Problems of Gauge Invariance in Composite-Particle Theory*

JOHN M. CORNWALL†
The Niels Bohr Institute, University of Copenhagen, Copenhagen, Denmark
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Ordinarily, a one-particle state in conventional field theory is said to be composite if $Z_1=0$, $Z_2=0$. (Z_1 and Z_2 are the usual vertex and wave-function renormalization constants.) The presence of neutral vector bosons coupled to a conserved current complicates the issues in two related aspects: (1) Z_1 and Z_2 become gauge-dependent, so that any equation like $Z_2=0$ is suspect, since the notion of compositeness should be gauge-invariant; (2) there is a theory (spinor-vector scattering) in which the spinor particle is automatically Reggeized, while one ordinarily expects only composite particles to lie on Regge trajectories. In this paper, we show that the notion of compositeness is indeed gauge-invariant. Further, we show that, in spinor-vector scattering, one may pick a special gauge in which the spinor shows some of the properties of a composite particle, and in which all manifestly non-Reggeized terms in the scattering amplitude vanish. Of course, the scattering amplitude is gauge-invariant, so there were no non-Regge terms to begin with.

I. INTRODUCTION

SOME time ago, Gell-Mann, Goldberger, and their collaborators[1–3] investigated the Regge behavior of conventional field theory, with at least one surprising result. In a world with spinors and massive vector bosons coupled to a conserved current, the spinor particle lies on a Regge trajectory, regardless of what values one chooses for the coupling constant and masses in the theory. This result does not hold for vector-scalar theory, or any renormalizable theory not involving vector bosons. It stands in apparent contradiction to the bootstrap principle and composite-particle philosophy, according to which only composite particles should lie on Regge trajectories. But the masses and coupling constants of composite particles are not arbitrary; they are to be determined by the forces which bind the composite particle together.

Mandelstam[4] has investigated this phenomenon. He shows that in the spinor channel of spinor-vector scattering, kinematic threshold constraints force the spinor particle to lie on a Regge trajectory (i.e., there are no Kronecker δ functions in the angular momentum for this channel). He further shows that there are (in general) such Kronecker δ functions in other channels (such as the channel with the bosons' quantum numbers). Thus, the bootstrap program is technically saved by defining a bootstrapped universe as one in which there are no Kronecker δ functions in any channel.

In this paper, we study the question of automatic Reggeization from the point of view of the requirements of gauge invariance which are imposed when neutral vector mesons are coupled to a conserved current. Specifically, we explore the implications of the freedom of choice of a gauge. This freedom of choice expresses itself in part through the fact that the spinor wave-function and vertex renormalization constants (Z_2 and Z_1, respectively) depend on the chosen gauge. We would like to connect this circumstance with the often expressed[5–7] idea that a composite (hence Reggeized) particle is characterized by the vanishing of Z_1 and Z_2. Crudely speaking, we might expect that we can choose a gauge such that $Z_1=0$, $Z_2=0$, and that in this gauge it will be manifest that the particle is Reggeized. Clearly, for this to make any sense, it must be possible to give finite expressions for Z_1 and Z_2. Cases where the Z's are undefined or identically zero because of divergent quantities are of no interest to us in this work, nor are they sensible in composite-particle theories. For spinor-vector scattering, it is possible to give finite expressions for the Z's; their vanishing for particular choices of gauge parameters then implies Reggeization of the spinor.

While this is indeed what happens for spinor-vector scattering, it is by no means the whole story. At least two questions raise themselves. First, why does this not work for scalar-vector scattering, where it is known[3] that the scalar particle does not lie on a Regge trajectory? The answer is the same in the $Z=0$ approach as it is in the approach of Ref. 3: The seagull diagrams interfere. These diagrams yield non-Regge terms in both sense and nonsense amplitudes, unlike elementary-particle poles which only appear in sense-sense amplitudes. As will become clear later on, it is impossible to choose a gauge in which the seagull diagrams do not contribute, hence impossible to conclude that scalar-vector scattering does Reggeize. (We cannot prove that it does not Reggeize, but explicit calculations[3] show that it does not.)

The second question strikes at the foundations of the composite-particle philosophy: If one can, in any

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† Alfred P. Sloan Foundation Fellow. On leave of absence from the University of California, Los Angles, Calif.
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[5] For a review article for the literature prior to 1967, see K. Hayashi, M. Kirayama, T. Muta, N. Saito, and T. Shirafuji, Fortschr. Physik **15**, 625 (1967). Two of the many articles which have appeared since then which are relevant to the methods of the present work are given in Refs. 6 and 7.
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[7] J. M. Cornwall and D. J. Levy, Phys. Rev. **178**, 2356 (1968).

theory involving vector mesons, manipulate the gauge so that Z_1 and Z_2 become equal to zero, how can the $Z=0$ criterion be meaningful? This question is not hard to answer, if one recalls the old work of Landau and Khalatnikov[8] (later reformulated by Zumino and Johnson[9,10]) concerning the effect of gauge transformations on renormalization constants. We define a class of covariant gauges by writing the vector-meson propagator as

$$\Delta_{\mu\nu}(k)=\Delta_{\mu\nu}^{(F)}(k)+k_\mu k_\nu \lambda(k^2). \tag{1}$$

The gauge for which $\lambda=0$ is the Feynman gauge, in which the free propagator is

$$\Delta_{\mu\nu}^{(F)}(k)=-g_{\mu\nu}/(k^2-m^2+i\epsilon). \tag{2}$$

The Feynman gauge is calculationally convenient both in quantum electrodynamics (QED) and massive-vector-boson theories. However, in massive-vector-boson theories there is a gauge which is physically distinguished because it is both manifestly covariant and contains no superfluous scalar components. This is the Proca gauge, where $-g_{\mu\nu}$ in Eq. (2) is replaced by $-g_{\mu\nu}+k_\mu k_\nu/m^2$. Corresponding gauges in QED, such as the radiation gauge, are not manifestly covariant, because the photon field is not an ordinary four-vector density; a change of reference frame implies a change of gauge if manifest covariance is desired. Later, we shall separate the vector propagator into the Proca gauge plus a remainder, but for now it is convenient to separate it into the Feynman gauge plus a remainder.

Now, consider a renormalizable field theory containing spinors, scalars, and the neutral vector meson. Let us denote, e.g., the spinor wave-function renormalization constant, as calculated to all orders of all coupling constants but in the Feynman gauge, by the symbol Z_2^F. It requires only a trivial modification of Landau and Khalatnikov's work[8] to show that, in any gauge λ (where $Z_2=Z_2^\lambda$),

$$Z_2^\lambda=Z_2^F G^\lambda, \tag{3}$$

where the gauge-dependent quantity G^λ depends only on the function $\lambda(k^2)$ in Eq. (1) and on no other details of the full theory. Furthermore, gauge transformations are compounded according to the formula

$$Z_2^{\lambda+\lambda'}=Z_2^F G^\lambda G^{\lambda'}=Z_2^\lambda G^{\lambda'}. \tag{4}$$

It is the crucial property of factorization in Eqs. (3) and (4) which allows us to give an essentially gauge-invariant definition of compositeness. We must first recognize two critical requirements for the $Z=0$ program to be meaningful: (a) The expressions for Z_1, Z_2, etc., must be finite in principle, and (b) these expressions must depend continuously in some domain on the renormalizable parameters of the theory (masses and coupling constants). In strong-interaction physics, one hopes that this is true, while in pure spinor-vector theory it is known[11,12] that it is possible to choose order by order in perturbation theory gauge functions λ so that requirements (a) and (b) are satisfied. Let us suppose, then, that it is possible to choose a gauge function λ, which depends on the observable parameters of the theory and possibly other parameters, such that Z_2^λ obeys conditions (a) and (b). If it is possible, *by varying only the observable parameters*, to make the Z's vanish for some choice of these observable parameters, we shall say that there is a composite particle in the theory. Clearly, by Eq. (4), Z_2 then vanishes in all gauges (for which $G^{\lambda'}$ is finite). In strong-interaction physics, one might hope that the gauge in which conditions (a) and (b) are true is the Feynman gauge.

On the other hand, it may be possible to modify the gauge arbitrarily in such a way that Z_2^λ vanishes, without regard to the particular values of the coupling constants and masses. As we shall see below, this simply corresponds to the fact that spinor-vector theory is automatically Reggeized in the spinor channel.

Our criterion of compositeness can distinguish between the case where vector-meson forces are actually sufficient to produce a bound-state or composite particle and the case where only automatic Reggeization occurs.

Observe that a change of vector-meson propagator induces a change not only in Z_2 but also in the cut structure of the propagator (see Secs. III and IV); this change would be exactly cancelled by the phase transformation of the charged fields which ordinarily accompanies a change of vector-meson gauge, but which we need not consider for the purposes of calculating scattering amplitudes. As Z_2 becomes small, the cut structure changes in such a way that a zero appears in the propagator near $W=M$, and in the $Z=0$ limit, the zero cancels the elementary-particle pole at $W=M$.[5–7] In this limit the unrenormalized propagator is gauge-invariant, as discussed in Ref. 7, which emphasizes the gauge transformation properties of the imaginary part of the propagator.

The main tool in our investigation of automatic Reggeization is the decomposition of the scattering amplitude into two parts: the one-particle irreducible graphs and the direct-channel Born term with full propagator and vertex functions. This decomposition is used extensively in the $Z=0$ theories[5–7] and has been studied thoroughly by Ida.[13] When vector mesons are present, this decomposition is not gauge-invariant, which allows us to see the effect of gauge changes on the one-particle irreducible amplitudes. Details of this are given in Sec. III, while Sec. II contains kinematic

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[13] Masakuni Ida, Phys. Rev. 135, B499 (1964); 136, B1767 (1964).

preliminaries. Section IV briefly recapitulates the Landau-Khalatnikov[8] arguments, with a derivation which differs somewhat from those in the literature.[8–10]

II. KINEMATICS

In this section, we give the kinematics necessary for the decomposition of a scattering amplitude for particles with spin into two parts, one of which is the one-particle irreducible part.[13] Our work uses the parity-conserving helicity amplitudes of Ref. 2, which should be consulted for further references. These kinematics are useful not only for the problem of automatic Reggeization, but also for the general program of studying compositeness for particles with spin, which has not received much attention in the literature.

Consider the elastic scattering process $a+b\to c+d$, where $a\cdots d$ are the helicities of the particles involved.[14] (The restriction to elastic scattering is inessential.) We write the T matrix as

$$T_{cd,ab}=[i(2\pi)^4/(\prod 2E)^{1/2}]\delta(p_f-p_i)M_{cd;ab}\,, \quad (5)$$

where M is the invariant amplitude. We treat the scattering in the s channel, where $s=W^2=(p_a+p_b)^2$; W is the c.m. energy. There is the usual Jacob-Wick[15] helicity expansion

$$M_{cd;ab}(W,\cos\theta)=\sum(2J+1)t_{cd;ab}{}^J(W)d_{\lambda\mu}{}^J(\theta)\,, \quad \lambda=a-b\,,\quad \mu=c-d\,. \quad (6)$$

We are interested in the case where a and c refer to a spin-$\frac{1}{2}$ particle, b and d to integral spins. Then, there is a MacDowell symmetry (*CTP* theorem) which says

$$M_{cd;ab}(W,\cos\theta)=\sum(2J+1)[f_{cd;ab}{}^J(W)d_{\lambda\mu}{}^J(\theta)+f_{cd;ab}{}^J(-W)d_{-\lambda-\mu}{}^J(\theta)]\,. \quad (7)$$

In other words, the T matrix is invariant under the exchange $W\leftrightarrow -W$, along with reversing all helicities in the d functions. Of course, this reversing of helicities is just shorthand for the phase relation

$$d_{\lambda\mu}{}^J(\theta)=e^{i\pi(\lambda-\mu)}d_{-\lambda-\mu}{}^J(\theta)\,. \quad (8)$$

Partity-conserving amplitudes which are free of kinematic singularities are defined by[2]

$$f_{cd;ab}{}^{\pm}=(\sqrt{2}\cos\theta)^{-|\lambda+\mu|}(\sqrt{2}\sin\theta)^{-|\lambda-\mu|}M_{cd;ab}\pm\eta_d(-)^{\lambda+\lambda_m+S_d}(\sqrt{2}\cos\theta)^{-|\lambda-\mu|}\times(\sqrt{2}\sin\theta)^{-|\lambda+\mu|}M_{-c-d;ab}\,, \quad (9)$$

where $\lambda_m=\max(|\lambda|,|\mu|)$, and η_d and S_d are the parity and spin of particle d.

Next, we discuss vertex functions which connect on-shell particles a and b to an off-shell spinor of momentum $(W,\mathbf{0})$. These are necessary to discuss the one-particle reducible terms (s-channel pole terms). We write, for the form factor,

$$F_{ab\rho}{}^{\pm}(W,\mathbf{p})=\bar{\chi}_\rho{}^{\pm}\langle 0|\psi(0)|\mathbf{p}a;\,-\mathbf{p}b\rangle(2E_a2E_b)^{1/2}\,, \quad (10)$$

where $\psi(0)$ is the renormalized Heisenberg field, and the four-spinors $\chi_\rho{}^{\pm}$ are defined in terms of two-component helicity spinors by

$$\chi_\rho{}^+=\begin{pmatrix}\chi_\rho\\0\end{pmatrix},\quad \chi_\rho{}^-=e^{i\pi(\rho-1/2)}\begin{pmatrix}0\\\chi_\rho\end{pmatrix}. \quad (11)$$

Rotational invariance allows us to write

$$F_{ab\rho}{}^{\pm}(W,\mathbf{p})=(E\mp M)^{1/2}D_{\pm\lambda\pm\rho}{}^{1/2}(U)F_{ab}{}^{\pm}(W)\,, \quad (12)$$

where E is the energy of the external spinor (particle a) in the c.m. frame, and $\lambda=a-b$. The rotation U is connected to the polar angles of $\mathbf{p}$ in the usual way.[15] MacDowell symmetry and parity invariance allow us to conclude

$$F_{ab}{}^-(W)=F_{ab}{}^+(-W)\,,\quad F_{-a-b}{}^{\pm}(W)=\pm\eta_b(-)^{S_b}F_{ab}{}^{\pm}(W)\,. \quad (13)$$

If b is an odd-parity particle, the factors $(E\mp M)^{1/2}$ remove all kinematic singularities from the form factors F; for even b parity, the signs would be reversed in this factor.

To define proper vertex functions, we need the kinematics for spinor propagators as discussed, for example, by Ida.[13] We write the propagator as $S(p)$; near its pole, $S(p)\sim(p-M)^{-1}$. Let us substitute W for p in S^{-1}, to define a function $Z(W)$:

$$S^{-1}(W)=(W-M)Z(W)\,,\quad Z(M)=1\,. \quad (14)$$

Then it is easy to show that

$$S(p)=\frac{W+p}{2W}[(W-M)Z(W)]^{-1}+\frac{W-p}{2W}[(-W-M)Z(-W)]^{-1}. \quad (15)$$

Observe the following relations:

$$\frac{W\pm p}{2W}\chi_\rho{}^{\pm}=\chi_\rho{}^{\pm},\quad \frac{W\pm p}{2W}\chi_\rho{}^{\mp}=0. \quad (16)$$

With the normalization given in Eq. (14), it is customary to define the wave-function renormalization constant Z_2 as the limit of $Z(W)$ as W approaches infinity—if this limit exists. In perturbation theory, this limit ordinarily does not exist, but, for a special class of gauges discussed in Sec. III, it does. $Z(W)$ obeys a dispersion relation, and Im$Z(W)$ is positive indefinite, if there are no vector mesons in the theory.

Proper vertex functions are defined by

$$\Gamma_{ab}{}^{\pm}(W)=F_{ab}{}^{\pm}(W)Z(\pm W)\,. \quad (17)$$

Because of parity conservation, it is sufficient[2] to set the spinor helicities a and c equal to $\frac{1}{2}$, and we write ex-

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[14] Other conventions: Our metric is such that $p^2=p_0{}^2-\mathbf{p}^2$; the Dirac equation is $(\gamma\cdot p-M)u(p)=0$, and Dirac spinors are normalized so that $\bar{u}u=2M$. We also use $p=\gamma\cdot p$.

[15] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

plicitly only the boson helicities b and d. Using Eqs. (9) and (15)–(17), we can write the parity-conserving amplitudes corresponding to the one-particle reducible graphs:

$$f_{db}^{\pm(1)}=-\sqrt{2}\frac{E\mp W}{W\mp M}\Gamma_d^{\pm}(W)\Gamma_b^{\pm}(W)Z^{-1}(\pm W). \quad (18)$$

Of course, these terms have no angular dependence, and hence correspond to Kronecker δ functions at $J=\frac{1}{2}$ (or $l=0$, if we set $J=l+\frac{1}{2}$). Furthermore, $f^{\pm}$ (see Ref. 1) vanishes except in sense-sense channels.

We record the following unitarity relations, valid in the elastic approximation, but easily generalized:

$$\mathrm{Im}F_b^+(W)=\sum_d \frac{q}{8\pi W}\tilde{f}_{db}^{\frac{1}{2}+}(W)F_d^+(W)^*, \quad W>W_0 \quad (19)$$

$$\mathrm{Im}\Gamma_b^+(W)=\sum_d \frac{q}{8\pi W}\tilde{f}_{db}^{\frac{1}{2}+}(W)\Gamma_d^+(W)^*, \quad W>W_0. \quad (20)$$

[For the definition of $\tilde{f}$, see Eq. (22) below.] Here we have introduced the partial-wave amplitudes $f^{J\pm}$ of the $f^{\pm}$, as given in Eq. (2.11) or Ref. 2. In Eqs. (19) and (20), W_0 is the positive threshold and q is the c.m. momentum. There are similar relations for F^- and Γ^-, for $W<-W_0$. Also, in the two-particle approximation to the propagator,

$$\mathrm{Im}Z(W)=\frac{q}{8\pi W}\frac{E-M}{W-M}\sum_b |\Gamma_b^+(W)|^2, \quad W>W_0 \quad (21)$$

with a similar expression for $W<-W_0$. Observe that Eq. (21) is not valid as it stands for intermediate states involving vector mesons; the appropriate modifications will be discussed in Sec. III. Equation (21) is valid in the Proca gauge [see remarks after Eq. (2)], which is the distinguished gauge for making unitarity calculations; however, in perturbation theory it leads to badly divergent expressions.

The full scattering amplitude f becomes

$$f_{db}^{\pm}(W,\theta)=f_{db}^{\pm(1)}(\omega)+\tilde{f}_{db}^{\pm}(W,\theta), \quad (22)$$

which defines the one-particle irreducible amplitude $\tilde{f}^{\pm}$. When vector mesons are not involved, Eqs. (17)–(20) allow us to conclude (just as in Ida's work[13]) that the $\tilde{f}^{\pm}$ are separately unitary scattering amplitudes, just as the $f^{\pm}$ are. This fact is used very often in discussing $Z=0$ theories,[5–7] where both f and $\tilde{f}$ are written in ND^{-1} form. The form factors are expressible in terms of D^{-1}, and the proper vertex functions in terms of $\tilde{D}^{-1}$. When an elementary particle becomes composite, $f^{(1)}$ disappears, which is the basis for saying that a theory with only composite particles is Reggeized. Because the decomposition in Eq. (22) is not gauge-invariant, it turns out to be possible to accomplish the vanishing of $f^{(1)}$ in a particular gauge, and thereby exhibit the phenomenon of automatic Reggeization. This lack of gauge invariance prevents us from concluding that $\tilde{f}$ is a unitary amplitude; in the next section, we rig up a pseudo-unitarity which allows us to pursue closely the ideas of the $Z=0$ theories in connection with automatic Reggeization, when vector mesons are involved.

III. AUTOMATIC REGGEIZATION

The decomposition (22) of the scattering amplitude into a one-particle irreducible part and a remainder is not gauge-invariant when vector mesons are present. We exploit this circumstance to inspect the inner machinery of gauge transformations as it works on the one-particle irreducible part $\tilde{f}$.

Because the full amplitude f is gauge-invariant, it follows that the only part of $\tilde{f}$ which changes under gauge transformations is the sense-sense $J=\frac{1}{2}$ sector, since $\tilde{f}=f$ otherwise. This sector of f or $\tilde{f}$ is the only one which might possibly be non-Reggeized. Ordinarily, when one discusses composite particles, one says[6] that $\tilde{f}$ is mainifestly Reggeized, since it gets contributions only from Feynman graphs which are know to be Reggeized in perturbation theory (i.e., which obey the Mandelstam representation). We emphasize that $\tilde{f}$ cannot be manifestly Reggeized in any arbitrary gauge, whether or not f is; in nearly all gauges $\tilde{f}$ necessarily has a non-Regge part in the sense-sense $J=\frac{1}{2}$ sector. Such a non-Regge part is easily extracted from, say, the fourth-order box diagram. The purpose of this section is to show that there is a class of gauges, in which Z_2, etc., vanish, where simultaneously the vertex contribution $f^{(1)}$ and the non-Regge part of $\tilde{f}$ vanish, thereby explicitly exhibiting the Regge properties of f (which equals $\tilde{f}$ in these gauges). The vertex contribution vanishes for just the reasons discussed in, e.g., Refs. 6 and 7: A "composite" particle appears in $\tilde{f}$, whose mass and coupling constant are equal to those of the elementary spinor.

To illustrate these ideas, consider the fourth-order uncrossed box diagram for vector-meson-spinor scattering. To define our gauge, we write the meson propagator as

$$-g_{\mu\nu}/(k^2-m^2+i\epsilon)+k_\mu k_\nu\lambda(k^2)/\gamma^2, \quad (23)$$

where γ is the vector-meson coupling constant, and the first term is the Feynman gauge propagator. The invariant scattering amplitude M of Sec. II is written as

$$M=\bar{u}_c(p')A^{\mu\nu}u_a(p)\epsilon_\mu{}^b(k)\epsilon_\nu{}^d(k')^*. \quad (24)$$

The contribution of the gauge term in (22) to the box diagram is

$$A^{\mu\nu}(\lambda)=-\frac{i\gamma^2}{(2\pi)^4}\int d^4q\,\lambda(q^2)\gamma^\nu S(\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{q})\gamma^\mu. \quad (25)$$

Here, $S(\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{q})$ is the free spinor propagator. This gauge-dependent contribution to A is, of course, canceled by terms which come from the radiative corrections to the s-channel $[s=(p+k)^2]$ pole term. In a similar way, the crossed box graph has a gauge-dependent part canceled out by the u-channel pole term. Note that $A(\lambda)$ is not Reggeized. In general, we can divide all non-gauge-invariant graphs in $\tilde{f}$ into two classes, direct and crossed, depending on whether the gauge terms are canceled by s- or u-channel pole terms. It is clear that all crossed graphs (including the u-channel pole terms) are Reggeized in the s channel; we may evaluate them, if we wish, in the Feynman gauge, since their sum is gauge-invariant. But for the direct-channel graphs, we seek special gauges in which the non-Regge terms drop out.

To this end, let us consider the vertex renormalization constant in perturbation theory. It is sufficient to consider only gauge functions λ which can be written

$$\lambda(k^2)=-\int_0^\infty \frac{\lambda(s)ds}{s-k^2-i\epsilon}. \tag{26}$$

We require, for reasons which will shortly be evident, that $\lambda(k^2)$ fall off like k^{-4} at infinity, which implies

$$\int_0^\infty ds\,\bar{\lambda}(s)=0. \tag{27}$$

A simple perturbation calculation shows that, if (27) holds,

$$Z_1=1-(16\pi^2)^{-1}\left[\ln\left(\frac{\Lambda^2}{M^2}\right)\right]\left[\gamma^2-\int ds\, s\lambda(s)\right] +\text{finite terms}, \tag{28}$$

where Λ^2 is a suitable cutoff. Therefore, if we choose

$$\int ds\, s\bar{\lambda}(s)=\gamma^2, \tag{29}$$

Z_1 is finite. A special case of (29) is the Landau gauge[11,12] in QED, where $\bar{\lambda}(s)=-\gamma^2\delta'(s)$.

Let us denote by $\Gamma(W)$ the coefficient of $\gamma\gamma^\mu$ in the renormalized proper vertex function $\Gamma^\mu(W)$. Of course, $\Gamma(W)$ is just a linear combination of the helicity vertices discussed in Sec. II. We can write

$$\Gamma(W)=1+\gamma^2[F(W)-F(M)]. \tag{30}$$

With Z_1 finite, $F(W)$ (obviously a gauge-dependent quantity) obeys an unsubtracted dispersion relation. In the usual way, we sum up an entire string of graphs by replacing (30) with

$$\Gamma(W)=[1-\gamma^2F(M)]/[1-\gamma^2F(W)]. \tag{31}$$

Because $F(W)\to 0$ as $W\to\infty$, we can recover Z_1 from Eq. (31) as

$$Z_1=\lim_{W\to\infty}\Gamma(W)=1-\gamma^2F(M), \tag{32}$$

a result which can be derived directly from the unrenormalized vertex, to $O(\gamma^2)$. Now we choose the gauge parameters so that Eqs. (27) and (28) are true, and also so that $1-\gamma^2F(W)$ has a zero for some $W=W_R\lesssim M$. Then, $\Gamma(W)$ has a pole at $W=W_R$; if we let $W_R\to M$, Eq. (32) tells us that $Z_1\to 0$, and the proper vertex function vanishes from Eq. (31). [It is clear that all components of $\Gamma_\mu(W)$, not just the coefficient of γ^μ, vanish as Z_1 vanishes, as long as Z_1 is finite and the unrenormalized vertices are well defined.] Clearly, these results are not confined to the approximate vertex function constructed here.

We can construct the propagator from the vertex with the aid of the Ward-Takahashi identity

$$\gamma^{-1}(p-p')_\mu\Gamma^\mu(p,p')=S^{-1}(p)-S^{-1}(p'), \tag{33}$$

letting, for example, p' be on the mass shell. It is easy to see that (1) Z_2 is finite if Z_1 is; (2) as $Z_1\to 0$, $Z_2\to 0$; and (3) $Z(W)$ has a zero at $W=W_R$, and as Z_2 goes to zero, $Z(W)$ vanishes like Z_2. If we were dealing with massless QED, these results would be trivial, for in such a theory $Z_1=Z_2$, and $\Gamma(W)$ in Eq. (30) is equal to $Z(W)$ in Eq. (14). But it is not trivial to discuss Reggeization in massless QED, because the propagator does not have a simple pole at $W=M$, in general, but a very complicated singularity.

So far, all of the properties of the vertex and propagator given above, in the special gauge where $\Gamma(W)$ has a pole, correspond precisely to the discussion in $Z=0$ theories[5–7] of the emergence of a composite particle of mass W_R. As $W_R\to M$, Z_1 and Z_2 vanish, and with them the entire one-particle reducible graphs (which vanish like Z_1^2/Z_2). Clearly, because the very existence of this pole depends on our choice of gauge, it is physically inappropriate to speak of this pole as a composite particle, but the mathematical manipulations are the same as if it were.

There is only one ingredient lacking to make the correspondence complete: In composite-particle theory, the one-particle irreducible amplitude $\tilde{f}$ is unitary. We give a particle interpretation to the gauge function λ, which allows us to think of $\tilde{f}$ as pseudo-unitary; that is, $\mathrm{Im}\tilde{f}$ is a sum of squared amplitudes, but is not positive definite.

Write the gauge function $\lambda(k^2)$ as

$$\lambda(k^2)=\frac{\gamma^2}{m^2(k^2-m^2)}+\sum_i \frac{C_i}{k^2-m_i^2+i\epsilon}. \tag{34}$$

The first term is added to put the vector meson into the usual Proca gauge, which is convenient for giving a unitary interpretation to the Cutkosky rules as applied to intermediate states containing vector particles. If all the C_i were positive, the other terms could be considered as physical scalar particles gradient-coupled to the spinor. However, the sum rule, (27) prevents this interpretation; at least one of the C_i is negative, so there is at least one scalar ghost in (34).

This is what prevents $\tilde{f}$ from enjoying positive-definite unitarity.

We think of unitarity for $\tilde{f}$ as presenting a multichannel problem, where the scalar states of mass m_i can scatter off the spinor, mathematically if not physically. It is not hard to convince oneself that the appropriate definition of the elastic scattering amplitude for spinor-scalar particle i is

$$M_{ii}=c_i\bar{u}(p')S^{-1}(p+k_i)u(p)\,, \qquad (35)$$

where k_i is the initial momentum of particle i. Likewise, the amplitude for spinor+vector $\rightarrow$ spinor+particle i is

$$M_{Vi}{}^{\mu}=(c_i)^{1/2}\bar{u}(p')\Gamma^{\mu}(p,\,p+k_i)u(p)\,. \qquad (36)$$

Clearly, both these amplitudes are non-Reggeized, and have only $J=\frac{1}{2}$ partial waves, corresponding to orbital angular momentum l of 0 and 1. For example, Eq. (35) can be subjected to the usual partial-wave analysis of scalar-nucleon scattering. The $l=0$ partial-wave amplitude, normalized to $e^{i\delta}\sin\delta$, is

$$a_0{}^i=(c_iq_i/8\pi W)(E_i+M)(W-M)Z(W)\,, \qquad (37)$$

where q_i is the c.m. momentum, and E_i the c.m. spinor energy. The $l=1$ amplitude is obtained by changing W into $-W$, by MacDowell symmetry. To check that Eq. (37) is indeed a pseudo-unitary amplitude, we apply the Cutkosky rules to the propagator and isolate the term with a threshold at $W=M+m_i$. With the aid of the Ward identity (23), we find, after a brief calculation,

$$\operatorname{Im}Z(W) \\ =(c_iq_i/8\pi W)(E_i+M)(W-M)|Z(W)|^2+\cdots\,, \qquad (38)$$

where the omitted terms have other thresholds. A similar relations holds for the part with threshold at $W=-(M+m_i)$. Equation (38) shows that Eq. (37) is unitary in the "elastic" approximation. A similar discussion proves the pseudounitarity of Eq. (36).

Of course, we do not have true physical unitarity, because the omitted terms in Eq. (38) are not positive definite (the C_i are not positive definite). If these omitted terms were positive definite, then we would be able to conclude that $|a_0{}^i|\leq 1$, and hence

$$|Z(W)|=O(W^{-2})\,, \quad \text{as} \quad W\rightarrow\infty\,. \qquad (39)$$

Then Z_2 would be identically zero, but $Z(W)$ is well defined, and the one-particle terms would not vanish. Such a scheme is not consistent with the composite-particle theories we are trying to imitate.

We are now in a position to combine the various elements of the problem. The only differences between our discussion and the usual composite-particle theory are that (a) $\tilde{f}$ is pseudo-unitary, instead of unitary, and (b) there are some manifestly non-Reggeized terms in $\tilde{f}$, such as those in Eqs. (35) and (36). But point (a) does not prevent us from writing $\tilde{f}$ in ND^{-1} form, which is all that the composite-particle theories need; as to point (b), when Z_1 and Z_2 approach zero, so do the non-Regge terms like (35) and (36), since $\Gamma_\mu(W)$ and $Z(W)$ vanish in this limit. As a result, both the one-particle terms and the gauge-scattering terms vanish simultaneously, leaving only a manifestly Reggeized $\tilde{f}$ (which is of course equal to the full amplitude f in the limit). Since f still has a spinor pole, it follows that the spinor lies on a Regge trajectory.

Observed that similar results cannot be derived for scalar-vector scattering, because there are non-Regge (seagull) parts coupled to nonsense channels. No amount of gauge transforming can affect these, because all nonsense amplitudes in f are gauge-invariant to begin with.

IV. GAUGE DEPENDENCE OF RENORMALIZATION CONSTANTS

In this section we give, for the sake of completeness, a derivation of the gauge transformation properties of the renormalization constants. The result is not new,[8–10] but the derivation differs from those in the literature, and is easily interpreted in terms of Feynman graphs.

Let us begin with a renormalizable field theory, say, pion-nucleon theory, to which we want to add a gauge-invariant massive-vector-boson coupling. Let $\psi(x)$ be the unrenormalized Heisenberg field for the nucleon, with the vector coupling constant γ set equal to zero. The complete propagator with all vector-boson couplings included is then

$$iS(x)=\langle 0|\left(\psi(x)\bar{\psi}(0)\exp\left[i\int d^4y\,\mathcal{L}_\gamma(y)\right]\right)_+|0\rangle\,, \qquad (40)$$

where it is understood that only connected graphs are to be saved, and

$$\mathcal{L}_\gamma(y)=-\gamma J_\mu(y)V^\mu(y)+\text{seagull terms}\,, \qquad (41)$$

with $V^\mu(y)$ the vector-boson field and $J_\mu(y)$ the conserved current. The seagull terms are, of course, those interaction terms involving $(V_\mu)^2$ which are necessary when scalar particles are in the theory. We make one mistake here and ignore seagull terms; a second mistake later on exactly cancels this first mistake.

By contracting all vector fields and doing some simple combinatorial analysis, we arrive at

$$iS(x)=\sum_{n=0}^{\infty}\frac{1}{n!}(-\tfrac{1}{2}i\gamma^2)^n\int dx_1\cdots dx_{2n} \\ \times\langle 0|(\psi(x)\bar{\psi}(0)J(x_1)\cdots J(x_{2n}))_+|0\rangle \\ \times\Delta(x_1-x_2)\cdots\Delta(x_{2n-1}-x_{2n})\,. \qquad (42)$$

For the sake of typographical convenience, we have omitted Lorentz indices; Δ should carry two such indices, and J one. Our convention is

$$J(x_1)J(x_2)\Delta(x_1-x_2)=J_\mu(x_1)J_\nu(x_2)\Delta^{\mu\nu}(x_1-x_2)\,, \qquad (43)$$

where $\Delta^{\mu\nu}$ is the vector-boson propagator. In any covariant gauge λ, we write

$$\Delta_{\mu\nu}(x_1-x_2)=\Delta_{\mu\nu}^{(F)}(x_1-x_2)+\gamma^{-2}\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}\lambda(x_1-x_2), \quad (44)$$

where $\Delta_{\mu\nu}^{(F)}$ is the free-vector propagator in the Feynman gauge.

Now we make repeated use of the Ward-Takahashi identity, after inserting Eq. (44) into Eq. (42) and integrating by parts. First, let us make our second mistake by assuming

$$[J_0(\mathbf{x},0),J_\mu(0)]=c \text{ number}. \quad (45)$$

This is not so in theories where seagull graphs are present. If Eq. (45) is true in theories where no seagulls exist, as is most likely, then it is not difficult to show that mistakes one and two cancel each other order by order in perturbation theory. With the assumption of Eq. (45), the Ward-Takahashi identity reads

$$\frac{\partial}{\partial x_1^\mu}\langle 0|(\psi(x)\bar{\psi}(0)J^\mu(x_1)J^\nu(x_2)\cdots)_+|0\rangle =[\delta_4(x_1)-\delta_4(x-x_1)] \times\langle 0|(\psi(x)\bar{\psi}(0)J^\nu(x_2)\cdots)_+|0\rangle. \quad (46)$$

This expression is used twice each time the function λ appears in Eq. (42). Thus, all the terms in Eq. (42) which contain the function λ to the first power are

$$\sum_{n=0}^{\infty}(-\tfrac{1}{2}i\gamma^2)^n\frac{n}{n!}\frac{2}{\gamma^2}[\lambda(0)-\lambda(x)]\int dx_3\cdots dx_{2n} \times\langle 0|(\psi(x)\bar{\psi}(0)J(x_3)\cdots)_+|0\rangle\Delta(x_3-x_4)\cdots -i[\lambda(0)-\lambda(x)]\langle 0|(\psi(x)\bar{\psi}(0))_+|0\rangle. \quad (47)$$

The n in the numerator comes from the n ways of inserting the gauge function. Equation (47) adds up to

$$-i[\lambda(0)-\lambda(x)]iS_{(F)}(x), \quad (48)$$

where $S_{(F)}(x)$ is the full propagator, to all orders in γ and any other coupling constants, but in the Feynman gauge. After some more combinatorial analysis, the general expression is

$$iS(x)=\sum_{nm}(-\tfrac{1}{2}i\gamma^2)^n\frac{1}{n!}\left(\frac{2}{\gamma^2}\right)^m\frac{n!}{m!(n-m)!}[\lambda(0)-\lambda(x)]^m \times\int dx_1\cdots\langle 0|(\psi(x)\bar{\psi}(0)J(x_1)\cdots)_+|0\rangle \times\Delta^{(F)}(x_1-x_2)\cdots=iS_{(F)}(x) \times\exp\{i-[\lambda(0)-\lambda(x)]\}. \quad (49)$$

The wave-function renormalization constant Z_2 can be identified[9] from the asymptotic behavior of Eq. (49), which yields

$$Z_2^\lambda=Z_2^F e^{-i\lambda(0)}, \quad (50)$$

where Z_2^F is computed in the Feynman gauge. Equation (50) immediately yields the factorization property

$$Z_2^{\lambda+\lambda'}=Z_2^\lambda e^{-i\lambda'(0)}, \quad (51)$$

which, as discussed previously, ensures the gauge-invariant meaning of the equations $Z=0$. A different proof of this gauge-invariant meaning has been given already[7]; in this work, the proof was based on a dispersion analysis of the propagator, and was valid only to first order in λ.

Similar conclusions about Z_1 can be drawn by observing that the ratio Z_1/Z_2 is gauge-invariant.

V. CONCLUSIONS

In this paper we have distinguished two different ways in which field-theoretic scattering amplitudes can show full Regge behavior. They are very similar in their mathematical aspects: In both cases, the renormalization constants approach zero, as a pole in $\tilde{f}$ moves to the spinor mass. As a consequence of the vanishing of the Z's all, non-Regge parts disappear from the scattering amplitude. In physical interpretation, the two ways are quite distinct. For automatic Reggeization, it is of course not necessary to appeal to the vanishing of the Z's in some particular gauge to prove Regge behavior. The full gauge-invariant amplitude f is Reggeized, no matter how the Z's behave, and the proofs of Refs. 2 and 4 may be invoked instead. Nevertheless, it is instructive to relate automatic Reggeization to true composite-particle theory, even if only because spinor-vector scattering is the only theory with spin in which the mathematics of composite-particle theory can be made plausible by virtue of the finiteness of the Z's.

Another important byproduct of studying the relation of vector mesons to $Z=0$ theories is the proof that the equations $Z=0$ are gauge-invariant, obviously a necessity for a meaningful theory. This is not merely an academic question, but quite a real one for the study of electromagnetic mass shifts of composite particles (a brief discussion of the relevent points is given in Ref. 7, along with a different proof of the gauge invariance of $Z=0$). Here one might expect to be faced with infrared problems, but strict adherence to gauge invariance will remove possibility of calculating an infrared-divergent electromagnetic mass shift.

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