Regge Cuts, Absorption Model, and Diffractive Effects in Inelastic Scattering*

F. HENYEY, G. L. KANE, JON PUMPLIN†, AND M. H. ROSS
Physics Department, University of Michigan, Ann Arbor, Michigan 48104
(Received 20 January 1969)

We propose a model for calculating Regge-cut contributions to scattering amplitudes in terms of a Regge-pole and elastic scattering. Physically, the cuts are caused by absorption effects. Our expression for the cuts contains only one parameter with limited range besides those associated with the Regge pole and elastic scattering. We can explicitly show that combining absorption and Regge poles leads to no double counting. In this model all Regge poles are evasive at $t=0$. This model, with π-exchange input, applied to the forward peaks in $\gamma p\rightarrow\pi^+n$, $\pi N\rightarrow$ (transverse $\rho)N$, $np\rightarrow pn$, and $\pi p\rightarrow\rho\Delta$, is qualitatively the same as the absorption model, which is successful for these reactions. On the other hand, the conspiring Regge-pole model with factorization fails for $\pi p\rightarrow\rho\Delta$. We also apply the model with ρ-exchange input to $\pi^-p\rightarrow\pi^0n$. The dip at $-t\approx0.6$ BeV^2 in πN charge exchange is a diffraction minimum which, in the Regge language, is an interference between the Regge pole and the Regge cut. The Regge-pole contribution to the amplitude, taken by itself, has no dip. We predict that the dip drifts to smaller values of $-t$ as the energy is raised and the forward peak shrinks. The crossover effect in the $\pi^\pm p$ differential cross sections is also obtained.

I. INTRODUCTION

WE present here a theoretical model for the calculation of high-energy quantum number exchange reactions.[1] The amplitude is represented as the sum of Regge-pole exchange amplitudes and a "principle-cut" amplitude associated with each pole. The cut is an absorption correction,[2] or equivalently a double-scattering correction, to the Regge-pole exchange, and is associated with elastic scattering in the initial and final states.[3–8]

A. Review of Results

In recent years a phenomenology of high-energy (quantum number exchange) reactions based on Reggeon exchange has enjoyed some success.[9] We claim, however, that recent detailed experimental information makes it clear that the description of high-energy reactions based on analytic properties in the angular momentum plane will not be simple mathematically: The Regge representation used must be determined by physical arguments[10] and experimental evidence, and not by mathematical simplicity. In particular, we show that multiple-scattering arguments and the evidence demand a large role for Regge cuts. We present an expression for the principal cut associated with any pole. The model which we use to determine this expression is an absorption model, with the single scattering input given by Regge-pole exchange. The expression involves only one (scale) parameter, in addition to the parameters needed to describe the pole. The scale parameter is closely tied to the physics, and it can only vary over a limited range.

The experimental evidence indicating a strong role for cuts is (a) forward peaking in π-exchange processes, and (b) dips and secondary maxima. There is also a variety of relatively detailed experimental indicators, such as polarization, in πN charge-exchange scattering[11] and the crossover in the $\pi^\pm p$ differential cross sections.[12] All these phenomena can probably be accurately understood in terms of the principal cut and its interference with the pole contribution. Detailed applications are presented. Good agreement with experiment is found. We do not consider elastic scattering as such, nor inelastic processes proceeding with no quantum number exchange.

After a qualitative discussion, we discuss the validity of the model in Sec. II. We develop in Sec. III the formula for the cut contribution. In Sec. IV we apply the formalism.

B. Qualitative Considerations

How do considerations (a) and (b) strongly support Regge cuts? Because the pion has parity $-(-1)^J$,

* Research supported in part by the U. S. Atomic Energy Commission.
† Present address: Stanford Linear Accelerator Laboratory, Stanford University, Stanford, Calif.
[1] For a brief preliminary version of this model, see F. Henyey, G. L. Kane, Jon Pumplin, and Marc Ross, Phys. Rev. Letters 21, 946 (1968).
[2] B. M. Udgaokar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962).
[3] R. C. Arnold, Phys. Rev. **153**, 1523 (1967); Argonne National Laboratory Report No. ANL/HEP 6804, 1968 (unpublished).
[4] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Nuovo Cimento **48A**, 1075 (1967).
[5] E. J. Squires, Phys. Letters **26B**, 461 (1968).
[6] J. Finkelstein and M. Jacob, Nuovo Cimento **56A**, 681 (1968); C. B. Chiu and J. Finkelstein *ibid.* **57**, 649 (1968).
[7] J. N. J. White, Phys. Letters **26B**, 461 (1968); F. Schrempp, Nucl. Phys. **B6**, 487 (1968).
[8] Several papers have appeared, introducing cuts with poles. The most important technical difference in the present formulation is the absence of nonsense wrong-signature zeros in the pole terms, and the presence of the "coherent inelastic factor," increasing the cut.
[9] See, e.g., the review article by L. Bertocchi, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience, New York, 1968), p. 197.
[10] M. Ross, in Proceedings of the Irvine Conference on Pion Nucleon Scattering, 1967 (unpublished).
[11] P. Bonamy *et al.*, Phys. Letters **23B**, 501 (1966).
[12] V. Barger and L. Durand, Phys. Rev. Letters **19**, 1925 (1967).

orbital angular momentum is usually involved in the creation or annihilation of a pion; thus most reactions proceeding via π exchange vanish at zero momentum transfer (which we call "forward" for convenience). Instead of zeros, forward structures (peaks) over a small momentum transfer interval, $\Delta t \approx m_\pi^2$, are observed in $pn \to np$,[13] $\gamma p \to \pi^+ n$,[14] and $\pi^+ p \to \rho^0 N^{*++}$.[15] The evidence is very strong that these peaks are associated with π exchange. It has been observed that the mathematics of the Regge representation allows the possibility of nonzero forward amplitudes in certain of these cases, if the π trajectory "conspires" with a trajectory of opposite parity.[16] This possibility is physically unmotivated, and also does not agree with all the known forward π-exchange reactions in its basic factorizable form.[17] On the other hand, the explanation in terms of cuts, or double scattering, is physically compelling: Forward reactions occur as a result of small-angle π exchange with compensating elastic, or diffractive, scattering. The absorption model includes such processes and has, indeed, been found successful in these cases.[18] (Of course, the small pion mass implies that there is little distinction between the elementary pion used in the absorption calculation and a π Reggeon.)

The cut contribution should be viewed as a diffraction phenomenon associated with the Regge pole involved in the quantum number exchange. The dips and secondary maxima observed in momentum-transfer distributions of a variety of reactions are a natural aspect of this diffraction phenomenon. To understand this statement, first consider multiple scattering in the eikonal approximation,[19] used for describing reactions on deuterium and other light nuclei, at high energy: The reaction occurs on one nucleon; the shadowing effects of a second nucleon need to be taken into account; they are taken into account by adding the double-scattering contribution (eikonal approximation) involving elastic scattering on the second nucleon; this term has phase roughly opposite to the single-scattering term and is broader in momentum transfer because it involves nucleons lined up spatially one behind the other. Now consider the relation of the multiple-scattering formalism to scattering from a homogeneous absorptive optical potential. Historically, we associate diffractive structure with the latter and, in particular, associate the diffraction minima and subsidiary maxima with edge effects of the optical potential. However, recent experiments[20] have shown that high-energy elastic scattering on deuterium, and other light nuclei, has a dip and secondary maxima at momentum transfer roughly related in the usual way to the deuteron radius. These observations cannot be explained by a homogeneous optical potential shaped like deuterium since the deuteron is too smeared out to yield the structure. Rather, the dip is beautifully explained[21] by approximate cancellation between the strongly peaked single-scattering amplitude and the broader double-scattering amplitude. For heavier nuclei, the homogeneous-optical-model picture eventually merges with the multiple-scattering picture. Thus, for these composite hadronic objects, the multiple-scattering formalism provides the most basic description of diffractive structure. It is this type of process which, we claim, naturally explains the dips observed in particle physics. These dips are diffraction dips. A collateral conclusion is that introduction of zeros in the Regge amplitudes at nonsense wrong-signature points[22] is physically artificial and is incorrect. In the conventional Regge-pole language, we expect then to find important multiplicative fixed poles at nonsense wrong-signature points in Regge-pole amplitudes. Excellent agreement is obtained in our approach with the observed structure of the π-charge-exchange scattering. The dip is predicted to move forward (in t) logarithmically as energy is increased because of the increasing range of the ρ-exchange amplitude (shrinkage). This effect is one of the most striking predictions of this paper.

Although we have not completed careful fits to the backward elastic $\pi^\pm p$ scattering data, we have found that it is easy to obtain a dip with appropriate properties in $\pi^+ p \to p\pi^+$, and no dip in $\pi^- p \to p\pi^-$. In addition, for some reactions the dips that would be expected from zeros in the Regge amplitudes at nonsense wrong-signature points are not present experimentally. From our point of view these reactions do not appear to require dips. Examples of these reactions include ω production and backward π^+ photoproduction.

It can be shown that the Gottfried-Jackson absorption model is essentially the same as single plus double scattering in the eikonal approximation. The quantum number exchange in their model is via an elementary particle, i.e., a Feynman propagator. The energy dependence predicted is wrong for all but π exchange. We propose instead to describe the quantum number exchange by Regge poles in the usual manner. Then the absorption correction or double-scattering term is calculated by the same method as that of Gottfried and Jackson. It is a Regge cut. Possible double counting is carefully discussed in Sec. II and is found not to exist.

[13] G. Manning *et al.*, Nuovo Cimento **41**, 167 (1967).

[14] A. M. Boyarski *et al.*, Phys. Rev. Letters **20**, 300 (1968); P. Heide *et al.*, Phys. Rev. Letters **21**, 248 (1968).

[15] Aachen-Berlin-CERN Collaboration, Phys. Letters **27B**, 174 (1968).

[16] The many papers in this field are reviewed in Ref. 9.

[17] M. LeBellac, Phys. Letters **25B**, 524 (1967).

[18] J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965); J. D. Jackson, J. Donohue, K. Gottfried, R. Keyser, and B. E. Y. Svensson, Phys. Rev. **139**, B428 (1965).

[19] R. J. Glauber in *Lectures in Theoretical Physics* (Wiley-Interscience, Inc., New York, 1959), Vol. 1.

[20] For example, the experiments on light nuclei by H. Palevsky *et al.*, Phys. Rev. Letters **18**, 1200 (1967).

[21] R. H. Bassel and C. Wilkin, Phys. Rev. **174**, 1179 (1968).

[22] The removal of the zeros is discussed by A. H. Mueller and T. L. Trueman, Phys. Rev. **160**, 1296 (1967); S. Mandelstam and L. L. Wang, *ibid.* **160**, 1490 (1967); C. E. Jone and V. L. Teplitz, *ibid.* **159**, 1271 (1967).

II. VALIDITY OF MODEL

In order for the proposed model to be valid, it is necessary that the physical effects involved in absorption and in Reggeization be distinct. If this were not the case it would be very likely that double counting would occur, and that the absorption correction would be too large a correction to Regge-pole exchange. Since our model generates Regge cuts, the physics of absorption should also agree with the physics of other approaches to Regge cuts.

Both Reggeization and absorption involve composite structures. What is required is that the composite structures be distinct. Regge-pole exchange involves two kinds of structure. It includes the composite structure of the exchange object and it includes that part of the structure of the scattering objects which

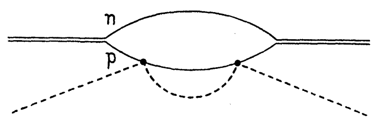

FIG. 1. Double scattering on a deuteron that will not occur at high energy.

gives rise to form factors. Absorption involves that part of the structure of the scattering objects which allows them to scatter more than once during a single collision. (The connection between absorption, multiple scattering, and Regge cuts is demonstrated in Sec. III.) Composite structure of the exchanged object is distinct from composite structure of the scattering objects. Since the absorption model with elementary-particle exchange lacks the former structure, it cannot be complete. This shows up in the wrong s dependence in such a model.

The two aspects of the structure of the scattering objects are also different. The form factor involved in single scattering represents the time-averaged structure of the object and does not require that part of the object which does the scattering to be composite. The multiple scattering involves the structure averaged in time only over the period in which the scattering occurs. If the relative velocity of the two scattering objects is much greater than the internal velocity within each object, this becomes the instantaneous structure. Existence of instantaneous structure requires that the part of the object causing the scattering be composite.

In order to clarify these ideas, consider the example of something scattering off a deuteron. In this example

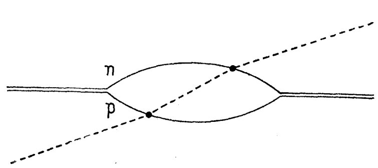

FIG. 2. Double scattering on a deuteron that persists at high energy when n and p are properly lined up.

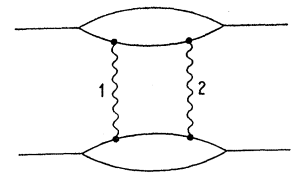

FIG. 3. Feynman diagram which has no high-energy double scattering, and which does not contribute to the cut. The wavy lines represent Reggeons or ladders.

assume that the proton and neutron are elementary. if the scattering is electromagnetic (assuming no magnetic moment for the neutron), it can only occur off the proton. There will be a deuteron form factor for this scattering, because the position of the proton within the deuteron is not determined. If the scattering velocity is greater than the proton's velocity, double scattering, as shown in Fig. 1, will not occur. If, on the other hand, the scattering is strong rather than electromagnetic, both a form factor and double scattering occur, even at high energies. The double scattering involves one scattering off the proton and one scattering off the neutron. This is illustrated in Fig. 2. This double scattering will occur whenever the proton and neutron are properly lined up. It is just the shadow-scattering correction to single scattering on the downstream nucleon. The imaginary part of the forward elastic-scattering amplitude on the upstream nucleon represents the loss of flux at the downstream nucleon. The double-scattering amplitude takes this into account, and the scattering is reduced under normal circumstances.

The double-scattering, or absorption correction is pure Regge cut, as shown in Sec. III. This represents further evidence that double counting does not occur. If the correction were simply a modification of the Regge-pole parameters, or contained such a modifica-

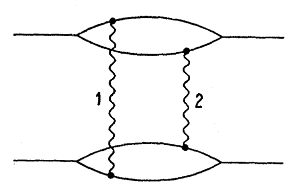

FIG. 4. Feynman diagram which has high-energy double scattering, and contributes to the cut.

tion, it would be probable that double counting did occur. However, since the mathematical structure of the correction is completely different from the single scattering, there is no double counting.

The aspects of composite structure discussed above also play a role in other approaches to Regge-cut theory. We shall compare our model with the Feynman-diagram approach.[23] The diagram shown in Fig. 3 is an example of a diagram in which the same two (elementary) particles scatter twice. We would then not expect it to contribute at energies high enough so that those two particles only come together once. This is also a diagram which does not contribute to the cut, because the two sides of it have no third double spectral function. The diagram shown in Fig. 4, on the other hand,

[23] S. Mandelstam, Nuovo Cimento **30**, 1127 (1963); **30**, 1148 (1963).

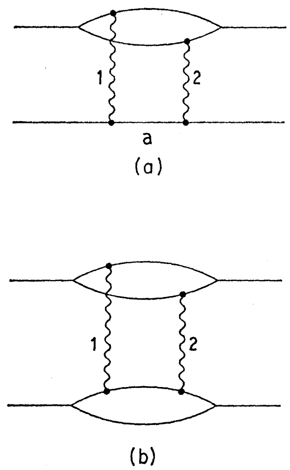

FIG. 5. (a) Feynman diagram corresponding to Fig. 2. However, it does not contribute to the cut because of a form factor for the intermediate particle a. (b) A similar diagram.

is an example of a diagram in which different particles scatter. We expect the double scattering to persist at all energies. It is also a diagram which contributes to the cut, since the two sides have third double spectral functions.

The diagram shown in Fig. 5 is an intermediate case. One particle scatters on two others. We expect double scattering to persist at high energy, yet this diagram does not contribute to the cut. The conditions that this not contribute are, first, that the intermediate particle at some point be the elementary particle a, governed by a Feynman propagator, and, second, that the intermediate particle have a form factor which vanishes for infinite four-momentum squares.[24] A specific example for the diagram in Fig. 5 is shown in Fig. 6. Physically the vanishing occurs because the projectile a is completely distorted by the initial scattering (i.e., is in state b) so that there is no elementary particle a for a distance of order $p/\Lambda^2 \propto \gamma$ behind the first target. Here Λ is the mass characteristic of the form factor and γ is the relativistic factor. Beyond this distance there is an increasing probability of finding state a (Fig. 7).

Thus Feynman processes of the type of Fig. 5 vanish at high energy because the projectile's form factor vanishing at infinity implies that none of the projectile is elementary in the initial part of the shadow behind the first target particle. Meanwhile, Feynman diagrams of the type of Fig. 4 give a contribution which persists at high energy. Thus the conventional method of estimating these processes simultaneously treats the particle as elementary (in the propagator) and composite (in the form factor that vanishes at infinity). It is not clear whether it should be considered relevant in an actual physical situation, where virtual particles of large momenta presumably behave in a subtle way; e.g., perhaps they are Reggeized with oscillating signature factor.

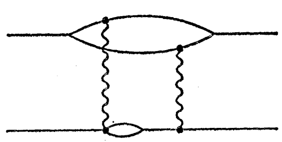

FIG. 6. Particular contribution to the form factor of Fig. 5.

Our expression is analogous in form to Fig. 5. The essential difference is that the intermediate particle is on the mass shell. Thus our expression represents the sum of all persistent diagrams associated with elastic scattering on the first target particle. There are additional double-scattering contributions associated with inelastic scattering at the first target. These are discussed later.

III. FORMULA FOR REGGE CUTS

In this section we construct the formula used for applications of our model, and we demonstrate the connection between absorption, Regge cuts, and multiple scattering.

We begin by showing very simply how an absorbed Regge-pole amplitude is equivalent to the same Regge-pole amplitude plus a moving cut. Assuming spinless particles for the moment, consider a process dominated by the exchange of a Regge pole. Although the detailed form of the Regge-pole amplitude does not play a role in these considerations, it might be useful to the reader to keep in mind that M^{ex} can be written in the form

$$M^{\text{ex}}(s,t)=\frac{\beta(t)}{\sin\pi\alpha(t)}\left(\frac{s}{s_0}\right)^{\alpha(t)}(1\pm e^{-i\pi\alpha(t)}). \qquad (1)$$

We apply the absorption by using the Sopkovitch procedure,[25] where we first expand M^{ex} in direct-channel partial waves and then multiply each partial wave by the square root of the S matrix for elastic scattering in that partial wave in the initial state, and again in the final state. For simplicity we assume that initial- and final-state elastic scattering are the same. Thus each partial wave of M^{ex} is multiplied by $e^{2i\delta_l(\text{el})}$.

Then we have (M is the total amplitude)

$$M^{\text{ex}}(s,t)=\sum_l (2l+1)P_l(z)M_l^{\text{ex}}(s),$$

$$M(s,t)=\sum_l (2l+1)P_l(z)M_l^{\text{ex}}(s)e^{2i\delta_l(\text{el})},$$

and putting $e^{2i\delta_l(\text{el})}=1-iqM_l^{\text{el}}/4\pi W$ in our normaliza-

FIG. 7. Intuitive sketch of spatial dependence of the scattering of an elementary particle a on a target when form factor as in Fig. 6 applies.

[24] The necessity of including form factors was brought out by H. Rothe, Phys. Rev. 159, 1471 (1967).

[25] N. J. Sopkovich, Nuovo Cimento 26, 186 (1962); K. Gottfried and J. D. Jackson, *ibid.* 34, 735 (1964).

tion (q and W are the magnitude of the c.m. three-momentum and total energy, respectively), this gives

$$M(s,t)=M^{\rm ex}(s,t)-\frac{iq}{4\pi W}\sum_l(2l+1)P_l(z)M_l^{\rm ex}(s)M_l^{\rm el}(s).$$

Now insert for $M_l^{\rm ex}$ and $M_l^{\rm el}$ their expression as a partial-wave projection of a full amplitude

$$M(s,t)=M^{\rm ex}(s,t)-\frac{iq}{4\pi W}\sum_l(2l+1)P_l(z)\times\left[\frac{1}{2}\int_{-1}^{1}dx\,P_l(x)M^{\rm ex}(s,x)\right]\times\left[\frac{1}{2}\int_{-1}^{1}dy\,P_l(y)M^{\rm el}(s,y)\right]$$

$$=M^{\rm ex}(s,t)-\frac{iq}{16\pi W}\int dx\,dy\times\left[\sum_l(2l+1)P_l(x)P_l(y)P_l(z)\right]\times M^{\rm ex}(s,x)M^{\rm el}(s,y).$$

It is a fairly conventional exercise to show that the sum in brackets is given by $2\theta(K)/\pi\sqrt{K}$, with $K=1-x^2-y^2-z^2+2xyz$. Then if one changes variables from y to φ, with $y=xz+(1-x^2)^{1/2}(1-z^2)^{1/2}\cos\varphi$, one finds

$$\int dx\,dy[\cdots]=\frac{1}{\pi}\int dx\,d\varphi=\frac{1}{\pi}\int d\Omega.$$

Thus, finally,

$$M(s,t)=M^{\rm ex}(s,t)-\frac{iq}{16\pi^2 W}\int d\Omega\,M^{\rm ex}(s,x)M^{\rm el}(s,y).$$

This is the essential result except for the coherent inelastic factor introduced later. Let us note some of its general properties before we discuss it in detail. $M^{\rm el}$ is the amplitude describing the initial or final elastic scattering, and to a good approximation can be taken as

$$M^{\rm el}=-ice^{At/2},$$

where A is the slope of the elastic differential cross section. In applications we will give $-ic$ the measured phase of elastic scattering at 0°, but here let us assume c if real (i.e., the diffraction elastic scattering is mainly imaginary) in which case it is determined by the optical theorem to be $c=2qW\sigma_T$. Thus we see that the relative magnitude of the pole and cut terms is fixed (with one qualification, to be discussed just below) and the relative phase is also determined and is mainly such that the two terms destructively interfere.

Next, we note that the first term is our original Regge pole, with no modification in value or redefinition of any parameters or function, while the second term is purely a Regge cut. Although this may not be obvious,

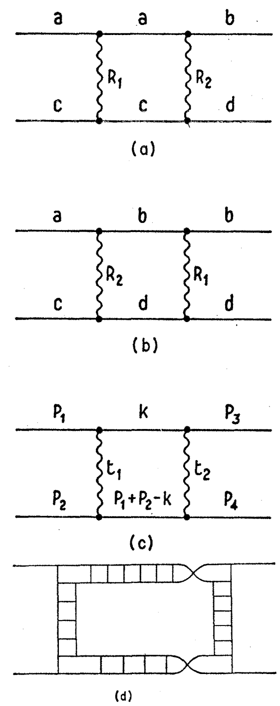

FIG. 8. (a) and (b) Multiple-scattering diagrams. The intermediate states are on mass shell. (c) Momentum labels for (a). (d) Feynman diagram which may correspond to double scattering.

recall that the first term is a pole in the complex l plane, and the second a superposition of two poles, which is a conventional way of obtaining a cut. In fact, the second term is a cut whether $M^{\rm el}$ is taken as above (a fixed pole) or as a moving Regge pole. These questions are discussed in more detail below. By choosing a simple form for $M^{\rm ex}$, with linear α, putting $\beta(t)/\sin\pi\alpha(t)$ = const over the region of integration (the scattering region), and giving the elastic scattering an $(s/s_0)^{\alpha_{\rm el}+\alpha_{\rm el}'t}$ dependence, the reader can convince himself that the cut term has the branch point at $t=0$, $\alpha_c=\alpha_{\rm ex}+\alpha_{\rm el}-1=\alpha_{\rm ex}$, the branch-point slope given by $1/\alpha_c'=1/\alpha_{\rm ex}'+1/\alpha_{\rm el}'$, and an energy dependence $1/\ln s$ relative to the pole term, all as in the usual Regge-cut formalism.[23] We will argue below, both theoretically and phenomologically, that the cut and pole contributions to the amplitude are comparable.

As discussed above, the mathematical pole-cut separation is one indication that there is no significant overlap in the physics of absorption and the physics of Reggeizing, so that it is sensible to do both; i.e., we are not including the same contributions twice. Thus a Regge cut is associated with every Regge pole in a definite way. Except for angular momentum and parity the pole and cut have the same quantum numbers. Since the cut corresponds to the simultaneous presence in the t channel of two space-time separated systems (e.g., see Fig. 4) there can be orbital angular momentum L exchanged, along with parity $(-1)^L$. Thus the cut always contains both natural- and unnatural-parity pieces; it can give rise to many interesting polarization effects, it can be self-conspiring, etc.

From another point of view which we will not pursue further here, the cut is in a bootstrap sense an important crossed-channel singularity. For example, it can give rise to circles on Argand diagrams in a definite partial wave. Analyses of the finite-energy sum rule sort will have to be reexamined.

So far there is one important physical effect that we have ignored. Imagine our cut term constructed as in Fig. 8(a), with a Pomeranchuk exchange representing the absorptive diffraction elastic scattering, followed by the quantum number exchange which causes the transition of interest. We should also, however, get a contribution from a process where a is replaced by a state a^*, with a^* any state that can be reached from a by Pomeranchon exchange—e.g., any Regge recurrence or diffraction inelastic product.[26] For the nucleon a^* would include all its Regge recurrences, the $N^*(1400)$ resonance, and all the isospin-$\frac{1}{2}$ resonances and continuum states. It appears to be sensible to assume that all of these processes contribute constructively, and contribute quite similar cut terms. It is difficult to estimate quantitatively their effect, but a typical estimate of one such term gives a contribution of the order of a few to 10% of the elastic cut term. Summing a number of such terms for both particles and for initial and final scattering, it would appear reasonable that these contributions should approximately double the elastic-cut term. To take account of these effects we introduce a "coherent inelastic" factor λ, multiplying the cut term. We expect λ to be of order 2 for most processes; presumably, it would vary somewhat depending on the particles involved. It is very encouraging that for a number of processes, including pion, vector meson, and baryon exchange, we find good fits to data (see below) for just such values of λ. The phenomenology is rather sensitive to λ, determining it to well within 20%.

The expression for the amplitude becomes

$$M=M^{\text{ex}}+M^{\text{cut}}, \tag{2}$$

where

$$M^{\text{cut}}=\lambda\delta M \tag{3}$$

and

$$\delta M=-\frac{iq}{16\pi^2\omega}\int d\Omega\, M^{\text{ex}}(s,x)M^{\text{el}}(s,y). \tag{4}$$

Note that we use M^{cut} for the entire cut and δM for that part including only the elastic intermediate states.

It is now of interest to examine the connection between the absorption model and Regge cuts. This can be understood by comparison with the model of Regge cuts originally proposed by Amati, Fubini, and Stanghellini (AFS).[27] The absorption formula has a similar form to the formula of AFS, although it differs in very important ways. AFS suggested that the cut contribution should be calculated by use of elastic unitarity. We are discussing inelastic processes, so elastic unitarity must be modified to include the initial and final states in the unitarity integral. We consider that cut in the AFS model generated by the Pomeranchon, with amplitude T_P, and another Regge pole, with amplitude T_R, exchanged either before or after the Pomeranchon. Then the amplitude of the AFS cut is (neglecting spin)

$$\text{Im}T_{\text{cut}}^{\text{AFS}}=\int d\Omega(T_P{}^*\rho_fT_R+T_R{}^*\rho_iT_P), \tag{5}$$

where the integration is over the angle of the momentum of the intermediate state.

Here ρ_f and ρ_i are the phase-space factors for the intermediate state the same as the final and initial states, respectively. If the amplitudes are expanded in partial waves, this equation becomes

$$\text{Im}T_{l\ \text{cut}}^{\text{AFS}}=T_{lP}{}^*\rho_{lf}T_{lR}+T_{lR}{}^*\rho_{li}T_{lP}. \tag{5'}$$

However, elastic unitarity is not a useful approximation at high energy. In fact, Mandelstam has shown that the cuts given by AFS are cancelled by other cuts. Therefore, $T_{\text{cut}}^{\text{AFS}}$ is not necessarily a good estimate of the actual cut contribution.

The absorption model combines an elastic-scattering amplitude T^{el}, with an exchange amplitude T^{ex}. The precise way in which these amplitudes are combined depends on the version of the absorption model. We use the Sopkovich formula[25]

$$T_l=(S^{\text{el}})^{1/2}T_l{}^{\text{ex}}(S_l{}^{\text{el}})^{1/2},$$

where

$$S_l{}^{\text{el}}=1+2iT_l{}^{\text{el}}\rho_l.$$

Defining the absorption correction by

$$\delta T_l=T_l{}^{\text{abs}}-T_l{}^{\text{ex}}$$

and expanding the square roots, we find

$$\delta T_l=iT_l{}^{\text{el}}\rho_{lf}T_l{}^{\text{ex}}+iT_l{}^{\text{ex}}\rho_{li}T_l{}^{\text{el}}+\cdots. \tag{6}$$

The lowest-order terms, explicitly exhibited in Eq. (6), are common to various versions of the absorption model. The higher terms differ among the models, but are generally small compared to the leading terms. We use only the explicitly exhibited terms in Eq. (6) for phenomenological fits.

Our model consists of identifying the exchange amplitude with the Regge-pole exchange, rather than with elementary-particle exchange as in the ordinary absorption model. We have

$$T^{\text{ex}}=T_R. \tag{7}$$

If, furthermore, we approximate elastic scattering with Pomeranchon exchange,

$$T^{\text{el}}\approx T_P, \tag{8}$$

[26] The same effect in scattering on nuclei has been discussed by Jon Pumplin and Marc Ross, Phys. Rev. Letters **21**, 1778 (1968).

[27] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962); Nuovo Cimento **26**, 896 (1962).

then the AFS model [Eq. (2)] and the present model [Eq. (6)] have similar forms. There are three very important differences, however. (a) There is no complex conjugation of any amplitude in Eq. (6). (b) Equation (6) gives the entire absorption correction, while Eq. (2) gives only the imaginary part of the AFS model cut. (c) Equation (8) is only a possible approximation. Our results are not changed if T^{el} consists of any combination of Regge poles, Regge cuts, and fixed poles.

The structure of Eqs. (6) and (5′) is the same, however, which leads to the following conclusions. (i) Both Eqs. (6) and (5′) give contributions that are purely Regge cut. The branch point trajectory is the same for both models. (ii) For any unitarity formula, an analogous absorption formula can be written. For example, the analog of Eq. (5) is

$$\delta T=\int d\Omega(iT^{\mathrm{el}}\rho_f T^{\mathrm{ex}}+iT^{\mathrm{ex}}\rho_i T^{\mathrm{el}}). \tag{9}$$

Another way of looking at the same physics is as a double-scattering phenomenon. The double-scattering amplitude can be evaluated using the diagrams in Fig. 8. As Feynman diagrams, the contribution M_{12} of Fig. 8(c) would be

$$M_{12}=i\int\frac{d^4k}{(2\pi)^4}R_1(t_1)R_2(t_2)\times\frac{1}{k^2+m^2-i\epsilon}\,\frac{1}{(p_1+p_2-k)^2+m^2-i\epsilon}. \tag{10}$$

As previously discussed, the off-mass-shell parts of the diagram do not properly represent the physical off-mass-shell particle. Only the on-mass-shell part should be kept.[28]

We take this into account by retaining only the singularities at $(p_1+p_2-k)^2+m^2=0$ and $k^2+m^2=0$. We work in the c.m. system where $p_1+p_2=(0,0,0,2iE)$, $k=(k\mathbf{\Omega},ik_0)$, and

$$\int d^4k=\int d\Omega\int_0^\infty k^2dk\int_{-\infty}^\infty dk_0.$$

Then M_{12} is given by

$$M_{12}=\frac{i}{(2\pi)^4}\int d\Omega\,R(t_1)R(t_2)\int_0^\infty k^2d'k\int d'k_0\frac{1}{k_0^2-k^2-m^2+i\epsilon}\,\frac{1}{(2E-k_0)^2-k^2-m^2+i\epsilon}, \tag{11}$$

where $d'k$, $d'k_0$ refer to the on-mass-shell part of the integrals:

$$M_{12}=\frac{i}{(2\pi)^4}\int d\Omega\,R(t_1)R(t_2)\int_0^\infty k^2d'k\left(\frac{-\pi i}{2k_0}\,\frac{1}{(2E-k_0)^2-k^2-m^2+i\epsilon}\bigg|_{k_0=(k^2+m^2)^{1/2}}+\frac{-\pi i}{2(2E-k_0)}\,\frac{1}{k_0^2-k^2-m^2+i\epsilon}\bigg|_{k_0=2E-(k^2+m^2)^{1/2}}\right) \tag{12}$$

$$=\frac{i}{(2\pi)^4}\int d\Omega\,R(t_1)R(t_2)(-\pi i)\int_0^\infty\frac{k^2d'k}{(k^2+m^2)^{1/2}4E[E-(k^2+m^2)^{1/2}+i\epsilon]}$$

$$=\frac{i}{(2\pi)^4}\int d\Omega\,R(t_1)R(t_2)(-\pi i)^2\frac{E^2-m^2}{4E^2}\,\frac{E}{(E^2-m^2)^{1/2}}=\frac{ip}{64\pi^2E}\int d\Omega\,R(t_1)R(t_2), \tag{13}$$

which is identical with one of the terms in Eq. (9). Figure 8(b) gives the other term.

If it is possible to use a composite-particle propagator rather than the elementary-particle propagator of Eq. (10), the integral can be taken off mass shell as well as on. A plausible way to do this is to let the intermediate particles a and c of Fig. 8(a) be "Reggeons." Then Figs. 8(a) and 8(b) become Regge-box diagrams, proposed by Arnold.[3] As shown by Rothe,[24] the version of Fig. 8 with elementary-particle sides has the on-mass-shell contribution cancelled by an off-mass-shell cut. The Reggeized version would have the same property if it were not for the phase factor associated with the signature of the Reggeon. This phase will vary over the cut, and make the contribution of the cut much smaller than it would be if it had constant phase. The only important contribution is the on-mass-shell part (including the Regge recurrences, which we only include phenomenologically, as one contribution in the coherent inelastic factor λ, in the present paper). For pictorial purposes, we replace Reggons by Feynman-diagram ladders with variable number of rungs. A typical diagram is shown in Fig. 8(d). Signature is essential in order to have the crosses on the two sides of the diagram,

[28] The opposite opinion has been given by M. E. Ebel and R. J. Moore, Phys. Rev. **177**, 2470 (1969). They treat the off-shell particles as elementary, which, as discussed in the text, we believe greatly overestimates the off-shell contribution.

since signature involves a half-twist of a ladder diagram. These crosses are exactly the necessary part of the Feynman diagram to make it contribute to the Regge cut.[29] Perhaps this provides a more precise connection between our formalism and the Feynman-diagram approach to the Regge cut.

In order to obtain a formula (for either double scattering or absorption, which are the same), spin must be included and the correct phase factor must be used. The phase-space factor appropriate to the Feynman amplitude is given in Eq. (12). The helicity is taken care of by summing over helicity indices. The formula for helicities λ, μ in the initial state and $\lambda'\mu'$ in the final state is

$$\delta M_{\lambda'\mu'\lambda\mu} = -\frac{iq}{32\pi^2 W}\int d\Omega \sum_{\lambda'',\mu''} (M^{\rm ex}{}_{\lambda'\mu'\lambda''\mu''}\, M^{\rm el}{}_{\lambda''\mu''\lambda\mu} + M^{\rm el}{}_{\lambda'\mu'\lambda''\mu''}\, M^{\rm ex}{}_{\lambda''\mu''\lambda\mu}). \quad (14)$$

Each M has a phase depending on its initial and final directions which must be included in the evaluation of Eq. (14). It is often convenient to change variables to the cosines of scattering angles in Eq. (14).

The absorption correction, then, is given by

$$\delta M_{\lambda'\mu'\lambda\mu}(z) = -\frac{iq}{16\pi^2 W}\int_{-1}^{1} dz_1 \int_{-1}^{1} dz_2 \frac{\theta(\Delta)}{(\Delta)^{1/2}} \times \sum_{\lambda'',\mu''} [M^{\rm el}{}_{\lambda'\mu'\lambda''\mu''}(z_1) M^{\rm ex}{}_{\lambda''\mu''\lambda\mu}(z_2) + M^{\rm ex}{}_{\lambda'\mu'\lambda''\mu''}(z_1) M^{\rm el}{}_{\lambda''\mu''\lambda\mu}(z_2)] \times \cos(h_1\varphi_1 + h_2\varphi_2 + h_3\varphi_3), \quad (15)$$

where

$$\Delta = 1 - z^2 - z_1^2 - z_2^2 + 2zz_1z_2, \quad (16)$$

$$e^{i\varphi_1} = (z_2 - zz_1 + i\Delta^{1/2})/(1-z^2)^{1/2}(1-z_1^2)^{1/2}, \quad (17)$$

$$e^{i\varphi_2} = (z_1 - zz_2 + i\Delta^{1/2})/(1-z_2^2)^{1/2}(1-z^2)^{1/2}, \quad (18)$$

$$e^{i\varphi_3} = (-z + z_1z_2 - i\Delta^{1/2})/(1-z_1^2)^{1/2}(1-z_2^2)^{1/2}. \quad (19)$$

The real part of these expressions are just the cosine addition theorem, so the angles φ_i have a simple geometric interpretation. Finally,

$$h_1 = \lambda' - \mu', \quad h_2 = \lambda - \mu, \quad h_3 = \lambda'' - \mu''. \quad (20)$$

The factor $\theta(\Delta)/\Delta^{1/2}$ is the Jacobian $\partial\Omega/\partial(z_1,z_2)$. A factor of 2 appears in Eq. (15) since φ_1, φ_2, φ_3 and $-\varphi_1$, $-\varphi_2$, $-\varphi_3$ correspond to the same z, z_1, z_2. The phase[30] is in the cosine instead of an exponential because the sine part integrates to zero as a result of this same symmetry. For small-angle scattering, one can put $\varphi_3 = -\varphi_1 - \varphi_2$ in Eq. (15); then it depends only on helicity flips. When the elastic scattering is diagonal in helicities $(\delta_{\lambda''\lambda'}\delta_{\mu''\mu'})$ the sums vanish and using $\varphi_1 + \varphi_2 + \varphi_3 = 0$ one gets just a factor $\cos n\varphi_2$, $n = (\lambda' - \mu') - (\lambda - \mu) =$ net helicity flip.

The above result is cast in several different, useful forms and discussed in some detail in the Appendix. To perform calculations the reader should proceed as follows. First it is necessary to obtain an appropriate set of Regge-pole exchange contributions to s-channel helicity amplitudes. As we discuss in our applications, it appears to be possible at the present time to describe experimental data with very simple Regge-pole amplitudes. The amplitudes need not vanish at wrong-signature points, nor at integral or half-integral J values for $t<0$, so long as they do not have poles at such points. Similarly, it appears that one can ignore possible conspiracies, always using the simplest alternatives, which will correspond to an evasive solution. Thus many of the subtleties introduced in the past few years concerning Regge amplitudes can probably be ignored, and it is not difficult to obtain a set of s-channel Regge-pole helicity amplitudes for any process in question. Having obtained these, the cut contribution associated with each one is constructed from Eq. (15), or the approximate versions of it [Eqs. (A11) and (A12)]. This construction is usually rather straightforward, particularly if the pole amplitudes can be taken as exponentials in t, when it can be done analytically; otherwise a numerical integration is generally required.

Finally, we remark that justification of our model requires that the Reggeon effecting the quantum number exchange be a "particle." It is not clear that such a description applies to elastic scattering (or "diffraction inelastic" scattering). There is no known particle definitely associated with the Pomeranchuk Regge trajectory, so that this may not be a simple Regge pole representing single elastic scattering. Since no quantum number exchange can be separated out to drive the elastic scattering, it may not make sense to consider elastic scattering as a single exchange with initial- and final-state interactions. Related models[3–7] have assumed a description for elastic scattering, involving single and multiple Pomeranchuk Regge-pole exchange. Although this assumption is plausible, our justification of our model does not extend to these models. Therefore, we apply our model only to inelastic (quantum number exchange) scattering.

IV. APPLICATIONS

A. Pion Exchange

In this section we discuss the application of our model to several processes which appear to be dominated by pion exchange at small momentum transfers. We restrict ourselves here to $-t \lesssim 0.1\,(\mathrm{GeV}/c)^2$; for larger $-t$ it is necessary to consider contributions from other exchanges (vector and tensor mesons). First let us review the experimental situation. The reactions $np \to pn$, $\gamma p \to \pi^+ n$, and $\pi^+ p \to \rho^0 \Delta^{++}$ all exhibit sharp forward

[29] C. Wilkin, Nuovo Cimento **31**, 377 (1964); also, Ref. 21.

[30] J. Charap, E. Lubkin, and A. Scotti, Ann. Phys. (N.Y.) **21**, 143 (1963).

peaks, with widths of order $-t \simeq m_\pi{}^2$. Separately, any of these can be treated in a Regge-pole theory, the first two with a conspiring pion trajectory, and the third with an evasive one. When considering them together one can show[17] that it is not possible for all three of these processes to have sharp forward peaks in a factorizable Regge-pole model.

From our point of view all of these reactions can be understood simply as resulting from an evasive-pion Regge pole and its associated Regge cut. There is no difficulty with LeBellac's argument[17] because the full amplitude (pole plus cut) does not factorize. In all cases the peak is due to a rapid variation of the pion-pole term (on the scale of $\Delta t = m_\pi{}^2$), relative to the associated (destructively interfering) cut with the same quantum numbers (apart from J^P, as discussed above). In addition to allowing us to deal with all three processes in a unified way, our treatment of any one of them involves fewer parameters than an ordinary Regge-pole treatment; we have only the pion Regge-pole parameters plus the coherent inelastic factor λ, and λ can vary only over a very limited range whose origin and approximate value is well understood physically (see Sec. III).

Consequently, although one cannot prove that the pion is evasive, there is no need to assume otherwise to understand the experimental data, and to us it appears to be the natural assumption.

We only discuss photoproduction in detail. The behavior of the other processes is qualitatively similar, and we defer their discussion to a future publication covering the full angular distribution and all relevant exchanges (at larger angles vector and tensor meson exchange will contribute).

For $\gamma p \to \pi^+ n$ we use a vector-dominance model in the sense that the photon is coupled to a ρ meson and the amplitude for $\rho p \to \pi^+ n$ is constructed by subtracting off the longitudinal ρ's to obtain the photoproduction cross section. That is,

$$d\sigma/dt(\gamma p \to \pi^+ n) = (1-\rho_{00}) d\sigma/dt(\rho^0 p \to \pi^+ n),$$

where ρ_{00} is the zero helicity density matrix element in the s-channel center of mass. In the pion-exchange region only the external ρ contributes (not ω or φ).

To understand the results qualitatively, consider $\pi N \to \rho N$. Define the quantum number n which measures the net helicity flip; for a process $a+b \to c+d$,

$$n = |(\lambda_c - \lambda_d) - (\lambda_a - \lambda_b)|.$$

From angular momentum conservation, any helicity amplitude must vanish as $(\sin\frac{1}{2}\theta)^n$ as $\theta \to 0$; we write this as $(-t)^{n/2}$, ignoring $-t_{\min}$. Then if we write the full amplitude M in the form

$$M = B + A,$$

when B is the pion Regge-pole term and A is the Regge

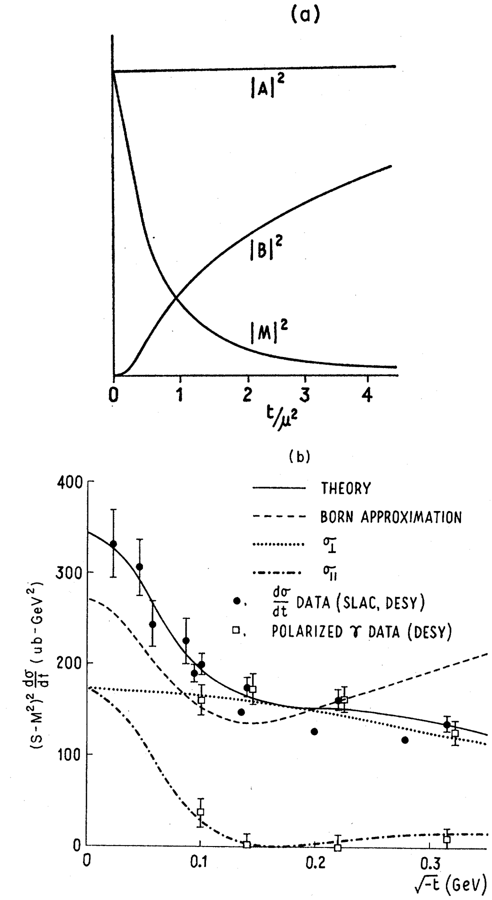

FIG. 9. (a) $\pi N \to \rho N$ amplitudes for $n=0$. (b) Comparison of our model with experiment for $\gamma p \to \pi^+ n$ for small t $(-t<0.1)$. See the discussion in the text for the interpretation of the various curves.

cut, near $t=0$ we have approximately

$$n=0:\quad B_{\frac{1}{2}-\frac{1}{2}}{}^{1} \approx 2egt/(t-\mu^2),$$
$$n=1:\quad B_{\frac{1}{2}-\frac{1}{2}}{}^{0} \approx 2eg(\sqrt{-t})(m_\rho + t/m^2)/\sqrt{2}(t-\mu^2),$$
$$n=2:\quad B_{\frac{1}{2}-\frac{1}{2}}{}^{-1} \approx -2egt/(t-\mu^2).$$

The amplitudes $M_{+\frac{1}{2}+\frac{1}{2}}{}^{\mu}$ fall off for pion exchange one power of s faster than these, so we do not need to consider them.

Compare the Regge-pole terms for $n=0$ and $n=2$. They are equal because the pion exchange is pure unnatural parity, which can be shown to give $B_{\frac{1}{2}-\frac{1}{2}}{}^{1} = -B_{\frac{1}{2}-\frac{1}{2}}{}^{-1}$. But there is no need for $M_{\frac{1}{2}-\frac{1}{2}}{}^{1}$ (with $n=0$) to vanish as $t \to 0$. Using the Regge-pole terms as they stand is known as choosing the evasive solution (or, in another language, giving the pion a Toller quantum number $M=0$).

On the other hand, because the cut term arises physically from the exchange of two space-time separated systems, both natural and unnatural parity are always present in its contribution, so $A_{\frac{1}{2}-\frac{1}{2}}{}^{1}$ need not and does not vanish as $t \to 0$.

Thus qualitatively what happens is as follows [Fig. 9(a)].[31] The cut term is only sizeable for the $n=0$ amplitude. There, at larger t the cut and pole have been destructively interfering and they are of similar magnitude; thus M is smaller than either one separately. Very near the forward direction $A_{\frac{1}{2}-\frac{1}{2}}{}^1$ is essentially constant, while suddenly $B_{\frac{1}{2}-\frac{1}{2}}{}^1$ drops to zero, and there is no longer any interference with A. Thus $A+B$ shows a forward peak. (We note that from this analysis we would expect the cross section for production of longitudinal ρ's to have a forward dip, while that for transverse ρ's would have the photoproduction forward peak.)

We have carried out quantitative calculations for $-t\leq 0.1$. The results are shown in Fig. 9(b); the fit is quite good. The pion Regge-pole parameters were chosen to be $\alpha(t)=t-m_\pi^2$, $s_0=1.0$ GeV^2, and $\beta(t)=2\sqrt{2}\,eg$ where $e^2/4\pi=1/137$ and $g^2/4\pi=14.7$. The elastic-scattering treatment was the same as for the πN charge-exchange analysis, described in Sec. IV B. The coherent inelastic factor λ was chosen to give a good fit to the data, giving a value $\lambda=2.7$. When ρ and ω exchange are included and the entire differential cross section fitted, this value will have to decrease somewhat. It is to be compared with $\lambda=1.5$ from the πN charge-exchange analysis and values of $\lambda\simeq 2.2$ in preliminary analysis of ω production and less in backward $\pi^\pm N$ elastic scattering.

This is perhaps a good place to emphasize two important practical differences between the present model and the conventional absorption model with elementary-particle exchange.[25] First, inclusion of the coherent inelastic states gives $\lambda>1$ and increases the effect of the destructive interference between the pole term and its absorption correction. Second, the Regge-pole term itself initially has the usual Regge exponential decrease $[e^{-i\pi/2}s/s_0]^{\alpha_0+\alpha' t}\sim e^{\alpha' t[\ln(s/s_0)-i\pi/2]}$ rather than the slow decrease or increase of the elementary-particle exchange. Thus (apart from the forward zero) the Regge-pole term is a decreasing exponential with a slope $\alpha'[\ln(s/s_0)-i\pi/2]$, while the cut is an exponential with a reciprocal slope given by $2/A+1/(\alpha'[\ln(s/s_0)-i\pi/2])$.

$$2/A+1/(\alpha'[\ln(s/s_0)-i\pi/2])=1.8\ (\text{GeV}/c)^2.$$

Without this decrease there would be a tendency for a dip near $-t\simeq 0.02$ rather than a break as is observed. The $n=2$ amplitudes fills in any dip structure as well, since it is mainly a pole term and varies smoothly $\sim te^{\alpha' t[\ln(s/s_0)-i\pi/2]}$.

We also note that the calculated magnitude is correct, as is to be expected from the nearness of the elementary pion to the forward direction.

Also shown in Fig. 9(b) are the cross section $\sigma_{||}$, and $\sigma_\perp$, corresponding to scattering with polarized photons with polarization parallel or perpendicular to the scattering plane, respectively. The break appears only in $\sigma_{||}$, corresponding to pure unnatural parity exchange, because there the pole and cut are interfering and the pole suddenly goes to zero as $-t\to 0$. The slowly varying natural parity contribution $\sigma_\perp$ is purely the cut contribution. The experimental data on π^+ photoproduction with polarized photons are in agreement with the curves in Fig. 9(b).[32]

The results for $np\to pn$ and $\pi^+p\to\rho^0\Delta^{++}$ are qualitatively the same as those for $\gamma p\to\pi^+n$ near the forward direction. In all cases it is now necessary to carry out detailed calculations over large angle and energy ranges to verify that the results continue to be so encouraging.[33] We are also analyzing the combined differential cross section and polarization data on $\gamma p\to\pi^+n$, $\gamma n\to\pi^-p$, and $\gamma p\to\pi^0p$.

B. Diffraction Dips: π-Charge Exchange and the π±p Crossover

We consider the reaction $\pi^-p\to\pi^0n$ via ρ exchange in order to illustrate the phenomenon of dip plus secondary maximum. The particular advantages of this reaction are that the data are relatively good and that there is only one well-established trajectory that can contribute. The amplitude is the sum of ρ Regge-pole exchange and the associated double-scattering term.

$$M_{\mu'\mu}=M_{\mu'\mu}{}^{\rho}+M_{\mu'\mu}{}^{\text{cut}}, \tag{21}$$

$$M_{\mu'\mu}{}^{\text{cut}}=\frac{-i\lambda_{\mu'\mu}}{32\pi^2}\int d\Omega\, M_{\mu'\mu}{}^{\rho}M_{\mu'\mu}{}^{\text{el}}. \tag{22}$$

For unpolarized nucleons,

$$d\sigma/dt=(64\pi q^2s)^{-1}(|M_{++}|^2+|M_{+-}|^2). \tag{23}$$

The subscripts are s-channel c.m. helicities.

The recoil-neutron polarization in the direction $\mathbf{p}_{\pi^-}\times\mathbf{p}_{\pi^0}$ is given by

$$P=2\,\text{Im}(M_{++}M_{+-}{}^*)/(|M_{++}|^2+|M_{+-}|^2). \tag{24}$$

The Regge-pole terms are taken to be

$$M_{++}{}^{\rho}=\frac{-m_\rho^2\gamma_{++}}{t-m_\rho^2}ie^{-i\pi\alpha_\rho(t)/2}(E/E_0)^{\alpha_\rho(t)}, \tag{25}$$

$$M_{+-}{}^{\rho}=\frac{-m_\rho^2\gamma_{+-}}{t-m_\rho^2}(-t)^{1/2}ie^{-i\pi\alpha_\rho(t)/2}(E/E_0)^{\alpha_\rho(t)}. \tag{26}$$

These forms satisfy all known requirements on Regge poles. They have the phase appropriate to the signature factor, the ρ pole, the appropriate energy dependence, etc. We have ignored various factors of $\sin\pi\alpha$, Γ functions, etc., which sometimes are included but mainly

[31] These corrections are an approximate evaluation of the formulas given by G. L. Kane, Phys. Rev. 163, 1545 (1967).

[32] P. Schmusser (private communication).

[33] Of course, we know that π exchange does not dominate at larger angles. In π^+, π^- photoproduction on deuterium, P. Heide, U. Kotz, R. Lewis, P. Schmusser, H. Skronn, and H. Wahl, Phys. Rev. Letters 21, 248 (1968) show substantial deviation from π exchange for $-t\gtrsim 0.01$.

serve to cancel one another. As far as is known the above representation is an entirely adequate Regge representation in the region where it is used, for $t \leq 0$. (For positive t one would have to include poles for the ρ recurrences.) Of course, for arbitrary $\gamma_{+\pm}$ this is obvious, but we shall assume that for the above form we can take $\gamma_{+\pm}=$ const. Slightly different forms were used in Ref. 1.

Note that the helicity-flip amplitude is not constructed to vanish when $\alpha_\rho = 0$. It appears to us more natural that the dip should arise as a pole-cut interference or diffraction minimum. Throughout our applications we have used Regge-pole amplitudes constructed as above, as simply as possible, with evasive solutions to conspiracy relations, and with no other zeros not required by general analyticity arguments. In particular, the conventional procedure of setting $M_{+-}=0$ when $\alpha_\rho=0$ is not necessary.[1,14]

To construct the cut terms, we make one further approximation which is good to an average of better than 2% over the range of integration where the pole terms are used. If the Regge-pole exchange were an exponential, the integration could be performed analytically, as given by Eq. (A12). Moreover, if the exponential is multiplied by a polynomial in t, $P(t)$, the integral still can be done explicitly as

$$P(t)e^{at}=P(d/da)e^{at},$$

and the operator $P(d/da)$ can be pulled outside the integral. The factor $1/(t-m_\rho^2)$ in our pole expression prevents this from being the case, so we approximate it by

$$\frac{1}{t-m_\rho^2}=\frac{e^{-t/m_\rho^2}e^{t/m_\rho^2}}{-m_\rho^2(1-t/m_\rho^2)}$$

$$=\frac{e^{t/m_\rho^2}}{-m_\rho^2}\frac{1-t/m_\rho^2+t^2/2m_\rho^4\cdots}{1-t/m_\rho^2}$$

$$\approx (e^{t/m_\rho^2}/-m_\rho^2)(1+t^2/2m_\rho^4). \quad (27)$$

The elastic-scattering amplitude is taken from the data to be fixed at $\alpha=1$, with constant phase, and helicity nonflip:

$$M_{\lambda'\lambda}{}^{\text{el}}=-\delta_{\lambda'\lambda}(i+\rho)s\sigma_T e^{Gt}, \quad (28)$$

where ρ is the ratio of the real to the imaginary part of the forward πN diffraction peak, and σ_T is the πN total cross section. All of these are taken from experiment, including their gradual energy dependence.[34]

The cut terms M^{cut} are then given by substituting (26) and (28) in (A12):

$$M_{++}{}^{\text{cut}}=\frac{-\lambda_{++}\sigma_T}{8\pi G}(1-i\rho)A_{++}$$

$$\times\left(1+\frac{1}{2m_\rho^4}\frac{d^2}{dB^2}\right)\frac{G}{G+B}\exp\frac{GBt}{G+B}, \quad (29)$$

[34] K. J. Foley *et al.*, Phys. Rev. Letters **11**, 425 (1963); **14**, 862 (1965); **15**, 45 (1965).

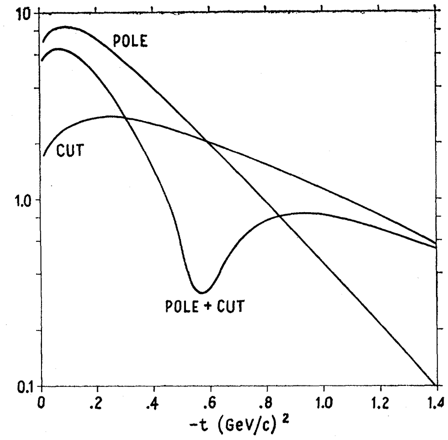

FIG. 10. Absolute value of spin-flip amplitudes in πN charge exchange (Solution II discussed below at 9.8 GeV/c). Arbitrary scale.

$$M_{+-}{}^{\text{cut}}=\frac{-\lambda_{+-}\sigma_T}{8\pi G}(\sqrt{-t})(1-i\rho)A_{+-}$$

$$\times\left(1+\frac{1}{2m_\rho^4}\frac{d^2}{dB^2}\right)\left(\frac{G}{G+B}\right)^2\exp\frac{GBt}{G+B}, \quad (30)$$

where

$$A_{+\pm}=\gamma_{+\pm}ie^{-i\pi\alpha_0/2}(E/E_0)^{\alpha_0}, \quad (31)$$

$$B=(1/m_\rho^2)+\alpha_1[\ln(E/E_0)-i\pi/2], \quad (32)$$

and we have assumed that the ρ trajectory is linear over the experimental region, $\alpha_\rho(t)=\alpha_0+\alpha_1 t$. The coherent inelastic factors $\lambda_{+\pm}$ are discussed for the spinless case in Sec. III.

Before proceeding with a quantitative discussion of our fit, we describe the qualitative features. The Regge-pole amplitudes are smooth as functions of t, having approximately an exponential form. The cut is about 180° out of phase with the pole. The double scattering can occur with two small-angle scatterings adding up to a relatively large angle. Therefore, at large angles the cut dominates the pole, although it gives a small correction to the integrated cross section. When the magnitude of the pole and cut are equal, their sum (interfering destructively) is very small. If one amplitude is dominant, as in $\pi^- p \to \pi^0 n$, this results in a dip in the cross section (see Fig. 10). The difference between the $\pi^+ p \to \pi^+ p$ and $\pi^- p \to \pi^- p$ amplitudes is the same as the $\pi^- p \to \pi^0 n$ amplitude by isospin conservation. Because these two amplitudes are mostly nonflip and imaginary, the crossover in their differential cross sections occurs close to the point at which the imaginary part of the nonflip charge-exchange amplitude vanishes, which in turn is close to the point at which the magnitudes of nonflip pole and cut are equal. In the expression for the cut there is a factor $[G/(G+B)]^{n+1}$ which is smaller for larger net helicity flip n, arising from the angular momentum conservation factor $(-t)^{n/2}$. Therefore, the crossover in the π^+

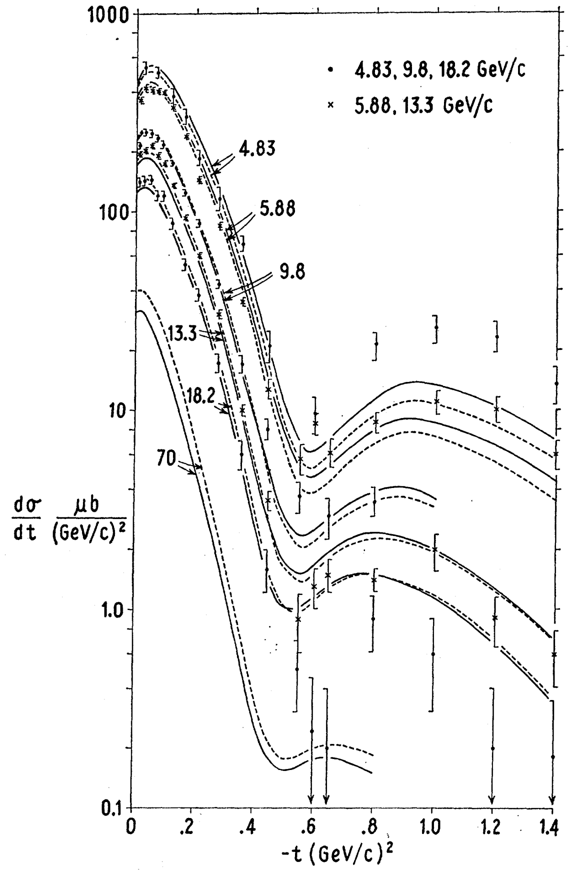

FIG. 11. Fit of $\pi^- p \to \pi^0 n$ using the parameters of Table I. The dip is a diffraction minimum. Neither the helicity-flip amplitude nor the Regge-pole term vanishes at the nonsense wrong-signature point where $\alpha_\rho = 0$. The data are from Ref. 35. The solid line refers to Solution II and the dashed line to Solution I.

and π^- differential cross sections occurs at a smaller t value than the $\pi^- p \to \pi^0 n$ dip. Both the crossover and the dip positions are functions of energy, since the

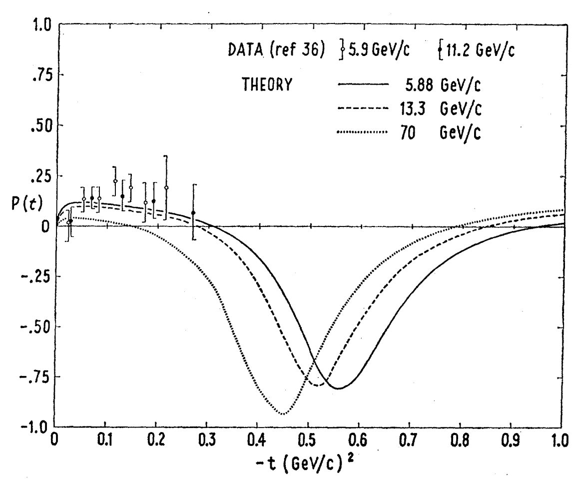

FIG. 12. Fit to polarization data of Ref. 36, using parameters of Solution II of Table I.

energy dependences of the pole and cut are different. At infinite energies these points would be at $t=0$, but at low energies the drift of the dip can be in either direction.

The pole and cut terms have different phases because the cut is constructed by integrating over the pole whose phase depends on the integration variable and because the elastic scattering has a small real part. Thus the helicity-flip and nonflip terms have different phases, and the polarization is nonzero.

A fit to the data on $d\sigma/dt$ for $\pi^- p \to \pi^0 n$,[35] the polarization at small angles,[36] and the total cross-section difference for $\pi^\pm p$ scattering[37] has been made. The seven parameters were varied. We find that the predictions are insensitive to the freedom of allowing different E_0's for flip and nonflip, so we set them equal. Also, a wide range of α_1 is permitted by the fit as long as

$$\alpha_1 \ln[(14\text{ GeV})/E_0] \approx 4\,, \qquad (33)$$

so we choose

$$\alpha_0 + m_\rho^2 \alpha_1 = 1\,. \qquad (34)$$

The fitting thus involves the six parameters: α_0, λ_{++}, λ_{+-}, E_0, γ_{++}, and γ_{+-}. We are not interested in fits

TABLE I. Parameters for πN charge-exchange scattering, energies in BeV units.

	α_0	α_1	E_0	λ_{+-}	λ_{++}	γ_{+-}	γ_{++}	χ^2
Solution I	0.473	0.9	0.165	1.511	1.292	85.7	−22.6	349
Solution II	0.418	1.0	0.269	1.547	1.311	129.3	−34.9	464

with unreasonable parameter values and so consider the rough ranges

$$M_\pi^2 \lesssim E_0 \lesssim M_N^2\,,$$
$$1.0 \lesssim \lambda \lesssim 2.0\,. \qquad (35)$$

We do not obtain an excellent fit, but rather a fairly good fit, comparable with all other attempts.[38] The results are significantly different from those reported earlier,[1] involving spin flip alone. Two kinds of fits are shown: Solution I is good in the low $-t$ region and poorer at high $-t$ (Fig. 11). Solution II is better than I at large $-t$ but poorer than I at low $-t$. The parameters are listed in Table I. If there are no systematic problems with the data or quoted errors we should obtain $\chi^2 \approx 100$. Instead, we obtain $\chi^2 \approx 350$.

The polarization results are shown in Fig. 12. The fits to the existing data are good. The general shape of the curve is insensitive to changes in the parameters,

[35] A. V. Stirling *et al.*, Phys. Rev. Letters **14**, 763 (1965); P. Somleregger *et al.*, Phys. Letters **20**, 75 (1966).
[36] P. Bonamy *et al.*, Phys. Letters **23**, 501 (1966).
[37] K. J. Foley *et al.*, Phys. Rev. Letters **19**, 330 (1967).
[38] See, e. g., W. Rarita, R. J. Ridell, Jr., Charles B. Chiu, and R. J. N. Phillips, Phys. Rev. **165**, 1615 (1968); see also Refs. 7 and 4, which show the same difficulty with large-angle data.

but reasonable fits to $d\sigma/dt$ can be obtained with maximum polarization ranging from 0.05 to 0.15 in the small-t region. A prediction for 70 GeV/c is shown. The polarization is small for small t because the helicity-flip amplitude dominates; there is no difficulty in obtaining different phases for the two amplitudes. Near the dip region the two amplitudes are comparable and the polarization approaches unity. The crossover in $\pi^{\pm}$ differential cross sections is assumed to satisfy

$$\mathrm{Re}(i+\rho)M_{++}=0. \tag{36}$$

It occurs at $-t=0.3$, while present data indicate $-t\approx 0.2$.

The theory appears to have one systematic problem when compared with differential cross-section data: The theoretical secondary maximum is too small at low energy and the theory decreases too slowly with increasing s for fixed t in the secondary maximum region; i.e., the theoretical effective α in the region where the cut dominates is too large. In Fig. 13, the energy dependence at three fixed t's is shown to illustrate the point.

We now discuss the sensitivity with which the parameters are determined and the significance of the values obtained. The fit is sensitive to the value of α_0. The effective α in the presence of the cut is higher than $\alpha(t)$ (of the pole) at low $-t$. We obtain fits comparable with I and II for $\alpha_0=0.46\pm 0.05$. Meanwhile, as remarked in connection with Eq. (33), α_1 is not well determined. The coherent inelastic factors are fixed within about 5% with $\lambda_{+-}=1.54$, $\lambda_{++}=1.28$. The value of γ_{++}/γ_{+-} is well determined at -0.27 while the absolute value of $|\gamma_{++}|$ varies roughly from 10 to 50. The "errors" are estimates of points where the fit becomes qualitatively worse, and, of course, depend on the very particular assumptions under consideration.

The values of the residue functions can be compared with the on-mass-shell coupling constants by extrapolating to the ρ pole. This gives $\gamma_{++}=-4g_{\rho\pi\pi}G_vE_0m/m_\rho^2$.

The values of the residue functions can be compared with the on-mass-shell coupling constants by extrapolating to the ρ pole. This gives

$$\gamma_{++}=-4g_{\rho\pi\pi}G_vE_0m/m_\rho^2 \text{ and } \gamma_{+-}=2g_{\rho\pi\pi}G_TE_0/m_\rho^2,$$

where a factor $G_V\gamma_\mu+G_T\sigma_{\mu\nu}K_\nu/2m$ goes at the ρ^-pn vertex and a factor $ig_{\rho\pi\pi}(p_{\pi\,\mathrm{in}}+p_{\pi\,\mathrm{out}})$ at the $\pi^-\rho^+\pi^0$ vertex. Using $g_{\rho\pi\pi}\simeq 5$ (corresponding to a ρ width of 105 MeV), $G_V=+3.5$ and $G_T=+3.7G_V$ (from vector dominance), we obtain $\gamma_{++}\simeq -112E_0$, $\gamma_{+-}\simeq 220E_0$. Using this relative sign for the γ's gives the polarization result that we have shown, of the same sign as the experimental data. One good fit to the data has $|\gamma_{++}|=22.6$, $|\gamma_{+-}|=85.7$, $E_0=0.165$; using this E_0 above gives γ_{++} only 20% too large and γ_{+-} just over a factor of 2 too large. This agreement is consistent with our assumptions about the slowly varying nature of the residue functions.

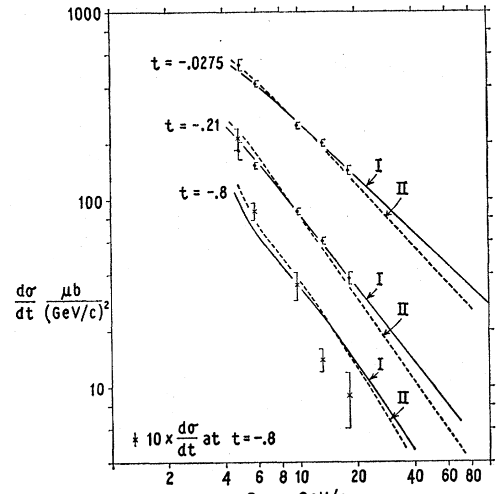

FIG. 13. Energy dependence of charge-exchange differential σ at three fixed momentum transfers. Data from Ref. 35.

The values of the λ's are reasonable. We have not attempted any detailed evaluation. Particular N^* and π^* contributions have been estimated by Ravenhall and Wyld[39] with contributions up to 0.2 from $N^*(1688)$ and A_1. Contrary to Ravenhall and Wyld, we take such numbers as indicating values of λ up to 2 or more, although the values of λ needed in our fit here are smaller than that. Our estimates of the coherent inelastic factor λ will be discussed in detail in future publications. Our opinion, stated very briefly, is that available experimental data imply the existence of a large number of contributions to λ all in the range 0.05–0.25. All of these contributions have the phase of the exchanged Regge pole, up to an over-all $\pm$ sign. If all have the same sign then we have complete coherence and we can get quite a large value of λ. If there is even partial coherence it is not difficult to get $\lambda\sim 2$.

The absolute values of λ involved here roughly correspond to total absorption in the s states, if one looks at $\lambda>1$ as an increase in absorption. That is, a value of one for the coefficient C in the Gottfried-Jackson absorption model[18] corresponds arithmetically to $\lambda=1.49$ for $\sigma_T=25$ mb, so we have effectively increased the absorption over that deduced empirically (as Gottfried and Jackson also found convenient). We believe that there is no reason in principle why λ could not be significantly larger since we regard the single- and double-scattering formalism as more general than the absorption model which originally motivated our amplitude. It is difficult to think about absorptive effects via optical analogies in the presence of coherent inelastic channels.

The relative values of λ_{++}, λ_{+-} make sense because the charge coupling of the ρ at an NN^* vertex should

[39] D. G. Ravenhall and H. W. Wyld, Phys. Rev. Letters 21, 1770 (1968).

be zero at $t=0$ according to the usual ideas about interaction with the conserved isospin current. Thus there will be contributions from the area under the form factors at these vertices, but the form factors will vanish at $t=0$. Since charge coupling corresponds to helicity nonflip at high energies, we expect λ_{++} to be only a little greater than 1. At an helicity- flip vertex it is the magnetic moment coupling that contributes at high energies. This can be as big as it wishes for all N^*'s, so we expect λ_{+-} to be significantly greater than 1.

The other parameter values are reasonable. The value for the trajectory slope α_1 is ≈ 1. The value of α_0 is significantly less than the effective value of $\alpha(0)\approx 0.55$, which, in the absence of consideration of cuts, has been considered the proper value for the ρ. There is not much to say about the values of E_0 except that they seem typical of other fits to the data.

Several possible reasons within the context of the present theory could be responsible for the difference between theory and experiment. The outstanding problem is lack of knowledge of the elastic amplitude near the break ($-t\approx 0.8$) and beyond. Instead of (28) this amplitude has less than half the slope, has destructive phase with respect to the forward term, and also has a considerable helicity-flip part at low energy.[40] The contribution of this region to the cut amplitude is large enough to account easily for the discrepancies in our fit. The destructive term will decrease the value of α. Perhaps the large flip amplitude at lower energies will have the sign to greatly increase the nonflip contribution to the secondary maximum at low energies.

Other possible problems are the following. (1) Our level of knowledge of the phase of the low $-t$ elastic amplitude is not sufficient. (Variations in the phase and phase dependence on t in the elastic amplitude were considered in the fitting, but the effects were not such as to promise great improvement.) (2) The trajectory $\alpha(t)$ may be curved. (3) The coherent inelastic factor λ may depend slightly on s and t. (4) The residue functions could depend on t [a linear dependence $(1+at)$ with $|a|<0.5$ was considered and was not found to be very helpful]. (5) t-channel unitarity corrections should be made.[41] (6) Lower-lying trajectories may be important.

It is not clear how to decide on the basis of experiments in the near future whether dips are diffraction minima or zeros in Regge amplitudes at nonsense wrong-signature points. Eventually, it will be possible to find out whether they move with energy, but not for some time. For many processes the two approaches will give similar results for differential cross sections. This is because, in the case where the pole amplitudes vanish at nonsense wrong-signature points, these points occur in the range of integration in constructing the cut, so cancellations occur and the cut contribution is smaller.

[40] R. J. Esterling *et al.*, Phys. Rev. Letters **21**, 1410 (1968).
[41] J. B. Bronzan and C. E. Jones, Phys. Rev. **160**, 1494 (1967).

To obtain large polarizations, however, a large cut contribution will be necessary in processes dominated by exchange of a single Regge pole, so good polarization data as a function of momentum transfer will probably suffice to distinguish; some processes under investigation include $\pi^-p\to\pi^0n$, $\pi^-p\to\eta n$, $\pi N\to\omega N$,[33] and $\pi N\to\omega\Delta$.

On the other hand, data may appear where a dip is required by a nonsense wrong-signature zero but is not present experimentally. There are two possibilities for this at the present time. One is in backward π^-p, where a dip should appear when $\alpha_\Delta=-\frac{3}{2}$. The data do not show such a dip, but there may be some other reason why it does not appear at such a large momentum transfer ($-u\simeq 2$). The second is in backward π^+ photoproduction, where the data show no dip. This can be interpreted by saying that Δ exchange dominates; since nucleon exchange is allowed this would appear to require considerable justification. Vector-dominance arguments appear to indicate that nucleon exchange does dominate.

From our point of view dips are not so easy to get. They generally require that only one Regge pole should dominate, and in addition that only one helicity amplitude should dominate; otherwise different helicity amplitudes have their diffraction minima at different places and fill in any dip.

Another experimental indicator could be the appearance of a dip associated with a given exchange at different momentum transfer. In backward π^0 photoproduction, it appears that nucleon exchange predominates. A dip is observed[42] at $u=-0.4$, rather than at $u=-0.15$, where it occurs in backward π^+p scattering.

C. Low-Lying Trajectories

One of the more uncomfortable features of conventional Regge-pole phenomenologies was the necessity to include trajectories associated with high-mass particles; these were presumably rather low-lying on a Chew-Frautschi plot, and should give contributions which fall off rapidly with energy. An example is the B meson, included conventionally in $\gamma p\to\pi^0p$ and $\pi N\to\omega N$, $\pi N\to\omega\Delta$. There is no guarantee that $\alpha_B(0)=-\frac{2}{3}$, as would follow from using straight-line trajectories with slopes of order unity, but to assume otherwise is an *ad hoc* procedure that is not very attractive. Such an intercept would give a contribution to a cross section of order s^{-2} relative to a vector meson (ρ,ω), and should presumably not be included.

We conjecture that, in general, our cut contributions will play the role of filling in dips and giving both natural- and unnatural-parity contributions to generate polarization. Preliminary results in ω production[43] and

[42] D. Ritson (private communication).
[43] F. Henyey, K. Kajantie, and G. L. Kane, Phys. Rev. Letters **21**, 1782 (1968).

π^0 photoproduction appear to be consistent with our conjecture. It appears at present that it will be possible to understand most reactions with only the highest-lying trajectories and their associated cuts.

The difference in π^0 photoproduction between our model, involving absorbed ω exchange, and a B-exchange model can be determined by a polarized-photon experiment, which distinguishes natural-parity and unnatural-parity exchanges.[44] If π^0 photoproduction were pure ω exchange, and if the ω coupled to the nucleon without any anomolous magnetic moment coupling, then it turns out that the absorbed amplitude would correspond to pure natural parity. These conditions on the ω should be approximately valid, so we expect a small unnatural-parity part. On the other hand, the B meson has unnatural parity. Therefore, the B-exchange model would predict a large unnatural-parity part. The recent Cambridge Electron Accelerator experiment[45] shows a small unnatural-parity part, strongly favoring our model. In ω production by ρ exchange,[43] on the other hand, the ρ will have a large ratio of anomalous moment to charge couplings, which introduces significant amounts of the unnatural-parity amplitudes and accounts for the large production of longitudinal ω's.

D. Further Discussion

Two areas of research on high-energy two-body reactions present themselves in connection with the present work. (a) Can the prescription be obtained in a more complete theoretical context? Techniques used to determine leading high-energy behavior of series of Feynman diagrams can be applied to find the types of diagrams that are relevant and to predict the form of the leading amplitude. If this first step is made, we may be in a position to determine general rules so that we can calculate, e.g., elastic scattering and $2\to3$ particle reactions. At this writing the first step is well in hand.[46] (b) Does the prescription fit the variety of inelastic two-body reactions now being observed? At present we are calculating $Ps+N\to Ps+N$ and $Ps+N\to V+N$ (in particular, photoprocesses), both forward and backward. One of the encouraging aspects of this phenomenology is that (in the one reaction studied, πN charge exchange) the coupling constants, assuming constant residues in a simple Regge-pole amplitude, agree rather well with the constants inferred from low-energy measurements. Thus we can hope to discuss high-energy experiments without arbitrary scale factors. Successful fitting of data will determine the form of the Regge-pole amplitude including the trajectory. It will also confirm (1) that there is no zero at a nonsense wrong-signature point, (2) that the coherent inelastic factor is greater than 1, and hopefully, (3) that a small set of exchanged Reggeons will suffice to describe all processes with high accuracy at all except lowest energies. It is in these matters that the present theory differs from other cut and conspiracy calculations.

Two important applications which will extend the theory will be to second dip and third maximum (from "triple scattering" or, equivalently, the cut arising from the second region in t of the elastic amplitude) and to double exchange (i.e., the cut due to exchange of two Reggeons rather than Reggeon plus elastic). Further empirical information on the elastic amplitude, on inelastic reactions at high $-t$, and on double-exchange processes are needed.

[44] P. Stickel, Z. Physik **180**, 170 (1964).

[45] D. Bellenger, R. Bordelon, K. Cohen, S. Deutsch, W. Lobar, D. Luckey, L. S. Osborne, E. Polhier, and R. Schwitters, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968).

[46] F. Henyey and C. Risk (private communication).

APPENDIX

Various forms for the absorption correction are given here. In particular, Eqs. (A11) and (A12) are useful for computations.

The entire amplitude for the exchange of a Regge pole and its associated cut is

$$M=M_{\text{pole}}+\lambda\delta M. \tag{A1}$$

δM is the absorption correction, given by

$$(\delta M)_l=-(iq/8\pi W)\times[(M_{\text{pole}})_l(M_{\text{el}})_l+(M_{\text{el}})_l(M_{\text{pole}})_l], \tag{A2a}$$

or, for the case with spin,

$$(\delta M_{\lambda'\mu',\lambda\mu})_j=-(iq/8\pi W)[(M_{\text{pole}\lambda'\mu',\lambda''\mu''})_j(M_{\text{el}\lambda''\mu'',\lambda\mu})_j+(M_{\text{el}\lambda'\mu',\lambda''\mu''})_j(M_{\text{pole}\lambda''\mu'',\lambda\mu})_j]. \tag{A2b}$$

In every equation (b) there is a sum over λ'', μ''. Inserting these expressions into a partial-wave expansion, and using partial-wave projections, we obtain

$$\delta M(z)=\frac{-iq}{32\pi W}\int dz_1dz_2\sum_l(2l+1)P_l(z)P_l(z_1)P_l(z_2)\times[M_{\text{el}}(z_1)M_{\text{pole}}(z_2)+M_{\text{pole}}(z_1)M_{\text{el}}(z_2)], \tag{A3a}$$

$$\delta M_{\lambda'\mu',\lambda\mu}(z)=\frac{-iq}{32\pi W}\int dz_1dz_2\sum_j(2j+1)d_{\lambda'-\mu',\lambda-\mu}{}^J(z)\times d_{\lambda'-\mu',\lambda''-\mu''}{}^J(z_1)d_{\lambda''-\mu'',\lambda-\mu}{}^J(z_2)\times[M_{\text{el}\lambda'\mu',\lambda''\mu''}(z_1)M_{\text{pole}\lambda''\mu'',\lambda\mu}(z_2)+M_{\text{pole}\lambda'\mu',\lambda''\mu''}(z_1)M_{\text{el}\lambda''\mu'',\lambda\mu}(z_2)]. \tag{A3b}$$

The sum over l or j can be explicitly evaluated:

$$\sum_l(2l+1)P_l(z_1)P_l(z_2)P_l(z)=(2/\pi)\theta(\Delta)/\Delta^{1/2}, \tag{A4a}$$

$$\sum_j(2j+1)d_{h_1h_2}{}^j(z)d_{h_1h_3}{}^j(z_1)d_{h_3h_2}{}^j(z_2)=(2/\pi)[\theta(\Delta)/\Delta^{1/2}]\cos(h_1\varphi_1+h_2\varphi_2+h_3\varphi_3), \tag{A4b}$$

where Δ is given in Eq. (16) and φ_1, φ_2, and φ_3 are given in Eqs. (17)–(19). At small angles and high energy these equations can be recast into forms useful for calculation purposes. We make the "impact parameter" replacements

$$l(j)\approx qr-\tfrac{1}{2},\quad \sum_{l=0}^{\infty}\approx\int_0^{\infty}q\,dr,\quad P_l(z)\approx J_0(r\sqrt{-t}),\quad d_{h_1h_2}{}^j(z)\approx J_{h_1-h_2}(r\sqrt{-t}). \tag{A5}$$

We also put $W\approx 2q$. Then we find

$$\delta M(t)\approx\frac{-i}{32\pi q^2}\int_{-\infty}^0\frac{dt'}{2}\int_{-\infty}^0\frac{dt''}{2}\times\int_0^{\infty}rdr\,J_0(r\sqrt{-t})J_0(r\sqrt{-t'})J_0(r\sqrt{-t''})\times[M_{\text{pole}}(t')M_{\text{el}}(t'')+M_{\text{el}}(t')M_{\text{pole}}(t'')], \tag{A6a}$$

$$\delta M_{\lambda'\mu',\lambda\mu}(t)\approx-\frac{i}{32\pi q^2}\int_{-\infty}^0\frac{dt'}{2}\int_{-\infty}^0\frac{dt''}{2}\times\int_0^{\infty}rdr\,J_{\lambda'-\mu'-\lambda+\mu}(r\sqrt{-t})J_{\lambda'-\mu'-\lambda''+\mu''}(r\sqrt{-t'})\times J_{\lambda''-\mu''-\lambda+\mu}(r\sqrt{-t''})\times[M_{\text{pole }\lambda'\mu',\lambda''\mu''}(t')M_{\text{el }\lambda''\mu'',\lambda\mu}(t'')+M_{\text{el }\lambda'\mu',\lambda''\mu''}(t')M_{\text{pole }\lambda''\mu'',\lambda\mu}(t'')]. \tag{A6b}$$

At least in Eq. (A6a), the r integral can be explicitly evaluated:

$$2q^2\int_0^{\infty}rdr\,J_0(r\sqrt{-t})J_0(r\sqrt{-t'})J_0(r\sqrt{-t''})=(2/\pi)\theta(\delta)/\delta^{1/2}, \tag{A7}$$

where

$$\delta=(-t^2-t'^2-t''^2+2tt'+2tt''+2t't'')/4q^4. \tag{A8}$$

The validity of this approximation can be seen by comparing δ with Δ[see Eq. (16)]:

$$\Delta=(-t^2-t'^2-t''^2+2tt'+2tt''+2t't''+tt't''/q^2)/4q^4. \tag{A9}$$

In case elastic scattering is given by an exponential

$$M_{\text{el }\lambda'\mu',\lambda\mu}=-4q^2\sigma_{\text{tot}}(i+\rho)e^{At/2}\delta_{\lambda'\lambda}\delta_{\mu'\mu}, \tag{A10}$$

the expression for the cut can be further simplified. The integrals in Eqs. (A6) involving r and the cosine of the elastic angle can be analytically evaluated. We find

$$\delta M=-(\sigma_{\text{tot}}/4\pi)(1-i\rho)e^{At/2}\times\int_{-\infty}^0\frac{dt'}{2}e^{At'/2}I_0(A\sqrt{(tt')})M_{\text{pole}}(t'), \tag{A11a}$$

$$\delta M_{\lambda'\mu',\lambda\mu}=-(\sigma_{\text{tot}}/4\pi)(1-i\rho)e^{At/2}\times\int_{-\infty}^0\frac{dt'}{2}e^{At'/2}I_n(A\sqrt{(tt')})M_{\text{pole}\lambda'\mu',\lambda\mu}(t'), \tag{A11b}$$

where the helicity flip $n=|\lambda'-\mu'-\lambda+\mu|$. In these expressions, I_n is the Bessel function of imaginary argument $I_n(z)=(-i)^nJ_n(iz)$.

If, furthermore, the pole amplitude can be approximated as

$$M_{\text{pole}\lambda'\mu',\lambda\mu}=(-t)^{n/2}\sum_i a_ie^{B_it/2}$$

[the $(-t)^{n/2}$ is a necessary kinematic factor], then the remaining integral can be analytically evaluated, yielding

$$\delta M=-\frac{\sigma_{\text{tot}}}{4\pi A}(1-i\rho)\sum_i a_i\frac{A}{A+B_i}e^{(AB_it/2)/(A+B_i)}, \tag{A12a}$$

$$\delta M_{\lambda'\mu',\lambda\mu}=-\frac{\sigma_{\text{tot}}}{4\pi A}(1-i\rho)(-t)^{n/2}\times\sum_i a_i\left(\frac{A}{A+B_i}\right)^{n+1}e^{(AB_it/2)/(A+B_i)}. \tag{A12b}$$

For unequal-mass scattering, t, t', and t'' in Eqs. (A6), (A11), and (A12) should be replaced by $t-t_0$, $t'-t_0$, and $t''-t_0$, where t_0 is the value of t for $\theta=0$.