Infinite-Component Field Theories, Fubini Sum Rules, Completeness, and Current Algebra. I. Discrete Spectra*

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It is shown that the Born-approximation scattering amplitudes in a class of infinite-component field theories satisfy Fubini sum rules. The contributions to the sum rules are analyzed, and completeness relations are obtained. These are found to differ radically from the naive expectations. Singularities associated with the vertices give rise to cuts in the scattering amplitudes; the discontinuities contribute to the sum rule and hence to the completeness relations. Such contributions are incompatible with current algebra and with locality of the second-quantized form of the theory. Spacelike solutions, on the other hand, seem to be less relevant than has been feared.

I. INTRODUCTION

LET

$$\mathcal{L}=\int d^4x\,\psi^\dagger(x)L\left(i\frac{\partial}{\partial x}\right)\psi(x) \tag{1.1}$$

be a Lagrangian for a theory of a set of fields ψ_σ, where $\sigma=1,2,\cdots$. The canonical conserved current is of the form

$$J_\mu(x)=\psi^\dagger(x)I_\mu\left(i\frac{\partial}{\partial x}\right)\psi(x)\,. \tag{1.2}$$

The operator $I_\mu(i\partial/\partial x)$, or its Fourier transform $I_\mu(p,q)$, may not be determined uniquely by the Lagrangian in all cases, but

$$I_\mu(p,p)=\partial L(p)/\partial p_\mu\,. \tag{1.3}$$

If ψ_i and ψ_j are two solutions of the field equation, then the orthonormalization condition (for the discrete energy spectrum) is

$$\psi_i^\dagger(\mathbf{p})I_0(E_i,\mathbf{p};E_j,\mathbf{p})\psi_j(\mathbf{p})=\delta_{ij}\,, \tag{1.4}$$

where E_i and E_j are the energy eigenvalues of the two states.

Naively, one expects a completeness relation of the form

$$\sum_i\frac{\psi_i(q)\psi_i^\dagger(q)}{\eta_i(q)}I_0(q,p)=1\,, \tag{1.5}$$

where

$$\eta_i(q)=\psi_i^\dagger(q)I_0(q,q)\psi_i(q)\,, \tag{1.6}$$

to be valid. This would be equivalent to the current algebra[1]

$$J(x)J_0(x')\,|_{x_0=x_0'}=J(x)\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,, \tag{1.7}$$

where $J_0(x)$ is the time component of the canonical conserved current (1.2), and $J(x)$ is the corresponding scalar density $\psi^\dagger(x)\psi(x)$. In practice, there are complications.

It has been shown[2] that, if the number of fields is finite, and if $L(p)$ is a second-order polynomial in p_μ, then the above simplified analysis can be made precise—provided only that the Lagrangian field equations have a sufficient number of independent solutions. (The required number is twice the number of fields.) First-order Lagrangians, and a class of fourth-order theories, were also treated. Here we attempt to extend the results to the case of infinitely many fields. Only field equations with discrete spectra are considered here. Continua will be treated in a subsequent paper.

II. SUMMARY AND CONCLUSIONS

In practice it is easy to evaluate the terms in the sum (1.5) but hopeless to carry out the summation by direct methods. A very convenient technique, which was applied successfully to finite-component theories,[2] is to derive (1.5) from a Fubini sum rule. We study the functions

$$T_\mu(p,p',q)=\psi_{\text{in}}^\dagger(p')\frac{1}{L(q)}I_\mu(q,p)\psi_{\text{fi}}(p)\,, \tag{2.1}$$

where ψ_{in} and ψ_{fi} are two particular solutions of the field equations. We show that T_0 is an analytic function of q_0, and that

$$\lim_{q_0\to\infty}q_0T_0(p,p',q)=\psi_{\text{in}}^\dagger(p')\psi_{\text{fi}}(p)\,. \tag{2.2}$$

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[1] The suggestion that quantities like $J(x)$ and $J_\mu(x)$, constructed by means of infinite unitary representations of noncompact groups, might be related to hadron form factors was first made in connection with relativistic $SU(6)$ theories; see Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965); C. Fronsdal, in *Proceedings of the International Seminar in High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965). The first evaluation of a form factor was carried out by C. Fronsdal, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1966). An application to hadrons was attempted by G. Cocho, C. Fronsdal, H. ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966). It was pointed out by us that problems related to gauge invariance could be solved by relating the currents to infinite-component wave equations [C. Fronsdal, Phys. Rev. **156**, 1653 (1967)], and that such currents naturally lead to models of Gell-Mann's current algebra [C. Fronsdal, *ibid.* **156**, 1665 (1967)].

[2] G. Cocho, C. Fronsdal, and R. White, Phys. Rev. **180**, 1547 (1969).

Consequently, T_0 satisfies a Fubini sum rule,[3]

$$\int dq_0 \,\mathrm{Abs}T_0(p,p',q) = F(p',p). \qquad (2.3)$$

This is equivalent to (1.5) if all the contributions to the integral are related to solutions of the field equations.

We next analyze the singularities of $T_0(q_0)$ and determine the poles and cuts that contribute to (2.3). The results are rather startling. The following summary is intended to show that our conclusions are not strongly model-dependent.

Suppose that there exists a set of real q_μ such that the matrix $L(q)$ has a purely discrete spectrum. (This is usually related to the existence of discrete points in the mass spectrum, whether or not a continuum is present in addition.) Let $\psi^{(\tau)}(q)$, with $\tau=0,1,2,\cdots$, be the eigenvectors of $L(q)$, and $L_\tau(q)$ the eigenvalues. Then[4]

$$T_0(p,p',q) = \sum_\tau V^{(\tau)}(p',q)\frac{1}{L_\tau(q)}V_0^{(\tau)}(q,p), \qquad (2.4)$$

where

$$V^{(\tau)}(p',q) = \psi_{\mathrm{in}}^\dagger(p')\psi^{(\tau)}(q), \qquad (2.5)$$

$$V_\mu^{(\tau)}(q,p) = \psi^{(\tau)\dagger}(q)I_\mu(q,p)\psi_{\mathrm{fi}}(p) \qquad (2.6)$$

are the scalar and vector form factors between the external states and the intermediary states. In any theory of physical interest, the form factors will have a singularity in q_0 for p_μ and $\mathbf{q}$ fixed. Suppose next that there exists a range of values of q_μ for which the sum (2.4) does not converge. [This is usually related to the fact that the spectrum of the matrix $L(q)$ depends on q.] Then a Sommerfeld-Watson transformation may be carried out,

$$T_0 = \tfrac{1}{2}i\int_{-i\infty}^{+i\infty} d\tau\,\csc\pi\tau\; V'^{(\tau)}(p',q)\frac{1}{L_\tau(q)}V_0'^{(\tau)}(q,p). \qquad (2.7)$$

The contour must pass to the right of all singularities of the integrand, with the exception of the poles of $\csc\pi\tau$ at $\tau=0,1,2,\cdots$. The primes on the form factors indicate the usual change of sign of the argument. From (2.7) it may be concluded that:

(a) T_0 has poles for all values of q_0 such that a zero of $L_\tau(q)$ coincides with one of the points $\tau=0,1,2,\cdots$, because the contour gets pinched between a pole of $1/L_\tau(q)$ and a pole of $\csc\pi\tau$. These poles of T_0 are related to the "timelike"[5] solutions of the field equations.

(b) T_0 has no additional poles related to any other zeros of $L_\tau(q)$. This means that T_0 has no poles that are related to "spacelike"[5] solutions of the field equations.[6]

(c) The poles of $1/L_\tau(q)$ give rise to "Regge terms" that can be separated from the integral. The residues, and hence T_0, have branch points at the values of q_0 at which the form factors are singular—even if the form factors have only poles. These branch points may lie on the physical sheet. If they do not, then this fact implies the existence of another branch point on the physical sheet. The associated branch cuts give contributions to the Fubini sum rule, and hence also to the completeness relations.

Conclusion (b) indicates that discrete spacelike solutions may, after all, be no obstacle to constructing reasonable saturations of Gell-Mann current algebras. We may note, in this connection, that "decoupling" of spacelike solutions (noncontribution to completeness) does not imply the vanishing of any current matrix elements.[7]

Conclusion (c) indicates that an exact model saturation of current algebra may not be possible within infinite-component field theories with purely discrete spectra. Equivalently, one may conclude that local field quantization is impossible, which would not be surprising in view of the results of Grodsky and Streater.[8] Probably this remains true in theories with partially continuous mass spectra. Nevertheless, it may be interesting to investigate how closely the ideal theory can be approached. Perhaps the δ function on the right-hand side of Eq. (1.7) has to be replaced by a rapidly decreasing function. This would not be unreasonable, since some infinite-component field theories are known to describe composite systems,[9] and the finite size of such systems may be expected to manifest itself in the form of a small departure from strict locality.

In order to judge the extent to which the extra, unwanted contributions to the completeness relations may be inevitable, it is helpful to explore the physical interpretation. The following discussion may not apply to all cases, but it is accurate in a number of examples.

The infinitely many states of the physical system

[3] V. de Alfero, S. Fubini, C. Rosetti, and G. Furlan, Phys. Rev. Letters **21**, 576 (1966).

[4] Equation (2.1) does not define T_μ completely, since the boundary conditions that must be imposed to select the appropriate Green's function have not been specified. Equation (2.4) contains the proper boundary conditions; they are expressed by the interpretation of $1/L_\tau(q)$ as an analytic function of q^2.

[5] We call these solutions "timelike" because the physical interpretation requires that they have $q^2>0$. This is not automatically true for an *ad hoc* Lagrangian, of course. Similarly, the "spacelike" solutions often have $q^2<0$, but this is not automatic. In the models considered in this paper, "timelike" solutions will be true solutions only if $q^2>0$, and "spacelike" solutions will be true solutions only if $q^2<0$.

[6] This rather surprising result depends, of course, on our choice (see Ref. 4) of boundary conditions for the Green's function. The crucial point is that the contour in (2.7) passes to the right of the zeros of $L_\tau(q)$. If T_0' is the function that is obtained by shifting the contour to the left of the zeros of $L_\tau(q)$, then the difference T_0-T_0' is just the "Regge-pole term"; this is given by (2.7) if the contour is a small closed curve surrounding the zeros of $L_\tau(q)$ only. Clearly this is a solution of the homogeneous equation $L(q)(T_0-T_0')=0$. An interesting discussion of this point has been given by T. Shirafuji, Progr. Theoret. Phys. (Kyoto) **39**, 1047 (1968).

[7] An analogy may be instructive. The Krönecker product of two unitary representations of a noncompact group can be reduced to a direct integral of unitary irreducible representations. The reduction is unique, and the irreducible representations that occur form a complete set. This does *not* imply that the Krönecker product cannot be coupled invariantly to a nonunitary irreducible representation.

[8] I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968).

[9] This has long been stressed by T. Takabayasi; see, for example, the review in Progr. Theoret. Phys. (Kyoto) **34**, 124 (1965).

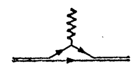

FIG. 1. Physical model of the interaction of a composite system with an external field system.

described by the Lagrangian can be thought of as the excitations of a particle with internal structure. The form factors, and their singularities in particular, give us information about this structure, and in some cases it is possible to interpret the information in considerable detail.[10] Suppose that our physical system consists of two particles interacting with each other and forming bound states and possibly scattering states. An external field quantum may be absorbed or emitted by one of the two constituents; this may be illustrated as in Fig. 1. The triangle diagram, in ordinary local-field theory, has a singularity—the anomalous threshold singularity. This singularity is found in potential theory as well, and it is this singularity that is always present in the form factors of infinite-component field theories.[10] It is *not* associated with any physical state; instead, it is the Landau singularity obtained by putting all three internal lines on their respective mass shells, running around in the direction of the arrows in Fig. 2.

Fubini and Furlan[11] proposed that their sum rule be saturated in the infinite-momentum frame, in order to avoid contributions from the "mass-type" singularity illustrated in Fig. 3(a). In infinite-component field theories, these particular mass-type singularities are absent[10] (as in potential theory); instead, we run afoul of the triangle singularities. These too can often be avoided by a judicious choice of reference frame; in the models studied in this paper, the preferred frame is the center-of-mass one. The triangle singularities do not become irrelevant in the infinite-momentum frame, and nothing is gained by specializing to that frame in the context of field-theoretic models. The reason that Fubini and Furlan[11] do not have trouble with triangle singularities is that they can tuck them away on an unphysical sheet.[12] The trouble with infinite-component field theories with discrete spectra is that the cuts that are supposed to hide the triangle singularity are absent, and the triangle singularity is always on the physical sheet. The missing cuts are illustrated in Fig. 3; one of them corresponds to physical states that lie outside the scope of the theory,[10] the other one is absent in the case of discrete spectra.

We believe that the "missing cuts" are the main obstructions against constructing model current algebras or infinite-component second-quantized local

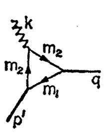

FIG. 2. Illustrating the Landau triangle singularity. When the binding is weak (see Ref. 12), it is possible for the three internal particles to be on the mass shells with the direction of motion indicated by the arrows.

FIG. 3. Diagram (a) illustrates the type of mass-type singularity that contributes to the Fubini sum rule in a finite-momentum frame, but is absent in field-theoretic models. Diagram (b) illustrates the normal branch point that will occur in field-theoretical models with a partly continuous mass spectrum.

fields. In a subsequent paper, we shall consider models with mixed (discrete and continuous) spectra. In such theories the ionization point supplies one of the missing branch points.

III. MODELS

To enumerate the fields, we use a set of N four-vector indices[13]:

$$\psi(x)\to\psi_{A_1\cdots A_N}(x)\,. \tag{3.1}$$

This tensor is supposed symmetric in all the indices but is subject to no other supplementary conditions. It is convenient to think of the field components as a basis for an irreducible representation $\mathfrak{D}(N)$ of the group $SU(3,1)$, with the generators $C_\mu{}^\nu$ defined by

$$C_\mu{}^\nu\psi_{A_1\cdots A_N}=\sum_{r=1}^{N}\delta_{A_r}{}^\nu\psi_{\mu A_1\cdots,\cdots A_N}\,. \tag{3.2}$$

The case of infinitely many fields is reached by analytic continuation in N; the representation $\mathfrak{D}(N)$ is unitary if N is negative real.[13]

Soluble field equations are obtained by taking the Lagrange operator $L(p)$ to be a polynomial $L_\tau(p^2)$ in $p^2\equiv p\cdot p$ and in $p^2\tau$, where τ is defined by

$$p^\mu C_\mu{}^\nu p_\nu=p^2(N-\tau)\,. \tag{3.3}$$

The eigenstates of mass are then the eigenstates of τ, and the mass values are found by solving the algebraic equation $L_\tau(p^2)=0$. It is important that τ always occur in the combination $p^2\tau$; otherwise $L(p)$ is not a differential operator, and one cannot hope to obtain a local-field theory.

The spectrum of τ is[13]

$$\begin{aligned}\tau&=0,1,2,\cdots, &&\text{if } p_\mu \text{ is timelike}\\ &=N,N-1,N-2,\cdots, &&\text{if } p_\mu \text{ is spacelike.}\end{aligned} \tag{3.4}$$

In the case of a non-negative integer N, the upper sequence terminates with the value N, and the lower sequence stops at zero; otherwise both sets are infinite. The reduction of the tensor (3.1) according to eigenvectors of τ may be written

$$\psi_{A_1\cdots A_N}(p)=\sum_{\tau=0}^{\infty}\psi_{A_1\cdots A_N}{}^{(\tau)}(p) \tag{3.5}$$

when p_μ is timelike. The details and the group-theoretical significance of this reduction are discussed in the Appendix and in Ref. 13.

[10] C. Fronsdal, Phys. Rev. **171**, 1811 (1968).
[11] S. Fubini and G. Furlan, Physics **1**, 229 (1964).
[12] According to local-field theory, the triangle amplitude of Fig. 2 has no anomalous singularity in q^2 if $m_2(p'^2-m_1^2-m_2^2)+m_1(k^2-2m_2^2)<0$.
[13] G. Cocho, C. Fronsdal, I. T. Grodsky, and R. White, Phys. Rev. **162**, 1662 (1967).

It is not hard to write field equations that have no spacelike solutions[14]; for example, one may take

$$L(p)=(p^2)^3-(p^\mu C_\mu{}^\nu p_\nu+Ap^2)^2. \qquad (3.6)$$

However, it turns out that spacelike solutions are much less important than anticipated, and we shall concentrate on simpler models.

For detailed investigation, we limit ourselves to the following type of Lagrangian:

$$L(p)=p^\mu C_\mu{}^\nu p_\nu+\alpha p^4+\beta p^2+\gamma, \qquad (3.7)$$

where α, β, and γ are real constants. In diagonal form this is

$$L_\tau(p^2)=-p^2\tau+b, \qquad (3.8)$$

$$b(p^2)=\alpha p^4+(\beta+N)p^2+\gamma. \qquad (3.9)$$

The conserved canonical current is given by (1.2), with[2]

$$I_\mu=\tfrac{1}{2}(C_{\mu\nu}+C_{\nu\mu})(p+q)^\nu+(\alpha p^2+\alpha q^2+\beta)(p+q)_\mu. \qquad (3.10)$$

When $\gamma=0$, this theory has a linearly rising mass spectrum.

IV. FUBINI SUM RULE

We shall write down a scattering amplitude that satisfies a Fubini sum rule when N is a positive integer, and show that it is possible to extend the validity of the sum rule, by analytic continuation in N, to the case of infinite-component fields.

To the Lagrangian (1.1), we add interactions with external scalar and vector fields,

$$\mathfrak{L}_I=\int d^4x[J(x)A(x)+J_\mu(x)A^\mu(x)], \qquad (4.1)$$

where $J(x)$ is the scalar density

$$J(x)=\psi^\dagger(x)\psi(x), \qquad (4.2)$$

and $J_\mu(x)$ is the conserved current defined by Eq. (1.2). The Feynman diagram of Fig. 4 illustrates the scattering amplitudes

$$T(p,p',q)=\psi_{\rm in}{}^\dagger(p')\frac{1}{L(q)}\psi_{\rm fi}(p), \qquad (4.3)$$

$$T_\mu(p,p',q)=\psi_{\rm in}{}^\dagger(p')\frac{1}{L(q)}I_\mu(q,p)\psi_{\rm fi}(p). \qquad (4.4)$$

Both external states will be taken to have $\tau=0$ (the "ground state") and $p^2>0$, $p'^2>0$.

First we calculate T (see the Appendix):

$$T=\frac{F(p,p')}{(-N-1)!z^N}\sum_{\tau=0}^{\infty}\frac{(-N-1+\tau)!}{\tau!L_\tau(q^2)}(1-z)^\tau. \qquad (4.5)$$

[14] The first infinite-component wave equation without spacelike solutions was written down by Y. Nambu, Phys. Rev. **160**, 1171 (1967). Another example may be found in Ref. 10.

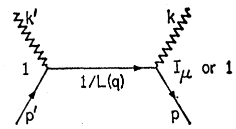

FIG. 4. Feynman diagram for the scattering amplitudes T_μ and T. Every solid line represents the propagation of the states of the infinite system.

Here

$$z=\frac{(p\cdot p')q^2}{(p\cdot q)(p'\cdot q)}, \qquad (4.6)$$

and

$$F(p,p')=\psi^{(0)\dagger}(p')\psi^{(0)}(p)=(p\cdot p'/mm')^N \qquad (4.7)$$

is the scalar form factor of the ground state. The series (4.5) converges when $q^2>0$; a more general representation is obtained by a Sommerfeld-Watson transformation:

$$T=\frac{F(p,p')}{(-N-1)!z^N}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\tau \times\frac{(-N-1+\tau)!(-\tau-1)!}{L_\tau(q^2)}(z-1)^\tau. \qquad (4.8)$$

This is valid in the cut plane

$$|\arg(z-1)|<\pi. \qquad (4.9)$$

The contour must pass to the left of the poles of $(-\tau-1)!$ and to the right of all other singularities of the integrand. When $L_\tau(q^2)$ is of the form (3.8), we have

$$T=\frac{F(p,p')}{b(q^2)z^N}\,{}_2F_1(-N,\,-b/q^2;\,1-b/q^2;\,1-z). \qquad (4.10)$$

The following calculations were carried out for this case only.

Next we derive a useful formula for T_μ in terms of T (see the Appendix):

$$T_\mu=\left[\frac{1}{2}(p+q)^\nu\left(p_\mu\frac{\partial}{\partial p^\nu}+p_\nu\frac{\partial}{\partial p^\mu}\right)+(\alpha p^2+\alpha q^2+\beta)(p+q)_\mu\right]T. \qquad (4.11)$$

This may be written

$$T_\mu=K_\mu T+K_\mu' F, \qquad (4.12)$$

where K_μ and K_μ' are given in the Appendix. As q_0 tends to infinity, K_0 behaves like $b(q^2)/q_0^2$, while (4.5) shows that T is analytic and tends to zero like $1/b(q^2)$. Furthermore, K_0' tends to q_0^{-1}, so that T_0 is dominated by the second term in (4.12), and

$$\lim_{q_0\to\infty}q_0T_0=F(p,p'). \qquad (4.13)$$

Although this result has been verified for Lagrangians of the form (3.7) only, we feel that it has much wider validity.

Let Γ be a closed contour that encircles all the singularities of T. Then it follows from the residue theorem that

$$\frac{1}{2\pi i}\int_{\Gamma} dq_0 T_0(p,p',q) = F(p,p'). \tag{4.14}$$

The integral may be written in terms of the absorptive part of T_0 and is a Fubini sum rule[3] for the scattering amplitude.

Our next job is to determine the poles and the cuts that contribute to the sum rule.

V. CONTRIBUTIONS TO THE SUM RULE. POLES

From (4.8) it is evident that T has poles at those values of q^2 for which $L_\tau(q^2)=0$ for non-negative integer τ,

$$q^2 = m_\tau^2, \quad \tau = 0, 1, 2, \cdots \text{ ("timelike spectrum").} \tag{5.1}$$

This is just the "timelike spectrum" [see Eq. (3.4)]. Of course, the equation $L_\tau(q^2)=0$ will generally have several solutions for q^2 for any given value of τ, but it is not necessary to complicate the notation by distinguishing them. The correspondence between the discrete solutions of the field equations and the poles of T is less than complete. First, all the points (5.1) are poles of T, including those that occur at negative values of q^2. Such anomalous points do not correspond to solutions of the field equations, since the only normalizable solutions for negative q^2 are [compare Eq. (3.4)]

$$q^2 = m_\tau^2, \quad \tau = N, N-1, N-2, \cdots \text{ ("spacelike spectrum").} \tag{5.2}$$

This is a familiar phenomenon; it occurs in the Dirac-Coulomb problem when the coupling constant Ze^2 is too large. Second, none of the points (5.2) are poles of T, whatever the sign of m_τ^2. The poles at (5.1) come from the pinching of the contour in (4.8) between the zeros of the denominator and the poles of $(-\tau-1)!$. Since the contour passes to the right of the zeros of $L_\tau(q^2)$, it cannot be pinched against the poles of $(-N-1+\tau)!$.

In the case of Lagrangians of the type (3.7), it has been demonstrated explicitly[2] that the sum of the residues of T_0 at the points (5.1) actually "saturates" the sum rule when N is a positive integer, provided $\gamma\neq 0$. When $\gamma=0$, the theory is not canonical, i.e., the field equations do not have a complete set of independent solutions. This manifests itself in the appearance of a pole of order $N+1$ at $q^2=0$; the contribution of this multiple pole must be included to saturate the sum rule. The poles at (5.1) also saturate the sum rule when N is noninteger and $\mathbf{q}=0$. In that case the residues of T_0 are

$$\frac{1}{2}\frac{\alpha p^2 q_0^2-\gamma}{\alpha q_0^4-\gamma}\binom{N}{\tau}(z-1)^\tau z^{-N} F. \tag{5.3}$$

Here we have left out terms that are odd in q_0, since these cancel out in the sum. The factor $\frac{1}{2}$ is removed by summing over the two signs of q_0, the fraction disappears when we add the contributions of the two poles with the same τ, and the next three factors sum to unity since z is independent of q_0 when $\mathbf{q}=0$. The fact that the timelike spectrum saturates the sum rule when $\mathbf{q}=0$ is elementary and well known.[15] However, this result also depends on γ being different from zero. Thus it appears that the distinction between canonical and noncanonical theories may be useful in the infinite-component case as well. Obviously we need a new criterion; we suspect that the distinguishing characteristic is the existence of solutions with vanishing norms, and we shall call theories canonical if they have no such solutions.

When N is not a positive integer, and $\mathbf{q}\neq 0$, then T_0 has singularities in addition to the poles of the timelike spectrum, and the real mass intermediary states do not saturate the sum rule. It has frequently been conjectured[16] that the spacelike spectrum supplies the additional residues needed for saturation, but we have seen that that is not always the case. Instead, the missing contribution is associated with branch cuts of T_0.

VI. CONTRIBUTIONS TO THE SUM RULE. CUTS

The representation (4.8) shows that T is a one-valued function of q_0 in the complex plane with the exceptions of the manifold

$$-\infty < z < 0. \tag{6.1}$$

Let $\mathbf{v}=\mathbf{p}/p_0$ and $\mathbf{v}'=\mathbf{p}'/p_0'$, and suppose for definiteness that

$$\mathbf{q}\cdot\mathbf{v} \leq \mathbf{q}\cdot\mathbf{v}'. \tag{6.2}$$

Then the image of (6.1) in the q_0 plane is the set

$$\{-|\mathbf{q}|<q_0<\mathbf{q}\cdot\mathbf{v}\}\cup\{\mathbf{q}\mathbf{v}'<q_0<|\mathbf{q}|\}, \tag{6.3}$$

consisting of two finite segments of the real axis. The discontinuity of T is

$$F\frac{(-z)^{-N}}{(-N-1)!}\int_{-i\infty}^{+i\infty} d\tau \frac{(-\tau-1)!}{(N-\tau)!}\frac{(1-z)^\tau}{L_\tau(q^2)} \tag{6.4}$$

across the right-hand line segment and the negative of this expression across the left segment. The contour of integration in (6.4) passes to the left of the poles of $(-\tau-1)$, but to the right of all other poles of the integrand. When N is a positive integer, the discontinuity vanishes.

In the center-of-mass frame, $\mathbf{q}=0$, the line segments (6.3) reduce to the point $q_0=0$. It was noted above that,

[15] Although many soluble field equations have this type of trivial completeness property in a special "frame," it is not always associated with the center-of-mass system.

[16] The existence of spacelike solutions was noted already by E. Majorana, Nuovo Cimento **9**, 335 (1932). Lack of completeness of the timelike solutions was first noted by E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. **159**, 1222 (1967). Many authors remark that "the spacelike solutions are needed for completeness," but nobody appears to have shown that inclusion of the spacelike solutions makes a complete set.

in the case of noncanonical Lagrangians of the type (3.7) with $\gamma=0$, the poles of the timelike spectrum fail to saturate the sum rule even in the center-of-mass system. We therefore expect to obtain saturation by including the contribution of a pole at the point $q_0=0$. Let us take

$$\alpha=1,\quad \beta+N=-m_0^2. \tag{6.5}$$

Then we get

$$L_\tau(q^2)=q^2(q^2-m_0^2-\tau), \tag{6.6}$$

$$b(q^2)=q^2(q^2-m_0^2), \tag{6.7}$$

and a linearly rising mass spectrum

$$m_\tau^2=m_0^2+\tau. \tag{6.8}$$

From (4.10) we get

$$T=\frac{F(p,p')}{q^2(q^2-m_0^2)z^N}\times{}_2F_1(-N,m_0^2-q^2;1+m_0^2-q^2;1-z). \tag{6.9}$$

Using (4.12), (A11), and (A12), we obtain, if $\mathbf{q}=0$,

$$\lim_{q_0\to 0} q_0T_0=p^2\lim q_0^2T+F$$
$$=F(p,p')[1-z^{-N}\times{}_2F_1(-N,m_0^2;1+m_0^2;1-z)]. \tag{6.10}$$

The sum of the residues (5.3) can easily be evaluated when $\gamma=0$ and $\mathbf{q}=0$, and is found to equal the negative of the second term in (6.10). Thus, as expected, the contribution of the pole at $q_0=0$, added to the contribution of the poles of the timelike spectrum, saturates the sum rule (4.14).

VII. COMPLETENESS RELATIONS

Let us write T_0 in the form

$$T_0=\sum_{\tau=0}^{\infty}\psi^{\dagger(0)}(p')\psi^{(\tau)}(q)\frac{1}{L_\tau(q^2)}\psi^{\dagger(\tau)}(q)\times I_0(q,p)\psi^{(0)}(p). \tag{7.1}$$

Each term has poles at the values of q_0 for which $L_\tau(q^2)$ vanishes, and the sum of the residues of T_0 at all the poles is

$$\sum_{\tau,\alpha}\psi^{\dagger(0)}(p')\frac{\psi^{(\tau)}(q)\psi^{\dagger(\tau)}(q)}{\eta_\tau(q)}I_0(q,p)\psi^{(0)}(p). \tag{7.2}$$

Here p_μ, p_μ', and $\mathbf{q}$ are fixed, while q_0 is to be evaluated at the various poles. Hence $q_0=q_0(\tau,\alpha)$, where the index α distinguishes the several solutions with the same τ. The quantity

$$\eta_\tau(q)=\frac{\partial}{\partial q_0}L_\tau(q^2)\bigg|_{q_0=q_0(\tau,\alpha)}=\psi^{\dagger(\tau)}(q)I_0(q,q)\psi^{(\tau)}(q) \tag{7.3}$$

is the physical probability norm.

The sum rule (4.14) can be written

$$\psi^{\dagger(0)}(p')\left[1-\sum_{\tau,\alpha}\frac{\psi^{(\tau)}(q)\psi^{\dagger(\tau)}(q)}{\eta_\tau(q)}I_0(q,p)\right]\psi^{(0)}(p)=\frac{1}{2\pi i}\int_{\Gamma'}dq_0T_0(p,p',q). \tag{7.4}$$

Here Γ' is a contour that encloses the two cuts, and the sum is over the poles that lie outside Γ'. If N is a positive integer (finite-component case), then the discontinuity (6.4) vanishes and we obtain the sum rule that expresses the completeness of the states of the discrete timelike spectrum. If N is negative (infinite-component case), then the integral gives a finite correction to the sum rule, except in the center-of-mass system, $\mathbf{q}=0$, if $\gamma\neq 0$.

VIII. CURRENT ALGEBRA AND FIELD QUANTIZATION

Consider the operator product

$$J(x)J_0(x)=\sum_p J(x)|n\rangle\langle n|J_0(x). \tag{8.1}$$

What is the "complete set of intermediary states" that is to be used here? The physical interpretation (more precisely, unitarity) requires that all contributions to the sum be associated with real intermediary states. If all the contributions to the Fubini sum rule were due to singularities of the propagator, then they would be attributable to real intermediary states. But this is not the case; the cuts are due to singularities of the vertex functions. Therefore the sum in (8.1) is not the same as the sum in the Fubini sum rule (unless N is a positive integer) and we do not obtain a current algebra. If the sum in (8.1) includes only the discrete states of the timelike spectrum, then

$$\langle p'|J(0)J_0(x)|p\rangle_{x_0=0}=\int d^3q\, e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\psi^{\dagger(0)}(p')\times\sum_{\tau,\alpha}\frac{\psi^{(\tau)}(q)\psi^{\dagger(\tau)}(q)}{\eta_\tau(q)}I_0(q,p)\psi(p)$$
$$=\langle p'|J(0)|p\rangle\delta^{(3)}(\mathbf{x})-\int d^3q\, e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\frac{1}{2\pi i}\int_{\Gamma'}dq_0T_0(p,p',q). \tag{8.2}$$

If the last term were absent, then we could obtain a Gell-Mann current algebra in the usual manner. Unfortunately the last term does not vanish, not even in the case of vanishing momentum transfer, nor in the infinite momentum frame. An exact saturation of current algebra is obtained in the two trivial cases only: when the number of states is finite, and when $\mathbf{q}=0$.

Attempts at canonical field quantization leads to the same difficulty. Quantizing the discrete timelike field modes, one obtains a theory that is unitary but nonlocal. Locality may be achieved by quantizing the field modes that correspond to the cuts, at the expense of unitarity. Clearly, the theory is of the type studied by Coleman and Norton,[17] which is characterized by runaway solutions, or ghosts. Let us emphasize, however, the rather unexpected result that the difficulties are *not* directly connected with the existence of spacelike solutions.

APPENDIX

Here we give some details of the calculations outlined in the text, and begin with the diagonalization of the operator τ defined by Eq. (3.3). In the rest system, $p_\mu=(m,0,0,0)$, Eq. (3.3) reduces to $\tau=C_1{}^1+C_2{}^2+C_3{}^3$. This is an invariant of the compact subgroup $SU(3)$ of $SU(3,1)$. The set of $SU(3)$ representations that occurs is precisely the same as in the ordinary three-dimensional harmonic oscillator, and τ is the total number of excitations,

$$\tau=0,1,2,\cdots,\quad \text{if } p^2>0\,. \tag{A1}$$

If p_μ is spacelike, e.g., $p_\mu=(0,0,0,p_3)$, then Eq. (3.3) reduces to $\tau=C_0{}^0+C_1{}^1+C_2{}^2$. This is an invariant of the noncompact subgroup $SU(2,1)$ of $SU(3,1)$. The spectrum is found by noting that $C_0{}^0+C_1{}^1+C_2{}^2+C_3{}^3$ is an $SU(3,1)$ invariant and has the value N, and that the excitation number $C_3{}^3$ is a non-negative integer; hence

$$\tau=N,N-1,N-2,\cdots,\quad \text{if } p^2<0\,. \tag{A2}$$

More details, including the reduction to the harmonic oscillator in the nonrelativistic limit, may be found in Ref. 13.

The reduction (3.5) is now recognized as a reduction of an $SU(3,1)$ representation according to an $SU(3)$ subgroup. In the rest system, $\psi_{A_1\cdots A_N}{}^{(\tau)}$ has τ spatial excitations (and $N-\tau$ temporal excitations), that is, τ indices have the values 1, 2, or 3 and $N-\tau$ indices have the value 0. More generally, for any timelike p_μ,

$$\psi_{A_1\cdots A_N}{}^{(\tau)}(p)=S\tilde{\psi}_{A_1\cdots A_\tau}{}^{(\tau)}\lambda_{A_{\tau+1}}\cdots\lambda_{A_N}, \tag{A3}$$

where S is a symmetrizer, λ_A is the velocity four-vector of p_μ, and $\tilde{\psi}^{(\tau)}$ is an arbitrary $SU(3)$ tensor, symmetric in all the indices and transverse with respect to λ. The normalization of $\tilde{\psi}^{(\tau)}$ is not of any intrinsic interest, but of course we must use a normalization consistent with (A3) when we derive the completeness relation. The invariant norm is

$$\psi^{\dagger A_1\cdots A_N}(q)\psi_{A_1\cdots A_N}(q)=\sum_\tau \binom{N}{\tau}^{-1}\tilde{\psi}^{\dagger(\tau)A_1\cdots A_\tau}(q)\tilde{\psi}_{A_1\cdots A_\tau}{}^{(\tau)}(q), \tag{A4}$$

and the completeness relation in a fixed-τ subspace is therefore

$$\sum_{\tilde{\psi}}\tilde{\psi}_{A_1\cdots A_\tau}{}^{(\tau)}(q)\tilde{\psi}^{\dagger(\tau)B_1\cdots B_\tau}(q)=\binom{N}{\tau}\tilde{T}_{A_1\cdots A_\tau}{}^{B_1\cdots B_\tau}(q), \tag{A5}$$

where $\tilde{T}(q)$ is the projection operator for symmetric transverse tensors with τ indices. With the help of (A3) it is completely straightforward to evaluate the scalar form factors[13]—e.g., (4.7) for the case $\tau=\tau'=0$.

To evaluate T we use the completeness relation in matrix space,

$$1=\sum_\tau \psi^{(\tau)}(q)\psi^{\dagger(\tau)}(q), \tag{A6}$$

to make a spectral analysis of the propagator,

$$\frac{1}{L(q)}=\sum_\tau \psi^{(\tau)}(q)\frac{1}{L_\tau(q)}\psi^{\dagger(\tau)}(q). \tag{A7}$$

We substitute this into (4.3), evaluate the vertex functions, and use (A5) in the form

$$p'^{A_1}\cdots p'^{A_\tau}\Big[\sum_{\tilde{\psi}}\tilde{\psi}_{A_1\cdots A_\tau}{}^{(\tau)}(q)\tilde{\psi}^{\dagger(\tau)B_1\cdots B_\tau}\Big]p_{B_1}\cdots p_{B_\tau}=\binom{N}{\tau}\big[(p'\cdot p)-(p'\cdot q)(p\cdot q)/q^2\big]^\tau. \tag{A8}$$

The expression for T_μ in terms of T was obtained as follows. From (3.2) and (A3)

$$C_\mu{}^\nu\psi^{(0)}(p)=p_\mu\frac{\partial}{\partial p_\nu}\psi^{(0)}(p). \tag{A9}$$

Equation (4.11) now follows from substitution of (A9) and (3.10) into (4.4).

The "kinematical factors" K_μ and K_μ' in Eq. (4.12) are found by direct evaluation of (4.11). Using the formula

$$\frac{\partial}{\partial z}\big[z^{c-1}\,{}_2F_1(a,b;c;z)\big]=(c-1)z^{c-2}\,{}_2F_1(a,b;c;z), \tag{A10}$$

one finds

$$K_\mu=\frac{N}{2(p\cdot q)}\big[p_\mu(p\cdot q+q^2)+q_\mu(p\cdot q+p^2)\big]+\big[(\alpha p^2+\alpha q^2+\beta)(p_\mu+q_\mu)-bK_\mu'\big], \tag{A11}$$

$$K_\mu'=\frac{1}{2p\cdot q}\left(p_\mu+\frac{p_\mu'(p\cdot q)-q_\mu(p\cdot p')}{(p\cdot q)(p'\cdot q)-q^2(p\cdot p')}(p^2+p\cdot q)\right). \tag{A12}$$

[17] S. Coleman and R. E. Norton, Phys. Rev. **125**, 142 (1962).