Three-Body Calculations with Local Potentials*

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A two-term separable approximation to the two-particle t matrix is developed which preserves exact behavior at each bound-state or resonance pole. While introducing no additional singularities, the approximation also produces the exact right-hand cut. This separable approximation is applied to the three-particle scattering problem with local potentials, and numerical results are obtained for the three-body binding energy in the case of the Yukawa and exponential potentials. A prescription is given for increasing the accuracy of this approach by systematically increasing the number of separable terms. This procedure is tested numerically, as is a method for estimating the error in such calculations. Binding-energy results are also presented for a modified pole approximation, and a simple explanation is given for the infinite-binding-energy phenomenon observed by Osborn for separable potentials.

I. INTRODUCTION

QUANTITATIVE results have never come easily in the three-body problem, and this remains true today despite significant advances in the mathematical theory. Although an exact and rigorous treatment of three-particle scattering has been formulated by Faddeev,[1] it is well known that formidable mathematical difficulties arise when one attempts to solve the Faddeev equations for an arbitrary potential. As a result, almost all of the calculations which have been based on the Faddeev formalism have relied on the use of separable potentials. From a phenomenological point of view, this approach has been reasonably successful. However, while they lead to enormous simplifications in the mathematics, we have considerable reason to believe that such nonlocal potentials are physically unrealistic.

In order to solve the Faddeev equations with more realistic potentials, it is necessary to develop new computational techniques. For this purpose, two possible approaches seem promising. The most direct method is to improve numerical quadrature techniques to such a degree that multidimensional integrals can be accurately evaluated with a minimum of integration points. This approach was recently employed by Osborn[2] to calculate three-particle binding energies in the case of a Yukawa or exponential potential. While of high accuracy, this method requires the manipulation of very large matrices ($N \geqslant 100$) and is suitable only for the largest of current computers. It is thus rather cumbersome for use in the type of model calculations currently performed with separable potentials. Consequently, it seems desirable to have an alternative method of somewhat less accuracy, but which is comparable in simplicity to a separable potential calculation and can be handled on a small computer.

For this purpose, several authors have suggested variations on an alternative approach to such calculations. This approach involves approximating the two-particle t matrix by a sum of separable terms. The simplification which results is equivalent to that produced by separable potentials. Depending on how the separable approximation is chosen, the resulting separable t matrix may or may not be derivable from a separable potential. One means of doing this was suggested by Lovelace[3] for the case where the two-body subsystems are dominated by a bound-state or resonance pole. Recently, a more general and systematic procedure has been proposed by Ball and Wong.[4]

In the following sections we consider the separable t-matrix approach in some detail. We begin in Sec. II by describing the reduction of the Faddeev equations to a set of one-dimensional integral equations under the approximations to be introduced. In Sec. III we give a brief review of separable approximations, and in the process develop a set of criteria to be applied to such approximations. These considerations lead us to the definition of a new separable approximation in Sec. IV. Using this approximation as a starting point, we define a procedure for calculating correction terms to successively improve its accuracy. The effectiveness of this approximation is tested numerically in Sec. V, where binding-energy results for the exponential and Yukawa potentials are compared with those obtained by Osborn. In addition, numerical results are obtained for a method introduced to estimate the error in such calculations. Section VI is devoted to a discussion of the general aims of our approach and analysis of the results obtained. Here we also present a simple explanation for a phenomenon observed by Osborn.

II. REDUCTION OF THE FADDEEV EQUATIONS

We choose to work with a modification of the Faddeev equations[5] which is obtained by expressing the three-particle scattering operator $T(W)$ in the form

$$T(W)=\sum_{\alpha=1}^{3} T_\alpha(W)+\sum_{\alpha,\beta=1}^{3} T_\alpha(W)X_{\alpha\beta}(W)T_\beta(W). \quad (1)$$

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[2] T. A. Osborn, Stanford Linear Accelerator Center Report No. SLAC 79, 1967 (unpublished).

[3] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[4] J. S. Ball and D. Y. Wong, Phys. Rev. **169**, 1362 (1968).

[5] A more detailed discussion of the theory and our notation is given in D. D. Brayshaw, Phys. Rev. **176**, 1855 (1968).

Here $T_\alpha(W)$ describes the two-body scattering of particles β and γ when α is undeflected ($\alpha\beta\gamma$ cyclic), and the operator $X_{\alpha\beta}(W)$ describes the propagation of the system between an initial scattering of particles α and γ and a final scattering of β and γ. One finds that $X_{\alpha\beta}(W)$ satisfies the equation

$$X_{\alpha\beta}(W)=\bar{\delta}_{\alpha\beta}G_0(W)+\sum_{\gamma=1}^{3}\bar{\delta}_{\alpha\gamma}G_0(W)T_\gamma(W)X_{\gamma\beta}(W), \quad (2)$$

where

$$\bar{\delta}_{\alpha\beta}=(\delta_{\alpha\beta}-1),$$

and $G_0(W)$ is the free three-particle propagator. By taking matrix elements of (2) between the usual three-body states $|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle$, one obtains a six-dimensional integral equation of the Faddeev type.

If we specialize to the case of three identical spinless particles there is only one amplitude, $\langle\mathbf{p}'\mathbf{q}'|X(W)|\mathbf{pq}\rangle$, to consider. Making an angular momentum decomposition along the lines suggested by Ahmadzadeh and Tjon,[6] we find that the $J=0$ amplitude satisfies the equation

$$X^{l'l}(p'q'|pq;W)=Z^{l'l}(p'q'|pq;W)+\sum_{l''=0,2,4,\cdots}\int_0^\infty p''^2dp''\int_0^\infty dq''K^{l'l''}\times(p'q'|p''q'';W)X^{l''l}(p''q''|pq;W). \quad (3)$$

Here the l indices label the relative angular momentum of any pair of particles, and the kernel is proportional to off-shell matrix elements of the two-body partial-wave scattering operator $t_l(s)$. The energy available to a two-particle subsystem is the total energy W less the energy $\frac{3}{4}q''^2$ of the third particle; hence $s=W-\frac{3}{4}q''^2$ in Eq. (3). (Here we choose units such that the mass of each particle is unity.)

The solution of Eq. (3), which is exact, completely determines the $J=0$ three-particle scattering amplitude through Eq. (1). However, the problems involved in obtaining this solution are far from trivial, as one must cope with an infinite set of coupled equations in two variables. We therefore consider two simplifying approximations which are commonly employed to reduce Eq. (3) to a form suitable for numerical solution. The first step is to restrict the index l to a finite set of values, say, $0\leq l\leq L$. In most cases, this is probably well justified, the reason being that the partial-wave amplitudes $t_l(p',p;s)$ tend to decrease quite rapidly with increasing l. For example, in the case of scattering from an exponential or a Yukawa potential, the $l=2$ amplitude is typically less than 1% of the $l=0$ amplitude. One would then expect that the error involved in taking $L=0$ would be quite small for such potentials. This has been checked in the case of the Yukawa potential by Ball and Wong,[4] who found the $l=2$ contribution to be negligible in calculating the three-particle binding energy and the free-particle–bound-state scattering amplitude. In the general case, we may not expect to be this fortunate, but it is reasonable to expect good accuracy with just a few partial waves included.

The second step is to assume that each $t_l(p',p;s)$ can be accurately approximated by a sum of terms separable in the initial and final momentum (p' and p) variables, i.e.,

$$t_l(p',p;s)\simeq\sum_{\lambda,\mu=1}^{N_l}A_\lambda{}^l(p',s)C_{\lambda\mu}{}^l(s)A_\mu{}^l(p,s). \quad (4)$$

From the symmetry of $t_l(p',p;s)$ under the exchange $p'\leftrightarrow p$ it follows that $C_{\lambda\mu}{}^l(s)=C_{\mu\lambda}{}^l(s)$, providing that we take the functions $A_\lambda{}^l(p',s)$ to be linearly independent. The approximation stated in Eq. (4), however, is considerably more difficult to justify than the neglect of the higher partial waves discussed above. A straightforward calculation of the off-shell t matrix for any simple local potential, and its comparison with the more obvious types of separable approximations, will soon convince one of this. However, it is still plausible that Eq. (4) can be made sufficiently accurate by taking N_l large enough and by choosing the $A_\lambda{}^l(p,'s)$ functions with maximum efficiency. It is this approach that we will be primarily concerned with in this paper and we therefore proceed under this assumption.

We define

$$X_{\lambda\mu}{}^{l'l}(q',q;W)=\int_0^\infty p^2dp\int_0^\infty p'^2dp'A_\lambda{}^{l'}(p',W-\tfrac{3}{4}q'^2)\times A_\mu{}^l(p,W-\tfrac{3}{4}q^2)X^{l'l}(p'q'|pq;W), \quad (5)$$

and similarly for $Z_{\lambda\mu}{}^{l'l}(q',q;W)$. It follows from Eq. (3) that the $X_{\lambda\mu}{}^{l'l}(q',q;W)$ satisfy the coupled equations

$$X_{\lambda'\mu}{}^{l'l}(q',q;W)=Z_{\lambda'\mu}{}^{l'l}(q',q;W)+\sum_{\lambda,l''}\int_0^\infty dq''K_{\lambda'\lambda}{}^{l'l''}(q',q'';W)X_{\lambda\mu}{}^{l''l}(q'',q;W). \quad (6)$$

Here

$$K_{\lambda'\lambda}{}^{l'l}(q',q;W)=q^2\sum_\mu C_{\lambda\mu}{}^l(W-\tfrac{3}{4}q^2)Z_{\lambda'\mu}{}^{l'l}(q',q;W). \quad (7)$$

Equation (6) is of a form that can be readily handled numerically, providing that L and the N_l's are not too large. In fact, it is just a slightly generalized version of many similar equations obtained by substituting separable t matrices into Eq. (2) directly. The path followed above, however, will be of some assistance when we consider how to choose our separable approximation [Eq. (4)] with maximum effectiveness. It is worth pointing out that Eq. (6) is also *formally* identical with the equations one obtains by assuming separable (non-local) potentials. The meaning of the word "formal" here is that the separable form chosen to approximate the off-shell t matrix in Eq. (4) need not correspond to

[6] A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **139**, B1085 (1965).

any separable t matrix one can derive from a separable potential. Our primary consideration is to fit the values of the function $t_l(p',p;s)$ over some domain of its variables, emphasizing the regions where accuracy is most important. The criteria used in employing separable potentials, which typically involve fitting some of the on-shell two-body data, may not be compatible with this objective. As an example, we observe that requiring the right side of Eq. (4) to satisfy off-shell unitarity is unnecessarily restrictive for calculations in which $s<0$.

One reason for working with Eq. (2) is that some of the amplitudes $X_{\lambda\mu}{}^{l'l}(q',q;W)$ can themselves take on physical significance in certain cases. The most common example is when the two-body system has a bound state or resonance of energy s_0 occurring in partial wave l. For s in the neighborhood of s_0, we have

$$t_l(p',p;s)=\frac{g(p')g(p)}{s-s_0}+R(p',p';s), \tag{8}$$

where $R(p',p;s)$ is regular at s_0. Here the function $g(p)$ has significance as the form factor for formation of the bound or resonant state. It is easy to show that if one of the $A_\lambda{}^l(p,s)$ functions defined in Eq. (4) has the property that

$$A_\lambda{}^l(p,s_0)=c_\lambda g(p),$$

then (9)

$$X_{\lambda\lambda}{}^{ll}(E^{1/2},E^{1/2};W)=c_\lambda{}^2\tau_0(E),$$

where $\tau_0(E)$ is the quasi-two-particle amplitude for scattering of the third particle off the bound state, and

$$E\equiv\tfrac{4}{3}(W-s_0).$$

III. SEPARABLE APPROXIMATIONS

In performing three-body calculations via the Faddeev equations, the usefulness of separable t matrices has been emphasized by many authors, including Mitra,[7] Amado,[8] and Lovelace.[3] Consequently, a great deal of effort has been devoted to the study of approximations which lead to separable off-shell amplitudes. In this section we briefly review some of the progress which has been made in this area. In the process, we will formulate a number of the conditions which we would like to impose on such approximations.

The simplest approximation is of the type proposed by Lovelace[3]; we shall refer to this as the "pole approximation," since it is applicable when the partial-wave amplitude has a single pole of energy $s=s_0$, and takes the form

$$t_l(p',p;s)\simeq\frac{g(p')g(p)}{s-s_0}. \tag{10}$$

This formula is based on the well-known factorization of the residue in terms of the form factor $g(p)$, and the expectation that the singular term should dominate for values of s not too far from s_0. Since the maximum value of s which occurs in the Faddeev equation is W, the total energy, one would expect the pole approximation to produce its most reliable results when $W\gtrsim s_0$. In this case the error would come almost entirely from the region where s is large and negative, and this region is suppressed by a $p'^2+\frac{3}{4}q'^2-W$ denominator in Eq. (3), as well as by the decline in the values of $t_l(p',p;s)$ itself as s moves away from the pole. Thus, if $s_0<0$, Eq. (10) should be suitable for calculating the low-energy scattering of the third particle from the two-body bound state. To a lesser extent, it might also produce reasonable accuracy in a three-body binding-energy calculation.

On the other hand, Eq. (10) will not produce the imaginary part of the amplitude for $s>0$, as it does not have the right-hand cut. In order to do three-body scattering or bound-state breakup calculations, one introduces more complicated formulas which reproduce the pole while satisfying unitarity. The prescriptions for doing this are essentially equivalent to working with a separable potential, and the imaginary part thus generated need not correspond to that of the actual amplitude. The approximation will also fail if the partial wave admits more than one bound or resonant state, as is the case in many interesting problems in nuclear physics. However, turning the argument for this approximation around, it is clear that we should exclude any approximation which does not correctly reproduce the behavior of the amplitude at its poles.

The ideas which led up to Eq. (10) can be generalized in a way which leaves us with more freedom with which to remedy its defects. What we are looking for is a separation of $t_l(p',p;s)$ into two parts,

$$t_l(p',p;s)=R_l(p',p;s)+t_l{}^s(p',p;s), \tag{11}$$

where $t_l{}^s(p',p;s)$ consists of a small number of separable terms and contains all poles of the amplitude. Thus $R_l(p',p;s)$ is a smooth function of s. A straightforward and systematic way of doing this has been investigated by Ball and Wong.[4] Their method is based on the observation that, for $s<0$, the eigenfunctions $\phi_n(p;s)$ of the Lippmann-Schwinger (LS) equation form a complete set. Hence, for negative s, $t_l(p',p;s)$ has the expansion

$$t_l(p',p;s)=\sum_{n=1}^{\infty}\frac{\lambda_n(s)}{1-\lambda_n(s)}\phi_n(p';s)\phi_n(p;s). \tag{12}$$

In this representation a pole of $t_l(p',p;s)$ corresponds to an eigenvalue $\lambda_n(s)$ passing through unity. If we keep only those terms in Eq. (12) for which this occurs, the resulting expression will be a suitable candidate for $t_l{}^s$. The number of separable terms is thus determined by the number of bound states supported by the potential. This number is small for simple short-range potentials of the Yukawa or exponential type, and one would

[7] A. N. Mitra, Nucl. Phys. **32**, 529 (1962); Phys. Rev. **131**, 832 (1963).
[8] R. D. Amado, Phys. Rev. **132**, 485 (1963).

therefore expect this procedure to be particularly useful for these potentials. Furthermore, the accuracy of the approximation at values of s away from the poles can be successively improved by taking additional terms of the series. In principle, this provides a means of performing a three-body calculation for $W<0$ to any desired degree of accuracy. In practice, of course, this will only be feasible if the terms fall off quite rapidly with increasing n.

It is clear that the disadvantages of this approach lie in the number of separable terms required, which is critical in many applications. For systems in which there are a large number of bound states, or potentials for which the convergence of the series expansion is slow, the number of terms can be prohibitive. Also, the approach as it stands is not suitable for calculations in which $W>0$.

These considerations suggest that we impose more stringent requirements on t_l^s. Namely, we might require that the number of separable terms contained in t_l^s be small and fixed, independent of the number of poles (bound-state or resonance) in the amplitude. We might also require that t_l^s contain the right-hand cut of the amplitude, so that the approximation $t_l \approx t_l^s$ gives the exact imaginary part of $t_l(p',p;s)$ for $s>0$. Although this seems at first glance to be quite a large order, there are actually several ways of obtaining such a t_l^s. One can see this most directly by noting a simple property of the LS equation, which we write in the form

$$t_l(p',p;s)=V_l(p',p) - \int_0^\infty \frac{dq\, q^2}{q^2-s-i\epsilon} V_l(p',q)t_l(q,p;s). \quad (13)$$

Alternatively, in operator notation,

$$t_l(s)=V_l-V_lG_0(s)t_l(s), \quad (14)$$

with

$$\langle p'|G_0(s)|p\rangle=\frac{\delta(p'-p)p^2}{p^2-s-i\epsilon}.$$

If we define a new "potential" $\tilde{V}_l(p',p;s)$ by the equation

$$\tilde{V}_l(p',p;s)=V_l(p',p)-\Delta V_l(p',p;s), \quad (15)$$

and require $\tilde{t}_l(s)$ to satisfy the LS equation with this potential, the solution $t_l(s)$ of Eq. (14) can be written

$$t_l(s)=\tilde{t}_l(s)+\Delta t_l(s),$$

where

$$\Delta t_l(s)=[1+\tilde{V}_l(s)G_0(s)]^{-1}\Delta V_l(s)[1+G_0(s)V_l]^{-1} = [1+V_lG_0(s)]^{-1}\Delta V_l(s)[1+G_0(s)\tilde{V}_l(s)]^{-1}. \quad (16)$$

Therefore, if $\Delta V_l(p',p;s)$ is chosen to be separable, i.e.,

$$\Delta V_l(p',p;s)=A_l(p';s)B_l(p;s), \quad (17)$$

it is clear from Eq. (16) that $\Delta t_l(p',p;s)$ will be separable also. We may therefore take

$$t_l^s(p',p;s)=\Delta t_l(p',p;s), \quad R_l(p',p;s)=\tilde{t}_l(p',p;s). \quad (18)$$

Furthermore, $\Delta V_l(p',p;s)$ can be chosen in such a way that the Born series for $\tilde{t}_l(p',p;s)$ converges for all s; this has been emphasized by Weinberg[9] in a series of papers on the "quasiparticle method." If this is done, the function $\Delta t_l(p',p;s)$ will clearly possess all the poles of $t_l(p',p;s)$. The trick, of course, is to guess a proper form for $\Delta V_l(p',p;s)$.

If we now impose the requirement that $\tilde{t}_l(p',p;s)$ does not have the right-hand cut, the possible choices of $\Delta V_l(p',p;s)$ are severely restricted. One possibility is that

$$\Delta V_l(p',p;s)=\frac{V_l(p',k)V_l(k,p)}{V_l(k,k)}, \quad (19)$$

where $k=s^{1/2}$ for $s>0$. Thus, for $s>0$, $\tilde{V}_l(p',q;s)$ vanishes linearly as $q\to s^{1/2}$, and hence the kernel of the LS equation for $t_l(p',p;s)$ has no singularity for positive s. As a result, $\tilde{t}_l(s)$ is continuous across the positive s axis, and $\Delta t_l(s)$ possesses the exact right-hand cut. It remains to be shown that for this choice of $\Delta V_l(p',p;s)$ all the poles of $t_l(s)$ occur in $\Delta t_l(s)$. This is actually quite simple, since $\Delta t_l(p',p;s)$ takes the explicit form

$$\Delta t_l(p',p;s)=\rho_l(p',s)t_l(k,k;s)\rho_l(p,s), \quad (20)$$

with

$$\rho_l(p,s)=t_l(p,k;s)/t_l(k,k;s).$$

If $t_l(p',p;s)$ has a pole at $s=s_0$, comparison with Eq. (10) shows that $\Delta t_l(p',p;s)$ has both the pole and the exact residue. Defining $t_l^s(p',p;s)$ by Eq. (18), we obtain the approximation suggested by Noyes[10] and by Kowalski.[11]

The Noyes-Kowalski approximation thus represents one possible solution for t_l^s, satisfying the conditions stated prior to Eq. (13). In many respects it constitutes a considerable improvement over the approximations considered earlier. On the other hand, it has some additional undesirable features which may be troublesome in certain applications. These arise from the presence of $t_l(k,k;s)$ in the denominator of Eq. (20). Here we note that the definition $k=s^{1/2}$ for $s>0$ suggests two possible definitions of k for $s<0$. The choice made in S-matrix theory is that $k=s^{1/2}$ for all s, and this leads to the familiar simple analytic properties of the amplitude as a function of s. In particular, $t_l(p',p;s)$ for $s<0$ is the analytic continuation of $t_l(p',p;s)$ for $s>0$, and this amplitude is analytic in s except for the right-hand cut. However, for this choice k is pure imaginary for $s<0$ and $t_l(k,k;s)$ will possess the usual left-hand singularities of the (analytically continued) on-shell amplitude. In addition, t_l^s may possess un-

[9] S. Weinberg, Phys. Rev. **130**, 776 (1963); **131**, 440 (1963).
[10] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).
[11] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965).

physical poles due to the possible vanishing of $t_l(k,k;s)$. In fact, such zeros of $t_l(k,k;s)$ are known to occur for sufficiently large potential strengths in many potentials. Thus, although $t_l^s(p',p;s)$ will reproduce the exact amplitude at each bound-state or resonance pole, it will also possess a number of additional singularities which do not appear in the actual amplitude. This forces us to rule out the choice $k=s^{1/2}$ in many cases.

As shown by Osborn,[2] the left-hand singularities of $t_l(k,k;s)$ can be eliminated if we take instead $k=|s|^{1/2}$. This prescription preserves the correct behavior of the approximation at the poles of t_l, and also along the right-hand cut. However, the amplitude for $s<0$ is no longer the analytic continuation of $t_l^s(p',p;s)$ to negative s, although the approximation is continuous at $s=0$. Furthermore, nonphysical poles can still arise due to zeros of $t_l(k,k;s)$. The former difficulty is probably not serious, but the latter appears to restrict the usefulness of the Osborn modification to values of the potential strength somewhat less than that required to create a second (two-body) bound state.

In passing, we note that the Noyes-Kowalski approximation possesses the interesting property that, for $s>0$, it gives the exact half-on-shell amplitude $t_l(p',s^{1/2};s)$ when either p' or p is equal to $s^{1/2}$ (on-shell). This property does not appear to have useful consequences in calculations for which $W<0$, but for $W>0$ this point is favored by the vanishing of the $p'^2+\frac{3}{4}q'^2-W$ denominator in Eq. (3).

Finally, in attempting to compare the relative accuracy of each of the above approximations over a range of values of s, it is important to note the following point. Our expectation that these approximations will reliably reproduce the behavior of the exact amplitude is based primarily on the pole-dominance argument, i.e., that a function which correctly reproduces the poles will be a good approximation elsewhere. However, there is an essential ambiguity inherent in this argument. This is due to the fact that we may replace the function $g(p)$ which occurs in Eq. (10) by

$$G(p,s)=g(p)+(s-s_0)h(p,s)\,, \tag{21}$$

where $h(p,s)$ is any function which is regular at $s=s_0$. The behavior at the pole will thus be preserved, but the behavior of the approximation in neighboring regions will depend on the choice of $h(p,s)$. Each of the above approximations gives rise to a different $h(p,s)$, and thus their relative effectiveness at some distance away from the pole is difficult to assess without actual computation.

From this point of view, a critical test of a separable approximation is provided by a three-body binding-energy calculation. If we choose a potential which supports a single two-body bound state of energy s_0, these calculations typically involve values of s which are considerably less than s_0. Thus we only need to evaluate $t_l(p',p;s)$ at some distance from the pole. Numerical results for the Yukawa potential by Ball and Wong, by Osborn, and by the author are compared in Sec. V for this purpose. The effectiveness of the Noyes-Kowalski approximation has also been investigated for the square-well potential by Reiner,[12] who explicitly calculated the off-shell matrix elements and compared them with the exact result and several other separable approximations.[13] In this case the Noyes-Kowalski formula was found to be orders of magnitude more accurate than the other separable forms considered.

IV. SEPARABLE APPROACH

In Sec. II we reviewed the well-known simplifications which occur in the Faddeev equations when a separable approximation is introduced for the off-shell two-body amplitude. Our purpose is to take advantage of these simplifications in performing three-body calculations with local potentials. To do so, we need a systematic and rapidly convergent procedure for generating accurate separable approximations. In this section we consider a particular approach to this problem.

The approximation we have in mind corresponds to the following explicit choice for the function $\Delta V_l(p',p;s)$ introduced in Eq. (15). We define

$$f_l(p,s)=\int_0^\infty \frac{q^2dq}{q^2-s-i\epsilon}V_l(q,p)$$

and

$$\Delta V_l(p',p;s)=\frac{f_l(p',s)}{f_l(s^{1/2},s)-1}V_l(s^{1/2},p)\,. \tag{22}$$

Although $f_l(s^{1/2},s)$ and $V_l(s^{1/2},p)$ are, properly speaking, only defined for s real and positive, we will allow s also to be negative, in which case these functions are to be regarded as their respective analytic continuations. We may now define two functions $\tilde{t}_l(p',p;s)$ and $\Delta t_l(p',p;s)$ as in Eq. (16). From the argument of Sec. III and Eq. (22), we see that $\Delta t_l(p',p;s)$ is separable in the variables p' and p. In fact,

$$\Delta t_l(p',p;s)=G_l(p',s)t_l(s^{1/2},p;s)\,,$$

where

$$G_l(p',s)=-[1-f_l(s^{1/2},s)]^{-1} \times\int_0^\infty dp\langle p'|[1+\tilde{V}_l(s)G_0(s)]^{-1}|p\rangle f_l(p,s)\,. \tag{23}$$

In other words, $G_l(p',s)$ satisfies the integral equation

$$G_l(p',s)=-[1-f_l(s^{1/2},s)]^{-1}f_l(p',s) -\int_0^\infty \frac{p^2dp}{p^2-s-i\epsilon}\tilde{V}_l(p',p;s)G_l(p,s)\,. \tag{24}$$

The above formulas thus provide a means of dividing t_l into a separable term Δt_l and a remainder term $\tilde{t}_l$.

[12] A. S. Reiner, Nuovo Cimento **51A**, 1 (1967).
[13] See, for example, J. Y. Guennégue's, Nuovo Cimento **42A**, 549 (1966).

We now observe that Δt_l contains the entire right-hand cut of t_l, i.e., $\tilde{t}_l$ is continuous across the positive s axis. A proof of this is given in the Appendix. As a consequence, $\tilde{t}_l$ is real for real s, and Δt_l has the exact imaginary part of t_l for positive s.

Furthermore, if our potential is such that it can be written as a superposition of Yukawa and exponential potentials, it can be shown that the "Born series" for $\tilde{t}_l$ converges absolutely for all real s, i.e., the infinite series

$$\tilde{t}_l(s)=\tilde{V}_l(s)-\tilde{V}_l(s)G_0(s)\tilde{V}_l(s)+\tilde{V}_l(s)G_0(s)\tilde{V}_l(s)G_0(s)\tilde{V}_l(s)\cdots \quad (25)$$

converges. The proof of this is given in [A],[14] where the off-shell amplitudes $t_l(p',p;s)$ generated by such potentials are expressed in terms of functions $r_l(p',p;s)$ defined by a convergent iterative procedure. Employing the notation of [A], the functions defined above have the following representations:

$$\begin{aligned} t_l(s^{1/2},p;s)&=[r_l(s^{1/2},p;s)+r_l(s^{1/2},-p;s)]/D_l(s)\,,\\ D_l(s)&=1+i\pi s^{1/2}(-1)^l r_l(s^{1/2},-s^{1/2};s)\,,\\ G_l(p',s)&=-(-1)^l i\pi s^{1/2} r_l(p',-s^{1/2};s)\,,\\ \tilde{t}_l(p',p;s)&=r_l(p',p;s)+(-1)^l r_l(p',-p;s)\,. \end{aligned} \quad (26)$$

Here $D_l(s)$ is the usual "D function" in partial wave l, and the above relations imply that

$$\begin{aligned} &G_l(s^{1/2},s)=1-D_l(s)\\ \text{and}\qquad &\tilde{t}_l(s^{1/2},p;s)=D_l(s)t_l(s^{1/2},p;s)\,. \end{aligned} \quad (27)$$

From the convergence of the r_l functions we immediately obtain the convergence of Eq. (25). It follows that Δt_l contains all the bound-state or virtual-state poles of t_l, and hence that the separable term correctly gives the residue at each such pole. From Eq. (27) we infer that this is also true if t_l has complex poles corresponding to resonances. We therefore conclude that for potentials of this class, the separable term $\Delta t_l(p',p;s)$ correctly gives the behavior at each bound-state or resonance pole, and gives the exact imaginary part of $t_l(p',p;s)$ for positive s.

We have not proved the former result for other local potentials, but that it is at least plausible can be seen from the following argument. From Eqs. (16) and (22) it is straightforward to show that

$$\tilde{t}_l(s^{1/2},p;s)=\left[1-\int_0^\infty \frac{q^2dq}{q^2-s-i\epsilon}t_l(s^{1/2},q;s)\right]^{-1}t_l\times(s^{1/2},p;s)\,. \quad (28)$$

Thus, for potentials of the class considered above, it follows from Eq. (26) that

$$D_l(s)^{-1}=1-\int_0^\infty \frac{q^2dq}{q^2-s-i\epsilon}t_l(s^{1/2},q;s)\,. \quad (29)$$

If Eq. (29) can be established independently for a given local potential, it will follow that Δt_l gives the correct behavior at each pole of t_l, and that $\tilde{t}_l$ has no poles of its own. This can be accomplished if we consider the function

$$F_l(s)\equiv\int_0^\infty \frac{q^2dq}{q^2-s-i\epsilon}t_l(s^{1/2},q;s)\,, \quad (30)$$

and show that it is analytic in s except for a right-hand cut, and vanishes as $s\to\infty$. This is plausible because $t_l(p,q;s)$ has only the right-hand cut in s, and the singularities in the p variable all depend on q for local potentials. The latter singularities will typically be washed out in the q integration. To show this for a given potential, it is easy to see that one may work with $V_l(s^{1/2},q)$ instead of $t_l(s^{1/2},q;s)$ in Eq. (30). Once this is established, the analytic properties in s of the right side of Eq. (29) uniquely determine it to be $D_l(s)^{-1}$. An example of a local potential not of the Yukawa or exponential type, but for which Eq. (29) is nevertheless valid, is the spherical "square well."

We wish to make use of the separable term Δt_l as a first approximation to t_l in performing three-body calculations. As noted above, it has most of the desired characteristics discussed in Sec. III. In particular, it exactly reproduces the behavior of t_l in the neighborhood of each bound-state or resonance pole. However, since $t_l(p',p;s)$ is symmetric under the interchange of p' and p and $\Delta t_l(p',p;s)$ is not, one would expect the approximation to fail except in the immediate vicinity of such a pole. This difficulty is avoided by symmetrizing in the p' and p variables, resulting in the separable approximation

$$t_l^s(p',p;s)=\tfrac{1}{2}[G_l(p',s)t_l(s^{1/2},p;s)+G_l(p,s)t_l(s^{1/2},p';s)]\,. \quad (31)$$

This is clearly of the form of Eq. (4) with $N_l=2$. The approximate amplitude $t_l^s(p',p;s)$ is related to $t_l(p',p;s)$ by the equation

$$t_l(p',p;s)=t_l^s(p',p;s)+R_l(p',p;s)\,, \quad (32)$$

with

$$R_l(p',p;s)=\tfrac{1}{2}[\tilde{t}_l(p',p;s)+\tilde{t}_l(p,p';s)]\,.$$

The first approximation to our three-body calculation is then obtained by employing the two-term separable approximation $t_l^s(p',p;s)$ in the Faddeev equations. Greater accuracy can be obtained by approximating $R_l(p',p;s)$ by a separable form, say,

$$R_l(p',p;s)\simeq\sum_{\lambda,\mu=3}^{N_l}A_\lambda{}^l(p',s)C_{\lambda\mu}{}^l(s)A_\mu{}^l(p,s)\,. \quad (33)$$

[14] D. D. Brayshaw, Phys. Rev. **167**, 1505 (1968), hereafter referred to as [A].

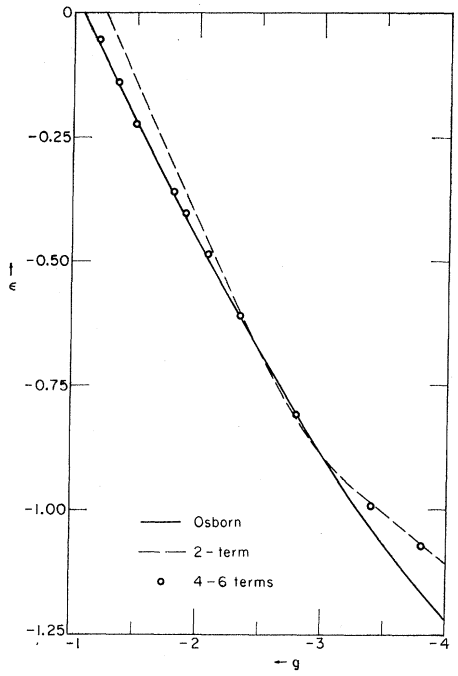

FIG. 1. Binding energies for the exponential potential as calculated from two- and more-term separable approximations.

Since the above-mentioned properties of $\tilde{t}_l$ imply that $R_l(p',p;s)$ is a smooth function of s, it is reasonable to expect that $R_l(p',p;s)$ will be considerably simpler to approximate than $t_l(p',p;s)$ itself. Furthermore, if the partial-wave amplitude has one or more bound-state or resonance poles, $R_l(p',p;s)$ should be small compared to $t_l^s(p',p;s)$ over a wide range of its variables, and hence even a crude approximation to $R_l(p',p;s)$ can improve the accuracy of the result. For example, in the case of a single Yukawa or exponential potential, numerical calculations by the author found $R_l(p',p;s)$ to be less than 10% of the total amplitude over a broad range of p', p, and s (the calculations were performed with unit range and couplings large enough to generate at least one bound state). In this case an approximation to $R_l(p',p;s)$ good to within 10% results in an over-all error of 1% or less.

Several methods of approximating $R_l(p',p;s)$ were tested by the author for these potentials, both by direct computation and by the three-body binding-energy calculations described in Sec. V. The main idea in all of these was to approximate $V_l(p',p)$ [and hence $\tilde{V}_l(p',p;s)$] by a separable form, and then to obtain a separable approximation to $\tilde{t}_l(p',p;s)$ by solving the LS equation with this separable potential. In approximating $V_l(p',p)$, the damping produced by the $p'^2+\frac{3}{4}q'^2-W-i\epsilon$ denominator in Eq. (3) for large p' suggests that it is most important to reproduce $\tilde{t}_l(p',p;s)$ exactly for small values of p'. Since we symmetrize in p' and p, we should also require accuracy for small p. This can be accomplished by choosing a separable form for $V_l(p',p)$ which is exact when either p' or p are small. To lowest order, this suggests the approximation

$$V_0(p',p)\simeq\frac{V_0(p',0)V_0(p,0)}{V_0(0,0)} \tag{34}$$

for the $l=0$ case, and corresponding expressions for the higher partial waves.

Although the approximation to $R_l(p',p;s)$ which results from using Eq. (34) can certainly be improved, it nevertheless should suffice for our purposes under the circumstances stated above, i.e., when the $t_l^s(p',p;s)$ term dominates. If necessary, one can add additional terms to Eq. (34) to reproduce $V_0(p',p)$ to higher orders in p'^2 or p^2. Given this approximation, one can proceed to calculate $\tilde{t}_l(p',p;s)$ in several different ways. The simplest is to take just the "Born approximation" $\tilde{t}_l(p',p;s)\simeq\tilde{V}_l(p',p;s)$. Secondly, one can solve for $\tilde{t}_l(p',p;s)$ as in an ordinary separable potential problem. The most accurate approach, however, is to just make the separable approximation to $\tilde{V}_l(p',p;s)$ in the inhomogeneous term of the LS equation, leaving the kernel exact.

At this point one can reduce the number of separable terms in the over-all approximation by making a slight modification in the way in which we define t_l^s and $\tilde{t}_l$. With the present definitions the approximation of Eq. (34) to $V_l(p',p)$ results in a three-term separable approximation to $R_l(p',p;s)$ or five terms over-all. This number can be reduced to four by replacing $t_l(s^{1/2},p;s)$ in Eq. (23) by $t_l(s^{1/2},p;s)-V_l(s^{1/2},p)$. The corresponding change in $\tilde{t}_l(p',p;s)$ requires it to be the solution of the equation

$$\tilde{t}_l(s)=V_l-\tilde{V}_l(s)G_0(s)\tilde{t}_l(s). \tag{35}$$

Of course, if one is willing to make use of separable approximations of four or more terms, there are alternative methods to choose from, such as the method of Ball and Wong discussed in Sec. III. However, our main interest here is to provide a simple means of calculating corrections to the results produced by the two-term formula given in Eq. (31). The numerical calculations to be discussed in Sec. V indicate that our two-term formula is highly accurate, and that even the crudest of the methods used to approximate the $R_l(p',p;s)$ correction term is sufficient to reduce the error significantly. The simplicity of these methods enables one to keep computer time to a minimum.

Finally, we note the explicit formulas for $f_0(p,s)$ in the two cases studied numerically. For the exponential potential with s-wave momentum-space representation $V_0(p',p)$, we have

$$\begin{aligned} V_0(p'p)&=\frac{4\mu G}{\pi}\frac{1}{(p'-p)^2+\mu^2}\frac{1}{(p'+p)^2+\mu^2},\\ f_0(p,s)&=\frac{G}{p^2+(\mu-is^{1/2})^2}. \end{aligned} \tag{36}$$

For the Yukawa potential,

$$V_0(p',p)=\frac{G}{2\pi\mu p' p}\ln\left[\frac{(p+p')^2+\mu^2}{(p-p')^2+\mu^2}\right],$$
$$f_0(p,s)=\frac{G}{2i\mu p}\ln\left[\frac{\mu-is^{1/2}+ip}{\mu-is^{1/2}-ip}\right]. \tag{37}$$

V. NUMERICAL CALCULATIONS

In order to check the effectiveness of the two-term separable approximation introduced in the last section, a number of numerical calculations were performed. Two typical local potentials (exponential and Yukawa) were considered. These potentials were parametrized by the coupling strength G and the inverse range μ, such that

$$V^{\exp}(r)=Ge^{-\mu r},$$
$$V^{\mathrm{Yuk}}(r)=Ge^{-\mu r}/\mu r. \tag{38}$$

As an initial check, off-shell t-matrix elements generated by the approximation were compared directly to the exact values. When μ and G were chosen so as to produce a two-particle bound state, the relative error was found to be on the order of 10% for p', $p\leq\mu$ and $100s_0<s<s_0$, s_0 being the energy of the bound state. These calculations were also performed with four- and six-term separable approximations obtained by approximating $R_l(p',p;s)$ in the manner described in Sec. IV; the relative error was then cut to 1%. The approximations were all considerably more accurate for the smaller values of p', p, and $|s|$ where the off-shell amplitude is largest numerically.

The separable approximations were then used to calculate the $J=0$ ground-state energy for a system of three identical spinless particles. In doing the calculation it was assumed that only the $l=0$ contribution is important in Eq. (3), and hence the results are directly comparable to those published recently by Humberston, Hall, and Osborn.[15] The calculation by Osborn, which corresponds to the direct inversion of Eq. (3) by a special numerical technique, provides the most direct test of the present work. The results for the exponential potential are plotted in Fig. 1, and those for the Yukawa potential in Fig. 2. The parameters ϵ and g are defined by

$$\epsilon=-(ME_B/\mu^2)^{1/2},$$
$$g=MG/\mu^2, \tag{39}$$

where E_B is the three-body binding energy and M is the common mass. Here we note that for the choice of μ and G defined in Eq. (38),

$$E_B(G,M,\mu)=(\mu^2/M)E_B(MG/\mu^2,1,1). \tag{40}$$

It is evident that the value of E_B predicted by the two-term approximation is in excellent agreement with Osborn's value for both potentials, except for very small couplings. Secondly, the four- and six-term results, which are generally indistinguishable from each other on the scale plotted, quite successfully reduce the error in the exponential case to a few percent. On the other hand, they significantly increase the discrepancy with Osborn's value for the Yukawa potential. The existence of this discrepancy is somewhat puzzling in view of the success of these approximations in actually reproducing the off-shell t-matrix elements. Although one would expect the method chosen to approximate $R_l(p',p;s)$ to be less effective for the Yukawa potential than the exponential potential [because of the less rapid falloff of $t_l(p',p;s)$ for large p' or p], it nevertheless seems good enough to give the *sign* of the correction to the two-term result. Hopefully, a calculation based on a more sophisticated way of approximating $R_l(p',p;s)$ will resolve this contradiction.

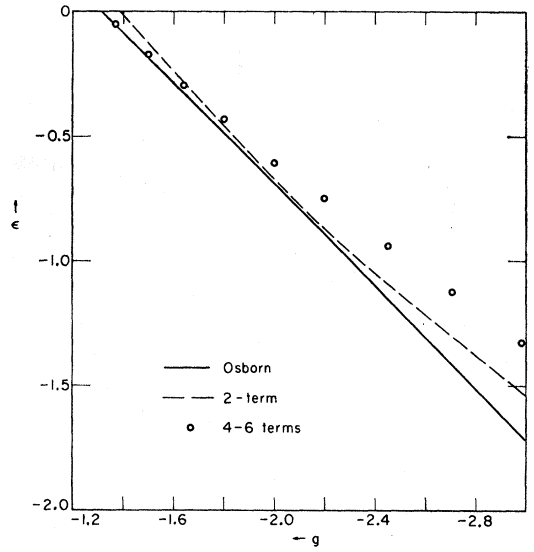

FIG. 2. Binding energies for the Yukawa potential as calculated from two- and more-term separable approximations.

In any case, the results seem to support the general practicality of the approach, and the two-term approximation appears suitable as a starting point for numerical calculations. It is important to point out that the numerical work required to obtain the two-term values quoted involved matrices of dimension 20 (allotting 10 points to the q integration), as compared to the matrices of dimension 105 employed by Osborn (allotting 7 points to the q integration). With respect to other calculations using separable approximations (such as those described in Sec. III) the use of Eq. (26) and the iterative formulas developed in [A] are particularly advantageous in reducing the amount of time required to compute each matrix element.

In order to use a computational procedure of the above type with confidence in performing physically interesting calculations, it is necessary to have some means of estimating the error introduced in the result by the separable approximation. With this end in mind,

[15] J. Humberston, R. Hall, and T. Osborn, Phys. Letters **27B**, 195 (1968).

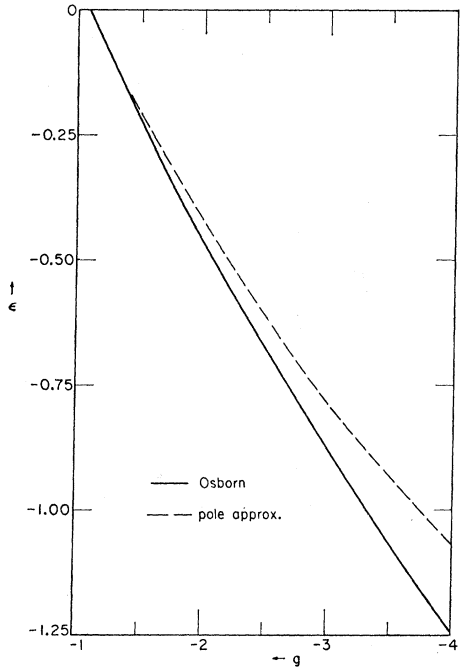

FIG. 3. Binding energies for the exponential potential as obtained from a simple pole approximation.

a particular type of error estimate was checked numerically against the binding-energy results described above. The method is applicable whenever the difference between $t_l(p',p;s)$ and $t_l^s(p',p;s)$ is small enough to be considered a perturbation, and is based on some results published by Yaes.[16] Assuming that the separable approximation produces a three-body bound state of energy W_0, we know that

$$X_{\lambda\mu}{}^{l'l}(q',q;W) \xrightarrow[W\to W_0]{} \frac{G_\lambda{}^{l'}(q')G_\mu{}^{l}(q)}{W-W_0}, \tag{41}$$

which defines the functions $G_\lambda{}^l(q)$. The change δW_0 in the bound-state energy due to a small perturbation of $t_l^s(p',p;s)$ is then given by

$$\delta W_0=\sum_{l',\lambda',\mu',l,\lambda,\mu}\int_0^\infty dq'q'^2\int_0^\infty dq\, q^2 G_\lambda{}^l(q)C_{\lambda\mu}{}^l \times(W-\tfrac{3}{4}q^2)\langle\mu ql|Z\delta TZ|\mu' q'l'\rangle C_{\mu'\lambda'}{}^{l'} \times(W-\tfrac{3}{4}q'^2)G_{\lambda'}{}^{l'}(q'). \tag{42}$$

Here the states $|\mu ql\rangle$ are defined in terms of the states $|pql\rangle$ appropriate to the $J=0$ three-body system by the equation

$$|\mu ql\rangle=\int_0^\infty dp\, p^2 A_\mu{}^l(p,W-\tfrac{3}{4}q^2)|pql\rangle,$$

with

$$\langle p'q'l|pql\rangle=\frac{\delta(p'-p)}{p^2}\frac{\delta(q'-q)}{q^2}\delta_{l'l}. \tag{43}$$

[16] R. J. Yaes, Phys. Rev. **170**, 1236 (1968).

Thus

$$Z^{l'l}(p'q'|pq;W)=\langle p'q'l'|Z|pql\rangle,$$

$$\langle p'q'l'|\delta T|pql\rangle=\frac{\delta(q'-q)}{q^2}\delta_{l'l} \times[t_l(p',p;W-\tfrac{3}{4}q^2)-t_l^s(p',p;W-\tfrac{3}{4}q^2)]. \tag{44}$$

The numerical results obtained through the use of Eq. (42) are given in Table I. The method is clearly successful in providing an order-of-magnitude estimate of the error, and thus should be a useful accessory to our calculational procedure.

Finally, some additional binding-energy calculations were performed to check the accuracy of a slightly modified pole approximation for these potentials. For values of μ and G such that the t matrix had a single bound-state pole at s_0, the approximation chosen was

$$t_l^p(p',p;s)=-\frac{g_l(p')g_l(p)}{D_l(s)}, \tag{45}$$

where $D_l(s)$ is the exact "D function" in partial wave l, and $g_l(p)$ is defined such that Eq. (45) gives the exact behavior of $t_l(p',p;s)$ as $s\to s_0$. For potentials which are superpositions of Yukawa and exponential potentials, we have the explicit formula

$$g_l(p)=[\pi^2 s_0 N_l(s_0)]^{1/2} r_l(p,-s_0{}^{1/2};s_0), \tag{46}$$

where $N_l(s)$ is the usual "N function" and can be evaluated from the relation

$$N_l(s)=r_l(s^{1/2},s^{1/2};s)+(-1)^l r_l(s^{1/2},-s^{1/2};s). \tag{47}$$

The numerical results for the exponential and Yukawa potentials are plotted in Figs. 3 and 4, respectively, where Osborn's curve is again reproduced for comparison. The agreement is reasonably good for small values of G, but the curves diverge as G increases.

The results plotted in Fig. 4 provide an interesting illustration of the ambiguity in pole approximations discussed at the end of Sec. III. These results can be directly compared with the results obtained for the same problem by Ball and Wong.[4] Their lowest-order approximation also involves one separable term giving

TABLE I. Error estimates for the exponential (exp) and Yukawa (Yuk) potentials based on Eq. (42). The calculations were performed with $M=\mu=1$.

Potential	G	No. of terms	W_0	δW_0	Rel. error (%)
exp	−1.9	2	−0.137	-6.04×10^{-3}	4.45
exp	−1.9	6	−0.163	-5.41×10^{-5}	0.033
exp	−2.1	2	−0.230	-9.88×10^{-3}	4.30
exp	−2.1	6	−0.233	-9.98×10^{-5}	0.043
Yuk	−2.2	2	−0.776	-3.97×10^{-2}	5.12
Yuk	−2.2	6	−0.554	-3.58×10^{-4}	0.065
Yuk	−3.0	2	−2.31	-9.37×10^{-2}	4.06
Yuk	−3.0	6	−1.75	-1.75×10^{-3}	0.100

exact behavior at the bound-state pole. However, the relative error of their results to Osborn's value is more than twice the error produced by the pole approximation above.

VI. DISCUSSION

In the previous sections we considered a particular two-term separable approximation and applied it to the calculation of three-particle binding energies. The approximation was so chosen as to produce the exact right-hand cut of the off-shell t matrix, to give exact behavior at each pole, and to introduce no additional singularities. It thus satisfies the general criteria developed in Sec. III for use in the Faddeev equation. In particular, its failure to possess any particular on-shell properties does not limit it for this purpose. With respect to numerical accuracy, it was pointed out earlier that binding-energy calculations of the type performed provide a severe test of such an approximation's effectiveness. For the potentials chosen, the numerical results indicate that the approximation is capable of reproducing some 90% of the exact amplitude. As compared to the simple pole approximation also investigated, the binding energy produced by the two-term formula remains close to the exact result over a much larger range of coupling strengths. The two-term approximation thus appears suitable for performing, with local potentials, the type of model calculation which is at present the province of separable potentials alone.

In addition, we also considered a procedure for improving on our approximation by successively adding more separable terms. By choosing these terms in the manner described in Sec. IV, it was possible, in the case of the exponential potential, to reduce the error in the binding energy to a few percent. To obtain this accuracy, two additional separable terms were sufficient, although calculations were also performed with four additional terms as a check on the convergence. On the other hand, the method employed produced results for the Yukawa potential in conflict with Osborn's values. It is worth pointing out that other methods, such as the approach of Ball and Wong, might well be applied to approximate $R_l(p',p;s)$ (which is always real and well behaved as a function of s) for this purpose. The important point is that even six- or eight-term approximations offer advantages in computation compared to two-dimensional numerical integration. The number of terms should be compared, for example, with the 15 points required by Osborn for the p integration. The advantage in fewer terms will be obviated, of course, if the functions required in the separable approximation are excessively complicated. In this respect the iterative formulas described earlier greatly simplify the computation required for potentials of the exponential or Yukawa type.

The philosophy expressed in this paper has clearly been that it is worthwhile going to some trouble to

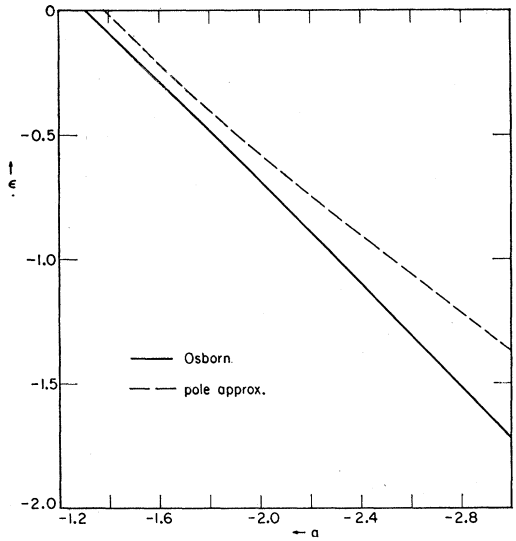

FIG. 4. Binding energies for the Yukawa potential as obtained from a simple pole approximation.

employ local potentials in three-body calculations, despite the obvious computational advantages of separable potentials. Aside from our fundamental reasons for believing that local potentials are the more "physical," the avoidance of local potentials in three-body calculations tends to negate one of the main reasons for doing these calculations, i.e., the ability to probe the off-shell matrix elements and hence discriminate between potentials which are equivalent in producing two-body behavior. It would indeed be unfortunate if local and separable potentials, with parameters chosen to fit the two-body data, were to produce identical three-body behavior. Hopefully, accurate separable approximations which are *not* equivalent to separable potentials will enable us to produce such a confrontation on a scale suitable to our present computers.

In this respect, the only evidence which is in to date concerns the comparison of binding-energy results for the Yamaguchi[17] and Bander[18] potentials with the local potential results of Osborn, of Ball and Wong, and of the present author. As shown by Osborn, the separable potentials, with parameters chosen to give the same (two-body) bound-state energy and scattering length as the local potential, produce a binding energy which is about 10% too large for couplings just slightly larger than the minimum required to support a two-body bound state. However, as the coupling is increased the separable potential result becomes rapidly much larger, and in fact goes to infinity at a coupling considerably smaller than that required to produce a second two-body bound state. This is certainly a dramatic contrast, but it is unfortunately not a three-body effect. Actually, the separable approximations can no longer

[17] Y. Yamaguchi, Phys. Rev. **95**, 1635 (1954).
[18] M. Bander, Phys. Rev. **138**, B322 (1965).

satisfy the *two-body* data for couplings greater than this critical value. This is easily seen by considering the particular case of the Yamaguchi potential,

$$V(\mathbf{P}',\mathbf{P})=-(\lambda/M)g(p')g(p)\,,\qquad g(p)=(p^2+\mu^2)^{-1}. \tag{48}$$

Here the scattering is purely s-wave, and we have

$$t_0(p',p;s)=\frac{4\pi V(\mathbf{p}',\mathbf{p})}{D(s)}\,,\qquad D(s)=1-(\lambda\pi^2/\mu)[\mu-i(Ms)^{1/2}]^{-2}. \tag{49}$$

Requiring that $D(s_0)=0$ for $s_0<0$, and that

$$t_0(s^{1/2},s^{1/2};s)\,|_{s=0}=T_0\,,$$

we obtain two equations from which to determine the two parameters λ and μ. They are equivalent to the relations

$$\lambda=(\mu/\pi^2)[\mu-i(Ms_0)^{1/2}]^2\,,\qquad (X-1)Y^2+\tfrac{1}{2}(X-4)Y-1=0\,, \tag{50}$$

where

$$Y\equiv\mu|Ms_0|^{-1/2}\,,\qquad X\equiv\tfrac{1}{2}M\pi T_0|Ms_0|^{1/2}. \tag{51}$$

Physically, y is required to be positive since μ is an inverse range. It is easy to verify that the quadratic equation for Y [Eq. (50)] permits a positive solution for Y only when $X>1$. We then get

$$Y=\frac{4-X+[(4-X)^2+16(X-1)]^{1/2}}{4(X-1)}\,,\quad X>1. \tag{52}$$

If we now determine X for the exponential or Yukawa potential, we find that for small couplings X is indeed larger than unity. Thus reasonable values for μ and λ result for small local couplings. However, as the local coupling strengths increase, X decreases, becoming smaller than unity and eventually negative. Clearly, as $X\to 1$, $\mu\to\infty$ and $\lambda/\mu^3\to\pi^{-2}$. From the relation

$$E_B(\lambda,\mu)=\mu^2E_B(\lambda/\mu^3,1)\,, \tag{53}$$

for the three-particle binding energy E_B, it follows that $E_B\to\infty$ as $X\to 1$. Thus, as observed, the binding energy becomes infinite at the critical value of the local coupling constant where $X=1$. For larger values of the coupling, the separable potential is completely meaningless since μ and λ both become negative. The discussion for Bander's potential is analogous and the critical value of the coupling is the same.

It is clear that the phenomenon is caused by the inability of the separable potential to produce simultaneously a finite binding energy and an arbitrarily small T_0 in the two-body problem. It is also apparent that a value of the local coupling constant for which $X<1$ *must* occur in the region observed since T_0 is known to change sign before a second bound state is produced. This, of course, is a general property and not restricted to the particular local potentials under consideration.

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Brookhaven National Laboratory for the use of its excellent computer facilities, and to Dr. R. F. Peierls for his help in performing the calculations.

APPENDIX: PROOF THAT t_l HAS NO RIGHT-HAND CUT

In this Appendix we prove the result quoted in Sec. IV to the effect that $\tilde{t}_l(s)$ does not have the right-hand cut of the partial-wave amplitudes. We begin with the defining equation for $\tilde{t}_l(s)$

$$\tilde{t}_l(s)=\tilde{V}_l(s)-\tilde{V}_l(s)G_0(s)\tilde{t}_l(s)$$

or

$$\tilde{t}_l(s)=[1+\tilde{V}_l(s)G_0(s)]^{-1}\tilde{V}_l(s)\,. \tag{A1}$$

Letting s be real and positive, defining

$$\Delta\tilde{t}_l(s)=\tilde{t}_l(s+i\epsilon)-\tilde{t}_l(s-i\epsilon)\,, \tag{A2}$$

and similarly for $\Delta\tilde{V}_l(s)$ and $\Delta G_0(s)$, it is straightforward to obtain

$$\Delta\tilde{t}_l(s)=[1+\tilde{V}_l(s+i\epsilon)G_0(s+i\epsilon)]^{-1}\mathcal{U}_l(s)G_0(s-i\epsilon)\times[1+\tilde{V}_l(s-i\epsilon)G_0(s-i\epsilon)]^{-1}G_0^{-1}(s-i\epsilon)\,,$$

with

$$\mathcal{U}_l(s)\equiv\Delta\tilde{V}_l(s)-\tilde{V}_l(s+i\epsilon)\Delta G_0(s)\tilde{V}_l(s-i\epsilon). \tag{A3}$$

Thus $\Delta\tilde{t}_l(s)$ will vanish providing that $\langle p'|\mathcal{U}_l(s)|p\rangle$ vanishes.

To show that this is the case, we make use of the formulas

$$\langle p'|\Delta\tilde{V}_l(s)|p\rangle=V_l(s^{1/2},p)\left[\frac{f_l(p',s+i\epsilon)}{1-f_l(s^{1/2},s+i\epsilon)}-\frac{f_l(p',s-i\epsilon)}{1-f_l(s^{1/2},s-i\epsilon)}\right],$$

$$\langle p'|\Delta G_0(s)|p\rangle=\pi i s^{1/2}\delta(p'-s^{1/2})\delta(p-s^{1/2}), \tag{A4}$$

$$f_l(p,s+i\epsilon)-f_l(p,s-i\epsilon)=\pi i s^{1/2}V_l(p,s^{1/2})\,.$$

From these it follows that

$$\langle p'|\Delta\tilde{V}_l(s)|p\rangle=i\pi s^{1/2}\tilde{V}_l((s-i\epsilon)^{1/2},p;s-i\epsilon)\tilde{V}_l\times(p',(s+i\epsilon)^{1/2};s+i\epsilon)=\langle p'|\tilde{V}_l(s+i\epsilon)\Delta G_0(s)\tilde{V}_l(s-i\epsilon)|p\rangle. \tag{A5}$$

From Eq. (A3) we see that $\langle p'|\mathcal{U}_l(s)|p\rangle$ vanishes and hence $\tilde{t}_l(s)$ is continuous across the positive s axis.