The Nucleon and the Roper Resonance

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A coupled two-channel bootstrap calculation is performed for the static system of πN and $\pi\pi N$ states in the positive-parity $I=J=\frac{1}{2}$ state. For the $\pi\pi N$ state, we use an analysis of Ball *et al.* which reduces the three-particle system to a two-particle system with a modified phase space. The calculations are made using the N/D method and the static Bethe-Salpeter equation. The results are in reasonable agreement with the experimental positions and partial widths, and imply that the Roper resonance is primarily a $\pi\pi N$ resonance.

I. INTRODUCTION

THE static model has been fairly successful in describing the low-energy P-wave πN scattering. The model was used by Chew[1] to suggest that in πN scattering, the N exchange gives rise to forces which are strong enough to produce the N^* resonance, while the forces due to N^* exchange can produce an N bound state. The calculations[2] with the static Bethe-Salpeter equation have since yielded results which essentially agree with those of Chew, who used the static N/D model with linearized D function.

This situation is, however, disturbed by the presence of the Roger resonance[3] at 1470 MeV and with a total width of 210 MeV. This resonance has the same quantum numbers as the nucleon except for its position and width, and it is very difficult indeed to find forces which can produce two nearby zeros in the D function for a single-channel calculation. Furthermore, the Roper resonance decays to a significant extent into $\pi\pi N$, the partial width being about $\frac{1}{2}m_\pi$ in spite of the small phase space for the $\pi\pi N$ channel. The situation therefore seems to demand a two-channel calculation involving P-wave πN and $\pi\pi N$ states in the $I=J=\frac{1}{2}$ channel.

In this paper, we present a two-channel πN and $\pi\pi N$ static calculation in the positive-parity $I=J=\frac{1}{2}$ state. On account of the small phase space for the $\pi\pi N$ state and the low-energy nature of our computations, it is reasonable to assume that the $\pi\pi$ are in the S-wave state. Furthermore, unless there is a low-energy S-wave $\pi\pi$ resonance, we will be justified in making a scattering-length approximation for the $\pi\pi$ interaction.[4] For the treatment of the three-particle system $\pi\pi N$, we use the unstable "particle" analysis given by Ball *et al.*,[5] which allows us to treat the $\pi\pi$ state as a single-particle state but with a weighted phase space. The input static forces in this simplified two-channel system are due to exchanges of the nucleon, the Roper resonance, and the N^*. The cutoff is fixed so as to give the correct position for the output nucleon. The calculations are first made in the N/D framework and then the static Bethe-Salpeter equation is solved using Pagels's approximation. Both the calculations yield more or less the same result that the bootstrap parameters for the masses and coupling constants agree with the experimental numbers quite well. The value of the coupling of the Roper resonance to the πN comes out somewhat smaller than the experimental value, but is not unreasonable considering the approximations involved. The bootstrap values are such that the Roper resonance comes out at about 1500 MeV with a partial decay width of $\frac{1}{2}m_\pi$ for decay into $\pi\pi N$ but a considerably smaller decay width for decay into πN, and the nucleon coupling to the πN agrees very well with the experimental number, while its "coupling" to the scalar $\pi\pi$ system is about two times smaller than the corresponding Roper-resonance coupling. The smallness of this coupling suggests a justification for the success of the current-algebra calculations[6] of the πN scattering lengths.

The model which we use is similar to the one which Ball, Shaw, and Wong[7] use in their analysis of the P_{11} πN partial-wave amplitude. Their analysis consists of a two-channel ND^{-1} calculation, with the two channels being the πN and the σN systems; σ is taken to be an S-wave resonance with a mass of $3.5m_\pi$. Our emphasis, however, is on the bootstrap aspect of the problem, that the forces due to the N^*, N, and Roper-resonance exchanges produce the same particles in the direct channel, whereas the Ball-Shaw-Wong analysis uses a one-pole parametrization in terms of four parameters to reproduce the general experimental features of the P_{11} πN partial wave. Furthermore, our analysis in terms of the $\pi\pi$ scattering-length approximation and the formalism of Ball, Frazer, and Nauenberg[5] is more general than the essentially single-particle σ approach of Ball, Shaw, and Wong,[7] especially in view of the highly dubious nature of the existence of the σ resonance.

The two-channel analyses do provide a bootstrap basis for the nucleon and the Roper resonance, with the nucleon being a bound state of πN and the Roper resonance being a resonance of primarily the $\pi\pi N$ state.

[1] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
[2] S. N. Biswas and L. A. P. Balázs, Phys. Rev. **156**, 1511 (1967).
[3] A. H. Rosenfeld *et al.*, University of California Radiation Laboratory Report No. UCRL-8030, 1967 (unpublished).
[4] This would be consistent with the S-wave resonance at higher energies, say, the S meson at 800 MeV.
[5] J. S. Ball, W. R. Frazer, and M. Nauenberg, Phys. Rev. **128**, 478 (1962).
[6] Y. Tomozawa, Nuovo Cimento **46A**, 707 (1966).
[7] J. S. Ball, G. Shaw, and D. Wong, Phys. Rev. **155**, 1725 (1967).

II. COUPLED-CHANNEL N/D PROBLEM

Let T_{11} denote the $I=J=\frac{1}{2}$ positive-parity amplitude for $\pi+N\rightarrow\pi+N$, T_{12} for $\pi+N\rightarrow\pi+\pi+N$, T_{21} for $\pi+\pi+N\rightarrow\pi+N$, and T_{22} for $\pi+\pi+N\rightarrow\pi+\pi+N$. For the energies in which we will be interested, it will be a good approximation to assume that the $\pi\pi$ in the $\pi\pi N$ channel are in an S-wave state. For the treatment of this $\pi\pi N$ state, we use the analysis of Ball *et al.*,[5] for which one first writes

$$\begin{aligned} T_{11}&=M_{11}\,,\\ T_{12}&=M_{12}f(t)\,,\\ T_{22}&=M_{22}f(t)f(t')\,, \end{aligned} \tag{1}$$

where

$$f(t)=t^{1/2}\frac{e^{i\delta_0}\sin\delta_0}{(t-4\mu^2)^{1/2}}\,, \tag{2}$$

with δ_0 being the S-wave $I=0$, $\pi\pi$ phase shift, t is the invariant square of the energy and momentum of the $\pi\pi$ system, and μ is the pion mass. The M_{12} and M_{22} thus defined are free from the singularities due to the $\pi\pi$ interaction. Furthermore, the t dependence of M_{ij} will be ignored. The T_{ij} are so normalized that the unitarity condition reads

$$\mathrm{Im}M_{ij}=M_{ik}{}^*\rho_k M_{kj}\,, \tag{3}$$

where the discontinuity is due to the singularities in the total energy variable, and for the static model

$$\rho_1=(\omega^2-\mu^2)^{3/2}$$

and

$$\rho_2=\frac{C\theta(\omega-2\mu)}{\pi}\int_{4\mu^2}^{\omega^2}(\omega^2-t)^{1/2}|f(t)|^2\left(\frac{t-4\mu^2}{t}\right)^{1/2}dt\,, \tag{4}$$

C being a constant which is so defined[8] as to give $\rho_2=(\omega^2-m^2)^{1/2}$ for the narrow-width approximation of an S-wave $\pi\pi$ resonance of mass m.

It is easy to solve (3) in the framework of the N/D method, for which one writes

$$M=ND^{-1}\,, \tag{5}$$

where N and D are 2×2 matrices, N has only left-hand singularities, and D has only right-hand singularities. The unitarized solution to M is then given in terms of N and D:

$$\begin{aligned} N(\omega)&=\frac{1}{\pi}\int_l\frac{\mathrm{Im}M^B(\omega')D(\omega')}{\omega'-\omega}d\omega'\,,\\ D(\omega)&=1-\frac{1}{\pi}\int_\mu^\Lambda\frac{\rho(\omega')N(\omega')}{\omega'-\omega}d\omega'\,, \end{aligned} \tag{6}$$

where the superscript B stands for the projected Born term, l stands for integration along the left, ρ is the diagonal matrix with diagonal elements ρ_1 and ρ_2, and Λ is a cutoff.

We solve the set of equations (6) with the input forces coming from the exchanges of the N, the N^*, and the Roper resonance. The static expressions for $M_{ij}{}^B$ are

$$\begin{aligned} M_{11}{}^B(\omega)&=-\frac{1}{9}\frac{\gamma_{N1}{}^2}{\omega}+\frac{1}{9}\frac{\gamma_{R1}{}^2}{\omega+\Delta_1}+\frac{16}{9}\frac{\gamma_{N^*1}{}^2}{\omega+\Delta_2}\,,\\ M_{12}{}^B(\omega)&=M_{21}{}^B(\omega)=\frac{\gamma_{N1}\gamma_{N2}}{\omega}+\frac{\gamma_{R1}\gamma_{R2}}{\omega+\Delta_1}\,,\\ M_{22}{}^B(\omega)&=\frac{\gamma_{N2}{}^2}{\omega}+\frac{\gamma_{R2}{}^2}{\omega+\Delta_1}\,, \end{aligned} \tag{7}$$

where γ_{N1} is the coupling of the nucleon to the πN channel, γ_{N2} is the "coupling" to the $(\pi\pi)N$ channel with the $\pi\pi$ being in an S-wave state, and similar definitions for the Roper resonance and the N^*; the Δ_i are the mass differences

$$\Delta_1=m_R-m_N \quad\text{and}\quad \Delta_2=m_{N^*}-m_N\,.$$

Equations (6) can now be solved using (7). For the evaluation of phase space ρ_2 we use scattering-length parametrization for $f(t)$:

$$f(t)=1\Big/\left[\frac{1}{a}-i\left(\frac{t-4\mu^2}{t}\right)^{1/2}\right], \tag{8}$$

where the scattering length a is taken[9] to be $|a|\approx1.2$ in units of the pion mass. Our results will be independent of the sign of the scattering length. The cutoff Λ is so determined as to give a zero in the determinant of $D(\omega)$ at $\omega=0$, corresponding to the nucleon pole. We then find that the experimental numbers for the various coupling constants and masses in (7) reproduce themselves as solutions of (6), i.e., the bootstrap parameters are close to the experimental numbers. The approximate bootstrap parameters are

$$\begin{aligned} &\Lambda=12\mu\,,\quad \gamma_{N1}\approx0.5\,,\quad \gamma_{N2}\approx1.4\,,\\ &\Delta_1\approx5\mu\,,\quad \gamma_{R1}\approx-0.05\,,\quad\text{and}\quad \gamma_{R2}\approx2.5\,, \end{aligned} \tag{9}$$

where we have taken $C\approx1/25$, compared to the experimental numbers

$$\begin{aligned} &|\gamma_{N1}|\approx0.5\,,\quad \Delta_1\approx3.8\,,\\ &|\gamma_{R1}|\approx0.14\,,\quad\text{and}\quad |\gamma_{R2}|\approx2.5\,, \end{aligned} \tag{10}$$

corresponding to Roper-resonance partial widths of $\frac{1}{2}m_\pi$ each for decays[10] into the πN and $(\pi\pi)N$ channels.

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[8] Taking a different value for C only redefines the matrix elements M_{12}, M_{21}, and M_{22} by constant factors.

[9] Our calculations are independent of the sign of a. The magnitude we have taken is consistent with the experiments: L. W. Jones *et al.*, Phys. Letters **21**, 590 (1966); W. D. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967); E. Malamud and P. Schlein, *ibid.* **19**, 1056 (1967).

[10] The comparison with the experimental widths has the difficulty due to the correction factors, which we take to be the same as for the N^*.

The mass of the Roper resonance comes out somewhat larger and the Roper-resonance coupling to πN is considerably smaller than the respective experimental values, but still the over-all agreement is reasonable considering all the approximations that go into the calculation. We have no direct experimental information on γ_{N2}, but the fact that the πN scattering lengths are correctly given by ignoring the $I=0$ state in the $\pi\pi \to N\bar{N}$ channel suggests that γ_{N2} is probably small and this conclusion is supported by our result.

III. STATIC BETHE-SALPETER EQUATION

In order to confirm the results of the N/D calculation, we will redo the calculations within the framework of a two-channel static Bethe-Salpeter equation. With the M_{ij} defined as before, the equation reads

$$M(\omega',\omega)=V(\omega',\omega)+\frac{1}{\pi}\int\frac{V(\omega',\omega'')\rho(\omega'')M(\omega'',\omega)}{\omega''-\omega}d\omega', \quad (11)$$

where $V_{ij}(\omega',\omega'')$ are the off-shell potentials which are the same as the Born projections in (7) except that ω is replaced by $\omega'+\omega''-\omega$, with ω being the on-shell energy. To facilitate the solution of (11), we generalize the Noyes[11] procedure and write

$$M(\omega',\omega)=f(\omega',\omega)M(\omega,\omega), \quad (12)$$

where $f(\omega',\omega)$ is a 2×2 matrix and $f(\omega,\omega)$ is the unit matrix. Then one can write

$$M(\omega,\omega)=\left[1+\frac{1}{\pi}\int\frac{V(\omega,\omega'')\rho(\omega'')f(\omega'',\omega)}{\omega-\omega''}d\omega''\right]^{-1}\times V(\omega,\omega) \quad (13)$$

and

$$f(\omega',\omega)=V(\omega',\omega)V^{-1}(\omega,\omega)\times\left[1+\frac{1}{\pi}\int\frac{V(\omega,\omega'')\rho(\omega'')f(\omega'',\omega)}{\omega-\omega''}d\omega''\right]-\frac{1}{\pi}\int\frac{V(\omega',\omega'')\rho(\omega'')f(\omega'',\omega)}{\omega-\omega''}d\omega''. \quad (14)$$

As a first approximation, one may take

$$f(\omega',\omega)=V(\omega',\omega)V^{-1}(\omega,\omega). \quad (15)$$

This leads to approximately the same results as those from the N/D method. However, this approximation has the serious shortcoming that the M is not a symmetric matrix, reminiscent of the similar difficulty arising in the determinantal approximation for the two-channel ND^{-1} solution.[12] Therefore, we use an alternative approximation, a generalization of Pagels's approximation to the two-channel problem. The analysis is standard[13] and reduces to writing in (14)

$$\rho_1(\omega)=C_1\delta(\omega-\omega_1), \quad \rho_2(\omega)=C_2\delta(\omega-\omega_2), \quad (16)$$

where C_i and ω_i are constants. Their values are obtained by fitting

$$\int_\mu^\Lambda\frac{\rho_i(\omega'')d\omega''}{\omega''+\alpha}=\frac{C_i}{\omega_i+\alpha} \quad (17)$$

for different values of α. For $\Lambda\approx 9\mu$ we have

$$\omega_1\approx\omega_2\approx 6.7\mu, \quad C_1\approx 1600, \quad C_2\approx 11. \quad (18)$$

There is, however, a difficulty in evaluating the residues in (13) arising from the pole in V at $\omega=0$ owing to the nucleon exchange. We can avoid this difficulty by shifting the mass of the exchanged nucleon to, say, the mass of the N^*. This modification is not serious since the forces due to the nucleon exchange are not important for the formation of the nucleon. Alternatively, one may interpret this as implying the superiority of the analytic ND^{-1} method for solving the nucleon problem. Anyway, with this modification we can solve Eqs. (13) and (14). The bootstrap solutions for these equations are

$$\Lambda\approx 9\mu, \quad \gamma_{N1}\approx 0.45, \quad \gamma_{N2}\approx 1.2, \quad \Delta_1\approx 4\mu, \quad \gamma_{R1}\approx -0.035, \quad \text{and} \quad \gamma_{R2}\approx 2.6. \quad (19)$$

These numbers are in close agreement with the results of the ND^{-1} method, thus giving us confidence that out solutions are essentially correct.

From the results, it is seen that in the bootstrap framework, the nucleon is primarily coupled to the πN channel, whereas the Roper resonance is coupled mainly to the $\pi\pi N$ state. The situation is similar to the ω-ϕ system where ϕ is coupled mainly to the $K\bar{K}$ system and ω to the $\rho\pi$ system.

[11] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).

[12] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

[13] See, for example, Ref. 2.