understood for a given collision amplitude, or set of amplitudes, one can begin to approximate (as in the narrow-resonance approximation, or in our unitarity hypothesis) in order to study special features of the amplitude. In particular, one may obtain constraints on the input assumptions such as the high-energy behavior or the masses of the resonances.

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Theory of High-Spin Fields

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A general method for the construction of wave functions and wave equations for higher spins is proposed. This method is based on the full use of projection operators, without any use of boosting and without explicit use of auxiliary fields. The wave equations are always expressed in the form of single matrix equations, so that their Lagrangians follow immediately. Then the entire program of quantization of the free fields can follow straightforwardly by using the so-called $d(\partial)$-operator technique which was developed previously. The present method is made up of two steps: The first step is to derive wave equations when the maximum spin is specified; the second step is to derive equations when the spin itself is specified. The technique employed in the first step, when it is applied to many-spinor representations, can be used to put the Bargmann-Wigner equations into the form of a single matrix equation. The same technique enables us to write explicitly the Harish-Chandra β matrices in terms of Dirac matrices. General arguments explain why the relativistic wave equations in general contain a certain number of arbitrary parameters, such as have been observed by several authors. The method is illustrated by several examples of relativistic wave equations.

1. INTRODUCTION

DERIVATION of relativistic wave functions and wave equations for particles of arbitrary spin is now a classic problem of quantum field theory. In the past three decades a variety of wave equations have been proposed, and the method employed in their derivation and quantization has been successively revised and generalized.[1–3] The problem has acquired realistic importance by the discovery of many resonances of high spins.

In the approach due to Dirac, Pauli, and Fierz[4–6] the relativistic wave equations of high spin were expressed by a set of differential equations, among which there were a certain number of subsidiary conditions. Such an expression made it difficult to construct the Lagrangian, which requires a compact, single matrix equation. Introduction of interactions was also difficult, because the interactions frequently contradicted subsidiary conditions.

Harish-Chandra[7] and Bhabha[8] studied in detail the algebraic aspects of wave equations of the special form

$$(\beta_\mu\partial_\mu+m)\psi=0. \tag{1a}$$

Although this equation has such a compact expression, it has a drawback: Its solutions correspond, not to unique spin, but to several spins. Furthermore, though we know that β matrices are the ones satisfying the condition

$$(\beta\cdot p)^{2s_{\max}-1}[(\beta\cdot p)^2-p^2]=0 \tag{1b}$$

(where $s_{\max}$ is the maximum spin of ψ and p is a four-dimensional vector), it has been an extremely difficult task to find an explicit expression for β_μ.

Derivation of various wave equations of high spin and their quantization (i.e., derivation of commutation relations, of Green's functions, of normalization conditions for wave functions, etc.) were put together in a simple formulation[9–11] by means of a differential

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[1] A detailed bibliography on classical works can be found, for example, in Refs. 2 and 3.

[2] E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave Equations* (Blackie & Son, Glasgow, Scotland, 1953).

[3] H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Co., Amsterdam, 1956).

[4] P. A. M. Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936).

[5] M. Fierz, Helv. Phys. Acta **12**, 3 (1939).

[6] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

[7] Harish-Chandra, Phys. Rev. **71**, 793 (1947); Proc. Roy. Soc. (London) **A192**, 195 (1947).

[8] H. J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945); **21**, 451 (1949).

[9] Y. Takahashi and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **9**, 1 (1953); Nucl. Phys. **51**, 193 (1964).

[10] H. Umezawa and A. Visconti, Nucl. Phys. **1**, 348 (1956).

[11] D. Lurié, Y. Takahashi, and H. Umezawa, J. Math. Phys. **7**, 1478 (1966).

operator [usually written as $d(\partial)$ and called the d operator], existence of which is due to the Klein-Gordon equation: When a wave equation

$$\Lambda(\partial)\psi=0$$

leads to the Klein-Gordon equation, there exists an operator $d(\partial)$ which satisfies the relation

$$d(\partial)\Lambda(\partial)=\Lambda(\partial)d(\partial)=(\Box-m^2). \quad (2)$$

It has been shown[10] that the Harish-Chandra condition (1b) is a simple result of the combination of (1a) and (2). The general formulation based on the use of the d operator has frequently been used in modern articles on high-spin particles.[12–17]

The $d(\partial)$ operator is known to be intimately related to the projection operators which have been extensively used in the current literature.[18–21] To be more specific, the d operator acts as a projection operator for energy momentum on the mass shell. This property of the d operator has been emphasized in the second article of Ref. 9 and also in Ref. 22. As was shown by Fronsdal,[18] when the energy momentum is not restricted by the mass-shell condition, the projection operators in general contain certain singular terms. Since one of the roles of wave equations is to confine the energy momentum to the mass shell, the construction of wave equations requires a knowledge of projection operators for arbitrary values of energy momentum, leading us to the appearance of singular terms in the expression of the wave equations themselves.

These singular terms have been eliminated by the introduction of auxiliary fields of the kind discussed by Fierz and Pauli in their early articles. This method has been revised and fully developed by Chang.[20] In this approach there naturally appears a certain number of arbitrary parameters in the expressions of wave equations. In this paper we follow a different approach, in which the singular terms are eliminated without any explicit use of auxiliary fields. As will be discussed later, we meet again some arbitrary parameters.

A complication in problems of relativistic wave equations of high spins is due to a huge variety of choice for expressions of field equations. Such a variety is due, first, to freedom in choosing the representation for the wave function ψ. For example, the spin-$\frac{1}{2}$ wave function can be realized in the spinor representation, the three-spinor representation, the vector-spinor representation, and so on. There is a further variety even when we stick to a special choice of representation of ψ. As a matter of fact, it has been shown[13,18,23,24] that the general form of the Lagrangian for spin $\frac{3}{2}$ in the vector-spinor representation contains one arbitrary parameter, even when we restrict the order of derivatives in the Lagrangian to be the lowest one (i.e., of the first order), and therefore that the Rarita-Schwinger equation[25] corresponds to a special value of such a parameter. In fact, there is no simple criterion in preferring one realization to any other. Although these two ambiguities, which are the price we pay in order to obtain simple covariant properties of wave functions, are irrelevant in the case of free fields because of the equivalence of the various theories, it is no longer trivial when an attempt is made to construct a quantum theory of interacting fields. Green's functions, in fact, will depend, in general, upon the choice of the representations, and in a given representation, they will depend upon a certain number of the arbitrary parameters mentioned above. The relevance of the choice of representations can be seen in studying the electromagnetic properties of a field of spin $\frac{1}{2}$. It has been shown,[20] for example, that a system of spin $\frac{1}{2}$ described by a third-rank spinor is not equivalent to the Dirac field in the presence of minimal electromagnetic interactions. This observation could be of interest in the physics of strong interactions, particularly in connection with the problem of nuclear magnetic moments.[20] In regard to the presence of parameters in the equations, it should be noted that they appear, as will be shown later, only in the off-shell part of the operator $d(\partial)$ and reflect the existence of fields of several spins, which manifest themselves, not in the form of free particles, but by their dynamical effects due to interactions. For example, it is for this reason that the spinless pion can decay through the intermediate vector meson without violating the law of conservation of angular momentum. An interesting question is how far the S matrix can be independent of the choice of parameters. It has been stated by several authors[12,14] that, as far as the minimal electromagnetic interaction is concerned, the S matrix does not depend upon the values of these parameters (equivalence theorem).

Interactions have usually been introduced through the Lagrangian in order to avoid contradiction between interactions and subsidiary conditions, because in such a formulation the subsidiary conditions themselves come out of the Lagrangian. Although such a method might well work in nonquantized theories, the question of internal consistency arises once commutation

[12] A. Kawakami and S. Kamefuchi, Nuovo Cimento **48A**, 239 (1967).
[13] S. Kamefuchi, H. Shimoidara, and H. Watanabe, Nucl. Phys. **B2**, 360 (1967).
[14] S. C. Bhargawa and H. Watanabe, Nucl. Phys. **87**, 273 (1966).
[15] L. M. Nath, Nucl. Phys. **68**, 660 (1965).
[16] S. Kamefuchi and Y. Takahashi, Nuovo Cimento **44**, 1 (1966).
[17] R. J. Rivers, Nuovo Cimento **34**, 386 (1966).
[18] C. Fronsdal, Nuovo Cimento Suppl. **9**, 416 (1958).
[19] W. K. Tung, Phys. Rev. **156**, 1385 (1967).
[20] S. J. Chang, Phys. Rev. **161**, 1308 (1967); *ibid.* **161**, 1316 (1967).
[21] D. L. Pursey, Ann. Phys. (N. Y.) **32**, 157 (1965).
[22] A. Aurilia and H. Umezawa, Nuovo Cimento **51A**, 14 (1967).
[23] P. A. Moldauer and K. M. Case, Phys. Rev. **102**, 279 (1956).
[24] K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N. Y.) **13**, 126 (1961).
[25] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).

relations are introduced.[15,19,24,26,27] In the present paper we are not concerned with the latter question. Since the main difficulties of the higher-spin theories are caused by the existence of redundant components of the wave functions, there have been other approaches[28,29] which are formulated in terms of only the independent components satisfying the Klein-Gordon equation. In such a formulation the wave function possesses only the $2S+1$ components which are necessary to describe an object of spin s. Weinberg in particular derived Feynman rules for particles of any spin by means of the boosting techniques. In this approach the explicit construction of the Lagrangian is not required at all. In the present paper we propose a manifestly covariant formulation in which the wave functions are classified in terms of irreducible representations of the inhomogeneous Lorentz group.

As was shown by Wigner and Bargmann,[30,31] the irreducible representations of the inhomogeneous Lorentz group are specified by two quantities (mass and spin). Therefore, wave functions of any spin can be explicitly written down when one knows the projection operator which sorts out corresponding irreducible representations. The projection operators and wave functions are constructed usually by the booster operator which transforms the wave function in the rest system into the one in the moving system. However, the construction of the projection operators and wave functions becomes much simpler when use is made of the Pauli-Lubanski matrix,[32] which makes it unnecessary to refer to the rest system and which therefore avoids use of the booster operator. We have shown in a previous paper[22] how to construct the projection-operator by means of the Pauli-Lubanski matrix.

Since an essential role of the relativistic wave equation is to pick out a wave function of a given spin, it is natural to expect that the knowledge of projection operators should be helpful to construct the wave equations. This paper is aimed at presenting a systematic method for derivation of relativistic wave equations (of unique spin) by means of the projection operators with no explicit use of auxiliary fields.

In Sec. 2 we shall present the general structure of the method. The method is made up of two steps. The first step is to derive a compact expression for relativistic equations of several spins. In particular, when this method is applied to a many-spinor representation, it enables us to rewrite the Bargmann-Wigner equations in a compact, single matrix equation.[33,34] This method helps us also to find an explicit expression of the Harish-Chandra β matrices, because no auxiliary fields appear in wave equations. The second step is to derive wave equations of a specific spin, from the wave equations obtained by the first step, by means of the spin-projection operators. It should be noted that the method can be applied not only to the maximum spin fields, but also to every kind of spin field in the representation concerned (e.g., to two kinds of spin-$\frac{1}{2}$ fields in the representation of $S_{\max}=\frac{3}{2}$). This method explains also why the general form of the Lagrangian for high spin contains the arbitrary parameters which have been discovered by many authors[12–15,17,18,23,24] in the case of spin $\frac{3}{2}$, 2, and $\frac{5}{2}$. In Sec. 3 we illustrate our method by several examples. In Sec. 4 we summarize the main results of our argument. In Sec. 4 we also show how to extend our method to cover the case of spin-dependent mass spectra.

2. GENERAL METHOD

A. Classification of Wave Functions

Let us begin with a covariant description of a wave function of given spin (say, s) in terms of two quantities which are invariant under the inhomogeneous Lorentz transformation:

$$x_\mu' = x_\mu + \epsilon_\mu + \delta\omega_{\mu\nu} x_\nu\,, \qquad \phi'(x') = (I + \tfrac{1}{2} S_{\mu\nu}\delta\omega_{\mu\nu})\phi(x)\,. \tag{3}$$

The two quantities are

$$P = p_\mu p_\mu \tag{4a}$$

and

$$W = w_\mu w_\mu = \tfrac{1}{2} S_{\mu\nu} S_{\mu\nu} p^2 - S_{\mu\rho} S_{\mu\sigma} p_\rho p_\sigma\,. \tag{4b}$$

Here p_μ is the energy-momentum vector and w_μ is the Pauli-Lubanski matrix[32] defined as

$$w_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho} S_{\nu\sigma} p_\rho\,, \tag{5}$$

where $\epsilon_{\mu\nu\sigma\rho}$ is the antisymmetric Ricci tensor. These two quantities commute with the ten generators of the inhomogeneous Lorentz transformation and, consequently, are multiples of the identity operator for each irreducible representation of the inhomogeneous Lorentz group. The eigenvalues of W have a form $(-p^2)s(s+1)$, where s is either an integer or half-integer and is called spin. The eigenvalues of P are usually written as $-m^2$, and m thus defined is called mass. Each irreducible representation of the inhomogeneous Lorentz group is specified by two numbers (m,s). Wave functions for m,s are the eigenfunctions

[26] P. Federbush, Nuovo Cimento **19**, 572 (1961).
[27] J. Schwinger, Phys. Rev. **130**, 800 (1963).
[28] S. Weinberg, Phys. Rev. **133**, B1318 (1964); **134**, B882 (1964).
[29] D. L. Weaver, C. L. Hammer, and R. H. Good, Jr., Phys. Rev. **135**, B241 (1964).
[30] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[31] W. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. 34, 211 (1948).
[32] J. K. Lubanski, Physica **9**, 310 (1942).
[33] In Refs. 16 and 34 are found detailed considerations of the Bargmann-Wigner equations of spin $\frac{3}{2}$. A more systematic and simpler version, based on the use of auxiliary fields, has been presented by S. J. Chang (Ref. 20).
[34] G. S. Guralnik and T. W. B. Kibble, Phys. Rev. **139B**, 712 (1965).

of P and W:

$$(p^2+m^2)\phi(p)=0, \qquad (6)$$

$$W\phi(p)=-s(s+1)p^2\phi(p). \qquad (7)$$

Equation (4) is the Klein-Gordon equation.[35] There are two sets of such $\phi(p)$ which are distinguished by the sign ($\epsilon=\pm$) of the energy p_0. Each set has $2s+1$ linearly independent components which are distinguished from each other by eigenvalues of the polarization operator defined by

$$\mathbf{S}\cdot\mathbf{p}=iw_4, \qquad (8)$$

where $\mathbf{S}$ is the spin vector $(-iS_{23}, -iS_{31}, -iS_{12})$. Thus, the relativistic wave functions are specified by four quantities: m, s, s_n, and ϵ. Here s_n denotes the polarization eigenvalue.

Let the wave functions $\phi(p)$ be realized in tensor representations, in spinor representations, or in their mixtures. Let us call such representations for $\phi(p)$ the basic representations. As is well known, pure tensor representations can be used only for particles of integer spins. The matrix $S_{\mu\nu}$ is fixed when the basic representation is specified. Each basic representation contains several spin spaces: $s=s_{\max}, s_{\max}-1, \cdots, s_{\min}+1, s_{\min}$). Here

$$\begin{aligned} s_{\min}&=0 \quad \text{for integer } s_{\max} \\ &=\tfrac{1}{2} \quad \text{for half-odd-integer } s_{\max}. \end{aligned} \qquad (9)$$

To obtain the wave function with a specific spin s, one needs a projection operator to select the representation of spin s. Such a projection operator can be easily constructed by means of (7). The result is

$$P(s)=\prod_{s'\neq s}\frac{W+s'(s'+1)p^2}{-s(s+1)p^2+s'(s'+1)p^2}. \qquad (10)$$

Here s' runs over the numbers $(s_{\max}, s_{\max}-1, \cdots, s_{\min})$, avoiding s. It is obvious that

$$\sum_s P(s)=1. \qquad (11)$$

Since the representation of the maximum spin (i.e., $s=s_{\max}$) appears only once, this representation is irreducible. In other words, $P(s_{\max})$ selects an irreducible representation. This is the projection operator derived in our previous article.[22] On the other hand, the representations of $s<s_{\max}$ are not irreducible. Let us denote their multiplicities by $n(s)$. Each irreducible representation of spin s can be selected by certain projection operators which will be denoted by $D_i(s)$ $[i=1, \cdots, n(s)]$. Obviously,

$$P(s)=\sum_i D_i(s). \qquad (12)$$

[35] Equations of the form (6) and (7) were considered by D. L. Pursey Ref. (21). Several methods of handling subsidiary conditions can be found in the same reference.

We have thus introduced the following projection operators:

$$\{D_i(s), \quad s=(s_{\max}, \cdots, s_{\min}), \quad i=1, \cdots, n(s)\},$$

each selecting respective irreducible representations.

B. Construction of Projection Operators

Our next task is to construct the projection operators $D_i(s)$. To do this we employ algebraic induction.

Let us begin with the basic representation of a single spinor (i.e., the Dirac space). Obviously,

$$s_{\max}=s_{\min}=\tfrac{1}{2},$$

and the basic representation itself is irreducible:

$$D(\tfrac{1}{2})=P(\tfrac{1}{2})=I. \qquad (13)$$

Here I is the unit matrix.

We shall now choose the vector representation as the basic one. Then we find that $s_{\max}=1$ and $s_{\min}=0$, and that each spin space is irreducible:

$$\begin{aligned} D(1)&=P(1)=-(1/2p^2)W, \\ D(0)&=P(0)=1+(1/2p^2)W. \end{aligned} \qquad (14)$$

Here use was made of relation (10).

Let us now consider two basic representations $R^{(1)}$ and $R^{(2)}$ and denote the projection operators $D_i(s)$ in $R^{(1)}$ and $R^{(2)}$ by $D_i^{(1)}(s)$ and $D_i^{(2)}(s)$, respectively. Then construct a new basic representation R, by the direct product of these two representations: $R=R^{(1)}\otimes R^{(2)}$. The projection operators $P(s)$ in the new representation R can be obtained by means of (10). Construct now all the products of the form

$$[D_i^{(1)}(s')\otimes D_j^{(2)}(s'')]P(s), \qquad (15)$$

with a fixed s and varying i, j, s', and s''. Each nonvanishing product of such a kind is the member of the set $\{D_i(s)\}$ (for the specified s) in the basic representation R.

Let us finally note that the method presented here does not require any use of the booster operator.

C. Construction of Wave Functions

Suppose now that a basic representation is given and let the symbol $f(p)$ stand for functions which belong to the basic representation and which satisfy the Klein-Gordon equation:

$$(p^2+m^2)f(p)=0. \qquad (16)$$

Then, wave functions of particles of (m,s) are given by $D_i(s)f(p)$, which will be written by $f_i^{(s)}(p)$.

However, there is a further complication when the basic representation is a direct product of a certain number (say, r) of spinor spaces, possibly together with a tensor representation. Let us denote the Dirac matrices in these spinor spaces by $\gamma_\mu^{(j)}$ $(j=1, \cdots, r)$. The linear space of wave function $f_i^{(s)}(p)$ is not irreducible, but is made up of several spaces (of spin s), which are distinguished by a different choice of signs in the following conditions:

$$(i\gamma_\mu^{(j)}p_\mu \pm m)f_i^{(s)}(p)=0, \quad j=1,\cdots,r.$$

In other words, the linear space spanned by $f_i^{(s)}(p)$ is the direct sum of 2^r spaces of spin s. Let us denote the wave functions belonging to an irreducible representation of (m,s) by $\phi_i^{(s)}(p)$. Such an irreducible representation can be selected, for example, by the condition

$$(i\gamma_\mu^{(j)}p_\mu + m)\phi_i^{(s)}(p)=0, \quad j=1,\cdots,r \tag{17}$$

which is called the Bargmann-Wigner equation.

It follows from the Klein-Gordon equation for $\phi_i^{(s)}(p)$ that general solutions of (17) can be defined by the following expression:

$$\begin{aligned}\phi_i^{(s)}(p)&=\left(\prod_{j=1}^{r}\frac{-i\gamma^{(j)}\cdot p+m}{2m}\right)f_i^{(s)}(p)\\&=\left(\prod_{j=1}^{r}\frac{-i\gamma^{(j)}\cdot p+m}{2m}\right)D_i(s)f(p).\end{aligned} \tag{18}$$

Let us rewrite this form by means of the projection operator[36]

$$\chi^{(j)}(p)=\tfrac{1}{2}[1+p^{-2}(\gamma^{(j)}\cdot p)(\gamma^{(1)}\cdot p)], \quad j=2,\cdots,r. \tag{19}$$

Note that

$$\begin{aligned}\chi^{(j)}(p)(-i\gamma^{(1)}\cdot p+m)f_i^{(s)}&=\tfrac{1}{2}[1-(i/m)\gamma^{(j)}\cdot p](-i\gamma^{(1)}\cdot p+m)f_i^{(s)}(p)\\&=\frac{-i\gamma^{(j)}\cdot p+m}{2m}(-i\gamma^{(1)}\cdot p+m)f_i^{(s)}(p),\end{aligned}$$

and that

$$\left(\frac{-i\gamma^{(1)}\cdot p+m}{2m}\right)^2 f_i^{(s)}(p)=\frac{-i\gamma^{(1)}\cdot p+m}{2m}f_i^{(s)}(p),$$

because of the Klein-Gordon equation (16). Using these relations, we can rewrite (18) in the following form:

$$\begin{aligned}\phi_i^{(s)}(p)&=\chi(p)\frac{-i\gamma^{(1)}\cdot p+m}{2m}f_i^{(s)}(p)\\&=\chi(p)\frac{-i\gamma^{(1)}\cdot p+m}{2m}D_i(s)f(p),\end{aligned} \tag{20}$$

where χ is the projection operator defined by

$$\chi(p)=\prod_{j=2}^{r}\chi^{(j)}(p). \tag{21}$$

Equation (18) [or (20)] defines the general expression for wave functions of (m,s). The normalization of these wave functions can be fixed by the method presented in Refs. 22 and 9.

D. Compact Expression for the Bargmann-Wigner Equations

Let us now rewrite the set of equations (17) in a compact form. To do this, we first note that (20) is the general expression for solutions of the following set of equations:

$$\begin{aligned}(i\gamma^{(1)}\cdot p+m)\phi_i^{(s)}(p)&=0,\\ [1-\chi(p)]\phi_i^{(s)}(p)&=0.\end{aligned} \tag{22}$$

In otherwords, (17) and (22) are equivalent to each other. On the other hand, the set of equations in (22) is equivalent to the following single equation:

$$\lambda(a,ip)\phi_i^{(s)}(p)=0, \tag{23}$$

where

$$\lambda(a,ip)=-(i\gamma^{(1)}\cdot p+m)\chi(p)-a[1-\chi(p)], \tag{24}$$

where a is an arbitrary parameter.[37] The equivalence between (22) and (23) can be proved by multiplying the projection operators $\chi(p)$ or $1-\chi(p)$ into both sides of (23). Equation (23) presents the single matrix equation which is equivalent to the Wigner-Bargmann equation (17).

Defining the operator

$$h(a,ip)=(-i\gamma^{(1)}\cdot p+m)\chi(p)+(1/a)(p^2+m^2)[1-\chi(p)], \tag{25}$$

we can show that

$$\lambda(a,ip)h(a,ip)=h(a,ip)\lambda(a,ip)=-(p^2+m^2). \tag{26a}$$

Therefore, the Green's function for Eq. (23) is given by $h(a,ip)/(p^2+m^2)$.

When $r>2$, Eqs. (23) and (25) have singular terms due to the p^{-2} term in (19). These singularities can be eliminated by multiplying λ with a certain nonsingular matrix $q(a,p)$:

$$\bar{\lambda}(a,ip)=q(a,p)\lambda(a,ip). \tag{27a}$$

Let us further define

$$\bar{h}(a,ip)=h(a,ip)q^{-1}(a,p), \tag{27b}$$

so that

$$\bar{\lambda}(a,ip)\bar{h}(a,ip)=\bar{h}(a,ip)\bar{\lambda}(a,ip)=-(p^2+m^2). \tag{26b}$$

Construction of the matrix $q(a,p)$ will be discussed later. The operator $q(a,p)$ does not influence the $\chi(p)$ term either in $\lambda(a,p)$ or in $h(a,p)$, when

$$q(a,p)\chi(p)=\chi(p) \tag{28a}$$

[36] We write $A^{(i)}\otimes A^{(j)}$ simply as $A^{(i)}A^{(j)}$.

[37] There is a more general expression for λ:

$$\lambda(a_1\cdots a_l:ip)=(i\gamma^{(1)}\cdot p+m)\chi(p)-\textstyle\sum_{i=1}^{l}a_i\chi_i(p).$$

Here $\{\chi_i(p)\}$ are the set of mutually orthogonal projection operators which are orthogonal also to χ: $\chi_i(p)\chi_j(p)=\delta_{ij}\chi_j(p)$, $\chi_i\chi(p)=0$. For simplicity, we use a special choice of parameters: $a_1=a_2=\cdots=a_l=a$.

or

$$\chi(p)q^{-1}(a,p)=\chi(p) \tag{28b}$$

is satisfied.

When (28b) is satisfied, then (25) and (27b) lead to

$$\bar{h}(a,ip)=(-i\gamma^{(1)}\cdot p+m)\chi(p)+(1/a)(p^2+m^2)[q^{-1}(a,b)-\chi(p)], \tag{29}$$

which shows that

$$[\bar{h}(a,ip)]_{p^2=-m^2}=(-i\gamma^{(1)}\cdot p+m)\chi(p), \tag{30}$$

and therefore that

$$\left[\left(\frac{1}{2m}\bar{h}(a,ip)\right)^2\right]_{p^2=-m^2}=\left(\frac{1}{2m}\bar{h}(a,ip)\right)_{p^2=-m^2}. \tag{31}$$

In other words, (28b) is a sufficient condition for $(1/2m)\bar{h}(a,ip)$ (with $p^2=-m^2$) to be a projection operator.

E. Derivation of Relativistic Wave Equation

The wave functions $\phi_i^{(s)}(p)$ for spin s are the solutions of the following set of equations:

$$\begin{aligned} F(p)D_i(s)\phi(p)&=0,\\ D_j(u)\phi(p)&=0 \quad \text{for} \quad (j,u)\neq(i,s). \end{aligned} \tag{32}$$

Here $(j,u)\neq(i,s)$ means that either $j\neq i$ or $u\neq s$. The operator $F(p)$ is defined by

$$\begin{aligned} F(p)&=-(p^2+m^2) \quad \text{for tensor representation},\\ &=\bar{\lambda}(a,ip) \quad \text{otherwise}. \end{aligned} \tag{33}$$

The operator $\bar{\lambda}$ was given in the previous section [cf. (24) and (27a)]. Since $D_i(s)$ and $D_j(u)$ are projection operators, we see that (29) is equivalent to the following equation:

$$\Lambda(ip)\phi(p)=0, \tag{34}$$

with

$$\Lambda(ip)=F(p)D_i(s)-\sum_{(j,u)\neq(i,s)} a_j^{(u)}D_j(u). \tag{35}$$

Here the parameters $a_i^{(u)}$ are arbitrary, unless zero. Equation (34) is a single matrix expression for a relativistic wave equation of (m,s). This form will be called the primitive form.[38] The operator $d(ip)$, which satisfies

$$d(ip)\Lambda(ip)=\Lambda(ip)d(ip)=-(p^2+m^2), \tag{36}$$

is given by

$$d(ip)=G(p)D_i(s)+\sum_{(j,u)\neq(i,s)}(p^2+m^2)\frac{1}{a_j^{(u)}}D_j(u), \tag{37}$$

where

$$\begin{aligned} G(p)&=1 \quad \text{for tensor representation},\\ &=\bar{h}(a,p) \quad \text{otherwise}. \end{aligned} \tag{38}$$

[38] A special case of such a form (i.e., $s=\frac{3}{2}$, $a_j^{(u)}=m$) was once derived by Fronsdal (cf. Ref. 18) in his consideration of the Pauli-Fierz equation. There the projection operator $D(\frac{3}{2})$ was constructed by means of the booster.

Obviously,

$$G(p)F(p)=F(p)G(p)=-(p^2+m^2). \tag{39}$$

The most general expression for a relativistic wave equation can be obtained by multiplying the primitive form $\Lambda(ip)$ by all possible nonsingular matrices:

$$\tilde{\Lambda}(ip)=\eta(p)\Lambda(ip). \tag{40}$$

The corresponding d operator is given by

$$\tilde{d}(ip)=d(ip)\eta^{-1}(p), \tag{41}$$

because

$$\tilde{d}(ip)\tilde{\Lambda}(ip)=\tilde{\Lambda}(ip)\tilde{d}(ip)=-(p^2+m^2) \tag{42}$$

due to (36). The form (37) shows where the off-shell terms [i.e., the (p^2+m^2) term] in the d operator come from.

It is convenient to choose $\eta(p)$ in such a way that the $D_i(s)$ term either in Λ or d does not change:

$$\eta(p)D_i(s)=D_i(s) \tag{43a}$$

or

$$D_i(s)\eta^{-1}(p)=D_i(s). \tag{43b}$$

When (43b) is satisfied, we find that

$$[\tilde{d}(ip)]_{p^2=-m^2}=[G(p)D_i(s)]_{p^2=-m^2}, \tag{44}$$

which further leads to

$$[(\tilde{d}(ip))^2]_{p^2=-m^2}=[\tilde{d}(ip)]_{p^2=-m^2}, \tag{45}$$

when (28b) is also satisfied. In other words (43b) and (28b) form a sufficient condition for $\tilde{d}(ip)$ (with $p^2=-m^2$) to be a projection operator.

It was proved in the second article of Ref. 9 that the d operator with p on the mass shell is a projection operator when the nonderivative term in the wave equation is a multiple of the unit matrix. Such a form of the wave equation was called the standard form. We have just seen that there is a much wider choice of wave equations, for which the d operator with p on the mass shell is a projection operator. Thus, generalizing the meaning of standard form, we say that wave equations are of the standard form when the corresponding d operator with p on the mass shell is a projection operator. We thus see that (43b) and (28b) form a sufficient condition for $\tilde{\Lambda}$ to be of the standard form. According to this generalized definition of the standard form, the Rarita-Schwinger equation for spin $\frac{3}{2}$, for example, is of the standard form although its nonderivative term is not the unit matrix. It is obvious that the primitive form is a special case $[\eta(p)=1]$ of the standard form.

It happens frequently that $\tilde{\Lambda}$ has singular terms of the p^{-2} form. To avoid such singular terms restricts the choice for $\eta(p)$. Elimination of singular terms is performed by repeated mixing of the terms in (35). This means that the elimination is performed by repeated multiplications of the following operators on $\Lambda(ip)$:

$$\eta_{kj}(v,u)=1+D_k(v)A_{kj}(v,u)D_j(u). \tag{46}$$

In other words, η is a product of operators of this kind. When $(k,v)\neq(j,u)$, $\eta_{kj}(v,u)$ in (46) is nonsingular whatever $A_{kj}(v,u)$ is; its inverse is given by

$$\eta^{-1}{}_{kj}(v,u)=1-D_k(v)A_{kj}(v,u)D_j(u) \quad \text{for} \quad (k,v)\neq(j,u). \tag{47}$$

When $(k,v)=(j,u)$, $A_{kj}(v,u)$ must satisfy certain conditions in order to make $\eta_{kj}(v,u)$ nonsingular. In particular, when $A_{kj}(v,u)$ is a c number (say, a), $\eta_{kj}(v,u)$ with $(k,v)=(j,u)$ takes the form $1+aD_k(v)$ which is nonsingular unless $a=-1$. The elimination of p^{-2} singular terms can be performed in the following way. First, choose a set of $A_{kj}(v,u)$ in such a way that the operators $D_k(v)A_{kj}(v,u)D_j(u)$ and their repeated products produce all the singular terms which appear in $\Lambda(ip)$ in (35). It is wise to begin with the choice $(k,v)\neq(j,u)$ and also with

$$(j,u)\neq(i,s) \tag{48a}$$

or

$$(k,v)\neq(i,s). \tag{48b}$$

The conditions (48a) and (48b) correspond, respectively, to (43a) and (43b). Instead of presenting a general but lengthy explanation, we shall illustrate the method in Sec. 3 D by means of the vector-spinor representation for $s=\frac{3}{2}$. The situation is quite similar when we want to avoid the singularities in $\tilde{\lambda}(a,p)$ [cf. (27)] by means of a suitable choice of $q(a,p)$; we again choose a product of operators of the form $(1+\chi_iC_{ij}\chi_j)$, where χ_i and χ_j are two projection operators chosen from the set $\{\chi_{kl}{}^{(\pm)}\}$ defined by $\chi_{kl}{}^{(\pm)}=1\pm p^{-2}(\gamma^{(k)}\cdot p)(\gamma^{(l)}\cdot p)$.

F. Construction of Lagrangian

The covariant form of $\tilde{\Lambda}$ is not Hermitian due to the indefinite nature of the Minkowski metric. To change $\tilde{\Lambda}$ into a Hermitian operator, we need a nonsingular matrix ρ of the form

$$\rho=(\prod_i \gamma_4{}^{(i)})(\prod_l g^{(l)}), \tag{49a}$$

where the first bracket is the direct product of γ_4 matrices associated with the spinor suffixes of the wave function, and the second bracket is the direct product of the metric tensors $g\equiv(g_{\mu\nu})$ due to all the vector suffixes:

$$\begin{aligned} &g_{ii}=1 \quad \text{for} \quad i=1,\,2,\,3;\\ &g_{44}=-1;\\ &g_{\mu\nu}=0 \quad \text{otherwise}. \end{aligned} \tag{49b}$$

We shall choose the arbitrary parameters appearing in $\tilde{\Lambda}$ in such a way that $\rho\tilde{\Lambda}$ becomes Hermitian:

$$\tilde{\Lambda}^{\dagger}(ip)\rho=\rho\tilde{\Lambda}(-ip). \tag{50}$$

The Lagrangian can be constructed as

$$L=\int d^4x\,\bar{\phi}(x)\tilde{\Lambda}(\partial)\phi(x), \tag{51a}$$

with

$$\bar{\phi}=\phi^{\dagger}\rho. \tag{51b}$$

3. EXAMPLES

In this section we shall illustrate the method presented in the previous section by means of several examples.

A. Scalar Basic Representation

The wave function has a single component $\phi(p)$. Since $S_{\mu\nu}=0$ in this representation, we see that $W=0$. The basic representation itself is irreducible and the relations (35) and (36) read as

$$\Lambda(ip)=-(p^2+m^2),$$
$$d(ip)=1.$$

Thus the wave equation is

$$(\Box-m^2)\phi(x)=0.$$

B. Vector Basic Representation

The wave function is a four-dimensional vector $\phi_\mu(p)$. This representation contains two representations ($s=1$ and $s=0$), each being irreducible. By inspection of the Lorentz transformation of ϕ_μ, we see that

$$(S_{\mu\nu})_{\sigma\rho}=\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma}. \tag{52}$$

Then (4b) leads to

$$W_{\sigma\rho}(p)=-2(p^2\delta_{\sigma\rho}-p_\sigma p_\rho). \tag{53}$$

The projection operators are given by relation (10):

$$D_{\sigma\rho}(1)=P_{\sigma\rho}(1)=-W_{\sigma\rho}/2p^2 =\delta_{\sigma\rho}-p_\sigma p_\rho/p^2, \tag{54}$$

$$D_{\sigma\rho}(0)=P_{\sigma\rho}(0)=p_\sigma p_\rho/p^2. \tag{55}$$

The primitive form (35) of the wave equation reads, in the case of unit spin $s=1$, as

$$\Lambda^{(1)}(ip)=-(p^2+m^2)D(1)-aD(0),$$

where a is an arbitrary parameter.

The p^{-2} singular terms do not appear when $a=m^2$. In this case, we find that

$$\Lambda_{\sigma\rho}{}^{(1)}(ip)=-[(p^2+m^2)\delta_{\sigma\rho}-p_\sigma p_\rho], \tag{56}$$

which gives the well-known Proca wave equation[39]:

$$(\Box-m^2)\phi_\sigma-\partial_\sigma\partial_\rho\phi_\rho=0.$$

The d operator is given by (37) as follows:

$$d_{\sigma\rho}{}^{(1)}(ip)=\delta_{\sigma\rho}+p_\sigma p_\rho/m^2. \tag{57}$$

The primitive form (35) for the spinless case is

$$\Lambda^{(0)}(ip)=-(p^2+m^2)D(0)-aD(1).$$

[39] A. Proca, Compt. Rend. **202**, 1490 (1936).

Here too, the p^{-2} singularities do not appear when $a=m^2$. This choice for the parameter a leads to

$$\Lambda_{\sigma\rho}^{(0)}(ip)=-[p_\sigma p_\rho+m^2\delta_{\sigma\rho}],$$

which leads us to the following equation for *spinless* particle[40]:

$$\partial_\sigma\partial_\rho\phi_\rho-m^2\phi_\sigma=0. \quad (58)$$

The corresponding d operator can be derived by means of (37). The result is[41]

$$d_{\sigma\rho}^{(0)}(ip)=(1/m^2)[(p^2+m^2)\delta_{\sigma\rho}-p_\sigma p_\rho]. \quad (59)$$

C. Two-Spinors Basic Representation

The wave functions in this representation are represented by $\phi_{\alpha\beta}$, where α and β are spinor suffixes. The γ_μ matrices associated with the suffixes α and β are denoted by $\gamma_\mu^{(1)}$ and $\gamma_\mu^{(2)}$, respectively. There are two spin states, i.e., $s=1$ and $s=0$. The representation for each spin is irreducible:

$$D(1)=P(1),\quad D(0)=P(0). \quad (60)$$

The Lorentz transformation of $\phi_{\alpha\beta}$ shows that

$$\begin{aligned}S_{\mu\nu}&=S_{\mu\nu}^{(1)}+S_{\mu\nu}^{(2)}\\&=[\beta_\mu,\beta_\nu],\end{aligned} \quad (61)$$

where

$$S_{\mu\nu}^{(i)}=\tfrac{1}{4}[\gamma_\mu^{(i)},\gamma_\nu^{(i)}] \quad (62)$$

and

$$\beta_\mu=\tfrac{1}{2}(\gamma_\mu^{(1)}+\gamma_\mu^{(2)}). \quad (63)$$

The relation (4b) leads to

$$W=p^2(\beta^2)^2-\beta^2[3p^2+2(\beta\cdot p)^2]+4(\beta\cdot p)^2. \quad (64)$$

The projection operators are obtained from (10) as

$$D(1)=-W/2p^2,\quad D(0)=1+W/2p^2. \quad (65)$$

Let us now construct the operator $\lambda(a,ip)$ introduced by (24). There is only one possibility for $\chi^{(j)}(p)$ defined by (19), i.e.,

$$\chi(p)=\chi^{(2)}(p)=\tfrac{1}{2}[1+p^{-2}(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)]. \quad (66)$$

The p^{-2} singularity in $\lambda(a,p)$ disappears when we choose the parameter as $a=m$. In this case we find that

$$\lambda(m,ip)=-\{\tfrac{1}{2}i(\gamma^{(1)}\cdot p)[1+p^{-2}(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)]+m\},$$

which can be rewritten as

$$\lambda(m,ip)=-[i(\beta\cdot p)+m], \quad (67)$$

which has the well-known Duffin-Kemmer-Petiau form[42–44]. Since $\lambda(m,ip)$ does not contain any p^{-2} singularity, we can use $\lambda(m,p)$ itself as $\bar{\lambda}(m,p)$ [cf. (33)].

[40] Equation (58) leads to the Klein-Gordon equation for $\partial_\mu\phi_\mu$: $(\Box-m^2)\partial_\mu\phi_\mu=0$. Denoting $\partial_\mu\phi_\mu$ by ϕ, Eq. (58) shows that $\phi_\sigma=(1/m^2)\partial_\sigma\phi$. Thus, $\phi_1=\phi_2=\phi_3=0$ in the rest system.

[41] It is interesting to note that

$$\Lambda^{(0)}=-m^2d^{(1)}\quad\text{and}\quad d^{(0)}=-(1/m^2)\Lambda^{(1)}.$$

[42] R. J. Duffin, Phys. Rev. **54**, 114 (1938).

[43] N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939).

[44] G. Petiau, thesis, Paris, 1936 (unpublished).

The primitive form (35) of the wave equation for the spin-1 case reads as

$$\Lambda^{(1)}(ip)=(1/2p^2)(i\beta\cdot p+m)W-a(1+W/2p^2), \quad (68)$$

where a is an arbitrary parameter. Because of the relation

$$(\beta\cdot p)^3=p^2(\beta\cdot p),$$

(64) leads to the relation

$$(\beta\cdot p)W=-2p^2P(\beta\cdot p), \quad (69)$$

where P is the matrix defined by

$$P=-\tfrac{1}{2}(\beta^2-4)(\beta^2-1). \quad (70)$$

As was shown in Ref. 22, this matrix satisfies the relations

$$\begin{aligned}&[P,\beta_\mu]=0,\\&P^2\beta_\mu=P\beta_\mu.\end{aligned} \quad (71)$$

We can compute $\Lambda^{(1)}$ in (68) by means of (69) and find that $\Lambda^{(1)}(ip)$ does not have any p^{-2} singularity when $a=m$; this choice for a leads to

$$\Lambda^{(1)}(ip)=-(i\tilde{\beta}\cdot p+m), \quad (72)$$

with

$$\tilde{\beta}_\mu=P\beta_\mu. \quad (73)$$

Therefore, the wave equation for ϕ with $s=1$ is

$$(\tilde{\beta}_\mu\partial_\mu+m)\phi(x)=0. \quad (74)$$

This is the equation which has been derived in our previous paper.[22] The derivation in this paper is more systematic and simpler than the previous one.

Let us now compute the d operator. To do this, we shall first construct $h(m,p)$ by means of (25) and (66):

$$\begin{aligned}\bar{h}(m,ip)&=h(m,p)\\&=-[i(\beta\cdot p)-(1/m)(i\beta\cdot p)^2\\&\qquad-(1/m)(p^2+m^2)].\end{aligned} \quad (75)$$

This agrees with the d operator[9,10] for the Duffin-Kemmer-Petiau equation, solutions of which have $s=1$ or 0. To derive the d operator for the case of unit spin $s=1$, we make use of the expression (37):

$$d^{(1)}(ip)=\bar{h}(m,ip)D(1)+(1/a)(p^2+m^2)D(0).$$

Note that we have chosen the parameter a so that $a=m$. Making use of (65), (69), and (71), we obtain

$$\begin{aligned}d^{(1)}(ip)=-[&i(\tilde{\beta}\cdot p)-(1/m)(i\tilde{\beta}\cdot p)^2\\&-(1/m)(p^2+m^2)].\end{aligned} \quad (76)$$

The primitive form (35) for the spinless case reads as

$$\Lambda^{(0)}(ip)=\lambda(m,p)D(0)-aD(1). \quad (77)$$

We shall choose $a=m$ to avoid the p^{-2} singularities. Then, using (65) and (67), we find that

$$\Lambda^{(0)}(ip)=-[i(\hat{\beta}\cdot p)+m], \quad (78)$$

with

$$\hat{\beta}_\mu = (1-P)\beta_\mu. \tag{79}$$

The d operator can be computed by (37). The result is

$$d^{(0)}(ip) = -[i(\hat{\beta}\cdot p) - (1/m)(i\hat{\beta}\cdot p)^2 - (1/m)(p^2+m^2)]. \tag{80}$$

D. Vector-Spinor Basic Representation

In this representation the wave functions are represented by $\phi_{\mu,\alpha}$, where μ and α are the vector and spinor suffixes, respectively. Such wave functions correspond to spin $\frac{3}{2}$ and spin $\frac{1}{2}$. There is only one irreducible representation for spin $\frac{3}{2}$:

$$D(\tfrac{3}{2}) = P(\tfrac{3}{2}). \tag{81}$$

There are two irreducible representations for spin $\frac{1}{2}$. We thus have two projection operators, $D_1(\frac{1}{2})$, and $D_2(\frac{1}{2})$, for spin $\frac{1}{2}$:

$$P(\tfrac{1}{2}) = D_1(\tfrac{1}{2}) + D_2(\tfrac{1}{2}). \tag{82}$$

Making use of the relation

$$(S_{\mu\nu})_{\sigma\rho} = (\delta_{\sigma\mu}\delta_{\rho\nu} - \delta_{\mu\rho}\delta_{\sigma\nu}) + \tfrac{1}{4}\delta_{\sigma\rho}[\gamma_\mu,\gamma_\nu], \tag{83}$$

we can derive W from (4b). The result is

$$W_{\sigma\rho} = -(15/4)p^2\delta_{\sigma\rho} + p^2\gamma_\sigma\gamma_\rho + 2p_\sigma p_\rho + (\gamma\cdot p)(\gamma_\sigma p_\rho - p_\sigma\gamma_\rho), \tag{84a}$$

which will be simply written, in the following, as

$$W = -(15/4)p^2 + p^2\gamma\bar{\gamma} + 2p\bar{p} + (\gamma\cdot p)(\gamma\bar{p} - p\bar{\gamma}). \tag{84b}$$

The rule of abbreviation is obvious.

The projection operators $P(\frac{3}{2})$ and $P(\frac{1}{2})$ are derived from (10) as

$$P(\tfrac{3}{2}) = 1 - \tfrac{1}{3}\gamma\bar{\gamma} - (2/3p^2)p\bar{p} - (1/3p^2)(\gamma\cdot p)(\gamma\bar{p} - p\bar{\gamma}) \tag{85}$$

and

$$P(\tfrac{1}{2}) = \tfrac{1}{3}\gamma\bar{\gamma} + (2/3p^2)p\bar{p} + (1/3p^2)(\gamma\cdot p)(\gamma\bar{p} - p\bar{\gamma}). \tag{86}$$

To compute $D_1(\frac{1}{2})$ and $D_2(\frac{1}{2})$, we shall use the products in (15), in which $R^{(1)}$ and $R^{(2)}$ are chosen to be a vector and a spinor representation, respectively. Nonvanishing terms of the form (15) with $s=\frac{1}{2}$ are the following:

$$D_1(\tfrac{1}{2}) = [D^{(1)}(1)\otimes D^{(2)}(\tfrac{1}{2})]P(\tfrac{1}{2}), \tag{87}$$

$$D_2(\tfrac{1}{2}) = [D^{(1)}(0)\otimes D^{(2)}(\tfrac{1}{2})]P(\tfrac{1}{2}). \tag{88}$$

Recalling that $D^{(2)}(\frac{1}{2})$ is the unit matrix in $R^{(2)}$, we find that

$$D_1(\tfrac{1}{2}) = \tfrac{1}{3}\gamma\bar{\gamma} - (1/3p^2)p\bar{p} + (1/3p^2)(\gamma\cdot p)(\gamma\bar{p} - p\bar{\gamma}), \tag{89}$$

$$D_2(\tfrac{1}{2}) = p^{-2}p\bar{p}. \tag{90}$$

Here were used the expressions (54) and (55) for $D^{(1)}(1)$ and $D^{(1)}(0)$, respectively.

According to (35) and (24) (with $\chi=1$), the primitive form for spin $\frac{3}{2}$ is

$$\Lambda^{(3/2)}(ip) = -(i\gamma\cdot p + m)D(\tfrac{3}{2}) - a_1 m D_1(\tfrac{1}{2}) - a_2 m D_2(\tfrac{1}{2}). \tag{91}$$

The corresponding d operator is given by (37) as

$$d^{(3/2)}(ip) = (-i\gamma\cdot p + m)D(\tfrac{3}{2}) + [(p^2+m^2)/m] \times ((1/a_1)D_1(\tfrac{1}{2}) + (1/a_2)D_2(\tfrac{1}{2})), \tag{92}$$

where (25) was taken into account. In the following, $D_1(\frac{1}{2})$ and $D_2(\frac{1}{2})$ will be simply denoted by D_1 and D_2.

The operator $\Lambda^{(3/2)}(ip)$ given by (91) contains four kinds of singular terms:

$$p^{-2}p\bar{p}, \quad p^{-2}(\gamma\cdot p)p\bar{p}, \quad (\gamma\cdot p)\gamma\bar{p}, \quad (\gamma\cdot p)p\bar{\gamma}. \tag{93}$$

The singular terms do not disappear simultaneously for any choice of the parameters a_1 and a_2. Therefore, we shall use the nonsingular matrices of the form (46) to eliminate the singular terms. For the sake of simplicity we shall choose the matrices $A_{kj}(u,v)$ in (46) in such a way that their derivatives are of the lowest order. We shall thus begin with $A_{kj}(u,v)$ with no derivatives. The possible form of such a kind is $\gamma\bar{\gamma}$. Then, note the following relations:

$$\begin{aligned} D_2\gamma\bar{\gamma}D_1 &= p^{-2}(\gamma\cdot p)p\bar{\gamma} - p^{-2}p\bar{p}, \\ D_1\gamma\bar{\gamma}D_2 &= p^{-2}p\bar{p} - p^{-2}(\gamma\cdot p)\gamma\bar{p}, \\ D_2\gamma\bar{\gamma}D_1\gamma\bar{\gamma}D_2 &= (3/p^2)p\bar{p}, \\ D_1\gamma\bar{\gamma}D_2\gamma\bar{\gamma}D_1 &= 3D_1. \end{aligned} \tag{94a}$$

Thus, $D_2\gamma\bar{\gamma}D_1$, $D_1\gamma\bar{\gamma}D_2$, and their products create all the singular terms in (93) except the second one (i.e., $(\gamma\cdot p)p\bar{p}/p^2$). Therefore, we shall introduce the first-order derivative through two matrices,[45] i.e., $(\gamma\cdot p)D_2\gamma\bar{\gamma}D_1$ and $(\gamma\cdot p)D_1\gamma\bar{\gamma}D_2$. The second term in (93) is now created by the following product:

$$(\gamma\cdot p)D_2\gamma\bar{\gamma}D_1\gamma\bar{\gamma}D_2 = (3/p^2)(\gamma\cdot p)p\bar{p}. \tag{94b}$$

Therefore, all the singular terms of (93) can be eliminated by means of the following nonsingular matrix:

$$\eta^{(1,2)}(p) = \eta_2(p)\eta_1(p), \tag{95a}$$

with

$$\begin{aligned} \eta_1(p) &= 1 + [(ig_1/m)(\gamma\cdot p) + g_2]D_1\gamma\bar{\gamma}D_2, \\ \eta_2(p) &= 1 + [(if_1/m)(\gamma\cdot p) + f_2]D_2\gamma\bar{\gamma}D_1. \end{aligned} \tag{95b}$$

Not all of the four parameters g_1, g_2, f_1, and f_2 are necessary for the purpose of eliminating the singular terms: Since both $(\gamma\cdot p)D_2\gamma\bar{\gamma}D_1\gamma\bar{\gamma}D_2$ and $D_2\gamma\bar{\gamma}D_1 \times(\gamma\cdot p)\gamma\bar{\gamma}D_2$ can create the singular term of the second form in (93), all of the four types of singular terms in (93) can be eliminated even when $f_1=0$ or $g_1=0$. We shall, however, continue our discussion with the four parameters in (95b) in order to obtain a wider class of wave equations. The matrix $\eta^{(1,2)}$ satisfies both of the conditions (43a) and (43b).

[45] Note that the projection operators $D(\frac{3}{2})$, D_1, and D_2 commute with $(\gamma\cdot p)$.

It is obvious from the relations in (94a) and (94b) that the product $\eta_2\eta_1$ creates no terms of the form $(\gamma\cdot p)\gamma\bar{\gamma}$, because the latter term requires the existence of the product $(\gamma\cdot p)D_1\gamma\bar{\gamma}D_2\gamma\bar{\gamma}D_1$, as is shown by the last line of (94a). Therefore, certain terms of the form $(\gamma\cdot p)\gamma\bar{\gamma}$ come out when we multiply $\eta_2\eta_1$ by a matrix which contains $D_1\gamma\bar{\gamma}D_2$. Since $D_1\gamma\bar{\gamma}D_2$ is a part[46] of $\gamma\bar{\gamma}$, it is much simpler to use $\gamma\bar{\gamma}$ itself instead of $D_1\gamma\bar{\gamma}D_2$. Because of the relation

$$D(\tfrac{3}{2})\gamma\bar{\gamma}=\gamma\bar{\gamma}D(\tfrac{3}{2})=0,$$

both of the conditions (43a) and (43b) are satisfied by the following nonsingular matrix:

$$\eta_3(p)=1+k\gamma\bar{\gamma},\quad k\neq-\tfrac{1}{4}, \tag{95c}$$

the inverse matrix of which is

$$\eta_3^{-1}(p)=1-[k/(4k+1)]\gamma\bar{\gamma}. \tag{95d}$$

We thus see that the number of parameters in the wave equations of the standard form increase even when we take $\eta_3(p)$ into account. Summarizing, we obtain a wide class of expressions for $\tilde{\Lambda}^{(3/2)}$ of the nonsingular standard form as

$$\tilde{\Lambda}^{(3/2)}(ip)=\eta(p)\Lambda^{(3/2)}(ip), \tag{96}$$

with

$$\eta(p)=\eta_3(p)\eta_2(p)\eta_1(p). \tag{97}$$

Practical computations are much simplified when use is made of the relations in (94a) and (94b) and of the following:

$$\begin{aligned}D_2\gamma\bar{\gamma}D_1(\gamma\cdot p)&=-(\gamma\cdot p)D_2\gamma\bar{\gamma}D_1,\\ D_1\gamma\bar{\gamma}D_2(\gamma\cdot p)&=-(\gamma\cdot p)D_1\gamma\bar{\gamma}D_2,\end{aligned} \tag{98}$$

which state that $D_2\gamma\bar{\gamma}D_1$ and $D_1\gamma\bar{\gamma}D_2$ anticommute with $(\gamma\cdot p)$.

When the parameters g_1, g_2, f_1, f_2, a_1, and a_2 satisfy the relations

$$\begin{aligned}a_2&=\tfrac{1}{3}[4-(1/a_1)],\\ f_2&=(1/3a_1)(a_1-1),\\ g_2&=(1/3a_2)(a_1-1),\\ f_1a_1&=g_1a_2-\tfrac{2}{3}a_1,\end{aligned} \tag{99}$$

no singular terms appear in $\tilde{\Lambda}$, which then takes the following form:

$$\begin{aligned}\tilde{\Lambda}^{(3/2)}(ip)=&-\{(i\gamma\cdot p+m)-\tfrac{1}{3}i[1-(2a_1-3g_1a_2)k]\\ &\times(\gamma\cdot p)\gamma\bar{\gamma}+\tfrac{1}{3}[(a_1-1)+(4a_1-1)k]m\gamma\bar{\gamma}\\ &-\tfrac{1}{3}i(1+3g_1a_2)(\gamma\bar{p}-p\bar{\gamma})-\tfrac{2}{3}i[a_1-(3g_1a_2-2a_1)k]p\bar{\gamma}\\ &-2i(2g_1a_2-\tfrac{1}{3})k\gamma\bar{p}\}-(3f_1g_1a_2/m)\\ &\times[(1+2k)p\bar{p}-k(\gamma\cdot p)\gamma\bar{p}].\end{aligned} \tag{100}$$

As is seen from (99), this form of $\tilde{\Lambda}$ depends on three parameters, i.e., a_1, k, and f_1. As was anticipated, elimination of singular terms in $\tilde{\Lambda}$ does not fix g_1 and f_1, but gives only a relation among them [i.e., the last of Eqs. (99)]. Since the parameter k does not appear in (99), it has nothing to do with the elimination of singular terms. Indeed, k was introduced, not for the purpose of elimination of singular terms, but to modify the coefficient of the $(\gamma\cdot p)\gamma\bar{\gamma}$ term in (100).

The d operator corresponding to $\tilde{\Lambda}$ in (100) is calculated by the relation

$$\tilde{d}^{(3/2)}(ip)=d(ip)\eta^{-1}(p), \tag{101}$$

with $d(ip)$ given in (92). The result is

$$\begin{aligned}\tilde{d}^{(3/2)}(ip)=&(-i\gamma\cdot p+m)\left[1-\tfrac{1}{3}\gamma\bar{\gamma}+\frac{i}{3m}(\gamma\bar{p}-p\bar{\gamma})+\frac{2}{3}\frac{p\bar{p}}{m^2}\right]+\frac{(p^2+m^2)}{m}\left\{\left[\left(\frac{1}{a_2}-\frac{2}{3}\right)-\frac{k}{4k+1}\frac{1}{a_1a_2}\right]\gamma\bar{\gamma}\right.\\ &+\left[\frac{g_1}{a_1}+\frac{2}{3}\left(1-\frac{1}{a_2}\right)\right]\frac{i}{m}\gamma\bar{p}+\left[\left(\frac{2}{3}-\frac{g_1}{a_1}\right)-\frac{2k}{4k+1}\left(1-\frac{2g_1}{a_1}\right)\right]\frac{i}{m}p\bar{\gamma}-\left.\left[\frac{2}{3}\left(1-\frac{1}{a_2}\right)-\frac{k}{4k+1}\left(2-\frac{2}{a_2}-\frac{g_1}{a_1}\right)\right]\frac{i}{m}\gamma(\gamma\cdot p)\bar{\gamma}\right\}\\ &+\frac{g_1f_1}{a_1}\frac{(p^2+m^2)}{m^3}\left[\left(1-\frac{3k}{4k+1}\right)p^2\gamma\bar{\gamma}-p\bar{p}+(\gamma\cdot p)\gamma\bar{p}-\left(1-\frac{3k}{4k+1}\right)(\gamma\cdot p)p\bar{\gamma}\right].\end{aligned} \tag{102}$$

In this computation use was made of the relations in (94a), (94b), (98), and (99).

As was shown by the last term in (100), in general, $\tilde{\Lambda}$ contains derivative terms of the second order. The wave equations of the first-order derivatives are obtained when[47]

$$f_1=0\quad\text{or}\quad g_1=0, \tag{103}$$

and their forms are given, respectively, by

$$\begin{aligned}\tilde{\Lambda}^{(3/2)}(ip)=&-\{(i\gamma\cdot p+m)-\tfrac{1}{3}(1-2a_1k)(\gamma\cdot p)\gamma\bar{\gamma}\\ &+\tfrac{1}{3}[(a_1-1)+(4a_1-1)k]m\gamma\bar{\gamma}\\ &-\tfrac{1}{3}i(1-2k)\gamma\bar{p}+\tfrac{1}{3}i(1-2a_1-4a_1k)p\bar{\gamma}\}\end{aligned} \tag{104a}$$

for $g_1=0$, and

$$\begin{aligned}\tilde{\Lambda}^{(3/2)}(ip)=&-[(i\gamma\cdot p+m)-\tfrac{1}{3}i(\gamma\cdot p)\gamma\bar{\gamma}+\tfrac{1}{3}(\bar{a}_1-1)m\gamma\bar{\gamma}\\ &-\tfrac{2}{3}i\bar{a}_1\gamma\bar{p}-\tfrac{1}{3}i(\gamma\bar{p}-p\bar{\gamma})]\end{aligned} \tag{104b}$$

[46] Note that $D_1\gamma\bar{\gamma}D_1=3D_1$ and $D_2\gamma\bar{\gamma}D_2=D_2$.

[47] Recall that $a_2=0$ is prohibited.

for $f_1=0$. Here we used the relations in (99). In (104b) the parameter $\bar{a}_1$ is defined by

$$\bar{a}_1=a_1+(4a_1-1)k. \tag{105}$$

We have thus obtained two sets of $\tilde{\Lambda}$ of the standard form with first-order derivatives: $\tilde{\Lambda}$ in (104a) contains two parameters a_1 and k, and $\tilde{\Lambda}$ in (104b) contains only one parameter $\bar{a}_1$.

Let us note that $\bar{a}_1=0$ when $a_1=n/(1+4k)$. Therefore, $\bar{a}_1=0$ is not prohibited, although $a_1=0$ is prohibited. The form of $\tilde{\Lambda}$ in (104b) with $\bar{a}_1=0$ gives the well-known Rarita-Schwinger equation.[25]

When $\bar{a}_1=1$ in (104b), $\tilde{\Lambda}$ takes the Harish-Chandra form (1). This is the form given in our previous paper [cf. Eq. (4.37) of Ref. 22]. Another expression of the Harish-Chandra form is given by (104a) with $k=(1-a_1)/(4a_1-1)$. This expression still contains one parameter, i.e., a_1.

Let us now construct the Lagrangian. To do this we need the nonsingular matrix ρ, which in the present case is given by

$$\rho_{\mu\nu}=\gamma_4 g_{\mu\nu}, \tag{106}$$

according to (49a). In the case of (104a), $\rho\tilde{\Lambda}$ becomes Hermitian [cf. the condition (50)] when the parameters a_1 and k satisfy the condition

$$k=-a_1^*. \tag{107}$$

The Lagrangian is

$$L=\int d^4x(\bar{\phi}(x)\tilde{\Lambda}(\partial)\phi(x))_{k=-a_1^*}, \tag{108}$$

with $\bar{\phi}=\phi^\dagger\rho$. This Lagrangian contains one complex parameter a_1. As a matter of fact, a Lagrangian with one parameter has been presented by several authors.[13,18,23,24] Introducing the symbol

$$A=\tfrac{1}{3}(2k-1)\neq-\tfrac{1}{2},$$

we can show that the latter Lagrangian agrees with (108). Thus the origin of the parameter in the Lagrangian is now explained.

The d operator corresponding to $\tilde{\Lambda}$ in (104a) with $k=-a_1^*$ is obtained from (102):

$$\tilde{d}^{(3/2)}(ip)=(-i\gamma\cdot p+m)\left(1-\tfrac{1}{3}\gamma\bar{\gamma}+\frac{1}{3m}(\gamma\bar{p}-p\bar{\gamma})+\frac{2}{3}\frac{p\bar{p}}{m^2}\right)+\frac{(p^2+m^2)}{m}\left\{\frac{1}{12}\left(1-\frac{9}{|(4a_1-1)|^2}\right)\gamma\bar{\gamma}+\frac{2}{3}\left(1-\frac{3a_1}{4a_1-1}\right)\frac{i}{m}\gamma\bar{p}\right.$$
$$\left.+\frac{2}{3}\left(1-\frac{3a_1^*}{4a_1^*-1}\right)\frac{i}{m}p\bar{\gamma}+\left[\frac{2}{3}-\frac{2}{|(4a_1-1)|^2}(5|a_1|^2-a_1-a_1^*)\right]\frac{i}{m}\gamma(\gamma\cdot p)\bar{\gamma}\right\}. \tag{109}$$

When we use $\tilde{\Lambda}$ in (104b), $\rho\tilde{\Lambda}$ becomes Hermitian when and only when $\bar{a}_1=0$, leading us to the Rarita-Schwinger Lagrangian, which does not contain any parameter.

Let us now turn our attention to the case of spin $\frac{1}{2}$ (i.e., $s=\frac{1}{2}$) in the vector-spinor representation. In this case too, we can derive quite a general form of wave equations which contain certain parameters. However, for simplicity, we shall fix the parameters from the beginning. There are two sets of equations corresponding to two irreducible representations of spin $\frac{1}{2}$.

According to the method [cf. Eq. (35)] presented in previous sections, the primitive form is

$$\Lambda_1^{(1/2)}(ip)=-[(i\gamma\cdot p)D_1+m]. \tag{110}$$

Here we have chosen the parameters in (35) as $a_2^{(1/2)}=a^{(3/2)}=m$. As is seen from (89), this form of Λ contains only one singular term, i.e., $(\gamma\cdot p)p\bar{p}/p^2$. According to (94a), such a singular term appears in $(\gamma\cdot p)D_1\gamma\bar{\gamma}D_2$. Thus, the singular term in $\Lambda_1^{(1/2)}$ can be eliminated by means of $\eta_1(p)$ in (95b) with $g_2=0$. The elimination requires, further, that $g_1=\frac{1}{3}$. The result is that

$$\begin{aligned}\tilde{\Lambda}_1^{(1/2)}(ip)&=\eta_1(p)\Lambda_1^{(1/2)}(ip)\\&=-[\tfrac{1}{3}(i\gamma\cdot p)\gamma\bar{\gamma}-\tfrac{1}{3}ip\bar{\gamma}+m],\end{aligned} \tag{111}$$

which gives the following wave equation for the case of spin $\frac{1}{2}$:

$$\tfrac{1}{3}[(\gamma\partial)\gamma_\mu\gamma_\nu-\partial_\mu\gamma_\nu]\phi_\nu+m\phi_\mu=0. \tag{112}$$

The corresponding d operator is given by (41) together with (37) as follows:

$$\begin{aligned}\tilde{d}_1^{(1/2)}(ip)=\tfrac{1}{3}(-i\gamma\cdot p+m)[\gamma\bar{\gamma}+(i/m)p\bar{\gamma}]\\+[(p^2+m^2)/m](1-\tfrac{1}{3}\gamma\bar{\gamma}).\end{aligned} \tag{113}$$

Another primitive form of wave equation for spin $\frac{1}{2}$ is given by

$$\Lambda_2^{(1/2)}(ip)=-[(i\gamma\cdot p)D_2+m]. \tag{114}$$

Here too, we have chosen the parameters in (41) as $a_1^{(1/2)}=a^{(3/2)}=m$. The operator $\Lambda_2^{(1/2)}$ also has a singular term of the form $(\gamma\cdot p)p\bar{p}/p^2$, which can be eliminated by $\eta_2(p)$ with $f_2=0$ and $f_1=1$ [cf. (95b) and (94a)]. The result is

$$\begin{aligned}\tilde{\Lambda}_2^{(1/2)}(ip)&=\eta_2(p)\Lambda_2^{(1/2)}(ip)\\&=-(ip\bar{\gamma}+m),\end{aligned} \tag{115}$$

which gives the following wave equation for spin $\frac{1}{2}$:

$$\partial_\mu\gamma_\nu\phi_\nu+m\phi_\mu=0. \tag{116}$$

The corresponding d operator is

$$\tilde{d}_1^{(1/2)}(ip)=(i/m)(i\gamma\cdot p-m)p\bar{\gamma}+(p^2+m^2)/m. \tag{117}$$

Although η_1 and η_2 do not satisfy the condition (43b) with $D_i(s)=D_1$ and D_2, respectively, the wave equations (112) and (116) are still of the standard form: $(1/2m)d_1^{(1/2)}(ip)$ and $(1/2m)d_2^{(1/2)}(ip)$ are projection operators when $p^2=-m^2$ [cf. (45)]. This is because any wave equation of first-order derivatives is of the standard form when the nonderivative term is the unit matrix with a constant coefficient.

A detailed analysis of wave equations of spin $\frac{1}{2}$ in the vector-spinor representation will be presented elsewhere. Here, Eqs. (112) and (116) are presented to illustrate the general method given in previous sections. These equations may be generalized by means of different choices of parameters.

E. Three-Spinor Basic Representation

The operator $\bar{\lambda}(a,p)$ defined by (27) together with (24) is the one which enables us to put the Bargmann-Wigner equations into one matrix equation as

$$\bar{\lambda}(a,\partial)\phi(x)=0. \qquad (118)$$

Such an operator is needed also in the derivation of wave equations of unique spin and mass [cf. (32) and (33)]. Already in Sec. 3A, we have illustrated the construction of $\bar{\lambda}$ by using the two-spinor representation. This example, however, was too simple to illustrate the general technique. Here we will construct $\bar{\lambda}$ in the three-spinor representation. The result presents a matrix equation which is equivalent to the Bargmann-Wigner equations (17) with $r=3$.

The wave function now has three-spinor suffixes $\phi_{\alpha_1\alpha_2\alpha_3}$. The Dirac matrices concerning the suffix α_i will be denoted by $\gamma^{(i)}$ ($i=1$, 2, and 3). The Bargmann-Wigner equations are

$$(i\gamma_\mu^{(i)}p_\mu+m)\phi=0, \quad i=1,2,3. \qquad (119)$$

To rewrite (119) in the compact form (118), we need the projection operator $\chi(p)$ in (21):

$$\chi(p)=\tfrac{1}{4}[1+p^{-2}(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)] \times[1+p^{-2}(\gamma^{(1)}\cdot p)(\gamma^{(3)}\cdot p)]. \qquad (120)$$

According to (24), the primitive form λ takes the form

$$\lambda(m,ip)=-(i\gamma^{(1)}\cdot p)\chi(p)-m, \qquad (121)$$

where, for simplicity, we took the parameter a as $a=m$. It may well be that a more general choice for a will lead us to a wider class of $\bar{\lambda}$ with certain parameters.

The operator λ in (121) contains only one singular term of the form $p^{-2}(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p)$. To eliminate this singular term by means of the nonsingular matrix $q(m,p)$ in (27), we shall introduce the following projection operators:

$$\chi_{23}^{\pm}=\tfrac{1}{2}[1\pm(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p)/p^2], \qquad (122)$$

and note the relations

$$\chi_{23}^{-}\chi(p)=0, \quad \chi(p)\chi_{23}^{+}=\chi(p), \qquad (123a)$$

$$\chi_{23}^{+}\gamma_\mu^{(1)}\gamma_\mu^{(2)}(\gamma^{(3)}\cdot p)\chi_{23}^{-}=\tfrac{1}{2}(\gamma^{(1)}\cdot p) +\tfrac{1}{2}(\gamma_\mu^{(1)}\gamma_\mu^{(2)})[(\gamma^{(3)}\cdot p)-(\gamma^{(2)}\cdot p)] -(1/2p^2)(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p). \qquad (123b)$$

Since the latter product contains also the singular term of the form $p^{-2}(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p)$, the singular term in λ can be eliminated by the following choice for $q(m,p)$:

$$q(m,p)=1+(ic/m)\chi_{23}^{+}\gamma_\mu^{(1)}\gamma_\mu^{(2)}(\gamma^{(3)}\cdot p)\chi_{23}^{-}. \qquad (124)$$

The elimination of the singular term requires that $c=\frac{1}{2}$. The result is

$$\begin{aligned}\bar{\lambda}(m,ip)&=q(m,p)\lambda(m,ip)\\ &=-\{\tfrac{1}{2}i(\gamma^{(1)}\cdot p)+\tfrac{1}{4}[(\gamma^{(2)}\cdot p)+(\gamma^{(3)}\cdot p)]\\ &\quad+\tfrac{1}{4}(\gamma_\mu^{(1)}\gamma_\mu^{(2)})[(\gamma^{(3)}\cdot p)-(\gamma^{(2)}\cdot p)]+m\}\\ &=-(i\beta\cdot p+m), \end{aligned} \qquad (125)$$

with

$$\beta_\mu=\tfrac{1}{2}\gamma_\mu^{(1)}+\tfrac{1}{4}(\gamma_\mu^{(2)}+\gamma_\mu^{(3)}) +\tfrac{1}{4}(\gamma_\nu^{(1)}\gamma_\nu^{(2)})(\gamma_\mu^{(3)}-\gamma_\mu^{(2)}). \qquad (126)$$

We have thus succeeded in rewriting the Bargmann-Wigner equations of $s_{\max}=\frac{3}{2}$ in the form of one matrix equation. The matrices β_μ in (126) present an explicit form of the Harish-Chandra β matrices (for $s_{\max}=\frac{3}{2}$) written in terms of the Dirac matrices. The β matrices in (126) satisfy the Harish-Chandra relation

$$(\beta\cdot p)^4=p^2(\beta\cdot p)^2, \qquad (127)$$

as they should [cf. (1b)].

The operator $\bar{h}(m,ip)$ satisfying

$$\bar{h}(m,ip)\bar{\lambda}(m,ip)=\bar{\lambda}(m,ip)\bar{h}(m,ip)=-(p^2+m^2)$$

can be obtained by (28) together with (25) and (124). The result is

$$\bar{h}(m,ip)=m-i(\beta\cdot p)-(1/m)[m^2+(\beta\cdot p)^2] +(i/m^2)[m^2+(\beta\cdot p)^2](\beta\cdot p) +[(p^2+m^2)/m][1-i(\beta\cdot p)/m]. \qquad (128)$$

It can be shown that

$$\begin{aligned}(\beta\cdot p)^2=&\tfrac{1}{8}p^2+\tfrac{1}{8}(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)+\tfrac{3}{8}(\gamma^{(1)}\cdot p)\\ &\times(\gamma^{(3)}\cdot p)+\tfrac{3}{8}(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p)+\tfrac{1}{8}p^2(\gamma^{(1)}\cdot\gamma^{(2)})\\ &+\tfrac{1}{8}(\gamma^{(1)}\cdot\gamma^{(2)})(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)\\ &-\tfrac{1}{8}(\gamma^{(1)}\cdot\gamma^{(2)})(\gamma^{(1)}\cdot p)(\gamma^{(3)}\cdot p)\\ &-\tfrac{1}{8}(\gamma^{(1)}\cdot\gamma^{(2)})(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p),\end{aligned}$$

$$\begin{aligned}(\beta\cdot p)^3=&\tfrac{3}{8}p^2(\gamma^{(1)}\cdot p)+\tfrac{3}{8}p^2(\gamma^{(2)}\cdot p)+\tfrac{1}{8}p^2(\gamma^{(3)}\cdot p)\\ &+\tfrac{1}{8}(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p)\\ &-\tfrac{1}{16}(\gamma^{(1)}\cdot\gamma^{(2)})(\gamma^{(1)}\cdot p)p^2-\tfrac{1}{8}(\gamma^{(1)}\gamma^{(2)})\\ &\times(\gamma^{(2)}\cdot p)p^2+\tfrac{1}{8}(\gamma^{(1)}\cdot\gamma^{(2)})(\gamma^{(3)}\cdot p)p^2\\ &+\tfrac{1}{16}(\gamma^{(1)}\cdot\gamma^{(2)})(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p)\\ &-\tfrac{1}{16}(\gamma^{(1)}\cdot\gamma^{(2)})^2(\gamma^{(1)}\cdot p)p^2\\ &+\tfrac{1}{16}(\gamma^{(1)}\cdot\gamma^{(2)})^2(\gamma^{(1)}\cdot p)(\gamma^{(2)}\cdot p)(\gamma^{(3)}\cdot p),\end{aligned}$$

where $(\gamma^{(1)}\cdot\gamma^{(2)})$ means $(\gamma_\nu^{(1)}\gamma_\nu^{(2)})$. The matrix $q(m,p)$ satisfies the condition (28a). Although the condition (28b) is not satisfied, still the wave equations (118) with (125) are of standard form, because $(1/2m)\bar{h}(m,ip)$ is a projection operator [cf. (31)]. This can be seen from the fact that the nonderivative term in $\bar{\lambda}(m,\partial)$ is the unit matrix with the coefficient m.

To select out the wave equations for spin $\frac{3}{2}$ and spin $\frac{1}{2}$ separately, we need to follow further the steps which were discussed in Sec. 3. Such equations will be discussed elsewhere.

4. CONCLUSIONS

In this paper we presented a general method for deriving relativistic wave functions and wave equations for arbitrary spins. According to this method, relativistic equations appear in the compact form of single matrix equations, thus leading immediately to the Lagrangian.

The method consists of two steps. The first step is to derive wave equations for several spins ($s_{\max}$, $s_{\max}-1$, $\cdots$, $s_{\min}$). These equations are written in the form of a single matrix equation $\bar{\lambda}\phi=0$ [cf. (27)]. When the basic representation is a many-spinor representation, this method enables us to rewrite the Bargmann-Wigner equations in the form of a single matrix equation. This method also presents a way of deriving the Harish-Chandra β matrices. An explicit expression for β_μ was given in Sec. 3 E.

The second step is to derive the wave equation of unique spin. This step is essentially based on use of the spin-projection operators $D_i(s)$. It should be noted that these operators are constructed not by the booster operator, but by means of the Pauli-Lubanski matrices. Since the general form of d operators is also given, the quantization of the wave equation is immediate. The wave equations thus obtained in general contain a certain number of arbitrary parameters. This explains why certain parameters had appeared in the equations for spin $\frac{3}{2}$ introduced previously by several authors.

According to our method, the wave equations are first written in the primitive form which contains certain p^{-2} singular terms. Since the existence of such terms is unpleasant, we presented a general method for elimination of these singular terms by means of a certain nonsingular matrix $\eta(p)$ rather than by means of the usual technique of using auxiliary fields. In the examples in Sec. 3, we took, for the sake of simplicity, $\eta(p)$ in such a way that the matrices $A_{kj}(v,u)$ in equation (46) do not contain any power of p higher than the first order. In general, we can introduce any higher-order derivatives in the wave equations by introducing higher powers of p in $\eta(p)$ [i.e., in $A_{kj}(v,u)$ in (46)]. It may be that elimination of the p^{-2} singular terms in the wave equation may be unnecessary because the effects of p^{-2} singular terms in the wave equations (i.e., in Λ) and those in the Green's functions (i.e., in d) are expected to compensate each other in each Feynman diagram. However, these p^{-2} singular terms may cause much trouble when we want to introduce the minimal electromagnetic interactions ($p\to p-eA$).

It has been known through many examples that the d operator in general contains terms proportional to (p^2+m^2), which contribute to the Green's function by the so-called off-shell term. The expression (37) explicitly explains why such terms exist. Furthermore, we see from (37) that the off-shell terms carry a variety of spins [expressed by u in (37)]. This is the reason, for example, why the spinless pion can decay through the intermediate vector meson without violating the conservation law of angular momentum.

Let us close our consideration by showing how to extend our method to cover more general cases where there exist multimasses, the values of which depend on spins: When the masses depend on (i,s) as $m(i,s)$, the equation of the primitive form [cf. (35)] is given by

$$\Lambda(ip)=\sum_{i,s} F^{(i,s)}(p)D_i(s)\,, \tag{129}$$

where

$$F^{(i,s)}(p)=[F(p)]_{m\to m(i,s)}\,.$$

Elimination of p^{-2} singular terms can be performed in the same way as in the single-mass case.