Recently, Sweig and Wada[14] have estimated the cross section for the production of A_1 meson by pions on heavy nuclei with the assumption that the diffraction dissociation is the most important mechanism by analogy to ρ^0 production. According to their estimate within the framework of the gauge theory, we get

$$d\sigma(\pi \to A_1)/dt = (16\pi)^{-1}\tfrac{1}{4}\sigma_{\mathrm{tot}}{}^2(A_1\text{—nucleus}),$$

whereas in the algebraic approach this relation will not hold.

[14] M. J. Sweig and W. W. Wada, Phys. Rev. Letters **21**, 414 (1968).

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Information Theory and Multiparticle Production*

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The secondary particles produced in high-energy inelastic collisions are treated statistically but in a manner distinct from the concept of one or more fireballs. We employ the classical grand-canonical approach, and in the spirit of information theory define an entropy for the system. It is postulated that this entropy is independent of the energy of the system for a sufficiently high incoming energy. Consequently, the energy dependence of the charged multiplicity N of the secondaries is determined to be $N \sim (E_0/\mu)^{2/3}$, where E_0 is the incoming center-of-mass energy and μ a normalizing mass. Under additional but natural assumptions, the ratios $N_{\pi^\pm}N_{K^\pm}N_{p\bar{p}}$ are predicted. Comparison of our results is made with experiment, and some limitations and consequences of the method are discussed.

1. INFORMATION-THEORY APPROACH

THERE is in the literature a number of statistical and thermodynamic models attempting to explain the properties of high-energy multiparticle production.[1] Since the early work of Fermi, these models have grown in sophistication, success, and generally in complexity. The basis of these models is the concept that incoming particles coalesce and/or transform into multiparticle systems (e.g., fireballs) which survive long enough to attain a state of equilibrium, and to which the analysis of an ideal quantum gas may be applied.

As far as we know, no attempt has been made to apply a statistical approach to the produced multiparticle system without this underlying concept of a coalescence. Indeed, this is natural, since we do not have a state at equilibrium but something more akin to an explosion. However, the information-theory approach to statistical mechanics (both classical and quantum) suggests a broader sense of the term equilibrium, and it is in this spirit that our analysis is made.

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[1] E. Fermi, Progr. Theoret. Phys. (Kyoto) **5**, 570 (1950); Phys. Rev. **81**, 683 (1951). See also, R. H. Milburn, Rev. Mod. Phys. **27**, 1 (1955); R. Hagedorn, Nuovo Cimento Suppl. **3**, 147 (1965); J. R. Wayland and T. Brown, *ibid.* **48**, 663 (1967); R. Hagedorn and J. Ranft, Nuovo Cimento Suppl. **6**, 169 (1968).

Information theory may be called the science of guessing. Given any system of which only a few observable constants of motion are known, the problem posed is to guess its distribution function f in phase space. Usually, there will be an infinite number of eligible distributions, each satisfying the conditions known of the system. Although it is the desire of any approach to make predictions, it is first necessary to select the most suitable f on the basis that it involves no spurious arbitrary information. To this end, the concept of entropy I is introduced. It can be shown that entropy measures the amount of missing information in our system, or, more precisely, for each f purporting to represent the system.[2] The preferred choice of f is then that f which maximizes I—i.e., which assumes the least *a priori.*

$$I = k \int f \ln f \quad (k = \text{arbitrary constant}), \qquad (1)$$

where $\int$ stands for integration (or summation) over phase space. The constants of motion, together with the normalization of f, are introduced into I via Lagrange multipliers.

$$I' = I + k(\Omega+1)\langle 1\rangle - k \sum_{i=1}^{n} \lambda_i \langle R_i\rangle, \qquad (2)$$

[2] For a very readable and fuller presentation of this approach, see Ammon Katz, *Principles of Statistical Mechanics* (W. H. Freeman and Co., San Francisco, 1967).

$\langle R_i \rangle$ being the expectation value of the observable corresponding to R_i.

It can then be shown that the f which maximizes I ($\partial I'/\partial f=0$) is given by

$$f=\exp(\Omega-\sum_{i=1}^{n}\lambda_i R_i). \quad (3)$$

The values of Ω and λ_i are determined by solving the following set of constraint equations:

$$\Omega(\lambda_1\cdots\lambda_n)=-\ln\int\exp(-\sum_{i=1}^{n}\lambda_i R_i), \quad (4)$$

$$\frac{\partial\Omega}{\partial\lambda_i}=\langle R_i\rangle\equiv r_i. \quad (5)$$

These constraint equations determine the λ_i (and hence f) uniquely if the R_i are linearly independent.

At this stage we have already achieved a good deal. For, although we have conspicuously taken care to select the f with the maximum amount of missing information, by its uniqueness we have a very specific discription of our system. For example, in the case of n arbitrary choices with no known bias (no $\langle R_i\rangle$), our analysis predicts a flat distribution, i.e., any particular choice is as likely as any other. This simplest of examples also tells us that any bias in the system will have to be put in as a constraint.

However, the real power in the approach comes via assumptions about the behavior of I under small modifications to our system. Consider an ideal gas. Its volume, energy, and particle number are assumed known. For this system an f can be found together with its corresponding entropy.

$$I=+k(-\Omega+\sum_{i=1}^{n}\lambda_i r_i). \quad (6)$$

It can now be argued that, under certain changes in the expectation values r_i, no additional information is gained or lost. Consequently, invariance of I under these infinitesimal variations dr_i leads to the ideal gas laws. In practice, such application implies some degree of stability for the system. Entropy is not conserved in irreversible processes, simply because more information is known about the initial state of the system than about the final state, namely the tendency for it to transfer from the former to the latter.

Can this concept be applied to multiparticle production? We have no intention of calculating the entropy of the incoming system and equating it to that of the final system. Experimentally it is found that the outcoming particles in a two-body scattering process are far from democratic. Two so-called primary particles,[3] often identical to the incoming pair, carry away the majority of the center-of-mass energy. The rest of the energy is predominantly distributed among pions, kaons, and nucleons. At machine energies (up to 30 GeV lab), pions dominate completely. In principle therefore, the energy and momenta of the primary particles can be measured, and even possibly an explicit model of their behavior given. In any case, we shall assume they contribute no appreciable entropy to the outcoming system. It has been postulated that the secondary particles remove a fixed percentage ($\sim$40%) of the total center-of-mass energy (E_0), on average.[4] We can therefore consider the average energy the secondaries in the center-of-mass frame ($E\simeq\frac{2}{5}E_0$) as an observable, together with the multiplicity (N) of the scattering. There are other constraints of the system also such as conservation of charge, total isotopic spin, angular momentum, etc. Thus, we may follow the first part of the information-theory prescription and define a suitable f. In fact, we should like to do this in a Lorentz-invariant, and dimensionally invariant manner, by a suitable choice of the phase integral and observables (e.g., total four-momentum P_ν rather than energy E).

It is known experimentally, however, that N is a function of E (and hence E_0). It is primarily this relationship between N and E that we seek. To this end we make the following postulate: *Asymptotically* ($E_0\gg$constant), *a production process reaches stability in the sense that the entropy is constant as a function of* E_0.

As suggestive of this assumption, note the universal properties of the Pomeranchukon (which, via unitarity, is related to the multiparticle-production processes). In particular, note the tendency of total cross sections to flatten out asymptotically. Our restriction of the above assumption to high energies lies both in the evidence from total cross sections, and in the belief that the secondaries are produced in a multitude of ways, some of which may (e.g., via resonance decays) involve energy thresholds. From a practical point of view the approximations in the classical grand-canonical formalism necessitates a high multiplicity. Also, we shall find our analysis can be performed explicitly in such a limit.

Our treatment of the produced particles will be classical. Generally, it is only when more than one particle carries the same quantum numbers (including its momentum), that the differences between various statistics is important. In our case, the occurrence of such a circumstance is considered negligible. Consequently, the fermions and bosons will be treated identically.

In Sec. 2 we shall present a calculation of the entropy of a system of produced pions only. We shall not

[3] We do not employ the word primary here in the usual sense of the incoming particles (as used in cosmic-ray physics) but, since we identify them closely with the incoming particles, the naming is not inappropriate. They are comparable to the outgoing particles in the two-body channel minus part of the incoming energy.

[4] See, for example, F. Turkot, in *Proceeding of the Topical Conference on High-Energy Collisions of Hadrons* (CERN, Geneva, 1968), Vol. 1, p. 316; K. Rybicki, Nuovo Cimento **49A**, 233 (1967).

impose conservation of isotopic spin, nor of total angular momentum. This analysis will extend in Sec. 3 to the case of pions and kaons, and pions, kaons, and nucleons, under some additional simplifying assumptions.

Finally, in Sec. 4, our results will be compared with experiment and a further discussion of our method made.

2. APPLICATION TO HIGH-ENERGY MULTIPION SYSTEM

Consider a system (secondary) consisting solely of pions. Experimentally, its basic observables are its average energy, in say its center-of-mass frame E_π, and the average number of charged pions produced N_π, both for a given incoming energy E_0. For simplicity, but without any loss of generality, we shall consider a system with equal numbers of positively and negatively charged pions. The experimental fact that only charged pions are in general observed will be taken into account in the usual way by setting the Lagrange multiplier for the number of neutral particles (γ) equal to zero. However, we shall carry it along with us for the time being to remind us that the corresponding phase-space integrals must still be performed. The incoming and primary particles are of course experimentally used to determine E_0 and E_π. For which purpose the assumption is made that the primaries, secondaries, and incoming system all share the same center-of-mass frame. Nevertheless, they are of no interest to us in calculating the entropy of the secondary system, and will be ignored in what follows.

We shall define the relativistically invariant phase-space integral per particle, a phase-space cell, as

$$\frac{1}{\mu^2}\int\frac{d^3P}{2P_0}. \tag{7}$$

The limits are over all 3-momenta, with $P_0{}^2=\mathbf{P}^2+m_\pi{}^2$. The normalizing mass μ is introduced to make Eq. (7) dimensionally invariant.

Our Lagrange multipliers will be (i) β^ν, corresponding to the total 4-momenta,

$$P_\nu=\sum_{i=1}^{2n+l}p_\nu{}^i;$$

(ii) α, corresponding to the total number of charged pions $2n$; and (iii) γ, corresponding to the number of neutral pions l. The conservation of charge relates the numbers of positively and negatively charged pions, and hence dictates that only one multiplier be used, namely α.

Our distribution for such a system with a given n and l, is

$$f_n=\exp[\Omega-\beta^\nu P_\nu/m-\alpha(2n)-\gamma l], \tag{8}$$

where again m is a mass introduced for dimensional reasons. In the grand-canonical formalism f is an infinite vector with components f_n

$$f=(f_0,f_1,\cdots f_n,\cdots). \tag{9}$$

Having insured a completely relativistic procedure, we now work in the center-of-mass frame of the secondaries $P_\nu\to(E_\pi,0,0,0)$ and

$$\beta^\nu\to(\beta,0,0,0). \tag{10}$$

Equation (10) can be proven from the symmetry of Eq. (7) under $\mathbf{P}\to-\mathbf{P}$ and the uniqueness of f.[5]

Consequently,

$$e^{-\Omega}=\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{n=0}^{\infty}\frac{1}{n!n!}\left\{\left(\frac{1}{\mu^2}\int e^{-\beta P_0/m}\frac{d^3P}{2P_0}\right)^{2n+l}e^{-2\alpha n-\gamma l}\right\}, \tag{11}$$

where the factorials allow for the indistinguishability among pions of the same charge. Such weighting would be expected to be somewhat modified if the conservation of the other two components of isotopic spin would be imposed. Note the infinite upper limits in the number summations. These can at best only be approximations,[6] since for a given E_0 there is a maximum $2n+l=E_0/m_\pi$. Similarly each phase integral should really have a cutoff imposed. However, both of these facts can be ignored for high enough energies and hence large N_π's if f_n is decreasing sufficiently rapidly as n (and hence $2n+l$, for any given l) approaches the physical upper limit.

Both from the identification of β as $1/kT$ for an ideal gas and by explicit calculation, it can be determined that β decreases as $E\pi$ increases. Consequently, being limited already to the high-energy region, we shall calculate each phase-space cell in the limit $\beta\to 0$,

$$\phi=\frac{1}{\mu^2}\int e^{-\beta P_0/m}\frac{d^3P}{2P_0}\sim\frac{2\pi m^2}{\beta^2\mu^2}, \tag{12}$$

which follows from the effective unimportance of the low-energy contribution to this integral, i.e., we can replace $P_0\simeq|\mathbf{P}|$ for sufficiently small β.

Now let us put $\gamma\equiv 0$, for the reasons already given; then Eq. (11) becomes

$$\Omega=-\ln[e^\phi I_0(2\phi e^{-\alpha})], \tag{13}$$

where

$$I_0(z)=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}z)^{2n}}{n!n!} \tag{14}$$

is the modified Bessel function of the first kind. Asymp-

[5] See Ref. 2, p. 64.

[6] The prescription for limits in this procedure is that they should define the available region in phase space within which the expectation values may lie. For an ideal gas there is no finite upper limit to the number of point particles one may enclose in a given volume; but even there, particles of a finite size would require an upper limit to $\sum_n$.

totically as $z \to \infty$ $(\beta \to 0)$,

$$I_0(z) \sim e^z/(2\pi z)^{1/2}. \quad (15)$$

Thus,

$$\Omega \sim -\phi - 2\phi e^{-\alpha}, \quad (16)$$

where we have dropped the $\ln(4\pi\phi e^{-\alpha})$ term coming from the denominator of (15). This fact will be considered later in estimating the energy region above which our approximations may be assumed to hold.

The constraint equations determining β and α are, therefore,

$$\frac{\partial\Omega}{\partial\alpha} = 2\phi e^{-\alpha} = N_\pi \quad (17)$$

and

$$\frac{\partial\Omega}{\partial\beta} = \frac{2}{\beta}(\phi + N_\pi) = \frac{E_\pi}{m}. \quad (18)$$

At this stage, we should solve these two simultaneous equations for α and β as a function of E_π and N_π. However, our intentions being more ambitious, we shall first require that the entropy for our system is a constant under any increment of E_π. This provides a relationship between E_π and N_π, given by

$$m\frac{dN_\pi}{dE_\pi} = -\frac{\beta}{\alpha}. \quad (19)$$

Equations (17)–(19) may be rewritten as

$$\beta^3 E_\pi/m_\pi - 2N_\pi\beta^2 = 4\pi m^2/\mu^2, \quad (20)$$

and

$$m\frac{dN}{dE_\pi} = \frac{\beta}{\ln(N_\pi\beta^2\mu^2/4\pi m^2)}, \quad (21)$$

where Eq. (17) has been used to eliminate α in Eq. (19), and give Eq. (21). The logarithmic term in Eq. (21) severely restricts the asymptotic energy dependence of N_π, and hence of β and α. Considering solutions of the kind

$$N_\pi = (E_\pi)^\eta(\ln E_\pi)^\epsilon, \quad \beta = (E_\pi)^\kappa(\ln E_\pi)^\omega, \quad (22)$$

only one satisfies both Eqs. (20) and (21), namely,

$$N_\pi = \lambda_\pi(E_\pi/\mu)^{2/3}, \quad (23)$$

$$\beta = b(m/\mu)(E_\pi)^{-1/3}, \quad (24)$$

$$-\alpha = \ln(\lambda_\pi b^2/4\pi), \quad (25)$$

where λ_π, b (and therefore α) are constants, independent of E_π, m, or μ. They are determined by solving numerically Eqs. (20) and (21) with the above energy dependences for the case when $m = \mu = 1$. They are

$$\lambda_\pi \simeq 4.0, \quad b \simeq 8.2, \quad \alpha \simeq -3.1. \quad (26)$$

Thus,

$$N_\pi = 4(E_\pi/\mu)^{2/3} \quad (27)$$

as $E_\pi \to \infty$. Incidentally, the probability P_n that $2n$-charged pions are produced, normalized by $\sum_{n=0}^{\infty} P_n = 1$, is given by

$$P_n = (\tfrac{1}{2}N_\pi)^{2n} I_0^{-1}(N_\pi)/(n!)^2. \quad (28)$$

This distribution, as with its related Poisson distribution, passes through its maximum at $2n = N_\pi$, i.e., at its mean value for n.

When the total charge of our system is nonzero, the modification of our solution, Eq. (27), is trivial. If this net charge is ν, we obtain I_ν in place of I_0, but the asymptotic limit of I_ν is independent of ν and is given by Eq. (15). Consequently, the same solution is obtained for N_π, and Eq. (28) is modified by replacing one of the $n!$ by $(n+\nu)!$ and I_0 by I_ν. Indeed, conservation of charge may be completely dropped without affecting the basic $(E_\pi/\mu)^{2/3}$ energy dependence of N_π.

Notice the consistency of our solution with the *a priori* assumptions that $\beta \to 0$ and $N_\pi \to \infty$ as $E_\pi \to \infty$. After $2n = N_\pi$, P_n decreases exponentially for sufficiently large n, and this, coupled with $N_\pi/E_\pi \to 0$ as $N_\pi \to \infty$, is consistent with the approximation of ∞ for the upper limits of our summations, which may be expected to go as E_π/N_π.

We have discarded a term in Ω of order $\frac{1}{2}\ln(2\pi N_\pi)$, whose effect is to change N_π into $N_\pi - 1/4\pi$ in Eqs. (17) and (18). Consequently, our solution can only be justified when $N_\pi \gg 1/4\pi$. This condition is very weak, and far outweighed by the limit approximations discussed above. Thus, a more realistic condition is when $N_\pi/E_\pi \ll 1/m_\pi$ (corresponding to $N_\pi \ll 2N^{\max}$).

3. SYSTEMS INCLUDING KAONS AND NUCLEONS

First consider the case when only pions and kaons are produced. We shall treat these pions and kaons as noninteracting systems in the sense that our statistical distribution is the product of the statistical distributions for each system.[7] Again for simplicity, we assume that the center-of-mass frames for each set of particles are one and the same (i.e., $\beta_\pi = \beta_K = 0$). Each set of particles is assumed neutral in total charge. The generalization of this last assumption to some statistical averaging over various $P_n(\nu)$, where ν corresponds to the net differences in total pion and kaon charges (e.g., $Q_\pi = \frac{1}{2}\nu$, $Q_K = -\frac{1}{2}\nu$), will not be considered here; but from what has been said earlier, each term in the averaging should produce asymptotically the same energy dependence that we derive below. Thus, our distribution is

$$f_{n,m} = \exp(\Omega - \beta_\pi{}^\nu P_\nu{}^\pi - 2\alpha n - \beta_K{}^\nu P_\nu{}^K - 2\alpha' m - \gamma l - \gamma' k), \quad (29)$$

[7] The mere fact that at 30 GeV in the lab, 96% of the secondary particles are pions, but that this percentage is dropping as the energy increases, implies either a bias towards pions at low energy, which must be put in by hand in our statistical approach, or used as a justification for the asymptotic form of our basic assumption concerning the entropy. In either case, the different growth rates of the multiplicity of pions and kaons suggests their treatment as independent systems.

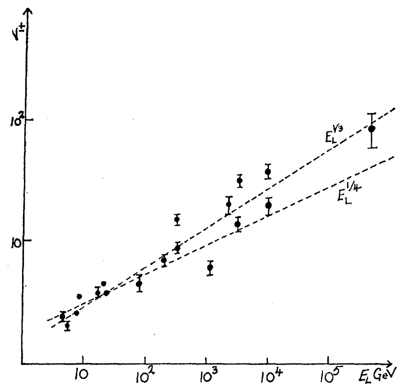

FIG. 1. Plot of total outgoing charged multiplicity $N^{\pm}$ as a function of lab energy E_L, for protons and pions interacting with nuclei (see Borashenkov *et al.*, Ref. 8). The dashed curves are interpolation curves, also taken from Ref. 8.

where m, α', γ', and β_K are the kaon counterparts of n, α, γ, and β_π for the pions. Consequently,

$$e^{-\Omega}\sim e^{\phi_\pi}I_0(2\phi_K)I_0(2\phi_\pi e^{-\alpha})I_0(2\phi_K e^{-\alpha'}). \tag{30}$$

We have distinguished ϕ_π from ϕ_K by allowing for a different μ in the two cases, namely μ_π and μ_K. The appearance of $I_0(2\phi_K)$, rather than e^{ϕ_K}, for the neutral kaons is a consequence of the existence of two neutral kaons related via hypercharge conservation.

Thus,

$$\Omega=-(\phi_\pi+\phi_\pi e^{-\alpha})-(2\phi_K+2\phi_K e^{-\alpha'}). \tag{31}$$

The entropy will be separable into two parts, one due to the pions, and one to the kaons. The extra ϕ_K in Eq. (31) causes λ_K to differ from λ_π, but only insignificantly. Consequently, the solution based upon the constancy of the entropy under separate variations of each set of particles is

$$N_\pi=4(E_\pi/\mu_\pi)^{2/3}, \tag{32}$$

$$N_K=4(E_K/\mu_K)^{2/3}. \tag{33}$$

Let us now assume that the energy available to the secondaries $(E_\pi+E_K)$ is asymptotically equally distributed among pions and kaons, i.e., $E_\pi=E_K=\frac{1}{2}E$. Then,

$$N_K/N_\pi=(\mu_\pi/\mu_K)^{2/3}. \tag{34}$$

If μ were a universal constant, then this ratio should be 1. Throughout we have naturally assumed that μ is independent of the total center-of-mass momentum of the system. But it is natural to propose that μ is proportional to the mass of the produced particle to which it corresponds. In which case,

$$N_K/N_\pi=(m_\pi/m_K)^{2/3}\simeq 0.43. \tag{35}$$

The case when nucleons $(p,\bar{p},n,\bar{n})$ are also produced may immediately be written down since the nucleons are treated identically to the kaons. Thus,

$$N_p/N_\pi=(m_\pi E_p/m_p E_\pi)^{2/3}. \tag{36}$$

If the equal subenergies assumption is again made, $E_\pi=E_K=E_p=\frac{1}{3}E$, we obtain

$$N_\pi:N_K:N_p=59:25:16. \tag{37}$$

Thus, we would predict the ratio of antiprotons to charged pions, produced at high energies, to be $\sim$13.5%. Notice that the assumption of equal subenergies does not necessarily imply equal energy per produced particle, since the ratio of neutral to charged particles is not expected to be the same for pions and kaons. Under the general assumption that

$$\begin{aligned} N_{\pi^\pm}&\equiv N_\pi\simeq 2N_{\pi^0},\\ N_{K^\pm}&\equiv N_K\simeq N_{K^0\bar{K}^0},\\ N_{p\bar{p}}&\equiv N_p\simeq N_{n\bar{n}}, \end{aligned} \tag{38}$$

and that on average a neutral particle has the same energy as its charged counterpart, we obtain

$$\langle\epsilon_\pi\rangle:\langle\epsilon_K\rangle:\langle\epsilon_p\rangle\simeq 21:31:48, \tag{39}$$

where ϵ_π is the average center-of-mass energy per pion, etc. At high enough energies these ϵ are identical to what is sometimes called the inelasticity per particle, the center-of-mass average 3-momentum per particle.

From our analysis, these ϵ's increase with energy as $E^{1/3}$, while Eq. (39) suggests that more than twice as much energy is needed per nucleon as per pion.

4. COMPARISON WITH EXPERIMENT AND CONCLUSIONS

The basic conclusion of our analysis, assuming that 40% of incoming energy goes into production, is the $E_0^{2/3}$ dependence of N. This corresponds to an $E_L^{1/3}$ dependence, where E_L is the laboratory energy. In comparing with experiment we should restrict ourselves to large N, and consequently to the cosmic-ray region. At present, the most popular fits in this region are made with $\log E_L$, and/or $E_L^{1/4}$. The latter drawing inspiration from the prediction of the Fermi model. However, in comparison with cosmic rays interacting with nuclei (10–10^6 GeV), and $E_L^{1/3}$ dependence is in very reasonable agreement with experiment[8] (see Fig. 1). The errors in experimental measurements, however, do not allow us to exclude any of these proposed energy dependences.[9] From this data, a crude estimate of μ_π (applying the analysis of Sec. 3) is

$$\mu_\pi\simeq 2\text{–}3\ \text{GeV}/c^2. \tag{40}$$

[8] V. S. Borashenkov, V. M. Maltsev, I. Patera, and V. D. Toneev, Fortsch. Physik **14**, 357 (1966). Further references are given in this review paper. C. B. A. McCusker, L. A. Peak, and R. L. S. Woolcott, Can. J. Phys. **46**, 655 (1967).

[9] In comparing our prediction with experiment, it should be noted that the total charged multiplicity $N^{\pm}$ usually quoted includes the charge of the primary particles, in our terminology. For high enough multiplicities, this difference may generally be neglected.

Until recently, the ratio of $N_{K^\pm}/N_{\pi^\pm}$ was believed to be around $\sim 20\%$.[10] However, a recent reanalysis of the determination of this ratio via the detection of γ rays, by Pilkuhn,[11] gives the values $N_{K^\pm}/N_{\pi^\pm}=0.44$, in extraordinary agreement with our result of 0.43 for the case when $\mu_\pi \sim m_\pi$ and $\mu_K \sim m_K$. In his analysis, Pilkuhn assumes only a very small percentage of heavier hadrons other than kaons. This suggests that the pion, kaon, and nucleon systems reach their asymptotic limits at different energies,[7] and that for certain energy regions we must consider the secondaries as being only pions, or only pions and kaons, and so forth.

Our prediction of $N_{p\bar{p}}/N_\pm^{\text{tot}} \simeq 16\%$ must therefore stand as a definite prediction for scattering in the higher TeV range. It is quite consistent with various experimental results on the multiplicity of antiprotons, which give upper limits of the order of

$$N_{\bar{p}} < 0.1 N_\pm^{\text{tot}}.$$

Thus, our simple approach gives quite reasonable results. Our analysis, of course, is far from complete. Where possible, we have clearly expressed the approximations involved, and the assumptions made. Although some, like the equality of the subenergies ($E_\pi = E_K = E_p$), are nothing more than a miniature application of information theory again. Another is the neglect of all other possible secondaries. However, the rate of growth of $N_\pm$ with energy is stubbornly independent of most modifications, being derivable basically from the condition that $\alpha \sim \ln(N\beta^2)$ be energy-independent.

The basic question must be the validity of our treatment of the particles as classical particles. We have given a justification for this, but a very practical difficulty underlies the choice. The quantum mechanical grand-canonical approach is not well behaved for negative α [as we required from Eq. (19)].[12] An embarrassment solved only partially by noting that the mass of the particles imply an $N^{\max}$, i.e., a cutoff. A final justification for our classical approach derives from the expression for $P_n(E')$, where E' is any given energy for an outcoming secondary system consisting (for simplicity) of n pions: m positive, m negative, and l neutral. The averaging over E' gives rise to the mean energy E. To obtain this $P_n(E')$, we introduce the integral

$$\int_0^\infty \delta(E' - \sum_{i=1}^n P_0{}^i) dE', \quad (E'>0)$$

into our usual expression for P_n, and compare the result with the definition of $P_n(E')$,

$$\int P_n(E') dE' = P_n. \tag{41}$$

This leads to

$$P_n(E') = \frac{e^{\Omega - \beta E' - 2\alpha m}}{\mu^{2n} m! m! l!} \int \frac{d^3P^1}{2P_0{}^1} \int \frac{d^3P^2}{2P_0{}^2} \cdots \int \frac{d^3p^n}{2P_0{}^n} \delta(E' - \sum_{i=1}^n P_0{}^i), \tag{42}$$

which is, up to a factor, the familiar center-of-mass phase-space integral for the production of $2m$-charged and l-neutral particles.[13]

Our approach is much more sympathetic to a bremsstrahlung concept of multiparticle production, than to the statistical-thermodynamic approaches, although $E_L{}^{1/3}$ energy dependences are also obtainable in some versions of these models.[14]

As always, we must rely upon experiment to distinguish between the various models, and unfortunately the experimental data are far from conclusive at this stage.

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[10] L. Briatone and M. Dardo, Nuovo Cimento **51B**, 475 (1967).

[11] H. Pilkuhn, Nucl. Phys. **B4**, 439 (1968).

[12] The problem arises due to the fact that conventionally, in the quantum approach, the summation from $n=0$ to ∞ is made before the phase-space integration. This sum becomes infinite for bosons if $\alpha<0$.

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