified absorptive part by (19), and obtaining the manifestly crossing-symmetric invariant amplitude by (20). Note that all parameters are determined except λ, the value at the symmetry point. At this time we see no way to determine λ within our approximate model. We observe, however, that one can recycle the calculation without the necessity of specifying the subtraction constant.

V. CONCLUSIONS

Inelastic unitarity is bound to become a more and more important factor as S-matrix calculations grow in sophistication. The various N/D prescriptions for incorporating partial-wave inelastic effects are well known,[2] and the larger problem of determining the invariant amplitudes is not even defined without a statement of inelastic unitarity. The Regge-pole phenomenology, which seems to provide an adequate description of high-energy scattering, directly yields[4–6] the asymptotic behavior of the partial-wave amplitudes in the physical region, and it is a short step to extract the asymptotic behavior of the inelasticity function $R_l(s)$. We have presented the results for the cases of pion-pion and pion-nucleon scattering.

The partial-wave asymptotic behavior, together with the assumed validity of partial-wave dispersion relations, also permits us to deduce the asymptotic behavior of the driving forces in the physical region. Since the N/D equations can be cast in a form[18] such that only physical-region information is required, these twin results determine completely the asymptotic behavior of the kernel in the N/D method. The findings in the R_l formalism thus complement those of Warnock in the η_l formalism.[6] Of course, the asymptotic information provided by the Regge theory still leaves a great deal of physics to be determined. One of the primary difficulties of partial-wave calculations has been the inability to incorporate crossing symmetry in a rigorous manner.

With the crossing-symmetry problem in mind, one notes that the Regge-pole inelasticity functions possess a striking feature. They are asymptotically (in energy) independent of l. This implies that at large energies the full unitarity condition for elastic processes assumes a form much like the exact statement of unitarity in the elastic energy range. That is, the Regge theory implies that to a good approximation all asymptotic inelastic effects can be lumped into a multiplicative factor times the solid-angle integral over elastic amplitudes.

This observation leads to an approximate formulation of the two-body S-matrix problem that we have discussed in detail in Sec. IV. The model requires the solution of nonlinear integral equations and the meeting of certain internal-consistency demands. It does not seem likely that exact solutions will be readily found. On the other hand, it should be possible to find the asymptotic behavior of the solutions without great difficulty and to judge thereby the eventual success or failure of the model.

[18] J. L. Uretsky, Phys. Rev. 123, 1459 (1961).

APPENDIX

In this Appendix we derive the asymptotic behavior of a class of integrals required in the analysis of Sec. III. We define the integrals

$$I_\beta(s)=P\int_{s_1}^{\infty}\frac{dx}{x(x-s)\ln^\beta(x)},$$
$$J_\beta(s)=\int_{s_1}^{\infty}\frac{dx}{x(x+s)\ln^\beta(x)},\qquad \text{(A1)}$$

and first consider $I_\beta(s)$. We break the principal-value integral at the singularity, according to the definition, and employ the binomial expansions in the convenient form

$$(x-s)^{-1}=-s^{-1}-\sum_{k=1}^{\infty}\frac{x^k}{s^{k+1}},\quad x<s$$
$$=\sum_{k=1}^{\infty}\frac{s^{k-1}}{x^k},\qquad s<x.$$

The change of variables $y=\ln(x)$ then yields

$$I_\beta(s)=-\frac{1}{s}\int_{(s_1)}^{\ln(s-\epsilon)}dy\,y^{-\beta}-\sum_{k=1}^{\infty}\frac{1}{s^{k+1}}\int_{\ln(s_1)}^{\ln(s-\epsilon)}dy\,y^{-\beta}e^{ky}$$
$$+\sum_{k=1}^{\infty}s^{k-1}\int_{\ln(s+\epsilon)}^{\infty}dy\,y^{-\beta}e^{-ky}.\qquad \text{(A2)}$$

The first integral is readily evaluated (the limit $\epsilon=0$ can be taken since the principal-value singularities now reside in the infinite sums). The first integral also turns out to dominate the asymptotic behavior of $I_\beta(s)$. To show this we denote the sum of the last two terms on the right-hand side of (A2) by $\tilde{I}_\beta(s)$, make the variable change $t=ky$, and obtain

$$\tilde{I}_\beta(s)=-\sum_{k=1}^{\infty}\frac{k^{\beta-1}}{s^{k+1}}\int_{\ln(s_1)^k}^{\ln(s-\epsilon)^k}dt\,t^{-\beta}e^{t}$$
$$+\sum_{k=1}^{\infty}k^{\beta-1}s^{k-1}\int_{\ln(s+\epsilon)^k}^{\infty}dt\,t^{-\beta}e^{-t}.\qquad \text{(A3)}$$

These integrals are related to the usual exponential and logarithmic integrals, but we do not need the full apparatus here. We require only the integration-by-parts identities

$$\int_a^b dt\, t^{-\beta}e^t=\sum_{p=0}^{q}\frac{\Gamma(p+\beta)}{\Gamma(\beta)}\left(\frac{e^b}{b^{p+\beta}}-\frac{e^a}{a^{p+\beta}}\right)+\frac{\Gamma(q+\beta+1)}{\Gamma(\beta)}\int_a^b dt\, t^{-q-\beta-1}e^t,$$

$$\int_a^\infty dt\, t^{-\beta}e^{-t}=\sum_{p=0}^{q}\frac{(-1)^p\Gamma(p+\beta)}{\Gamma(\beta)a^{p+\beta}e^a}-(-1)^q\times\frac{\Gamma(q+\beta+1)}{\Gamma(\beta)}\int_a^\infty dt\, t^{-q-\beta-1}e^{-t}. \qquad \text{(A4)}$$

The last integrals on the right-hand sides of (A4) we denote henceforth by the general symbol $R(q+1)$. These remainder terms are always smaller in magnitude than the terms obtained by extending the p sums one unit.

We use the identities (A4) in (A3), interchange the orders of the infinite and finite sums, set $\epsilon=0$ where it does not affect convergence, and obtain

$$\tilde{I}_\beta(s)=\frac{1}{s}\sum_{p=0}^{q}\frac{\Gamma(p+\beta)}{\Gamma(\beta)\ln^{p+\beta}(s)}\sum_{k=1}^{\infty}\frac{1}{k^{p+1}}\left[\frac{(-1)^p s^k}{(s+\epsilon)^k}-\frac{(s-\epsilon)^k}{s^k}\right]+\frac{1}{s}\sum_{p=0}^{q}\frac{\Gamma(p+\beta)}{\Gamma(\beta)\ln^{p+\beta}(s_1)}\sum_{k=1}^{\infty}\frac{s_1{}^k}{k^{p+1}s^k}+R(q+1). \qquad \text{(A5)}$$

The sum involving s_1 in (A5) is obviously convergent and the whole term is of order s^{-2}. In the first sum on the right-hand side of (A5) only the $p=0$ term involves sums divergent in the limit $\epsilon\to 0$. These are the usual $\ln(\epsilon)$ terms in principal-value integrals and are readily seen to cancel. The k sums with $p\geqslant 1$ are convergent in the limit $\epsilon=0$ and are simply Riemann ζ functions. We also note that only the terms with p odd survive. As a consequence we can write $\tilde{I}_\beta(s)$ in the form

$$\tilde{I}_\beta(s)=-\frac{2}{s}\sum_{n=0}^{q}\frac{\Gamma(2n+\beta+1)\zeta(2n+2)}{\Gamma(\beta)\ln^{2n+\beta+1}(s)}+R(q+1)+O(s^{-2}), \qquad \text{(A6)}$$

where $\zeta(2n+2)$ is the Riemann ζ function.

We now return to (A2) and obtain the final results

$$I_{\beta=1}(s)=-s^{-1}\ln[\ln(s)/\ln(s_1)]+\tilde{I}_{\beta=1}(s),$$
$$I_{\beta\neq 1}(s)=[(\beta-1)s]^{-1}[\ln^{1-\beta}(s)-\ln^{1-\beta}(s_1)]+\tilde{I}_{\beta\neq 1}(s),$$

with $\tilde{I}_\beta(s)$ given by (A6). The cases in which β is zero or a negative integer are special (and trivial) and do not concern us here. The evaluation of $J_\beta(s)$ [Eq. (A1)] proceeds in a similar way. It is convenient to break the integral at the point $x=s$ and to use the binomial expansions

$$(x+s)^{-1}=s^{-1}+\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{s^{k+1}}, \quad x\leqslant s$$
$$=-\sum_{k=1}^{\infty}\frac{(-1)^k s^{k-1}}{x^k}, \quad s\leqslant x.$$

The same analysis as before then leads to the results

$$J_{\beta=1}(s)=s^{-1}\ln[\ln(s)/\ln(s_1)]+\tilde{J}_{\beta=1}(s),$$
$$J_{\beta\neq 1}(s)=[(1-\beta)s]^{-1}[\ln^{1-\beta}(s)-\ln^{1-\beta}(s_1)]+\tilde{J}_{\beta\neq 1}(s),$$

where

$$\tilde{J}_\beta(s)=-\frac{2}{s}\sum_{n=0}^{q}\frac{\Gamma(2n+\beta+1)\eta(2n+2)}{\Gamma(\beta)\ln^{2n+\beta+1}(s)}+R(q+1)+O(s^{-2}),$$

and $\eta(n)$ is the η function defined by

$$\eta(n)=\sum_{k=1}^{\infty}(-1)^{k-1}k^{-n}$$

and related to the Riemann ζ function by

$$\eta(n)=(1-2^{1-n})\zeta(n).$$

In the analysis of Sec. III the integrals involve the subtraction point s_0. The fact that the subtraction point does not affect the asymptotic behavior of the integrals (as is intuitively obvious) is readily shown by expanding $(s'-s_0)^{-1}$ binomially and using partial-fraction identities. The s_0-dependent terms are a factor s^{-1} smaller than the asymptotic results presented above.